We study noncooperative game theory to deal with situations where multiple players make strategically interdependent decisions, which means that one’s payoff depends not only on his own decision but also on others’. To describe a game, we need to know the followings:

(1) Players: people who are involved in the strategic situation

(2) Rules of the game:
   - Timeline: Who moves when
   - Information: What they know when they move
   - Strategy: What they can do

(3) Outcomes: What will happen as a result of each possible combination of action

(4) Payoffs: Preference over the possible outcomes

All games are modeled in two ways depending on whether players move simultaneously or sequentially. The former case is represented by the *normal form games* while the latter by the *extensive form games*. As will be seen later, each extensive form game also has a normal form representation, which motivates us to first study the normal form game.

*I thank Bill Sanholm at the UW-Madison for generously allowing me to borrow from his lecture note.*
1 Normal Form Game

Let us first take an example of a simultaneous game.

Example 1.1. (Matching Pennies) A familiar game called ‘matching pennies’ is described as follows:

(1) Players: Two players denoted 1 and 2

(2) Rules: Each player simultaneously puts a penny down, either heads up or tails up.

(3) Outcomes: If the two pennies match, player 1 pays 1 dollar to player 2; otherwise, player 2 pays 1 dollar to player 1.

This game can be expressed by the following matrix.

<table>
<thead>
<tr>
<th>Player 1</th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>-1,1</td>
<td>1,-1</td>
</tr>
<tr>
<td>T</td>
<td>1,-1</td>
<td>-1,1</td>
</tr>
</tbody>
</table>

The first (second) number in a cell represent player 1 (2)’s payoff

- In general, we write a normal form game as $\Gamma_N = [I, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$, where
  - $I = \{1, \cdots, I\}$: Set of players
  - $S_i$: Set of pure strategies for player $i$. $s = (s_1, \cdots, s_I) \in S := \prod_{i \in I} S_i$
  - $u_i: S \rightarrow R$ : Player $i$’s payoff or utility.

  A player can also choose randomly out of his strategy set, which gives rise to what is called a mixed strategy.

- Given player $i$’s pure strategy set $S_i$, a mixed strategy for player $i$, $\sigma_i: S_i \rightarrow [0,1]$, assigns to each pure strategy $s_i \in S_i$, a probability $\sigma_i(s_i) \geq 0$ with $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$. 

2
− If \( S_i = \{s_{i1}, \cdots, s_{Mi}\} \), then player \( i \)'s set of all possible mixed strategies is a simplex given by

\[
\Delta(S_i) = \{(\sigma_{i1}, \cdots, \sigma_{Mi}) \in \mathbb{R}^M : \sigma_{mi} \geq 0, \forall m = 1, \cdots, M \text{ and } \sum_{m=1}^{M} \sigma_{mi} = 1\},
\]

or the set of probability distributions on \( S_i \).

− We denote \( \Sigma_i := \Delta(S_i) \) and \( \Sigma := \prod_{i \in I} \Sigma_i \).

− Given a profile of mixed strategies \( \sigma = (\sigma_1, \cdots, \sigma_I) \), each player \( i \)'s expected (or von Neumann-Morgenstern) utility is

\[
u_i(\sigma) = \sum_{s \in S} u_i(s) \left( \prod_{j \in I} \sigma_j(s_j) \right).
\]

**Example 1.2. (Example 1.1 continued)** Suppose that in the Example 1.1, player 1 uses \( H \) and \( T \) with probabilities 1/3 and 2/3 while player 2 with probabilities 3/4 and 1/4. That is, \( \sigma = (\sigma_1, \sigma_2) = (\left(\frac{1}{3}, \frac{2}{3}\right), (\frac{3}{4}, \frac{1}{4})) \). Then, this strategy profile generates the following probability distribution on \( S \):

<table>
<thead>
<tr>
<th></th>
<th>( H )</th>
<th>( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H )</td>
<td>1/4</td>
<td>1/12</td>
</tr>
<tr>
<td>( T )</td>
<td>1/2</td>
<td>1/6</td>
</tr>
</tbody>
</table>

For instance, the expected payoff of player 1 is

\[
1/4 \times (-1) + 1/12 \times 1 + 1/2 \times 1 + 1/6 \times (-1) = 1/6.
\]

Suppose now that two players use a ‘die’ as a common randomization device so their strategy is for both to choose \( H \) if an odd number comes up while choosing \( T \) otherwise. This strategy generates the following distribution on \( S \):

<table>
<thead>
<tr>
<th></th>
<th>( H )</th>
<th>( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H )</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>( T )</td>
<td>0</td>
<td>1/2</td>
</tr>
</tbody>
</table>

This would have been impossible if two players had randomized independently. □

The concept of correlated strategy captures the idea that players use the common randomization device.
• A correlated strategy is a probability distribution $\mu \in \Delta(S)$.

