The multiple fields created by two perfectly-conducting punches on piezoelectric/piezomagnetic materials with anisotropy

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This article proposes an analytical model of two perfectly conducting punches acting on piezoelectric/piezomagnetic materials with anisotropy. For six types of distinct or repeated roots, related auxiliary function are obtained. The closed-form solutions of the reduced integral equations are derived for two electrically-conducting and magnetically-conducting flat punches. The stress, electric displacement and magnetic induction intensity factors at both edges of each punch are given explicitly in terms of the distance between the two punches and the width of each punch. Numerical test is done to demonstrate the contributions of the distances between the two flat punches and various material properties on the contact behaviors. The obtained results delineate that at the outer edges, there has stronger strength for the singularities of the surface contact stress, surface electric displacement and surface magnetic induction compared with that at the inner edges.

1 Introduction

Most materials exhibit anisotropic behavior [1], either due to crystalline structure or because the material possesses more complex structural composition at the nanoscale. Evolution of anisotropy has a significant impact on material properties [2]. Indentation technique is a powerful tool to study materials properties, which is widely used in the semiconductor, biomedical, and magnetic recording industries [3–6], and etc. Developing indentation technique needs the solution of contact problem of anisotropic materials, which is an important and interesting subject concerned by many researchers. Clements and Ang [7] solved generalized plane contact problems for an anisotropic half-space and layer in which the elastic moduli vary quadratically with the spatial variables. Kuchytsky-Zhyhailo and Rogowski [8] investigated the rigid spherical indenter contacting with an anisotropic linear elastic layer bonded to a rigid substrate. Boffy et al. [9] performed a contact analysis of deformable bodies loaded normally and tangentially against graded layers bonded to heterogeneous substrates. Recently, Barber and Ciavarella [10] proposed an approximate JKR solution for the adhesive contact of anisotropic materials, and found that the dimensionless degree of anisotropy can significantly increase the pull-off force.

The growing application of new and especially anisotropic composite materials in modern engineering gives rise to new challenges in mechanics. For example, because of exhibiting a remarkable cross-coupling between the electric and magnetic ferroic orders [11–13], piezoelectric/piezomagnetic materials have great potentials to be widely used in smart materials/intelligent structures. The effect of anisotropy on the performances of piezoelectric/piezomagnetic materials is of interest. Zhao et al. [14] derived the analytical solutions in an anisotropic piezoelectric/piezomagnetic bi-material to reveal the effect of different non-uniform loadings (dislocations and tractions) on the induced fields. Problems of anisotropic piezoelectric/piezomagnetic materials under contact loading were also concerned. For anisotropic piezoelectric materials, on kind of single phase piezoelectric/piezomagnetic materials, Figueiredo and Stadler [15] proposed an asymptotic model for the equilibrium state of a thin anisotropic piezoelectric plate in frictional contact with a rigid obstacle. Chung [16] constructed the Green’s function for an anisotropic piezoelectric half-space bonded to a thin piezoelectric layer under the action of a generalized line force and a generalized line dislocation. Recently, Zhou and Zhong [17] conducted an...
exact analysis of anisotropic magneto-electro-elastic materials loaded by a frictionally sliding punch. Besides one single punch case mentioned above, there are two punch cases in practical application, such as the double-headed constant temperature hot press machine [18], which has a high efficiency compared with the single-headed constant temperature hot press machine. Then, the interaction of two collinear punches attracts researchers’ attention, especially when smart materials with coupling effects are involved. Wang et al. [19] examined two identical and collinear surface punches on the piezoelectric layer, and found that the relative distance between two flat collinear punches has a significant influence on the punch tip fields. Punch profiles have a significant effect on contact behaviors [20]. Two semi-cylindrical punches acting on anisotropic piezoelectric materials [18] were also concerned. In [18, 19], materials involved are piezoelectric. To the authors’ knowledge, piezoelectric/piezomagnetic materials with magneto-electric coupling subject to two punches have been received few attentions. How two perfectly-conducting punches as well as the anisotropy affect the distributions of various field quantities of piezoelectric/piezomagnetic materials is an interesting issue to be studied.

The present article is devoted to addressing the interaction of two perfectly conducting punches over the surface of anisotropic piezoelectric/piezomagnetic materials. The analysis is organized as follows: In Sect. 2, basic equations and general solutions are presented. In Sect. 3, the stated problem of two perfectly conducting punches is equipped with boundary conditions. In Sects. 4 and 5, integral equations are obtained for general cases, and analytical solutions for two perfectly conducting flat punches are derived. Section 6 reduces the results to a single flat punch, which covers the classic ones. In Sect. 7, Numerical analysis is conducted. Conclusions are drawn in Sect. 8.

## 2 Basic equations and general solutions

There are two rigid punches acting on the surface of anisotropic piezoelectric/piezomagnetic materials (PEPMs). The equilibrium equations with vanishing body forces and free charges are given as follows:

\[ \sigma_{ik,k} = 0, \]
\[ D_{i,i} = 0, \]
\[ B_{i,i} = 0, \]

where the comma stands for the differentiation with respect to the corresponding coordinate variables.

The generalized Hooke’s law for piezoelectric/piezomagnetic materials considering both piezoelectric and piezomagnetic material properties may be written in the following form:

\[ \sigma_{ij} = c_{ij,k}(u_{k,i} + u_{k,i})/2 + \epsilon_{ijkl}\phi_k + h_{kij}\psi_k, \]
\[ D_{i} = \epsilon_{ijk}(u_{k,i} + u_{k,i})/2 - \epsilon_{ijk}\phi_k - d_{ik}\psi_k, \]
\[ B_{i} = h_{ijk}(u_{k,i} + u_{k,i})/2 - d_{ik}\phi_k - u_{ik}\psi_k. \]

In Eqs. (1)–(6), \( u, \phi, \) and \( \psi \) denote the mechanical displacements, electric potential, and magnetic potential; \( \sigma_{ij}, \]
\( D_{i}, \) and \( B_{i} \) stand for the stress components, electric displacements and magnetic inductions; \( c_{ij,k}, \epsilon_{ijkl}, \) and \( \epsilon_{ijk} \) are the elastic coefficients, piezoelectric coefficients, and dielectric coefficients; and \( h_{kij}, d_{ik}, \) and \( \mu_{ij} \) represent the piezomagnetic coefficients, magnetoelastic coefficients, and magnetic permeability coefficients.

From Eqs. (1)–(6), one gets

\[ (c_{i,j,k}u_{j,i} + \epsilon_{i,k,j}\phi + h_{k,i,j}\psi)_{,k_i} = 0, \]
\[ (\epsilon_{i,j,k}u_{j,i} - \epsilon_{i,j,k}\phi - d_{i,j,k}\psi)_{,k_i} = 0, \]
\[ (h_{i,j,k}u_{j,i} - d_{i,j,k}\phi - \mu_{i,j,k}\psi)_{,k_i} = 0. \]

For a two-dimensional case, Eqs. (7)–(9) can be written in the following operator forms:

\[ \Omega U = 0, \]

where \( U = (u \ v \ w \ \phi \ \psi)^T \) with superscript \( T \) denoting the transpose, mechanical displacements \( u_i \) are replaced as \( u, v \) and \( w, \) and operator matrix \( \Omega = (\Omega_{i,j})_{5,5} \) are given in the Appendix. In what follows, \( c_{i,j,k}, \epsilon_{i,k,j}, \) and \( h_{k,i,j} \) are rewritten as \( c_{ik}, \)
\( e_{ki,j}, \) and \( h_{kij} \) for simplicity.

