THERMAL STRESS INTENSITY FACTORS FOR AN INTERFACIAL CRACK ON A CUSP-TYPE INCLUSION

KANG YONG LEE and YONG HOON JANG
Department of Mechanical Engineering, Yonsei University, Seoul 120–749, Korea

Abstract—Under a uniform heat flow, thermal stress intensity factors for an interfacial crack on a rigid cusp-type inclusion are determined by Hilbert formulation. The thermal stress intensity factors are expressed as functions of the heat flow angle. The variations of interfacial thermal stress intensity factors for the interfacial crack and the cusp crack with respect to the locations of the interfacial crack are illustrated. In particular, when the tip of the interfacial crack approaches that of the cusp crack, the corresponding thermal stress intensity factors are observed to experience abrupt changes in their magnitudes. Furthermore, the complex potential functions and the thermal stress intensity factors for the cusp-type inclusion without the interfacial crack are derived with the cusp surface boundary conditions insulated or fixed to zero relative temperature.

INTRODUCTION

In the analysis of the cusp crack under a uniform heat flow, Lee and Choi [1, 2] determined the thermal stress intensity factors (TSIFs) for the cusp cracks insulated or fixed to zero relative temperature with traction-free or rigid cusp crack surfaces. Lee and Cho [3] also obtained the numerical solutions of TSIFs for traction-free cusp cracks in a finite body with the aforementioned thermal boundary conditions.

In studies on the interfacial crack along the inclusion boundary, England [4] obtained the stress field in the vicinity of the interfacial crack tip on an elastic circular inclusion under the mechanical loading from the Hilbert problem. Toya [5, 6] obtained the crack growth criteria for elastic circular and rigid elliptical inclusions partially bonded to an elastic plate subjected to uniform biaxial loads applied to infinity. By solving the Hilbert problem, Tamate and Yamada [7] obtained the stress intensity factor (SIF) of mode III for a partially bonded elastic circular inclusion under an anti-plane shear stress at infinity. For two symmetric cracks on a rigid elliptic inclusion embedded in an infinite matrix under in-plane biaxial loading, Viola and Piva [8] found the SIFs and fracture criterion. Berezhntskii and Sakhnenko [9] obtained the SIFs for the interface crack on a rigid hypocycloid inclusion subjected to mode III mechanical loading at infinity. Sendeckyj [10–12] found the SIFs for the rigid elliptic and hypotrochoid inclusions partially bonded to an elastic infinite body under various mechanical loading conditions and showed that the SIFs have singularity phenomena in the case where the tip of the interface crack meets that of the cusp crack. However, studies on the TSIFs for the interfacial crack along the inclusion boundary seem to be relatively rare.

In this paper, the complex potentials and TSIFs for the cusp crack and the interfacial crack on a symmetric lip or an airfoil-type rigid inclusion insulated or fixed to zero relative temperature are obtained from the Hilbert formulation.

THEORY

Determination of the complex potentials

As shown in Fig. 1, the region $S^+$ in the $z$ plane for the inclusion with two cusps is mapped into the exterior region $S^+$ of the unit circle $|\zeta| = 1$ by the following conformal mapping function $W(\zeta)$ [13]:

$$z = W(\zeta) = R \left( \zeta + \sum_{m=1}^{n=3} C_m \zeta^{-m} \right), \quad (1)$$

where $R$ and $C_m$ ($m = 1, 2, 3$) are the complex constants.
The tips \( t_1 = r_1 \exp(i\theta_1') \) and \( t_2 = r_2 \exp(i\theta_2') \) of the interfacial crack part \( \Gamma_\delta^c \) and the bonded part \( \Gamma_\beta^c \) in the \( \zeta \) plane are mapped into \( \sigma_1 = \exp(i\theta_1), \sigma_2 = \exp(i\theta_2), \Gamma_\delta^c \) and \( \Gamma_\beta^c \), respectively.

The stress components \( \sigma_{\zeta \zeta} \) and \( \sigma_{\zeta \eta} \) of the elliptical coordinates \( (\zeta, \eta) \) and the displacement components \( u \) and \( v \) of the Cartesian coordinates \( (x, y) \) are expressed in terms of the two complex potentials \( \Phi(\zeta), \Psi(\zeta) \) holomorphic in \( S^+ \) in the forms [14]

\[
\sigma_{\zeta \zeta} + i\sigma_{\zeta \eta} = \Phi(\zeta) + \bar{\Phi}(\zeta) - \frac{\zeta W'(\zeta)}{W'(\zeta) - \bar{W}'(\zeta)} \Phi'(\zeta) - \frac{\bar{\zeta} W'(\bar{\zeta})}{\bar{W}'(\zeta) - \bar{W}'(\zeta)} \Psi(\zeta) \tag{2}
\]

\[
2\mu \frac{\partial(u + iv)}{\partial \eta} = i\zeta W'(\zeta) [\kappa \Phi'(\zeta) + \bar{\Phi}(\zeta)] + \bar{i}\zeta [W'(\zeta) \Phi'(\zeta) + \bar{W}'(\zeta) \bar{\Phi}(\zeta)] + 2i\mu \zeta G(\zeta) W'(\zeta), \tag{3}
\]

where

\[
G(\zeta) = \beta \chi(\zeta) \tag{4}
\]

\[
\beta = \left\{ \begin{array}{ll}
(1 + \nu)\alpha & \text{(plane strain state)} \\
\chi & \text{(plane stress state)}
\end{array} \right.
\]

\[\kappa = \begin{cases}
3 - 4\nu & \text{(plane strain state)} \\
3 - \nu & \text{(plane stress state)}
\end{cases}
\]

\(G(\zeta)\) is defined in \( S^+\) and \( \chi(\zeta)\) is the complex temperature potential in the \( \zeta \) plane. \( \alpha, \nu \) and \( \mu \) are the coefficients of linear thermal expansion, Poisson's ratio and shear modulus, respectively.