  – Obviously, a mixed strategy is a special case of correlated strategy.
2 Solution Concepts for Normal Form Games

What should we expect to observe in a game played by rational players who are fully knowledgeable about the structure of the game and each others’ rationality?

2.1 Dominance

• A strategy $s_i$ is strictly dominant if $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$ for all $s'_i \neq s_i$ and all $s_{-i} \in S$.
  
  - If a player has a dominant strategy, then we should expect him to choose it.
  
  - There are few games that possess strictly dominant strategies.

Example 2.1. (Prisoner’s Dilemma) Two suspects in a bank robbery case are being examined in separate cells. Each has to choose either ‘Confess’ or ‘Deny’. If both confess, then they are sentenced to 2 years in prison. If only one confesses, then he is let go free while the other is sentenced to 3 years in prison. If both deny, then they are convicted of a lesser crime and sentenced to 1 year in prison. This game can be described by the matrix:

<table>
<thead>
<tr>
<th></th>
<th>Deny</th>
<th>Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deny</td>
<td>$-1, -1$</td>
<td>$-3, 0$</td>
</tr>
<tr>
<td>Confess</td>
<td>$0, -3$</td>
<td>$-2, -2$</td>
</tr>
</tbody>
</table>

To confess is a strictly dominant strategy for both players. □

• A strategy $\sigma_i \in \Sigma_i$ is strictly dominated if there exists $\sigma'_i \in \Sigma_i$ such that

\[ u_i(\sigma_i, \sigma_{-i}) < u_i(\sigma'_i, \sigma_{-i}), \forall \sigma_{-i} \in \Sigma_{-i}. \]

  - To check whether a strategy is strictly dominated, it suffices to check against pure strategies of opponents since

\[ (\star) \iff u_i(\sigma_i, s_{-i}) < u_i(\sigma'_i, s_{-i}), \forall s_{-i} \in S_{-i}. \]

  - It is possible for a strategy to be dominated by no pure strategy but by a mixed strategy. Consider, for instance, the following game:
B is dominated by neither $T$ nor $M$ but it is dominated by $\frac{1}{2}T + \frac{1}{2}M$.

- If a pure strategy $s_i$ is strictly dominated, then so is any mixed strategy which uses $s_i$ with a positive probability.

Sometimes, we can eliminate dominated strategies iteratively.

**Example 2.2. (Iterated Strict Dominance)** Consider the following game:

<table>
<thead>
<tr>
<th></th>
<th>$L$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>3, -</td>
<td>0, -</td>
</tr>
<tr>
<td>$M$</td>
<td>0, -</td>
<td>3, -</td>
</tr>
<tr>
<td>$B$</td>
<td>1, -</td>
<td>1, -</td>
</tr>
</tbody>
</table>

(1) Eliminate $B$ since it is strictly dominated by $\frac{1}{2}T + \frac{1}{2}M$.

(2) Once $B$ is eliminated, then $L$ is strictly dominated by $R$ so eliminate it.

(3) Once $B$ and $L$ are eliminated, then $T$ is strictly dominated by $M$ so eliminate it.

After successive elimination, only $(M, R)$ survives. □

- The process of eliminating strictly dominated strategies iteratively until no further strategy can be deleted, is called *iterated strict dominance (ISD).*

- We can only apply ISD when the rationality of players is commonly known among themselves (or *common knowledge*): In the above example, for instance, (1) holds since 1 is rational. (2) holds since 2 is rational and he knows that 1 is rational. (3) holds since 1 is rational and he knows that 2 is rational, and also knows that 2 knows that he is rational.

**Remark 2.1. (Common Knowledge)** A fact is *common knowledge* among players if all the statements of the form “$i$ knows that $j$ knows that … $k$ knows the fact” hold true. □
A convenient procedure to find strategies that survive ISC is (1) to iteratively eliminate all strictly dominated pure strategies and (2) to check all remaining mixed strategies, when no further pure strategies can be eliminated.