If there is function \( h(x, z) \) such that

\[ |\Omega| h(x, z) = 0, \]

then one may write the general solutions of Eq. (10) as

\[ (u \ v \ w \ \phi \ \psi)^T = (\Delta_{1} \ \Delta_{2} \ \Delta_{3} \ \Delta_{4} \ \Delta_{5})^T h, \]
with \(\Delta_{ij}(i, j = 1, \ldots, 5)\) denoting the cofactors of \(\mathbf{\Omega} = (\Omega_{ij})_{5 \times 5}\). Choosing \(i = 3\) in Eq. (12) results in a set of general solutions

\[
\begin{pmatrix}
    u \\
    v \\
    w \\
    \phi \\
    \psi
\end{pmatrix} = \begin{pmatrix}
    \alpha_{11} \frac{\partial^8}{\partial x^8} + \alpha_{12} \frac{\partial^8}{\partial x^7 \partial z} + \alpha_{13} \frac{\partial^8}{\partial x^6 \partial z^2} + \alpha_{14} \frac{\partial^8}{\partial x^5 \partial z^3} + \alpha_{15} \frac{\partial^8}{\partial x^4 \partial z^4} + \alpha_{16} \frac{\partial^8}{\partial x^3 \partial z^5} + \alpha_{17} \frac{\partial^8}{\partial x^2 \partial z^6} + \alpha_{18} \frac{\partial^8}{\partial x \partial z^7} + \alpha_{19} \frac{\partial^8}{\partial z^8} \\
    \alpha_{21} \frac{\partial^8}{\partial x^8} + \alpha_{22} \frac{\partial^8}{\partial x^7 \partial z} + \alpha_{23} \frac{\partial^8}{\partial x^6 \partial z^2} + \alpha_{24} \frac{\partial^8}{\partial x^5 \partial z^3} + \alpha_{25} \frac{\partial^8}{\partial x^4 \partial z^4} + \alpha_{26} \frac{\partial^8}{\partial x^3 \partial z^5} + \alpha_{27} \frac{\partial^8}{\partial x^2 \partial z^6} + \alpha_{28} \frac{\partial^8}{\partial x \partial z^7} + \alpha_{29} \frac{\partial^8}{\partial z^8} \\
    \alpha_{31} \frac{\partial^8}{\partial x^8} + \alpha_{32} \frac{\partial^8}{\partial x^7 \partial z} + \alpha_{33} \frac{\partial^8}{\partial x^6 \partial z^2} + \alpha_{34} \frac{\partial^8}{\partial x^5 \partial z^3} + \alpha_{35} \frac{\partial^8}{\partial x^4 \partial z^4} + \alpha_{36} \frac{\partial^8}{\partial x^3 \partial z^5} + \alpha_{37} \frac{\partial^8}{\partial x^2 \partial z^6} + \alpha_{38} \frac{\partial^8}{\partial x \partial z^7} + \alpha_{39} \frac{\partial^8}{\partial z^8} \\
    \alpha_{41} \frac{\partial^8}{\partial x^8} + \alpha_{42} \frac{\partial^8}{\partial x^7 \partial z} + \alpha_{43} \frac{\partial^8}{\partial x^6 \partial z^2} + \alpha_{44} \frac{\partial^8}{\partial x^5 \partial z^3} + \alpha_{45} \frac{\partial^8}{\partial x^4 \partial z^4} + \alpha_{46} \frac{\partial^8}{\partial x^3 \partial z^5} + \alpha_{47} \frac{\partial^8}{\partial x^2 \partial z^6} + \alpha_{48} \frac{\partial^8}{\partial x \partial z^7} + \alpha_{49} \frac{\partial^8}{\partial z^8} \\
    \alpha_{51} \frac{\partial^8}{\partial x^8} + \alpha_{52} \frac{\partial^8}{\partial x^7 \partial z} + \alpha_{53} \frac{\partial^8}{\partial x^6 \partial z^2} + \alpha_{54} \frac{\partial^8}{\partial x^5 \partial z^3} + \alpha_{55} \frac{\partial^8}{\partial x^4 \partial z^4} + \alpha_{56} \frac{\partial^8}{\partial x^3 \partial z^5} + \alpha_{57} \frac{\partial^8}{\partial x^2 \partial z^6} + \alpha_{58} \frac{\partial^8}{\partial x \partial z^7} + \alpha_{59} \frac{\partial^8}{\partial z^8}
\end{pmatrix} h(x, z),
\]  

(13)

where \(\alpha_i, (i, j = 1, \ldots, 5)\) are given in the Supplementary Materials.

Performing Fourier transform to Eq. (11) with respect to \(x\), one has

\[
\lambda_0 \frac{\partial^8 H}{\partial x^8} + \lambda_1 r^2 \frac{\partial^8 H}{\partial z^8} + \lambda_2 r^4 \frac{\partial^8 H}{\partial x^4 \partial z^4} + \lambda_3 r^6 \frac{\partial^8 H}{\partial x^2 \partial z^6} + \lambda_4 r^8 \frac{\partial^8 H}{\partial z^8} + \lambda_5 r^{10} H = 0,
\]  

(14)

where \(\lambda_j (j = 0, 1, \ldots, 5)\) are rather lengthy expressions and are omitted here for brevity, and \(H(\tau, z)\) is given as

\[
H(\tau, z) = \int_0^\infty h(x, z) \cos(\tau x) dx.
\]  

(15)

One may obtain the characteristic equation associated with Eq. (14) as follows:

\[
\lambda_0 \varsigma_{10}^8 + \lambda_1 \varsigma_{08}^8 + \lambda_2 \varsigma_{06}^8 + \lambda_3 \varsigma_{04}^8 + \lambda_4 \varsigma_{02}^8 + \lambda_5 = 0,
\]  

(16)

with \(\varsigma\) being the eigenvalue.

Considering the properties of the eigenvalue \(\varsigma\) with the real part, one may express the auxiliary function \(H(\tau, z)\) as follows.

**Case I:** five distinct roots, which can be further detailed as:

**Case IA:** two pairs of conjugate roots and one real root, i.e., \(\varsigma_1 = \theta_1 + i \vartheta_1, \varsigma_2 = \theta_1 - i \vartheta_1, \varsigma_3 = \theta_2 + i \vartheta_2, \varsigma_4 = \theta_2 - i \vartheta_2, \) and \(\varsigma_5 = \rho_5 (\theta_1, \vartheta_1 > 0, n = 1, 2, \rho_5 > 0)\), then

\[
H(\tau, z) = \sum_{n=1}^{2} [\ell_{2n-1} \cos(\theta_n \tau z) + \ell_{2n} \sin(\theta_n \tau z)] e^{\theta_n \tau z} + \ell_{5} e^{\rho_{5} \tau z},
\]  

(17)

where and thereafter unknown functions \(\ell_j (j = 1, \ldots, 5)\) will be determined from the boundary conditions.

**Case IB:** one pair of conjugate roots and three real roots, i.e., \(\varsigma_1 = \theta_1 + i \vartheta_1, \varsigma_2 = \theta_1 - i \vartheta_1, \varsigma_3 = \rho_k (k = 3, 4, 5) (\theta_1, \vartheta_1 > 0, \rho_3, \rho_4, \rho_5 > 0)\), then

\[
H(\tau, z) = [\ell_1 \cos(\theta_1 \tau z) + \ell_2 \sin(\theta_1 \tau z)] e^{\theta_1 \tau z} + \sum_{k=3}^{5} \ell_k e^{\rho_k \tau z}.
\]  

(18)

**Case IC:** five real roots, i.e., \(\varsigma_j = \rho_j > 0 (j = 1, \ldots, 5)\), then

\[
H(\tau, z) = \sum_{j=1}^{5} \ell_j e^{\rho_j \tau z}.
\]  

(19)