By the stress continuation method [14], the definition of \( \Phi(\zeta) \) is extended into the interior \( S^- \) as follows:

\[
W'(\zeta) \Phi(\zeta) = -W'(\zeta) \Phi(1/\zeta) + \frac{W'(\zeta) \bar{\Phi}'(1/\zeta)}{\zeta^2} + \frac{\bar{W}'(1/\zeta) \bar{\Phi}(1/\zeta)}{\zeta^2} \quad (\zeta \in S^-). \tag{5}
\]

From eq. (5), the following equation is obtained:

\[
W'(\zeta) \Psi(\zeta) = -\bar{W}'(1/\zeta) \Phi(\zeta) + \bar{W}'(1/\zeta) \Phi'(\zeta) + \frac{\bar{W}'(1/\zeta) \bar{\Phi}'(1/\zeta)}{\zeta^2} \quad (\zeta \in S^+). \tag{6}
\]

Substituting eq. (6) into eqs (2) and (3), we obtain the following stress and displacement field equations of \( \Phi_1(\zeta) \) in the case where \( \zeta \) approaches a point \( \sigma = \exp(i\theta) \) of the circle \( \Gamma \) from \( S^+ \):

\[
\sigma W'(\sigma) [\sigma_{\zeta \zeta} + i\sigma_{\zeta \eta}] = \sigma [\Phi_+^+(\sigma) - \Phi_-^-(\sigma)]
\]

\[
2\mu \frac{\partial(u + iv)}{\partial \sigma} = \kappa \Phi_+^+(\sigma) + \Phi_-^-(\sigma) + 2\mu G(\sigma) W'(\sigma), \tag{7}
\]

where

\[
\Phi_1(\sigma) = W'(\sigma) \Phi(\sigma) \tag{8}
\]

\[
G(\sigma) W'(\sigma) = [G(\sigma) W'(\sigma)]^+. \tag{9}
\]
Provided that the boundary surface of a rigid cusp-type inclusion is maintained either at the insulated state or fixed to zero relative temperature state and that the uniform heat flow with uniform temperature gradient $\tau$ in the $z$ plane is directed at an angle of $\varphi$ to the $x$-axis, the complex temperature potential $\chi(\zeta)$ in the $\zeta$ plane is given by an image method [15] in the form

$$\chi(\zeta) = Q \left\{ \lim_{|\zeta| \to \infty} W'(\zeta) \right\} \zeta + \gamma Q \left\{ \lim_{|\zeta| \to \infty} \overline{W'(\zeta)} \right\} \frac{1}{\zeta},$$

(10)

where $Q = \tau \exp(-i\varphi)$ and $\gamma = \pm 1$ in the cases of thermally insulated and zero temperature rigid inclusion, respectively.

Because of the condition of single-valuedness at the rigid inclusion boundary, the complex potentials $\Phi_1(\zeta)$, $\Psi_1(\zeta)$, $\Phi(\zeta)$ and $\Psi(\zeta)$ are expressed at $\zeta \to \infty$ as follows:

$$\Phi_1(\zeta) = \frac{A}{\zeta} + O\left(\frac{1}{\zeta^2}\right), \quad \Psi_1(\zeta) = \frac{B}{\zeta} + O\left(\frac{1}{\zeta^2}\right),$$

$$\Phi(\zeta) = \frac{A}{R_0^2} + O\left(\frac{1}{\zeta^2}\right), \quad \Psi(\zeta) = \frac{B}{R_0^2} + O\left(\frac{1}{\zeta^2}\right),$$

(11)

where

$$\Psi_1(\zeta) = W'(\zeta) \Psi(\zeta).$$

(12)

By using the thermal dislocation concept, Lee and Choi [1, 2] formulated the following relation between the constants $A$ and $B$ of eq. (11):

$$A = B = -\frac{2\mu \beta R^2 (\gamma Q - C_1 Q)}{\kappa + 1}.$$  

(13)

From eq. (7), the traction-free condition on the interfacial crack surface $\Gamma_s$ and the displacement-free condition on the bonded surface $\Gamma_b$, the following nonhomogeneous Hilbert problem is formulated:

$$\Phi^+(\sigma) - \Phi^-(\sigma) = 0 \quad (\sigma \in \Gamma_s)$$

$$\kappa\Phi^+(\sigma) + \Phi^-(\sigma) = -2\mu G(\sigma) W'(\sigma) \quad (\sigma \in \Gamma_b).$$

(14)

(15)

The solution [14] is as follows:

$$\Phi_1(\zeta) = -\frac{\mu X(\zeta)}{\pi \kappa i} \int_{\Gamma_b} \frac{G(\sigma) W'(\sigma)}{X^+(\sigma)} \frac{d\sigma}{\sigma - \zeta} + P(\zeta) X(\zeta),$$

(16)

where

$$X(\zeta) = [(\zeta - \sigma_1)(\zeta - \sigma_2)]^{-1/2}[(\zeta - \sigma_3)/(\zeta - \sigma_1)]^{\lambda}$$

$$\lambda = \ln \kappa / 2\pi$$

$$P(\zeta) = F_1 \zeta + F_2 + \frac{F_3}{\zeta} + \frac{F_4}{\zeta^2} + \frac{F_5}{\zeta^3} + \frac{F_6}{\zeta^4}.$$  

(17)

The positive direction of the contour integration is clockwise. The Plemelj function $X(\zeta)$ is holomorphic in the whole $\zeta$ plane cut along the arc $\Gamma_b$ on which $X^+(\sigma) = -X^-(\sigma)/\kappa$ and satisfies the condition

$$\lim_{\zeta \to \infty} [\zeta X(\zeta)] = 1$$

at the branch of the function $X(\zeta)$. $F_i (i = 1, \ldots, 6)$ are the constants to be determined. In order that $\Phi_1(\zeta)$ is to be holomorphic, $P(\zeta)$ should be the order of $O(\zeta)$ at $\zeta \to \infty$ since the order of $X(\zeta)$ is $O(1/\zeta)$ and be the order of $O(1/\zeta^4)$ at $\zeta \to 0$ since the order of $W'(\zeta)$ is $O(1/\zeta^4)$ [15].