- In fact, the order of elimination does **not** matter.
- A game is *dominance solvable* if there is a unique strategy profile which survives the iterated strict dominance.
- But, only a few games are dominance solvable.

A strategy $\sigma_i \in \Sigma_i$ is *weakly dominated* if there exists $\sigma'_i \in \Sigma_i$ such that

$$u_i(\sigma_i, s_{-i}) \leq u_i(\sigma'_i, s_{-i}), \forall s_{-i} \in S_{-i},$$

with strict inequality for some $s_{-i} \in S_{-i}$.

- A strategy is weakly dominant if it weakly dominates all other strategies.
- A weakly dominated strategy is not ruled out by rationality alone but by cautiousness.

Consider the following game:

<table>
<thead>
<tr>
<th></th>
<th>$L$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>1, −</td>
<td>0, −</td>
</tr>
<tr>
<td>$B$</td>
<td>0, −</td>
<td>0, −</td>
</tr>
</tbody>
</table>

→ A weakly dominated strategy becomes strictly dominated if opponents are restricted to completely mix their strategies.

- With iterated weak dominance, the order of elimination does matter. Consider the following game:

<table>
<thead>
<tr>
<th></th>
<th>$L$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>5, 1</td>
<td>4, 0</td>
</tr>
<tr>
<td>$M$</td>
<td>6, 0</td>
<td>3, 1</td>
</tr>
<tr>
<td>$B$</td>
<td>6, 4</td>
<td>4, 4</td>
</tr>
</tbody>
</table>

If $T$ is eliminated first, then we end up with $(B, R)$ surviving while if $M$ is eliminated first, then we end up with $(B, L)$ surviving.
2.2 Rationalizable Strategies

What is the tightest prediction that we can make given only the common knowledge of rationality? Rational players not only don’t use strictly dominated strategies; they also don’t use strategies which are never a best response. If we apply this idea iteratively, we obtain the sets of rationalizable strategies.

- Strategy $\sigma_i \in \Sigma_i$ is a best response (BR) to $\sigma_{-i} \in \Sigma_{-i}$, denoted $\sigma_i \in B_i(\sigma_{-i})$, if
  \[ u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}), \forall \sigma_i' \in \Sigma_i. \]
  
  - Note that for some $\sigma_{-i}$, $B(\sigma_{-i})$ may contain more than one strategy, so the mapping $B : \Sigma_{-i} \to \Sigma_i$ is called best response correspondence.
  
  - Strategy $\sigma_i$ is never a best response to $C \in \Sigma_{-i}$ if for all $\sigma_{-i} \in C$, $\sigma_i \notin B_i(\sigma_{-i})$.

- Rationalizable strategies are those which remain after we iteratively eliminate all strategies which are never a BR.

**Example 2.3. (Rationalizability)** Consider the following game:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>3,3</td>
<td>0,0</td>
<td>0,2</td>
</tr>
<tr>
<td>M</td>
<td>0,0</td>
<td>3,3</td>
<td>0,2</td>
</tr>
<tr>
<td>B</td>
<td>2,2</td>
<td>2,2</td>
<td>2,0</td>
</tr>
</tbody>
</table>

Mixing $T$ and $M$ is never a BR for 1. So, eliminate any mixture of $T$ and $M$. But the pure strategies $T$ and $M$ are not eliminated.

Q: Can we eliminate $R$ once the mixtures of $T$ and $M$ are eliminated?

A: No. $R$ is a BR if 2’s belief puts one half on 1 using $T$ and the other half on 1 using $M$.

\[\square\]

- A convenient procedure to determine rationalizable strategies is (1) to iteratively remove pure strategies which are never a BR (to any mixture) and (2) to check all remaining mixed strategies when no further pure strategies can be removed.

  - If a (mixed) strategy is strictly dominated, then it is never a BR, which implies the following:
    
    Set of rationalizable strategies $\subset$ Set of strategies that survive ISD.
With two players, two sets are equal since a (mixed) strategy is strictly dominated if and only if it is a never BR.

Letting $R_i$ denote the set of player $i$’s rationalizable strategies, we have the following result:

**Theorem 2.1.** For each player $i$, $R_i$ is nonempty and contains at least one pure strategy. Further, for each $\sigma_i \in R_i$, there exist $\sigma_{-i} \in \prod_{j \neq i}$ convex hull $(R_j)$ such that $\sigma_i \in B_i(\sigma_{-i})$.