**Case II:** only double root, which can be further detailed as:

**Case IIA:** one pair of conjugate roots, one single real root and one double real roots, i.e., \(\varsigma_1 = \theta_1 + i \vartheta_1, \varsigma_2 = \theta_1 - i \vartheta_1, \varsigma_3 = \rho_3 > 0, \) and \(\varsigma_4 = \varsigma_5 = \rho_4 > 0\), then

\[
H(\tau, z) = [\ell_1 \cos(\theta_1 \tau z) + \ell_2 \sin(\theta_1 \tau z)] e^{\theta_1 \tau z} + \sum_{k=3}^{4} \ell_k e^{\rho_k \tau z} + \ell_5 \tau z e^{\rho_5 \tau z}.
\]  

(20)
\(\mathbf{Case II B)}\): three distinct real roots and one double real root, i.e., \(\zeta_n = \rho_n > 0\) \((n = 1, 2, 3)\) and \(\zeta_4 = \zeta_5 = \rho_4 > 0\), then
\[
H(\tau, z) = \sum_{n=1}^{4} \ell_n e^{\rho_n \tau z} + \ell_5 \tau z e^{\rho_4 \tau z}.
\]  
(21)

\(\mathbf{Case II C)}\): one single real root and two double real roots, i.e., \(\zeta_1 = \rho_1 > 0\), \(\zeta_2 = \zeta_3 = \rho_3 > 0\), and \(\zeta_4 = \zeta_5 = \rho_4 > 0\), then
\[
H(\tau, z) = \sum_{n=1}^{2} \ell_n e^{\rho_n \tau z} + \ell_3 \tau z e^{\rho_3 \tau z} + \ell_4 e^{\rho_4 \tau z} + \ell_5 \tau z e^{\rho_5 \tau z}.
\]  
(22)

\(\mathbf{Case III)}\): only one triple root, which can be further detailed as:

\(\mathbf{Case III A)}\): one pair of conjugate roots, and one triple real roots, i.e., \(\zeta_1 = \theta_1 + i \theta_1\), \(\zeta_2 = \theta_1 - i \theta_1\), and \(\zeta_3 = \zeta_4 = \zeta_5 = \rho_3 > 0\), then
\[
H(\tau, z) = [\ell_1 \cos (\theta_1 \tau z) + \ell_2 \sin (\theta_1 \tau z)] e^{\rho_3 \tau z} + \sum_{k=3}^{5} \ell_k (\tau z)^{k-3} e^{\rho_k \tau z}.
\]  
(23)

\(\mathbf{Case III B)}\): two distinct real roots and one triple real roots, i.e., \(\zeta_n = \rho_n > 0\) \((n = 1, 2)\) and \(\zeta_3 = \zeta_4 = \zeta_5 = \rho_3 > 0\), then
\[
H(\tau, z) = \sum_{n=1}^{2} \ell_n e^{\rho_n \tau z} + \ell_3 (\tau z)^3 e^{\rho_3 \tau z}.
\]  
(24)

\(\mathbf{Case IV)}\): one double real root and one triple real root, i.e., \(\zeta_1 = \zeta_2 = \rho_1 > 0\) and \(\zeta_3 = \zeta_4 = \zeta_5 = \rho_3 > 0\), then
\[
H(\tau, z) = \sum_{n=1}^{2} \ell_n (\tau z)^{n-1} e^{\rho_n \tau z} + \sum_{k=3}^{5} \ell_k (\tau z)^{k-3} e^{\rho_k \tau z}.
\]  
(25)

\(\mathbf{Case V)}\): one real root and one quadruple root, i.e., \(\zeta_1 = \rho_1 > 0\) and \(\zeta_2 = \zeta_3 = \zeta_4 = \zeta_5 = \rho_2 > 0\), then
\[
H(\tau, z) = \ell_1 e^{\rho_1 \tau z} + \sum_{k=2}^{5} \ell_k (\tau z)^{k-2} e^{\rho_k \tau z}.
\]  
(26)

\(\mathbf{Case VI)}\): one quintuple root, i.e., \(\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \zeta_5 = \rho_1 > 0\), then
\[
H(\tau, z) = \sum_{j=1}^{5} \ell_j (\tau z)^{j-1} e^{\rho_j \tau z}.
\]  
(27)

The displacements, electric potential, magnetic potential, stresses, electric displacements, and magnetic influxes are expressed on the basis of the auxiliary function \(H(\tau, z)\)

\[
\begin{pmatrix}
  u \\
  v \\
  w \\
  \phi \\
  \psi
\end{pmatrix}
= \frac{2}{\pi} \int_{0}^{\infty} \sum_{j=1}^{5} \ell_j \tau^j \begin{pmatrix}
  T_U^{(j)} (\tau, z) \sin (\tau x) \\
  T_V^{(j)} (\tau, z) \sin (\tau x) \\
  T_W^{(j)} (\tau, z) \cos (\tau x) \\
  T_{\phi}^{(j)} (\tau, z) \cos (\tau x) \\
  T_{\psi}^{(j)} (\tau, z) \cos (\tau x)
\end{pmatrix} d\tau,
\]  
(28)

\[
\begin{pmatrix}
  \sigma_{xx} \\
  \sigma_{xz} \\
  \sigma_{zx} \\
  \sigma_{zz}
\end{pmatrix}
= \frac{2}{\pi} \int_{0}^{\infty} \sum_{j=1}^{5} \ell_j \tau^j \begin{pmatrix}
  T_{XX}^{(j)} (\tau, z) \cos (\tau x) \\
  T_{XZ}^{(j)} (\tau, z) \cos (\tau x) \\
  T_{ZX}^{(j)} (\tau, z) \sin (\tau x) \\
  T_{ZZ}^{(j)} (\tau, z) \sin (\tau x)
\end{pmatrix} d\tau,
\]  
(29)

\[
\begin{pmatrix}
  D_x \\
  D_z
\end{pmatrix}
= \frac{2}{\pi} \int_{0}^{\infty} \sum_{j=1}^{5} \ell_j \tau^j \begin{pmatrix}
  T_{DX}^{(j)} (\tau, z) \sin (\tau x) \\
  T_{DZ}^{(j)} (\tau, z) \cos (\tau x)
\end{pmatrix} d\tau,
\]  
(30)
\[
(B_x, B_z) = \frac{2}{\pi} \int_0^\infty \sum_{j=1}^5 \xi_j \tau^9 \left( T_{Bx}^{(j)} (\tau, z) \sin(\tau x) \right) \left( T_{Bz}^{(j)} (\tau, z) \cos(\tau x) \right) d\tau,
\]
where \( T_{Bm}^{(j)} (\tau, z) (j = 1, \ldots, 5 \ m = U, V, W, \Phi, \Psi, XX, ZZ, XY, YZ, DX, DZ, BX, BZ) \) are given in the Appendix.

3 Modeling of two punches over the surface

Two perfectly conducting (electrically-conducting and magnetically-conducting), symmetrical punches are indenting over the surface of anisotropic piezoelectric/piezomagnetic materials. Let two contact regions be \([-r_2, -r_1]\) and \([r_1, r_2]\). The considered problem will be equipped with boundary conditions in the \(x > 0\) part due to the symmetry.

The regularity conditions fulfill at infinity
\[
u(x, z), v(x, z), w(x, z), \varphi(x, z), \psi(x, z) \to 0, \quad \sqrt{x^2 + z^2} \to \infty.
\]

The punch foundation has the following form:
\[
w(x, 0) = w_f(x), \quad x \in [r_1, r_2],
\]
where \(w_f(x)\) is known a priori.

The surface shear stresses remain free
\[
\sigma_{xz}(x, 0) = 0, \quad \sigma_{yz}(x, 0) = 0.
\]

While the surface normal stress has the following properties:
\[
\sigma_{zz}(x, 0) = -p_1(x), \quad x \in [r_1, r_2],
\]
\[
\sigma_{zz}(x, 0) = 0, \quad x \notin [r_1, r_2],
\]
\[
\int_{-r_2}^{r_1} p_1(x) dx = P_1,
\]
where \(p_1(x)\) is unknown, and \(P_1\) denotes the indentation force.