The integration of eq. (16) gives the following results:

$$\Phi_1(\zeta) = [H(\zeta) + P(\zeta)] X(\zeta),$$

(18)
where

\[
H(\zeta) = -\frac{2\mu}{\kappa + 1} \left[ \frac{G(\zeta)W'(\zeta)}{X(\zeta)} - G_1(\zeta) - G_2(\zeta) \right]
\]

\[
G(\zeta)W'(\zeta) = \beta R \left[ \mathcal{Q} \left( \zeta - 3 \sum_{m=1}^{3} mC_m \zeta^{-m} \right) + \frac{\gamma \mathcal{Q}}{\zeta} \left( 1 - \sum_{m=1}^{3} mC_m \zeta^{-m-1} \right) \right]
\]

\[
G_1(\zeta) = \beta R \left[ \mathcal{Q} \left( \zeta^2 - A_1 \zeta + A_2 - C_1 \right) + \gamma \mathcal{Q} \right]
\]

\[
G_2(\zeta) = \frac{\beta R^2}{X(0)} \left[ \left( \gamma \mathcal{Q} \left( 1 - A_4 C_1 - 2A_4 C_2 - 3A_4 C_3 \right) - \mathcal{Q} \left( C_1 - 2A_4 C_2 + 3A_4 C_3 \right) \right) \frac{1}{\zeta^4} + \right.
\]

\[
+ \left( \gamma \mathcal{Q} \left( A_3 C_1 - 2A_4 C_2 - 3A_4 C_3 \right) + \mathcal{Q} \left( 3A_4 C_1 - 2C_2 \right) \right) \frac{1}{\zeta^2} + \right.
\]

\[
+ \left( \gamma \mathcal{Q} \left( -C_1 + 2A_4 C_2 - 3A_4 C_3 \right) - 3C_3 \mathcal{Q} \right) \frac{1}{\zeta^3} + \gamma \mathcal{Q} \left( -2C_2 + 3A_4 C_3 \right) \frac{1}{\zeta^4} - 3\gamma C_3 \mathcal{Q} \frac{1}{\zeta^3} \right]
\]

\[
A_1 = \left( \cos \frac{\omega}{2} + 2\lambda \sin \frac{\omega}{2} \right) e^{i\theta_0}
\]

\[
A_2 = \left( \frac{1}{4} + \lambda^2 \right) \left( 1 - \cos \omega \right) e^{2i\theta_0}
\]

\[
A_3 = \left( \cos \frac{\omega}{2} - 2\lambda \sin \frac{\omega}{2} \right) e^{-2i\theta_0}
\]

\[
A_4 = \left( \frac{1}{4} + \lambda^2 \right) \left( 1 - \cos \omega \right) e^{-2i\theta_0}
\]

\[
A_5 = \left( \frac{1}{4} + \lambda^2 \right) \left( -\frac{1}{2} \cos \frac{3\omega}{2} - \frac{\lambda}{3} \sin \frac{3\omega}{2} + \frac{1}{2} \cos \frac{\omega}{2} + \lambda \sin \frac{\omega}{2} \right) e^{-3i\theta_0}
\]

\[
A_6 = \left[ \frac{1}{12} \left( -\frac{15}{16} - \frac{7\lambda^2}{2} + \lambda^4 \right) \cos 2\omega - \frac{\lambda}{2} (1 + 4\lambda^2) \sin 2\omega \right.
\]

\[
+ \frac{1}{3} \left( \frac{3}{16} + \frac{\lambda^2}{2} - \lambda^4 \right) \cos \omega + \frac{\lambda}{12} \left( \frac{1}{2} + 2\lambda^2 \right) \sin \omega + \left( \frac{1}{16} + \frac{\lambda^2}{2} + \lambda^4 \right) \right] e^{-4i\theta_0}
\]

\[
X(0) = \kappa e^{-i(\theta_0 + 2\omega)} = e^{(2n - \omega) i - \theta_0}
\]

\[
\omega = \theta_2 - \theta_1, \quad \theta_0 = (\theta_1 + \theta_2)/2.
\]  

(19)

\[G_1(\zeta) \text{ and } G_2(\zeta) \text{ are the principal parts of the function } G(\zeta)W'(\zeta)/X(\zeta) \text{ at } \zeta \to \infty \text{ and } \zeta \to 0, \text{ respectively.}
\]

The series expansion of the Plemelj function \(X(\zeta)\) at \(\zeta \to \infty\) is in the form

\[
X(\zeta) = \frac{1}{\zeta^4} \left[ 1 + \frac{A_1}{\zeta^4} + \frac{B_1}{\zeta^2} + \frac{B_2}{\zeta^3} + O \left( \frac{1}{\zeta^4} \right) \right],
\]  

(20)

where

\[
B_1 = \left[ \left( \frac{3}{4} - \lambda^2 \right) \cos \omega + 2\lambda \sin \omega + \left( \frac{1}{4} + \lambda^2 \right) \right] e^{2i\theta_0}
\]

\[
B_2 = \left[ \frac{1}{2} \left( \frac{5}{4} - 3\lambda^2 \right) \cos \frac{3\omega}{2} + \frac{\lambda}{3} \left( \frac{23}{4} - \lambda^2 \right) \sin \frac{3\omega}{2} + \frac{3}{2} \left( \frac{1}{4} + \lambda^2 \right) \cos \frac{\omega}{2} + \lambda \left( \frac{1}{4} + \lambda^2 \right) \sin \frac{\omega}{2} \right] e^{2i\theta_0}.
\]  