### 2.3 Nash Equilibrium

ISD and rationalizability only rely on the common knowledge of rationality but they often fail to give us tight prediction. For a tighter prediction, we need to impose restrictions on players’ beliefs about opponents’ behavior.

- A strategy profile $\sigma \in \Sigma$ is a Nash equilibrium if

$$\sigma_i \in B_i(\sigma_{-i}), \forall i \in I.$$  

Two requirements are underlying the concept of Nash equilibrium.

- Each player has correct beliefs about opponents’ strategies.
- Each behaves rationally (or chooses their best response) given their beliefs.

**Example 2.4. (Battle of Sexes)** Consider the following game:

<table>
<thead>
<tr>
<th></th>
<th>Soccer</th>
<th>Drama</th>
</tr>
</thead>
<tbody>
<tr>
<td>Man</td>
<td>(2,1)</td>
<td>(0,0)</td>
</tr>
<tr>
<td></td>
<td>(0,0)</td>
<td>(1,2)</td>
</tr>
</tbody>
</table>

All the strategies are rationalizable but there are only three Nash equilibria: $(S, S)$, $(D, D)$, and $(\frac{2}{3}S + \frac{1}{3}D, \frac{2}{3}S + \frac{2}{3}D)$. For instance, $(\frac{2}{3}S + \frac{1}{3}D, \frac{1}{3}S + \frac{2}{3}D)$ is an NE: For the man, given that the woman plays $\frac{1}{3}S + \frac{2}{3}D$, he is indifferent between $S$ and $D$ since

$$u_M(S, \frac{1}{3}S + \frac{2}{3}D) = 1 = u_M(D, \frac{1}{3}S + \frac{2}{3}D),$$
and thus mixing between $S$ and $D$ is optimal. Similarly, mixing is optimal for the woman. □

A simple test to check if a given strategy profile constitutes NE is provided by the following proposition.

**Proposition 2.1.** The following statements are equivalent:

1. $σ ∈ Σ$ is an NE.
2. For every player $i$, $u_i(σ) = u_i(s_i, σ_{-i})$ for every $s_i ∈ S_i$ given positive weight by $σ$, and $u_i(σ) ≥ u_i(s_i, σ_{-i})$ for every $s_i$ given zero weight by $σ_i$.
3. For every player $i$, $u_i(σ) ≥ u_i(s_i, σ_{-i})$ for every $s_i ∈ S_i$.

- Nash equilibrium can be interpreted as
  - Consequence of rational inference
  - Viewed by players as an obvious way to play games
  - Focal points (Schelling)
  - Self-enforcing agreement prior to games
  - Stable social convention

One crucial feature of NE is its existence in broad circumstances.

**Theorem 2.2.** A Nash equilibrium exists in game $Γ_N = [I, \{S_i\}, \{u_i\}]$ if for all $i = 1, \cdots, I$,

1. $S_i$ is a nonempty, convex, and compact subset of some Euclidean space.
2. $u_i(s_1, \cdots, s_I)$ is continuous in $s$ and quasiconcave in $s_i$.

**Proof.** The proof is based on the Kakutani’s fixed point theorem and can be found in p. 260 of MWG. □
Thanks to this result, we can prove the existence of NE in every finite game.

**Theorem 2.3.** Every game \( \Gamma_N = [I, \{S_i\}, \{u_i\}] \) in which each \( S_i \) is finite, has a mixed strategy Nash equilibrium.

**Proof.** Consider a game \( \Gamma'_N = [I, \{\Delta(S_i)\}, \{u_i\}] \). We show that \( \Gamma'_N \) has a pure strategy equilibrium by verifying that it satisfies (1) and (2) of the Theorem 2.2. First, (1) follows easily from the fact that if \( S_i \) is finite, then \( \Delta(S_i) \) is convex and compact. For (2), remember that \( u_i(\sigma) = \sum_{s \in S} (\prod_{j=1}^{I} \sigma_j(s_j)) u_i(s) \), which implies that \( u_i \) is continuous in \( \sigma \) while being linear in \( s_i \) and thus quasiconcave in \( \sigma_i \).