The constant electric potential and constant magnetic potential within the contact region \(x \in [r_1, r_2]\) lead to
\[
\frac{\partial \phi(x, 0)}{\partial x} = 0, \quad x \in [r_1, r_2],
\]
\[
\frac{\partial \psi(x, 0)}{\partial x} = 0, \quad x \in [r_1, r_2].
\]

The perfectly conducting property makes the electric displacement \(p_2(x)\) and magnetic influx \(p_3(x)\) unknown inside the contact area
\[
D_z(x, 0) = -p_2(x), \quad x \in [r_1, r_2],
\]
\[
B_z(x, 0) = -p_3(x), \quad x \in [r_1, r_2],
\]
\[
D_z(x, 0) = 0, \quad x \notin [r_1, r_2],
\]
\[
B_z(x, 0) = 0, \quad x \notin [r_1, r_2],
\]
\[
\int_{-r_2}^{-r_1} p_2(x) dx = \int_{r_1}^{r_2} p_2(x) dx = P_2,
\]
\[
\int_{-r_2}^{-r_1} p_3(x) dx = \int_{r_1}^{r_2} p_3(x) dx = P_3,
\]
where \(P_2\) and \(P_3\) represent the accumulated electric charge and the accumulated magnetic induction acting on the fully perfectly conducting punch.
4 Integral equations

Differentiating the vertical displacement, electric potential, and magnetic potential on the surface leads to the following integral equations:

\[
\begin{align*}
\left( \frac{\partial w(x, 0)}{\partial x}, \frac{\partial \phi(x, 0)}{\partial x}, \frac{\partial \psi(x, 0)}{\partial x} \right) &= \frac{2}{\pi} \int_0^\infty \int_{r_1}^{r_2} \left( \frac{\Upsilon_{1k}}{\Upsilon_{2k}} \right) \left( \frac{\Upsilon_{2k}}{\Upsilon_{3k}} \right) \cos(tu) \sin(tx) p_k(u) du dt, \\
&= \left( \frac{\partial w(x, 0)}{\partial x}, \frac{\partial \phi(x, 0)}{\partial x}, \frac{\partial \psi(x, 0)}{\partial x} \right),
\end{align*}
\]

where \( p_k(u)(k = 1, 2, 3) \) are defined in Eqs. (36), (41), and (42), and \( \Upsilon_{ik}(i, k = 1, 2, 3) \) are given as

\[
\begin{align*}
\Upsilon_{11} &= \sum_{j=1}^{5} (-1)^{j+i+1} \Upsilon_{ij}^{(C)} \frac{R_{ij}^{(0)}}{R^{(0)}}, \\
\Upsilon_{12} &= \sum_{j=1}^{5} (-1)^{j+i} \Upsilon_{ij}^{(C)} \frac{R_{ij}^{(0)}}{R^{(0)}}, \\
\Upsilon_{13} &= \sum_{j=1}^{5} (-1)^{j+i+1} \Upsilon_{ij}^{(C)} \frac{R_{ij}^{(0)}}{R^{(0)}},
\end{align*}
\]

where

\[
\Upsilon_{ij}^{(C)} = T_{W}^{(j)}(\tau, 0), \quad \Upsilon_{1j}^{(C)} = T_{\Phi}^{(j)}(\tau, 0), \quad \Upsilon_{3j}^{(C)} = T_{\Psi}^{(j)}(\tau, 0),
\]

and \( R_{ij}^{(0)}(t, j = 1, \ldots, 5) \) are the complement minors of the following matrix \( R^{(0)} \):

\[
R^{(0)} = \begin{bmatrix} T_{WZ}^{(j)}(\tau, 0) & T_{XZ}^{(j)}(\tau, 0) & T_{YZ}^{(j)}(\tau, 0) & T_{DZ}^{(j)}(\tau, 0) & T_{BZ}^{(j)}(\tau, 0) \end{bmatrix}_{j=1,\ldots,5}.
\]

One may recast the integral equations in Eq. (47) to singular integral equations as follows:

\[
\frac{1}{\pi} \int_{r_1}^{r_2} \sum_{k=1}^{3} \left( \frac{\Upsilon_{1k}}{\Upsilon_{2k}} \right) \frac{p_k(u)}{u-x} du - \frac{1}{\pi} \int_{r_1}^{r_2} \sum_{k=1}^{3} \left( \frac{\Upsilon_{1k}}{\Upsilon_{3k}} \right) \frac{p_k(u)}{u+x} du = - \left( \frac{\partial w_f(x)}{\partial x}, \frac{\partial \phi(x, 0)}{\partial x}, \frac{\partial \psi(x, 0)}{\partial x} \right).
\]

Analytical solutions of singular integral equations in Eq. (53) can be obtained with considering Eqs. (38), (45), and (46) for some special cases.

5 Analytical solutions of two perfectly conducting flat punches

For two punches with flat foundation and constant electric potential and constant magnetic potential, one has Eqs. (39), (40) and

\[
\frac{\partial w_f(x)}{\partial x} = 0, \quad 0 < r_1 < x < r_2.
\]

The integral equations in Eq. (53) become

\[
\int_{r_1}^{r_2} \sum_{k=1}^{3} \left( \frac{\Upsilon_{1k}}{\Upsilon_{2k}} \right) \frac{2x p_k(u)}{u^2 - x^2} du = 0.
\]

Define the following quantities:

\[
s = \frac{u^2 - r_2}{r_d}, \quad t = \frac{x^2 - r_2}{r_d}, \quad p_k(t) = \frac{p_k(x)}{x} (k = 1, 2, 3),
\]

\[
r_d = \frac{(r_2)^2 - (r_1)^2}{2}, \quad r_c = \frac{(r_2)^2 + (r_1)^2}{2}.
\]
One may recast Eq. (55) and Eqs. (38), (45), and (46) in terms of \( p_k(t) \) \((k = 1, 2, 3)\) as follows:

\[
\int_{-1}^{1} \sum_{k=1}^{3} \left( \frac{Y_{ik}}{s - t} \right) \frac{p_k(s)}{s - t} ds = 0, \quad (58)
\]

\[
\int_{-1}^{1} \hat{p}_k(s) ds = 2P_k/r_d \quad (k = 1, 2, 3). \quad (59)
\]

Further, various surface stresses, surface electric displacements and surface magnetic inductions can be obtained in terms of \( p_k(x) \) or \( p_k(t) \) \((k = 1, 2, 3)\) by using Eqs. (29)–(31)

\[
\sigma_{xx}(x, 0) = \begin{cases} 
\sum_{k=1}^{3} T_{XX}^{(0k)} p_k(x), & r_1 < |x| < r_2 \\
0, & |x| < r_1 \text{ or } |x| > r_2
\end{cases}, \quad (60)
\]

\[
\sigma_{zz}(x, 0) = \begin{cases} 
\sum_{k=1}^{3} T_{ZZ}^{(0k)} p_k(x), & r_1 < |x| < r_2 \\
0, & |x| < r_1 \text{ or } |x| > r_2
\end{cases}, \quad (61)
\]

\[
\sigma_{xz}(x, 0) = \begin{cases} 
-\frac{x}{\pi} \sum_{k=1}^{3} T_{XZ}^{(0k)} \int_{-1}^{1} \frac{1}{s - t} \hat{p}_k(s) ds, & r_1 < |x| < r_2 \\
0, & |x| < r_1 \text{ or } |x| > r_2
\end{cases}, \quad (62)
\]

\[
\sigma_{zx}(x, 0) = \begin{cases} 
-\frac{x}{\pi} \sum_{k=1}^{3} T_{ZX}^{(0k)} \int_{-1}^{1} \frac{1}{s - t} \hat{p}_k(s) ds, & r_1 < |x| < r_2 \\
0, & |x| < r_1 \text{ or } |x| > r_2
\end{cases}, \quad (63)
\]