(21)
Substituting eq. (20) into eq. (18) and using eq. (8), the following equation of $\Phi(\zeta)$ can be obtained:

\[
\Phi(\zeta) = \frac{1}{R} \left\{ F_1 + \left[ \frac{2 \mu R^2 Q}{k + 1} \left( B_1 - A_1^2 + A_2 \right) + A_1 F_1 + F_2 \right] \frac{1}{\zeta} + \left[ \frac{2 \mu R^2}{k + 1} \left( B_2 Q - A_1 B_1 Q + 2 C_2 Q + (\gamma Q - C_1 Q + A_2 Q) A_1 \right) + \frac{1}{X(0)} \left( \frac{1}{\zeta^3} \right) \right]\right\} + O\left(\frac{1}{\zeta^3}\right)
\]

(22)

The series expansion of $X(\zeta)$ at $\zeta = 0$ is as follows:

\[
X(\zeta) = X(0)(1 + J_1 \zeta + J_2 \zeta^2 + J_3 \zeta^3 + J_4 \zeta^4 + J_5 \zeta^5 + O(\zeta^6)),
\]

(23)

where

\[
J_1 = A_3
\]

\[
J_2 = \left[ \frac{3}{4} - \frac{\lambda^2}{4} \right] \cos \omega - 2 \lambda \sin \omega + \frac{1}{4} + \frac{\lambda^2}{2} \right] e^{-2\omega_0}
\]

\[
J_3 = \left[ \frac{1}{2} \left( 4 - 3 \lambda^2 \right) \cos 2 \omega - \frac{1}{3} \left( \frac{23}{4} - \lambda^2 \right) \sin 2 \omega + \frac{1}{4} + \frac{\lambda^2}{2} \right] e^{-3\omega_0}
\]

\[
J_4 = \left[ \frac{1}{12} \left( 6 - \frac{43 \lambda^2}{2} + \lambda^2 \right) \cos 2 \omega - \frac{1}{6} \left( 11 - 4 \lambda^2 \right) \sin 2 \omega + \frac{1}{4} \left( \frac{15}{16} + \frac{7 \lambda^2}{2} - \lambda^4 \right) \cos \omega \right.
\]

\[
\left. - \frac{1}{3} \left( 1 + 4 \lambda^2 \right) \sin \omega + \frac{1}{4} \left( \frac{15}{16} + \frac{5 \lambda^2}{2} + \lambda^4 \right) \right] e^{-4\omega_0}
\]

\[
J_5 = \left[ \frac{1}{24} \left( 6 - \frac{95 \lambda^2}{2} + 5 \lambda^2 \right) \cos 5 \omega - \frac{1}{6} \left( \frac{93}{2} + 115 \lambda^2 \right) \sin 5 \omega + \frac{1}{4} \left( \frac{9}{16} + \frac{5 \lambda^2}{2} + \lambda^4 \right) \cos \omega \right.
\]

\[
\left. + \frac{1}{4} \left( \frac{9}{16} + \frac{5 \lambda^2}{2} + \lambda^4 \right) \cos 5 \omega - \frac{1}{6} \left( \frac{9}{16} + \frac{5 \lambda^2}{2} + \lambda^4 \right) \sin 5 \omega \right] e^{-5\omega_0}.
\]

(24)

Substituting eq. (23) into eq. (18) gives the following equation:

\[
\Phi_1(\zeta) = \frac{2 \mu R^2}{k + 1} \left( L_1 J_1 + L_2 J_2 + L_3 J_3 + L_4 J_4 + L_5 J_5 + X(0)(A_2 Q + \gamma Q - C_1 Q) \right)
\]

\[
+ X(0)(F_2 + F_3 J_1 + F_4 J_2 + F_5 J_3 + F_6 J_4) + \left( \frac{2 \mu R^2}{k + 1} \right) \left( C_1 Q - \gamma Q + L_1 + L_2 J_1 + L_3 J_2 \right)
\]

\[
+ L_4 J_3 + L_5 J_4) + X(0)(F_3 + F_4 J_1 + F_5 J_2 + F_6 J_3) \right] \frac{1}{\zeta} + \left( \frac{2 \mu R^2}{k + 1} \right) \left( 2 C_2 Q + L_2 + L_3 J_1 + L_4 J_2 + L_5 J_3 + X(0)(F_4 + F_5 J_1 + F_6 J_2) \right) \frac{1}{\zeta^2}
\]

\[
+ \left( \frac{2 \mu R^2}{k + 1} \right) \left( 3 C_3 Q + C_1 \gamma Q + L_3 + L_4 J_1 + L_5 J_2) + X(0)(F_3 + F_4 J_1 + F_6 J_2) \right) \frac{1}{\zeta^3}
\]

\[
+ \left( \frac{2 \mu R^2}{k + 1} \right) \left( 2 C_2 \gamma Q + L_4 + L_5 J_1 + X(0)F_6 \right) \frac{1}{\zeta^4} + O(\zeta),
\]

(25)
where
\[
\begin{align*}
L_1 &= \gamma \hat{Q}(1 - A_4 C_1 - 2A_5 C_2 - 3A_6 C_3) - Q(C_1 - 2A_5 C_2 + 3A_4 C_3) \\
L_2 &= \gamma \hat{Q}(A_4 C_1 - 2A_4 C_2 - 3A_5 C_3) - Q(2C_1 - 3A_5 C_3) \\
L_3 &= \gamma \hat{Q}(-C_1 + 2A_5 C_2 - 3A_4 C_3) - 3C_1 Q \\
L_4 &= \gamma \hat{Q}(-2C_1 + 3A_3 C_3) \\
L_5 &= -3\gamma C_3 \hat{Q}.
\end{align*}
\]  

(26)