**Example 2.5.** *(Cournot Duopoly Game)* Suppose that two firms produce an identical product with no cost and face a market demand given by \( p(Q) = \max\{1 - Q, 0\} \), where \( Q = q_1 + q_2 \). So, firm \( i \)'s profit is given by \( \pi_i(q_1, q_2) = q_i p(Q) = q_i(1 - Q) \). The best response function of firm \( i \) is obtained by solving

\[
\max_{q_i \geq 0} q_i(1 - q_i - q_{-i}) \quad \rightarrow \quad B_i(q_{-i}) = \max\{(1 - q_{-i})/2, 0\}.
\]

In Nash equilibrium, we must have \( q_i \leq 1 \) for all \( i \) so that

\[
q_1 = B(q_2) = (1 - q_2)/2 \\
q_2 = B(q_1) = (1 - q_1)/2,
\]

which yields \( q_1 = q_2 = 1/3 \). In fact, this is also the only rationalizable strategy profile. First, any quantity outside \([B(1), B(0)] = [0, 1/2]\) is never a BR for either firm. Thus, only \([0, 1/2]\) survives. Given this, any quantity outside \([B(1/2), B(0)] = [1/4, 1/2]\) is never a BR. Thus, only \([1/4, 1/2]\) survives. Given this, any quantity outside \([B(1/2), B(1/4)] = [1/4, 3/8]\) is never a BR. Thus, only \([1/4, 3/8]\) survives. Repeating this way, only \(1/3\) survives. \( \square \)

### 2.4 Correlated Equilibrium

Remember that if a common randomization device is available, then we can define a correlated strategy as a probability distribution \( \mu \) on \( S \).
• \( \mu \) is a correlated equilibrium (CE) if for all \( i \), and all \( s_i \) and \( s'_i \),

\[
\sum_{s_{-i} \in S_{-i}} \mu(s_i, s_{-i})u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \mu(s_i, s_{-i})u_i(s'_i, s_{-i}),
\]

that is, it is (weakly) better to follow the instruction of the randomization device.

– For any given NE, \( \sigma = (\sigma_1, \cdots, \sigma_I) \), we can define a correlated strategy \( \mu(s) = \prod_{i \in I} \sigma_i(s_i) \) and show that \( \mu \) is a CE.

– In this sense, the set of NE is smaller than the set of CE.

• Why do we often use CE?

– Simplicity: The set of CE is a polytope since it is defined by a finite number of linear inequality.

– Applicability: In some applications, players may better use correlation devices, which are provided by the world in multitudes.

– Learnability: There are procedures which enable players to learn to play correlated equilibria.

– Rationality: Aumann (1986) shows that the correlated equilibrium is what we should expect players to play if they are ‘Bayesian rational’.

Example 2.6. (Correlated Equilibrium) Consider the following game:

|     | Lee 
|-----|-----
| Kim |     |
| T   | L   | R  |
|     | 5, 5| 2, 6|
| B   | 6, 2| 1, 1|

This game has three NE: \((T, R), (B, L), \) and \((\frac{1}{2}T + \frac{1}{2}B, \frac{1}{2}L + \frac{1}{2}R)\). The resulting payoffs are \((2, 6)\), \((6, 2)\), and \((3.5, 3.5)\), respectively.

Players can achieve a higher payoff using a correlated strategy such as \( \mu(T, R) = 1/3 \), \( \mu(T, L) = 1/3 \), and \( \mu(B, L) = 1/3 \), which results in the payoffs \((13/3, 13/3)\). Let us verify that this is a correlated equilibrium: For player 1, \((\text{♦})\) requires

\[
7/3 = \frac{1}{3}U(T, L) + \frac{1}{3}U(T, R) \geq \frac{1}{3}U(B, L) + \frac{1}{3}U(B, R) = 7/3
\]

\[2 = \frac{1}{3}U(B, L) \geq \frac{1}{3}U(T, L) = 5/3.
\]

Similarly, \((\text{♦})\) is satisfied for player 2. \[\square\]
3 Games of Incomplete Information

So far, we have assumed that every player is perfectly informed of the payoffs of all other players. There are, however, many real-life situations where players are not certain about others’ payoffs. For instance, in the Cournot game, one firm may well be imperfectly informed of the other’s cost of production.

These situations are modeled as follows.

- A Bayesian game is a tuple $\Gamma_B = [I, \{\Theta_i\}_{i \in I}, \{S_i\}_{i \in I}, \{u_i\}_{i \in I}, p]$.