\[
D_x(x, 0) = \begin{cases} 
\sum_{k=1}^{3} T_{XX}^{(0k)} p_k(x), & r_1 < |x| < r_2 \\
0, & |x| < r_1 \text{ or } |x| > r_2
\end{cases}, \quad (64)
\]

\[
D_z(x, 0) = \begin{cases} 
\sum_{k=1}^{3} T_{ZZ}^{(0k)} p_k(x), & r_1 < |x| < r_2 \\
0, & |x| < r_1 \text{ or } |x| > r_2
\end{cases}, \quad (65)
\]

\[
B_x(x, 0) = \begin{cases} 
-\frac{x}{\pi} \sum_{k=1}^{3} T_{XZ}^{(0k)} \int_{-1}^{1} \frac{1}{s - t} \hat{p}_k(s) ds, & r_1 < |x| < r_2 \\
0, & |x| < r_1 \text{ or } |x| > r_2
\end{cases}, \quad (66)
\]

\[
B_z(x, 0) = \begin{cases} 
-\frac{x}{\pi} \sum_{k=1}^{3} T_{ZX}^{(0k)} \int_{-1}^{1} \frac{1}{s - t} \hat{p}_k(s) ds, & r_1 < |x| < r_2 \\
0, & |x| < r_1 \text{ or } |x| > r_2
\end{cases}, \quad (67)
\]

where \( T_{m}^{(0k)} \) \((k = 1, 2, 3 \ m = XX, ZZ, XZ, YZ, DX, DZ, BX, BZ)\) are given as

\[
T_{m}^{(01)} = \sum_{j=1}^{5} (-1)^j T_{m}^{(j)} (r, 0) R_{1j}^{(0)}, \quad (68)
\]

\[
T_{m}^{(02)} = \sum_{j=1}^{5} (-1)^{j+1} T_{m}^{(j)} (r, 0) R_{sj}^{(0)}. \quad (69)
\]
Solving the normalized integral equations Eqs. (58) and (59), one obtains

\[
\hat{p}_k(t) = \frac{P_k}{r_{ij} \pi \sqrt{1 - t^2}} (k = 1, 2, 3), \quad |t| < 1.
\]  

(71)

Considering Eqs. (56) and (57) leads to:

\[
p_k(x) = \frac{2P_k |x|}{\pi \sqrt{[(r_2)^2 - x^2][x^2 - (r_1)^2]}} (k = 1, 2, 3), \quad r_1 < |x| < r_2.
\]  

(72)

Setting \( P_2 = 0 \) and \( P_3 = 0 \) leads to a fully insulating punch or a mechanical punch. The related integral equations include the first expression of Eq. (55) and Eq. (38), and the solution is \( p_1(x) = \frac{2P_1 |x|}{\pi \sqrt{[(r_2)^2 - x^2][x^2 - (r_1)^2]}}, \) i.e., the first expression of Eq. (72). This solution is just the same as the second expression of Eq. (6.8) given in [21] for a mechanical punch acting on the surface of anisotropic materials.

Consideration of the properties about minors of the matrix \( \mathbf{R}^{(0)} \) in Eq. (52) yields

\[
T_{ZZ}^{(1)} = -1, \quad T_{ZZ}^{(2)} = 0, \quad T_{ZZ}^{(3)} = 0,
\]  

(73)

\[
T_{XZ}^{(k)} = 0 (k = 1, 2, 3),
\]  

(74)

\[
T_{YZ}^{(k)} = 0 (k = 1, 2, 3),
\]  

(75)

\[
T_{DZ}^{(1)} = 0, \quad T_{DZ}^{(2)} = -1, \quad T_{DZ}^{(3)} = 0,
\]  

(76)

\[
T_{BZ}^{(1)} = 0, \quad T_{BZ}^{(2)} = 0, \quad T_{BZ}^{(3)} = -1.
\]  

(77)

One may conclude that the required boundary conditions Eqs. (34)–(46), (41), and (42) hold with Eqs. (61)–(63), (65), and (67) considered.

Checking Eqs. (64) and (66) and considering the formula

\[
\frac{1}{\pi} \int_{-1}^{1} \frac{1}{(s - t) \sqrt{1 - s^2}} ds = 0,
\]  

(78)

one may find that the surface electric displacement and surface magnetic induction in the \( x \) direction under the flat punch always keep free.

Surface in-plane stress \( \sigma_{xx}(x, 0) \) can be determined by considering Eqs. (60) and (72), which plays a key role in causing the surface damage.

The stress, electric displacement and magnetic induction intensity factors at both edges of each punch can be introduced and simplified as follows:

\[
K_I (\pm r_1) = \lim_{x \to \pm r_1} \sqrt{2\pi |x \mp r_1|} p_1(x) = 2P_1 \frac{r_1}{\pi \sqrt{[(r_2)^2 - (r_1)^2]}},
\]  

\[
K_I (\pm r_2) = \lim_{x \to \pm r_2} \sqrt{2\pi |r_2 \mp x|} p_1(x) = 2P_1 \frac{r_2}{\pi \sqrt{[(r_2)^2 - (r_1)^2]}},
\]  

\[
K_D (\pm r_1) = \lim_{x \to \pm r_1} \sqrt{2\pi |x \mp r_1|} p_2(x) = 2P_2 \frac{r_1}{\pi \sqrt{[(r_2)^2 - (r_1)^2]}},
\]  

\[
K_D (\pm r_2) = \lim_{x \to \pm r_2} \sqrt{2\pi |r_2 \mp x|} p_2(x) = 2P_2 \frac{r_2}{\pi \sqrt{[(r_2)^2 - (r_1)^2]}},
\]  

(79)
\[ K_B(\pm r_1) = \lim_{x \to \pm r_1} \sqrt{2\pi |x|} \cdot p_3(x) = 2P_1 \left[ \frac{r_1}{\pi (r_2^2 - r_1^2)} \right]. \] (81)

\[ K_B(\pm r_2) = \lim_{x \to \pm r_2} \sqrt{2\pi (r_2^2 - x)} \cdot p_3(x) = 2P_1 \left[ \frac{r_2}{\pi (r_2^2 - r_1^2)} \right]. \]

Let \( l_d \) and \( l_w \) denote the distance between the two punches and the width of each punch, i.e.,
\[ l_d = r_2 + r_1, \quad l_w = r_2 - r_1. \] (82)

Then, Eqs. (79)–(81) can be rewritten as
\[ K_I(\pm r_1) = P_1 \sqrt{\frac{2(l_d - l_w)}{\pi l_d l_w}}, \quad K_I(\pm r_2) = P_1 \sqrt{\frac{2(l_d + l_w)}{\pi l_d l_w}}, \] (83)
\[ K_D(\pm r_1) = P_2 \sqrt{\frac{2(l_d - l_w)}{\pi l_d l_w}}, \quad K_D(\pm r_2) = P_2 \sqrt{\frac{2(l_d + l_w)}{\pi l_d l_w}}, \] (84)
\[ K_B(\pm r_1) = P_3 \sqrt{\frac{2(l_d - l_w)}{\pi l_d l_w}}, \quad K_B(\pm r_2) = P_3 \sqrt{\frac{2(l_d + l_w)}{\pi l_d l_w}}. \] (85)

Eqs. (83)–(84) clearly demonstrate how the interaction of two punches contributes to the stress, electric displacement and magnetic induction intensity factors through the distance between the two punches \( l_d \). When two punches keep far away, i.e., \( l_d \to \infty \), Eqs. (83)–(85) take the forms
\[ K_I(\pm r_1) = K_I(\pm r_2) = 2P_1 \sqrt{\frac{1}{2\pi l_w}}. \] (86)
\[ K_D(\pm r_1) = K_D(\pm r_2) = 2P_2 \sqrt{\frac{1}{2\pi l_w}}. \] (87)
\[ K_B(\pm r_1) = K_B(\pm r_2) = 2P_3 \sqrt{\frac{1}{2\pi l_w}}. \] (88)

Eqs. (86)–(88) illustrate that the interaction between the two punches become weaker with \( l_d \) becoming larger.