Comparing the coefficients of $\Psi_1(z)$ in eq. (11) with those of $\Psi_1(z)$ obtained by using eqs (6), (22) and (25), and the coefficients of $\Phi(z)$ of eq. (11) with those of eq. (22), gives the following $F_i (i = 1, \ldots, 6):$

\[
\begin{align*}
F_1 &= 0 \\
F_2 &= -\frac{2\mu R^2}{\kappa + 1} (\gamma \hat{Q} + Q(2A_4 + A_2 + B_1 - C_1)) \\
F_3 &= \frac{M_1 \cos \theta_0 + N_1 \sin \theta_0}{e^{(2\pi - \omega)\lambda} + C_3 P_1} + i \frac{M_1 \sin \theta_0 - N_1 \cos \theta_0}{e^{(2\pi - \omega)\lambda} - C_3 P_1} = Q_1 + iR_1 \\
F_4 &= \frac{M_2 \cos 2\theta_0 + N_2 \sin 2\theta_0}{P_1 e^{(2\pi - \omega)\lambda} + C_3 P_1} + i \frac{M_2 \sin 2\theta_0 - N_2 \cos 2\theta_0}{P_1 e^{(2\pi - \omega)\lambda} - C_3 P_1} = Q_2 + iR_2 \\
F_5 &= -\frac{2\mu R^2 C_3}{(\kappa + 1)X(0)} (3\gamma \hat{Q} P_2 e^{-2\theta_0} + 2(\gamma \hat{Q} - C_1 \hat{Q})) \\
F_6 &= 0,
\end{align*}
\]  

(27)

where

\[
\begin{align*}
M_1 &= -\frac{2\mu R^2}{\kappa + 1} \left\{ \begin{array}{c}
((\gamma - C_1) - C_3 P_1 (\gamma C_1 + 3C_3) e^{-(2\pi - \omega)\lambda}) Re(Q) \\
- C_3 P_1 [(\gamma - C_1) Re(Q) \cos \theta_0 + (\gamma + C_1) Im(Q) \sin \theta_0] \\
+ 2C_2 C_3 (Re(Q) \cos \theta_0 - Im(Q) \sin \theta_0) [P_1 + (P_1 - \gamma P_2) e^{-(2\pi - \omega)\lambda}] \\
+ 2C_3 P_5 [(\gamma - C_1) Re(Q) \cos 2\theta_0 + (\gamma + C_1) Im(Q) \sin 2\theta_0] \\
- C_3 (Re(Q) \cos 2\theta_0 + Im(Q) \sin 2\theta_0) [3P_2 - \gamma P_1 e^{-(2\pi - \omega)\lambda}] \\
- (Re(Q) \cos 2\theta_0 - Im(Q) \sin 2\theta_0) [\gamma C_3 P_2 + C_3 (\gamma C_1 P_3 + 3C_3 P_5) e^{-(2\pi - \omega)\lambda}] \\
- 2\gamma C_3 (Re(Q) \cos 3\theta_0 - Im(Q) \sin 3\theta_0) (P_1 - P_1 P_2) + C_3 (Re(Q) \cos 4\theta_0 \\
- Im(Q) \sin 4\theta_0) (3\gamma P_2 P_3 + 3\gamma P_1 P_7 - 3\gamma P_8 + P_8 - 2P_{10} + P_{11}) \end{array} \right\}
\end{align*}
\]

\[
M_2 = \frac{2\mu R^2}{\kappa + 1} \left[ (C_1 - \gamma) Re(Q) + 2C_3 P_5 (\gamma - C_1) Re(Q) \cos 2\theta_0 \\
+ (\gamma + C_1) Im(Q) \sin 2\theta_0 + 3C_3 P_2 (Re(Q) \cos 2\theta_0 + Im(Q) \sin 2\theta_0) \\
+ \gamma C_1 P_2 (Re(Q) \cos 2\theta_0 - Im(Q) \sin 2\theta_0) + 2\gamma C_2 P_2 (Re(Q) \cos 3\theta_0 \\
- Im(Q) \sin 3\theta_0) + 3\gamma C_3 (P_8 + P_{10}) (Re(Q) \cos 4\theta_0 - Im(Q) \sin 4\theta_0) \right]
\]  

\[+ (P_1 \cos \theta_0 + R_1 \sin \theta_0) e^{(2\pi - \omega)\lambda}.\]
\[ N_1 = -\frac{2\mu \beta R^3}{\kappa + 1} \left\{ (\gamma + C_1) + C_3 P_3 (\gamma - 3C_3) e^{-i(2\pi - \omega)\theta} \right\} \text{Im}(Q) \]
\[ - C_2 P_1 (\gamma + C_1) \text{Re}(Q) \sin \theta_0 - (\gamma + C_1) \text{Im}(Q) \cos \theta_0 \]
\[ + 2C_2 C_3 (\text{Re}(Q) \sin \theta_0 + \text{Im}(Q) \cos \theta_0) [P_1 + (P_1^2 + \gamma P_4) e^{-(2\pi - \omega)\theta}] \]
\[ + 2C_3 P_3 (\gamma - C_1) \text{Re}(Q) \sin 2\theta_0 - (\gamma + C_1) \text{Im}(Q) \cos 2\theta_0 \]
\[ - C_3 (\text{Re}(Q) \sin 2\theta_0 - \text{Im}(Q) \cos 2\theta_0) [3P_2 - \gamma P_1 e^{-(2\pi - \omega)\theta}] \]
\[ - (\text{Re}(Q) \sin 2\theta_0 + \text{Im}(Q) \cos 2\theta_0) [3\gamma C_2 P_2 + C_3 (C_1 P_1 + 3\gamma C_3 P_4) e^{-(2\pi - \omega)\theta}] \]
\[ - 2\gamma C_2 (\text{Re}(Q) \sin 3\theta_0 + \text{Im}(Q) \cos 3\theta_0) (P_1 - P_1^2 + P_1 P_1 P_1 + C_3 (\text{Re}(Q) \sin 4\theta_0) \text{Im}(Q) \cos 4\theta_0) \text{Re}(Q) \sin 3\theta_0 + \text{Im}(Q) \cos 3\theta_0) \text{Im}(Q) \sin 3\theta_0 \]
\[ + (\text{Re}(Q) \sin 3\theta_0 + \text{Im}(Q) \cos 3\theta_0) (P_3 - P_3 + P_3 P_3 P_3 + C_3 (\text{Re}(Q) \sin 4\theta_0 + \text{Im}(Q) \cos 4\theta_0) \text{Re}(Q) \sin 4\theta_0 + \text{Im}(Q) \cos 4\theta_0) \]