  - $\theta_i \in \Theta_i$: Player $i$’s type, which is randomly chosen by nature and only observed by player $i$. $\theta = (\theta_1, \cdots, \theta_I) \in \Theta := \prod_{i \in I} \Theta_i$.

  - $u_i : S \times \Theta_i \to R$: Player $i$’s payoff function, depending on the strategy profile and his type.

  - $p \in \Delta(\Theta)$: Probability distribution over $\Theta$, referred to as common prior.

Example 3.1. (Entry Deterrence I) Consider an industry with 2 firms: an incumbent (player 1) and a potential entrant (player 2). Player 1 decides whether to build a new plant, and simultaneously player 2 decides whether to enter. Imagine that player 2 is uncertain whether player 1’s cost of building is high or low, while player 1 knows his cost. The payoffs are depicted as follows:

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>Not E</th>
<th></th>
<th>E</th>
<th>Not E</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>0, -1</td>
<td>2, 0</td>
<td>Not B</td>
<td>2, 1</td>
<td>3, 0</td>
</tr>
<tr>
<td>Not B</td>
<td>2, 1</td>
<td>3, 0</td>
<td></td>
<td>2, 1</td>
<td>3, 0</td>
</tr>
</tbody>
</table>

Payoffs when 1’s cost is high      Payoffs when 1’s cost is low

Let $p$ denote the prior probability that player 2 assigns to player 1’s being high. In this game, “don’t build” is a strictly dominant strategy for player 1 of the high cost type. Letting $x$ denote the probability that player 1 of low cost type builds, $y$ denote the probability that player 2 enters, the search for equilibrium boils down to finding a pair $(x, y)$. The optimal
strategy for player 1 of low type is

\[ x = 1 \text{ (build)} \quad \text{if } y < \frac{1}{2} \]
\[ = 0 \quad \text{if } y > \frac{1}{2} \]
\[ \in [0, 1] \quad \text{if } y = \frac{1}{2}. \]

Also, the optimal strategy of player 2 is

\[ y = 1 \text{ (enter)} \quad \text{if } x < \frac{1}{2(1-p)} \]
\[ = 0 \quad \text{if } x > \frac{1}{2(1-p)} \]
\[ \in [0, 1] \quad \text{if } x = \frac{1}{2(1-p)}. \]

For instance, \((x = 0, y = 1)\) is an equilibrium for any \(p\), and \((x = 1, y = 0)\) is an NE if and only if \(p \leq 1/2\). \(\qed\)

As seen in the Example 3.1, each player’s strategy depends on his type. In general, the (pure) strategy of player \(i\) in a Bayesian game is a function \(s_i : \Theta_i \to S_i\). Let \(S_i\) be the set of all such functions.

- A (pure strategy) Bayesian Nash equilibrium for \(\Gamma_B = [I, \{\Theta_i\}, \{S_i\}, \{u_i\}, p]\) is a profile of functions, \((s_1(\cdot), \ldots, s_I(\cdot))\) satisfying

\[
(♣) \quad \mathbb{E}_{θ}[u_i(s_i(θ_i), s_{−i}(θ_{−i}), θ_i)] \geq \mathbb{E}_{θ}[u_i(s_i'(θ_i), s_{−i}(θ_{−i}), θ_i)], \forall s_i'(\cdot) \in S_i.
\]

- The condition (♣) holds if and only if the following holds: for all \(θ_i\) occurring with positive probability,

\[
(♠) \quad \mathbb{E}_{θ_{−i}}[u_i(s_i(θ_i), s_{−i}(θ_{−i}), θ_i)|θ_i] \geq \mathbb{E}_{θ_{−i}}[u_i(s_i'(θ_i), s_{−i}(θ_{−i}), θ_i)|θ_i], \forall s_i' \in S_i.
\]

**Proof.** This follows immediately from noting that

\[
\mathbb{E}_{θ}[u_i(s_i(θ_i), s_{−i}(θ_{−i}), θ_i)] = \sum_{θ\in Θ} p(θ)u_i(s_i(θ_i), s_{−i}(θ_{−i}), θ_i)
\]
\[
= \sum_{θ_i \in Θ_i} p(θ_i) \sum_{θ_{−i} \in Θ_{−i}} p(θ_{−i}|θ_i)u_i(s_i(θ_i), s_{−i}(θ_{−i}), θ_i)
\]
\[
= \sum_{θ_i \in Θ_i} p(θ_i)\mathbb{E}_{θ_{−i}}[u_i(s_i(θ_i), s_{−i}(θ_{−i}), θ_i)|θ_i].
\]
The condition (♠) is more convenient to check than (♣).