### 6 Results reduced to a single flat punch

If \( r_1 = 0 \), the two perfectly conducting flat punches each with the external loadings \( P_k (k = 1, 2, 3) \) integrates as a single flat punch with contact region \( x \in (-r_2, r_2) \) and external loadings \( 2P_k (k = 1, 2, 3) \), which take the following forms with considering Eqs. (38), (45) and (46):
\[ \int_{-r_2}^{r_2} p_k(x)dx = \int_0^{r_2} p_k(x)dx + \int_0^{r_2} p_k(x)dx = P_k + P_k = 2P_k (k = 1, 2, 3). \] (89)

For a single flat punch with contact region \( x \in (-r_2, r_2) \) and external loadings \( 2P_k (k = 1, 2, 3) \), the surface stresses, surface electric displacement and surface magnetic induction independent of material properties are
\[ p_k(x) = \frac{2P_k}{\pi \sqrt{(r_2^2 - x^2)}} (k = 1, 2, 3). \] (90)

When two perfectly conducting flat punches integrates as a single flat punch, the expressions in Eq. (90) are the same as those for a single flat punch with contact region \( x \in (-r_2, r_2) \) and external loadings \( 2P_k (k = 1, 2, 3) \) given in [22]. Various intensity factors are
\[ K_I(\pm r_2) = 2P_1 \sqrt{\frac{1}{\pi r_2^2}}. \] (91)
\[ K_D (\pm r_2) = 2P_r \sqrt{\frac{1}{\pi r_2}}, \quad (92) \]
\[ K_B (\pm r_2) = 2P_r \sqrt{\frac{1}{\pi r_2}}. \quad (93) \]

Letting \( r_1 = 0 \) in Eq. (53) yields
\[
\frac{1}{\pi} \int_{r_2}^{r_2} 3 \left( \frac{Y_{1k}}{Y_{2k}} - \frac{Y_{3k}}{\pi} \right) p_k(\nu) d\nu = - \left( \frac{\partial \psi (x)}{\partial x} \right),
\]
where the even properties of \( p_k(x) (k = 1, 2, 3) \) in the interval \((-r_2, r_2)\) and the following expressions are employed:
\[
\int_0^{r_2} \frac{1}{\nu + x} p_k(\nu) d\nu = - \int_{-r_2}^0 \frac{1}{\nu - x} p_k(\nu) d\nu \quad (k = 1, 2, 3). \quad (95)\]

Addressing Eqs. (94) and (89) arrives at Eq. (90).

Or more directly, setting \( r_1 = 0 \) in Eqs. (72) and (79)–(81) for two flat punches just also produces the results in Eqs. (90)–(93) for a single flat punch. All these reductions show the correctness of present derivation.

Introduce \( \omega_j (j = 1, \ldots, 5) \) as
\[
\omega_j = 2(-1)^j \frac{R_{ij}^{(0)} P_j - R_{ij}^{(0)} P_c + R_{ij}^{(0)} P_j}{|R^{(0)}|}. \quad (96)\]

Various field quantities of PEPMs under a single conducting flat punch can be given in terms of elementary functions as follows:
\[
\left( \sigma_{xx}, \sigma_{zz}, \sigma_{xz}, \sigma_{xc}, \sigma_{zc}, D_x, D_z, B_c, B_x, B_z \right)^T = \sum_{j=1}^5 \omega_j \left( \Theta_{XX}^{(j)}, \Theta_{ZZ}^{(j)}, \Theta_{XZ}^{(j)}, \Theta_{YX}^{(j)}, \Theta_{DX}^{(j)}, \Theta_{DZ}^{(j)}, \Theta_{BX}^{(j)}, \Theta_{BZ}^{(j)} \right)^T, \quad (97)\]

where \( \Theta_{\alpha}^{(j)} (x, z) (j = 1, \ldots, 5, m = XX, ZZ, XZ, YZ, DX, DZ, BX, BZ) \) are elementary functions and depend on eigenvalue properties. Since anisotropic piezoelectric/piezomagnetic materials usually generate distinct roots, results for Case I, i.e., five distinct roots case, are given below.

**Case I A:** two pairs of conjugate roots and one real root

\[
\Theta_0^{(2n-1)} (x, z) = \frac{1}{2} \left[ I_o^{(c)} (\theta_n, \vartheta_n) A^{(c)} (x, z, \theta_n, \vartheta_n) - I_o^{(s)} (\theta_n, \vartheta_n) A^{(s)} (x, z, \theta_n, \vartheta_n) \right],
\]
\[
\Theta_0^{(2n)} (x, z) = \frac{1}{2} \left[ I_o^{(c)} (\theta_n, \vartheta_n) A^{(c)} (x, z, \theta_n, \vartheta_n) + I_o^{(s)} (\theta_n, \vartheta_n) A^{(s)} (x, z, \theta_n, \vartheta_n) \right], \quad (98)\]
\[
n = 1, 2, \quad \alpha = XX, ZZ, DZ, BZ,
\]
\[
\Theta_0^{(5)} (x, z) = K_0 (\rho_3) \Delta (x, z, \rho_3), \quad \alpha = XX, ZZ, DZ, BZ,
\]
\[
\Theta_l^{(2n-1)} (x, z) = \frac{1}{2} \left[ I_l^{(c)} (\theta_n, \vartheta_n) B^{(c)} (x, z, \theta_n, \vartheta_n) - I_l^{(s)} (\theta_n, \vartheta_n) B^{(s)} (x, z, \theta_n, \vartheta_n) \right],\]
\[
\Theta_l^{(2n)} (x, z) = \frac{1}{2} \left[ I_l^{(c)} (\theta_n, \vartheta_n) B^{(c)} (x, z, \theta_n, \vartheta_n) + I_l^{(s)} (\theta_n, \vartheta_n) B^{(s)} (x, z, \theta_n, \vartheta_n) \right], \quad (100)\]
\[
n = 1, 2, \quad l = XZ, YZ, DX, BX,
\]
\[
\Theta_l^{(5)} (x, z) = K_l (\rho_5) \Lambda (x, z, \rho_5), \quad l = XZ, YZ, DX, BX,
\]

where \( A^{(c)} (x, z, \theta, \vartheta), A^{(s)} (x, z, \theta, \vartheta), \Delta (x, z, \rho), B^{(c)} (x, z, \theta, \vartheta), B^{(s)} (x, z, \theta, \vartheta), \Lambda (x, z, \rho) \) are given as
\[
A^{(c)} (x, z, \theta, \vartheta) = \sum_{n=1}^{\infty} 2M^{(1)} \left( x(n)(\xi), z, \zeta \right), \quad (102)\]
\[
A^{(s)} (x, z, \theta, \vartheta) = \sum_{n=1}^{\infty} M^{(2)} \left( x(n)(\xi), z, \zeta \right). \quad (103)\]
\[
B^{(c)}(x, z, \theta, \vartheta) = M^{(2)}\left((x^{(1)}(\zeta), z, \zeta)ight) - M^{(2)}\left((x^{(2)}(\xi), z, \xi)\right),
\]
\[
B^{(s)}(x, z, \theta, \vartheta) = -M^{(1)}\left((x^{(1)}(\zeta), z, \zeta)\right) + M^{(1)}\left((x^{(2)}(\zeta), z, \zeta)\right),
\]
\[
\Delta(x, z, \rho) = M^{(1)}(x, z, \rho),
\]
\[
\Lambda(x, z, \rho) = M^{(2)}(x, z, \rho),
\]
where
\[
x^{(1)}(\zeta) = \text{Im}(\zeta)z + x, \quad x^{(2)}(\zeta) = \text{Im}(\zeta)z - x,
\]
\[
M^{(1)}(x, z, \zeta) = \frac{\left[\chi_3(x, z, \text{Re}(\zeta))^2 - x^2\right]^\frac{1}{2}}{\chi_2(x, z, \text{Re}(\zeta))^2 - \left[\chi_1(x, z, \text{Re}(\zeta))\right]^2},
\]
\[
M^{(2)}(x, z, \zeta) = \frac{\text{sign}(x)\left[\chi_3(x, z, \text{Re}(\zeta))^2 - x^2\right]^\frac{1}{2}}{\chi_2(x, z, \text{Re}(\zeta))^2 - \left[\chi_1(x, z, \text{Re}(\zeta))\right]^2},
\]
where \(\text{sign}(\cdot)\) stands for the sign function, and
\[
\chi_n(x, z, \rho) = \frac{1}{2} \sqrt{\left(\frac{x + r_2}{2}\right)^2 + (\rho z)^2 + (-1)^n \sqrt{\left(x - r_2\right)^2 + (\rho z)^2}}, \quad n = 1, 2.
\]