\[ P_1 = \cos \frac{\omega}{2} - 2\lambda \sin \frac{\omega}{2} \]

\[ P_2 = \frac{1}{2} + 2\lambda^2 + \left( \frac{1}{2} - 2\lambda^2 \right) \cos \omega - 2\lambda \sin \omega - P_1^2 \]

\[ P_3 = \left( \frac{1}{4} + \lambda^2 \right) (1 - \cos \omega) \left( \cos \frac{\omega}{2} - 2\lambda \sin \frac{\omega}{2} \right) \]

\[ P_4 = \left( \frac{1}{4} + \lambda^2 \right) \left( -\frac{1}{2} \cos \frac{3\omega}{2} - \frac{\lambda}{3} \sin \frac{3\omega}{2} + \frac{1}{2} \cos \omega + \lambda \sin \omega \right) P_1 \]

\[ P_5 = P_1 - \left[ \left( \frac{3}{4} - \lambda^2 \right) \cos \omega - 2\lambda \sin \omega + \left( \frac{1}{4} + \lambda^2 \right) \right] \]

\[ P_6 = P_1 \left[ \frac{1}{12} \left( -\frac{15}{16} + \frac{7\lambda^2}{2} + \lambda^4 \right) \cos 2\omega - \frac{\lambda}{12} (1 + 4\lambda^2) \sin 2\omega + \frac{1}{3} \left( \frac{3}{16} + \frac{\lambda^2}{2} - \lambda^4 \right) \cos \omega \right. \]
\[ + \left. \frac{\lambda}{3} \left( \frac{1}{2} + 2\lambda^2 \right) \sin \omega + \frac{1}{4} \left( \frac{1}{16} + \frac{\lambda^2}{2} + \lambda^4 \right) \right] \]

\[ P_7 = \left( \frac{1}{2} - 2\lambda^2 \right) \cos \frac{3\omega}{2} - \left. \frac{1}{3} \left( \frac{23}{4} + \frac{\lambda}{4} - \lambda^2 + \lambda^3 \right) \sin \frac{3\omega}{2} + \left( \frac{1}{2} + 2\lambda^2 \right) \cos \frac{\omega}{2} \right. \]
\[ - \left. P_1 (\cos \omega - 2\lambda \sin \omega) \right] \]
\[
P_s = \frac{1}{2} \left( \frac{15}{16} \frac{25\lambda^2}{6} + \frac{\lambda^4}{3} \right) \cos 2\omega - \lambda \left( \frac{23}{4} - \lambda^2 \right) \sin 2\omega + P_1 \left[ \frac{3}{4} + \lambda^2 \right] \cos \frac{3\omega}{2} + \frac{1}{3} \left( \frac{23}{4} - \frac{\lambda}{4} - \lambda^2 - \lambda^3 \right) \sin \frac{3\omega}{2} - \left( \frac{1}{4} + \lambda \right) \cos \omega + \lambda \left( \frac{1}{2} + 2\lambda^2 \right) \sin \omega \\
+ \left( \frac{1}{2} + \frac{4\lambda^2}{3} + \frac{8\lambda^4}{3} \right) \cos \omega - \frac{2\lambda}{3} (1 + 4\lambda^2) \sin \omega - \left( \frac{3}{16} + \frac{\lambda^2}{2} - \lambda^4 \right) \cos^2 \omega \\\n+ \lambda \left( \frac{1}{2} + 2\lambda^2 \right) \sin \omega \cos \omega + \frac{7}{32} + \frac{5\lambda^2}{4} + \frac{3\lambda^4}{2}\

P_s = P_1 \left[ \frac{1}{2} \left( \frac{5}{4} - 3\lambda^2 \right) \cos \frac{3\omega}{2} + \frac{\lambda}{3} \left( \frac{23}{4} - \lambda^2 \right) \sin \frac{3\omega}{2} + \frac{3}{2} \left( \frac{1}{4} + \lambda \right) \cos \frac{\omega}{2} + \lambda \left( \frac{1}{4} + \lambda^2 \right) \sin \frac{\omega}{2} \right] \\

P_{10} = P_1 \left[ \frac{3}{4} - \lambda^2 \right] \cos \omega + 2\lambda \sin \omega + \left( \frac{1}{4} + \lambda^2 \right) \left( \cos \frac{\omega}{2} + 2\lambda \sin \frac{\omega}{2} \right) \\

P_{11} = P_1 \left( \cos \frac{\omega}{2} + 2\lambda \sin \frac{\omega}{2} \right)^3 \\
P_{12} = P_1 \left[ \frac{3}{4} - \lambda^2 \right] \cos \omega - 2\lambda \sin \omega + \left( \frac{1}{4} + \lambda^2 \right) \\
P_{13} = P_{12}/P_1.
\] (28)

**Determination of TSIFs**

The TSIFs at the cusp crack tip on the positive \( x \)-axis can be expressed by

\[
K = K_1 - iK_{11} = 2 \sqrt{\frac{\pi}{W^\prime(1)}} \Phi_1(1).
\] (29)

Introducing eq. (18) into eq. (29), the following equation is obtained:

\[
K = K_1 - iK_{11} \\
= 2 \sqrt{\frac{\pi}{2R(C_1 + 3C_2 + 6C_3)}} \left[ \frac{-2\mu v R^2}{\kappa + 1} \left( Q + \gamma \bar{Q} \right) (1 - C_1 - 2C_2 - 3C_3) - \left( Q(1 - A_1 + A_2 - C_1) \\
+ \gamma \bar{Q} + \frac{1}{X(0)} [\gamma \bar{Q}(1 - C_1(1 - A_3 + A_4) - 2C_2(1 - A_3 + A_4 + A_5) \\
- 3C_3(1 - A_3 + A_4 + A_5 + A_6)) + Q(-C_1 - 2C_2(1 - A_3) - 3C_3(1 - A_3 + A_4)))] \right) \\
+ F_2 + F_3 + F_4 + F_5 \right) (1 - e^{i(\theta_0 + 0.5\omega)})^{-0.5 + i} (1 - e^{i(\theta_0 - 0.5\omega)})^{-0.5 - i},
\] (30)

where \( K_1 \) and \( K_{11} \) represent mode I and mode II TSIFs, respectively.