**Case I B):** one pair of conjugate roots and three real roots
\[
\Theta_o^{(1)}(x, z) = \frac{1}{2} \left[1^{(c)}(\theta_1, \vartheta_1)A^{(c)}(x, z, \theta_1, \vartheta_1) - I_1^{(s)}(\theta_1, \vartheta_1)A^{(s)}(x, z, \theta_1, \vartheta_1)\right],
\]
\[
\Theta_o^{(2)}(x, z) = \frac{1}{2} \left[1^{(s)}(\theta_1, \vartheta_1)A^{(c)}(x, z, \theta_1, \vartheta_1) + I_2^{(c)}(\theta_1, \vartheta_1)A^{(s)}(x, z, \theta_1, \vartheta_1)\right],
\]
\[
o = XX, ZZ, DZ, BZ,
\]
\[
\Theta_o^{(k)}(x, z) = K_o(\rho_k)\Delta(x, z, \rho_k), \quad k = 3, 4, 5, \quad o = XX, ZZ, DZ, BZ,
\]
\[
\Theta_i^{(l)}(x, z) = \frac{1}{2} \left[1^{(c)}(\theta_1, \vartheta_1)B^{(c)}(x, z, \theta_1, \vartheta_1) - I_3^{(s)}(\theta_1, \vartheta_1)B^{(s)}(x, z, \theta_1, \vartheta_1)\right],
\]
\[
\Theta_i^{(l)}(x, z) = \frac{1}{2} \left[1^{(s)}(\theta_1, \vartheta_1)B^{(c)}(x, z, \theta_1, \vartheta_1) + I_4^{(c)}(\theta_1, \vartheta_1)B^{(s)}(x, z, \theta_1, \vartheta_1)\right],
\]
\[
l = XZ, YZ, DX, BX,
\]
\[
\Theta_i^{(k)}(x, z) = K_i(\rho_k)\Lambda(x, z, \rho_k), \quad k = 3, 4, 5, \quad l = XZ, YZ, DX, BX.
\]

**Case I C):** five real roots
\[
\Theta_o^{(j)}(x, z) = K_o(\rho_j)\Delta(x, z, \rho_j), \quad j = 1, \ldots, 5 \quad o = XX, ZZ, DZ, BZ,
\]
\[
\Theta_i^{(j)}(x, z) = K_i(\rho_j)\Lambda(x, z, \rho_j), \quad j = 1, \ldots, 5 \quad l = XZ, YZ, DX, BX.
\]

In Eqs. (112)–(117), \(A^{(c)}(x, z, \theta, \vartheta), A^{(s)}(x, z, \theta, \vartheta), \Delta(x, z, \rho), B^{(c)}(x, z, \theta, \vartheta), B^{(s)}(x, z, \theta, \vartheta),\) and \(\Lambda(x, z, \rho)\) are defined in Eqs. (102)–(107). In Eqs. (98)–(101) and (112)–(117), \(I_m^{(c)}(\theta, \vartheta), I_m^{(s)}(\theta, \vartheta),\) and \(K_m(\rho)\) \((m = \Psi, XX, ZZ, XZ, YZ, DX, DZ, BX, BZ)\) are given in the Supplementary Materials.

Various surface stresses, surface electric displacements, and surface magnetic inductions for \(r_1 = 0\) given in Eqs. (60)–(67) equals those given in Eq. (97) when \(z = 0.\)
7 Numerical results

The piezoelectric and piezomagnetic coefficients used in numerical computation are given as follows:

\[
\begin{align*}
\varepsilon_{11} &= 214 \text{ GPa}, \quad \varepsilon_{13} = 115 \text{ GPa}, \quad \varepsilon_{16} = 214 \text{ GPa}, \quad \varepsilon_{33} = 205 \text{ GPa}, \quad \varepsilon_{36} = 115 \text{ GPa}, \\
\varepsilon_{44} &= 43.92 \text{ GPa}, \quad \varepsilon_{45} = 115.4 \text{ GPa}, \quad \varepsilon_{55} = 43.92 \text{ GPa}, \quad \varepsilon_{66} = 49.3 \text{ GPa}, \\
\varepsilon_{31} &= -2.64 \text{ Cm}^{-2}, \quad \varepsilon_{33} = 11.16 \text{ Cm}^{-2}, \quad \varepsilon_{36} = -2.64 \text{ Cm}^{-2}, \\
\mu_{33} &= 6.96 \text{ Cm}^{-2}, \quad \varepsilon_{15} = 6.96 \text{ Cm}^{-2}, \\
h_{31} &= 232.12 \text{ NAm}^{-1}, \quad h_{33} = 279.88 \text{ NAm}^{-1}, \quad h_{36} = 232.12 \text{ NAm}^{-1}, \\
h_{14} &= 220 \text{ NA}^{-1}m^{-1}, \quad h_{15} = 220 \text{ NA}^{-1}m^{-1}, \\
\varepsilon_{11} &= 6.752 \times 10^{-9} \text{C}^2\text{N}^{-1}m^{-2}, \quad \varepsilon_{33} = 7.5972 \times 10^{-9} \text{C}^2\text{N}^{-1}m^{-2}, \\
\mu_{11} &= 239 \times 10^{-6} \text{Ns}^2\text{C}^{-2}, \quad \mu_{33} = 68.8 \times 10^{-6} \text{Ns}^2\text{C}^{-2}.
\end{align*}
\]

(118)

Fig. 1 shows the surface contact stress, surface electric displacement and surface magnetic induction, \( p_k(x) / \sigma_{k,0}(\sigma_{1,0} = P_1 / \pi, k = 1, 2, 3) \), under two flat punches with different distances between them.

Fig. 1 shows the surface contact stress, surface electric displacement and surface magnetic induction, \( p_k(x) / \sigma_{k,0}(\sigma_{1,0} = P_1 / \pi, k = 1, 2, 3) \), under two flat punches with different distances \( l_d = r_2 + r_1 \). Near each edge, there exist serious contact stress, electric displacement, and magnetic induction singularities under two perfectly conducting flat punches. The fact that the stronger strength of the singularities prevails at the outer edges than at the inner edges is reflected in Fig. 1. As the two perfectly conducting flat punches near, the strength of the singularities at the inner edges become attenuated and vanishes when two flat punches integrates as one single flat punch, while the opposite trend is found at the outer edges. Note that the solid line in Fig. 1 can be plotted either from Eqs. (61), (65), and (67) or from Eq. (90) or from Eq. (97), which validates the present derivation and programming.

To measure the criticality of the singular nature of the surface contact stress, surface electric displacement and surface magnetic induction inherent in perfectly conducting flat punches, the respective intensity factors are provided in Fig. 2. It is demonstrated that the intensity factors at the outer edges decrease, while those at the inner edges increase as the distances between the two flat punches \( l_d \) increases, which agrees with those seen in Fig. 1. Moreover, Fig. 2 delineates that a wider contact region tends to alleviate the severity of the surface contact stress, surface electric displacement, and surface magnetic induction at the edges.