To determine TSIFs when both \( \theta_0 \) and \( \omega \) are zero, we substitute zero into \( \theta_0 \) and \( \omega \) of eq. (18), obtain the complex potential and then substitute the obtained complex potential into eq. (29) to give the result.

When a line crack lies on the bonded part in a bimaterial, the SIFs are defined as follows [16]:

\[
K_1 - iK_{11} = 2 \sqrt{(2\pi) e^{2i} \lim_{z \to a} (z - a)^{0.5 + i} \phi'(z)},
\] (31)

where \( \phi(z) \) is defined in the \( z \) plane and \( a \) is the half crack length.
To apply eq. (31) to the curvilinear crack, we transform the coordinate in the $z$ plane into that in the $\zeta$ plane and use the conformal mapping function to give the following SIFs at the right-side tip of the interfacial crack:

$$K_1 - iK_\Pi = 2\sqrt{(2\pi)e^{i\phi}}(1 + i)\left[W'(\zeta)\right]^{1/2 + i\theta^*} \lim_{\zeta \to \sigma_i} (\zeta - \sigma_i)^{1/2 + i\theta^*} \Phi(\zeta),$$  \hspace{1cm} (32)

where

$$\theta^* = \arg[\sigma_i W'(\sigma_i)].$$ \hspace{1cm} (33)

Obtaining $\Phi(\zeta)$ from eqs (8) and (18) and substituting it into eq. (32), the TSIFs are expressed in the forms

$$K = K_1 - iK_\Pi
= 2\sqrt{(2\pi)e^{i\phi}}(1 + i)\left[R(\sigma_i - \lambda_2)\left(1 - \frac{C_1}{\sigma_i^2} + \frac{2C_2}{\sigma_i^3} - \frac{3C_3}{\sigma_i^4}\right)\right]^{-0.5 + i\theta^*}
\cdot \left\{\frac{2\mu \beta \pi R^2}{\kappa + 1}\left[Q(\sigma_i^2 - A_1\sigma_i + A_2 - C_1) + \gamma \bar{Q} + \frac{1}{X(0)}\left(\gamma \bar{Q}(1 - A_4 \sigma_i - 2A_3 \sigma_i - 3A_2 \sigma_i) + Q(-2 \sigma_i + 3A_2 \sigma_i)\right)\right]
+ (\gamma \bar{Q}(C_1 + 2A_3 \sigma_i - 3A_2 \sigma_i) - 3C_3 \bar{Q})\frac{1}{\sigma_i^3} + \gamma \bar{Q}(2 \sigma_i - 3A_2 \sigma_i)\frac{1}{\sigma_i^3} - \frac{3\gamma C_3 \bar{Q}}{\sigma_i^4}\right]\right\}
\cdot \left\{F_3 + F_4 + \frac{F_5}{\sigma_i^2 + \frac{F_3}{\sigma_i^2}}\right\},$$ \hspace{1cm} (34)

DISCUSSION

The conformal mapping functions [13] for the rigid symmetric lip and airfoil cusp cracks shown in Figs 2 and 3, respectively, are obtained by introducing the following complex constants into eq. (1):

$$R = \frac{R_0(l + 2)}{4}, \hspace{0.5cm} C_1 = \frac{2(1 - l)}{l + 2}, \hspace{0.5cm} C_2 = 0, \hspace{0.5cm} C_3 = \frac{l}{l + 2}, \hspace{0.5cm} C_m = 0 (m \geq 4, \ 0 \leq l \leq 1; \ lip \ type)$$ \hspace{1cm} (35)

---

Fig. 2. Configuration of symmetric lip cusp crack for configuration parameter $l$. 
Fig. 3. Configuration of symmetric airfoil cusp crack for configuration parameter $\delta$.

\[
R = \frac{R_0(1 + \delta)}{2}, \quad C_1 = \frac{1 - \delta}{1 + \delta}, \quad C_2 = \frac{\delta}{1 + \delta}, \quad C_m = 0 \quad (m \geq 3, 0 \leq \delta < 1; \text{airfoil type}), \quad (36)
\]

where $2R_0$ is an equivalent crack length, and $l$ and $\delta$ are the configuration parameters for the lip and airfoil cusp cracks, respectively.

Introducing eqs (35) and (36) into eq. (18) and zero into $\theta_1$ and $\theta_2$ of eq. (18) gives the following complex potentials without the interfacial crack:

\[
\phi_1(\zeta) = \frac{2\mu \beta R^2}{\zeta} \left[ -\frac{\gamma \bar{Q} - C_1 Q}{\kappa + 1} + \left(\frac{2(\gamma \bar{Q} - C_1 Q)}{\kappa + 1} + 3C_1 Q + \gamma C_1 \bar{Q}\right) \frac{1}{\kappa \zeta^2} + \frac{3\gamma C_1 \bar{Q}}{\kappa \zeta^4} \right] \quad \text{(lip type)} \quad (37)
\]

\[
\phi_1(\zeta) = \frac{2\mu \beta R^2}{\zeta} \left[ -\frac{\gamma \bar{Q} - C_1 Q}{\kappa + 1} + \left(\frac{\gamma \bar{Q} - C_1 Q}{\kappa + 1} + 2Q\right) \frac{C_1}{\kappa \zeta} + \left(C_1 - \frac{8C_2}{3}\right) \frac{\gamma \bar{Q}}{\kappa \zeta^3} \right] \quad \text{(airfoil type).} \quad (38)
\]