The surface in-plane stress \( \sigma_{1,0}(x, 0) / |\sigma_{1,0}(\sigma_{1,0} = P_1 / \pi) \) under two collinear flat punches as illustrated in Fig. 3 clearly depicts that its magnitude is also unbounded at each edge of the two punches. The effect of the distances between the two flat punches on the surface in-plane stress is the same as that on the surface contact stress, surface electric displacement, and surface magnetic induction as shown in Fig. 1.

Checked in Fig. 4 is how the material property ratios affect the surface in-plane stress under two perfectly conducting flat punches. To this end, one of the two piezoelectric/piezomagnetic coefficient ratios \( (\varepsilon_{150}/\varepsilon_{15}, h_{150}/h_{15}) \) is taken to be variable and other coefficients keep the same as those given in Eq. (118). It is obvious from Fig. 4 a) that the increase in the piezoelectric coefficient ratio \( \varepsilon_{150}/\varepsilon_{15} \) relieves the concentration strength in the vicinities of all edges. From Fig. 4 b), the surface in-plane stress appears to be quite insensitive to the variations of the piezomagnetic coefficient ratio when \( h_{150}/h_{15} \) in no larger than 7.5.
Fig. 2  Intensity factors of the surface contact stress, surface electric displacement and surface magnetic induction at: a) the inner edges \( x = \pm r_1 \) and b) the outer edges \( x = \pm r_2 \).

Fig. 3  The surface in-plane stress \( \sigma_{xx}(x,0)/\sigma_{1,0} \) \((\sigma_{1,0} = P_1/\pi)\) under two collinear flat punches with different distances between them with \(|P_1| = 10^5|P_2| = 10^5|P_3|\).

Fig. 4  The influences of material property ratios: a) \( \varepsilon_{150}/\varepsilon_{15} \) and b) \( h_{150}/h_{15} \) on the surface in-plane stress \( \sigma_{xx}(x,0)/\sigma_{1,0} \) \((\sigma_{1,0} = P_1/\pi)\) under two collinear flat punches with \(|P_1| = 10^5|P_2| = 10^5|P_3|\).

8 Conclusions

To gain an insight of the interaction of multiple punches on the contact behaviors of advanced materials, an exact analysis of two perfectly conducting punches acting on anisotropic piezoelectric/piezomagnetic materials is performed in this article. Six kinds of expressions of the auxiliary function are given for distinct or repeated roots. For two perfectly conducting flat punches, the closed-form solutions of the reduced integral equations are presented. Numerical results are carried out to show the influences of the distances between the two flat punches and various material properties on the physical objectives.
The results reveal that the stronger strength of the singularities of the surface contact stress, surface electric displacement and surface magnetic induction prevails at the outer edges than at the inner edges.

A Appendix

1. Expressions of \( \Omega = (\Omega_{ij})_{5 \times 5} \) appearing in Eq. (10)

\[
\Omega_{ij} = \Omega_{ji} (i, j = 1, \ldots, 5),
\]

\[
\Omega_{11} = c_{11} \frac{\partial^2}{\partial x^2} + c_{55} \frac{\partial^2}{\partial z^2}, \quad \Omega_{12} = c_{16} \frac{\partial^2}{\partial x^2} + c_{45} \frac{\partial^2}{\partial z^2}, \quad \Omega_{1k} = \beta_{k-2} \frac{\partial^2}{\partial x \partial z} (k = 3, 4, 5),
\]

\[
\Omega_{22} = c_{60} \frac{\partial^2}{\partial x^2} + c_{44} \frac{\partial^2}{\partial z^2}, \quad \Omega_{2k} = \beta_{k+1} \frac{\partial^2}{\partial x \partial z} (k = 3, 4, 5),
\]

\[
\Omega_{33} = c_{59} \frac{\partial^2}{\partial x^2} + c_{33} \frac{\partial^2}{\partial z^2}, \quad \Omega_{34} = e_{15} \frac{\partial^2}{\partial x^2} + e_{33} \frac{\partial^2}{\partial z^2}, \quad \Omega_{35} = h_{15} \frac{\partial^2}{\partial x^2} + h_{33} \frac{\partial^2}{\partial z^2},
\]

\[
\Omega_{44} = -e_{11} \frac{\partial^2}{\partial x^2} - e_{33} \frac{\partial^2}{\partial z^2}, \quad \Omega_{45} = -d_{11} \frac{\partial^2}{\partial x^2} - d_{33} \frac{\partial^2}{\partial z^2},
\]

\[
\Omega_{55} = -\mu_{11} \frac{\partial^2}{\partial x^2} - \mu_{33} \frac{\partial^2}{\partial z^2},
\]

where

\[
\beta_1 = c_{13} + c_{35}, \quad \beta_2 = e_{31} + e_{15}, \quad \beta_3 = h_{31} + h_{15},
\]

\[
\beta_4 = c_{36} + c_{45}, \quad \beta_5 = e_{36} + e_{45}, \quad \beta_6 = h_{36} + h_{45}.
\]

2. Expressions of \( T_{m}^{(j)} (\tau, z) (j = 1, \ldots, 5 \ m = U, V, W, \Phi, \Psi, XX, ZZ, XZ, YZ, DX, DZ, BX, BZ) \) appearing in Eqs. (28)–(31)

Here, results for Case I, i.e., five distinct roots case, are presented (The other cases can be obtained using a similar method and are omitted in the present paper).

Case I A: two pairs of conjugate roots and one real root

\[
T_{m}^{(2k-1)} (\tau, z) = \left[ T_{m}^{(1)} (\theta_1, \vartheta_1) \cos (\theta_1 \tau z) - T_{m}^{(1)} (\vartheta_1, \theta_1) \sin (\theta_1 \tau z) \right] e^{\theta_1 \tau z}, \quad k = 1, 2,
\]

\[
T_{m}^{(2k)} (\tau, z) = \left[ T_{m}^{(1)} (\theta_1, \vartheta_1) \cos (\theta_1 \tau z) + T_{m}^{(1)} (\vartheta_1, \theta_1) \sin (\theta_1 \tau z) \right] e^{\theta_1 \tau z}, \quad k = 1, 2,
\]

\[
T_{m}^{(5)} (\tau, z) = K_{m} (\rho_5) e^{\rho_5 \tau z}.
\]

Case I B: one pair of conjugate roots and three real roots

\[
T_{m}^{(1)} (\tau, z) = \left[ I_{m}^{(1)} (\theta_1, \vartheta_1) \cos (\theta_1 \tau z) - I_{m}^{(1)} (\vartheta_1, \theta_1) \sin (\theta_1 \tau z) \right] e^{\theta_1 \tau z},
\]

\[
T_{m}^{(2)} (\tau, z) = \left[ I_{m}^{(1)} (\theta_1, \vartheta_1) \cos (\theta_1 \tau z) + I_{m}^{(1)} (\vartheta_1, \theta_1) \sin (\theta_1 \tau z) \right] e^{\theta_1 \tau z},
\]

\[
T_{m}^{(k)} (\tau, z) = K_{m} (\rho_k) e^{\rho_k \tau z}, \quad k = 3, 4, 5.
\]

Case I C: five real roots

\[
T_{m}^{(k)} (\tau, z) = K_{m} (\rho_k) e^{\rho_k \tau z}, \quad k = 1, \ldots, 5.
\]

In Eqs. (A.9)–(A.11), \( I_{m}^{(1)} (\theta, \vartheta), \ I_{m}^{(1)} (\vartheta, \theta), \) and \( K_{m} (\rho) \ (m = U, V, W, \Phi, \Psi, XX, ZZ, XZ, YZ, DX, DZ, BX, BZ) \) are given in the Supplementary Materials.

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References