Substituting the above potentials into eq. (29) gives the following TSIFs for the rigid symmetric lip and airfoil cusp cracks without an interfacial crack:

\[
K = 4\mu \beta R^2 \sqrt{\frac{\pi}{2R(C_1 + 6C_2)}} \left[ \frac{1}{\kappa + 1} \left(-\frac{\gamma \bar{Q} + C_1 Q}{\kappa} + \frac{2C_1(\gamma \bar{Q} - C_1 Q)}{\kappa} \right) \right.
\]

\[
+ \left(\frac{1}{\kappa} \left(3C_1 Q + \gamma \bar{Q}(C_1 + 3C_2)\right) \right] \quad \text{(lip type)} \quad (39)
\]

\[
K = 4\mu \beta R^2 \sqrt{\frac{\pi}{2R(C_1 + 3C_2)}} \left[ -\frac{\gamma \bar{Q} + C_1 Q}{\kappa + 1} + \left(\frac{\gamma \bar{Q} - C_1 \bar{Q}}{\kappa + 1} + 2Q\right) \frac{C_1}{\kappa} \right.
\]

\[
+ \left(\frac{1}{\kappa} \left(3C_1 Q - \gamma \bar{Q}(C_1 + 3C_2)\right) \right] \quad \text{(airfoil type).} \quad (40)
\]

Equations (37)–(40) are identical to the previous solutions [2]. From eqs (30) and (34), it is noted that both the mode I and mode II TSIFs are derived in the forms of the periodic functions of the uniform heat flow direction. To obtain the nondimensional TSIFs, the following equations are defined:

\[
K^* \equiv K \left/ \frac{2\sqrt{\pi \mu \beta \tau R_0^{1.5}}}{\kappa + 1} \right. \quad (\text{cusp crack}) \quad (41)
\]

\[
K^* \equiv K \left/ \frac{2\sqrt{(2\pi) \mu \beta \tau R_0^{1.5} + \nu}}{\kappa + 1} \right. \quad (\text{interfacial crack}). \quad (42)
\]
Fig. 4. Variation of dimensionless TSIFs for an insulated symmetric lip-type rigid inclusion with various interface crack locations. (a) $K_1^*$. (b) $K_2^*$. 

Specifically, $t = \delta = 0.5$, $\kappa = 2.5$, and $\varphi = 45^\circ$ are considered. The variations of $K_1^*$ and $K_2^*$ for a rigid symmetric lip cusp inclusion with the various interface crack locations ($\omega, \theta_0$) are shown in Figs 4 and 5, respectively. For $\gamma = 1$, TSIFs undergo sharp changes in the case where the tip of the interfacial crack approaches that of the cusp crack. This shows a trend similar to the result of Sendeckyj [11], obtained under mechanical loading conditions. For $\gamma = -1$, similar results are obtained except for the opposite signs of TSIFs with respect to the interfacial crack location. The tendency of $K_1^*$ and $K_2^*$ for the rigid symmetric air foil cusp inclusion as shown in Figs 6 and 7 is similar to that of the rigid symmetric lip cusp inclusion.

The variations of $K_1^*$ and $K_2^*$ for the interfacial crack on a rigid symmetric lip cusp inclusion with various interface crack locations ($\omega, \theta_0$) are shown in Figs 8 and 9, respectively. The TSIFs' singularity phenomena can be also seen in Fig. 8a and b. Unlike the cusp crack case, Fig. 8a and b show that the values of the TSIFs for the interfacial crack tip away from the cusp crack tip have an undulation with the location of the interfacial crack tip. The trends of TSIFs for $\gamma = -1$ in Fig. 9a and b are similar to the case of $\gamma = 1$ except for the opposite signs of TSIFs with respect to the interfacial crack. For the interfacial crack on a rigid symmetric airfoil cusp inclusion (Figs 10 and 11), the tendency of TSIFs is similar to that of a rigid symmetric lip cusp inclusion.

Fig. 5. Variation of dimensionless TSIFs for a symmetric lip-type rigid inclusion under zero temperature boundary condition with various interface crack locations. (a) $K_1^*$. (b) $K_2^*$. 
Fig. 6. Variation of dimensionless TSIFs for an insulated symmetric airfoil-type rigid inclusion with various interface crack locations. (a) $K_1^*$. (b) $K_II^*$.

Fig. 7. Variation of dimensionless TSIFs for a symmetric airfoil-type rigid inclusion under zero temperature boundary condition with various interface crack locations. (a) $K_1^*$. (b) $K_II^*$.

Fig. 8. Variation of dimensionless TSIFs for an interface crack on an insulated symmetric lip-type rigid inclusion with various interface crack locations. (a) $K_1^*$. (b) $K_II^*$. 
Fig. 9. Variation of dimensionless TSIFs for an interface crack on a symmetric lip-type rigid inclusion under zero temperature boundary condition with various interface crack locations. (a) $K_1^\infty$. (b) $K_2^\infty$.

Fig. 10. Variation of dimensionless TSIFs for an interface crack on an insulated symmetric airfoil-type rigid inclusion with various interface crack locations. (a) $K_1^\infty$. (b) $K_2^\infty$.

Fig. 11. Variation of dimensionless TSIFs for an interface crack on a symmetric airfoil-type rigid inclusion under zero temperature boundary condition with various interface crack locations. (a) $K_1^\infty$. (b) $K_2^\infty$. 
CONCLUSIONS

In the study on the determination of TSIFs for the interfacial crack along a cusp-type inclusion boundary insulated or fixed to zero relative temperature under a uniform heat flow in a two-dimensional elastic body, the following results were obtained.

1. The TSIFs for the cusp crack and the interfacial crack on a cusp crack were derived.
2. The TSIFs are expressed in the forms of the periodic functions of the uniform heat flow direction.
3. When the tip of the interfacial crack meets with that of the cusp crack, the TSIFs have singularities.
4. The TSIFs for the interfacial crack tip away from the cusp crack have an undulation with the location of the interfacial crack.
5. The complex potential functions and the TSIFs for the cusp-type inclusion without the interfacial crack were derived with the cusp surface boundary conditions insulated or fixed to zero relative temperature. These solutions are identical to the previous results.

REFERENCES


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