**Example 3.2. (Bayesian Nash Equilibrium)** Two firms, firm 1 and 2, forms a research consortium to develop a new product. The rule of consortium stipulates that if one firm successfully develops the product, then it is shared by the other. It is known that the value of product is \( v \in (0, 1) \) to both firms and the developing cost of each firm \( i \), \( c_i \), is drawn from a uniform distribution on \([0, 1]\). But the magnitude of \( c_i \) is only known to firm \( i \) while the other only know its distribution.

We look for a BNE of this game. Let \( s_i : [0, 1] \to \{0, 1\} \) denote firm \( i \)'s strategy according to which \( s_i(c_i) = 1 \) if it develops while \( s_i(c_i) = 0 \) if it does not develop. Then, firm \( i \) prefers to develop if

\[
v - c_i \geq v \text{Prob}(s_j(c_j) = 1) \quad \text{or} \quad c_i \leq v[1 - \text{Prob}(s_j(c_j) = 1)],
\]

which implies that the firm \( i \)'s decision will follow a cutoff rule: For some \( \hat{c}_i \in [0, 1] \), it develops if and only if \( c_i \leq \hat{c}_i \). At \( c_i = \hat{c}_i \), it is indifferent between developing and not developing so \( \hat{c}_i = v[1 - \text{Prob}(s_j(c_j) = 1)] \). Thus, we have

\[
\hat{c}_1 = v(1 - \hat{c}_2) \\
\hat{c}_2 = v(1 - \hat{c}_1),
\]

which implies that \( \hat{c}_1 = \hat{c}_2 = v/(1 + v) \).
4 Extensive Form Games

We begin with an example of extensive form game.

**Example 4.1. (A Simple Card Game)** player 1 and 2 each bet $1. Player 1 is given a card which is high or low; each is equally likely. Player 1 can raise the bet to $2 or fold. If player 1 raises, player 2 can call or fold. We can describe the situation by the following ‘tree’:

![Tree Diagram]

- An extensive form game is described by $\Gamma_E \{X, N, p(\cdot), \mathcal{H}, H, A, \alpha(\cdot), \iota(\cdot), \rho(\cdot), u\}$:

  (i) $X = D \cup T$: A set of nodes. $x \in X$ is a decision (terminal) node if $x \in D$ ($T$). $x_0$ is an initial node.

  (ii) $N = \{0, 1, \cdots, I\}$: Players. player 0 is called ‘nature’.

  (iii) $p : X \rightarrow X \cup \{\emptyset\}$: A function assigning to a node $x$ its single immediate predecessor $p(x)$.

  (iv) $\mathcal{H} = \{\cdots, h, \cdots\}$: A collection of information sets or a partition of $D$.

  (v) $H = \{H_0, H_1, \cdots, H_I\}$: A partition of $\mathcal{H}$. $H_i$ is a collection of player $i$’s information sets.

  (vi) $\alpha : X \setminus \{x_0\} \rightarrow A$: A function assigning action, $\alpha(x)$, that leads to a node $x$ from its immediate predecessor.
(vii) \( \iota : \mathcal{H} \to N \): A function assigning to each information set the player (including nature) who moves at the decision nodes in that set. \( C_h \) is the set of actions available at information set \( h \).

(viii) \( \rho : \mathcal{H} \times A \to [0, 1] \): A function assigning probabilities to actions at information sets where nature moves.

(ix) \( u_i : T \to \mathbb{R} \): Player \( i \)'s utility function which assigns payoffs to final nodes. \( u = (u_1(\cdot), \cdots, u_I(\cdot)) \).

• We will always assume that \( \Gamma \) exhibits perfect recall:

  - A node cannot have a successor in its own information set.

\[ \text{Diagram 1} \]

  - If nodes \( x \) and \( y \) are in the same information set of player \( i \), then choices made by \( i \) leading to \( x \) and \( y \) are identical.

\[ \text{Diagram 2} \]

• Player \( i \)'s strategy specifies how he acts at each of his information sets.

  - We must specify his actions at an information set \( h \).

(i) even when other players' actions prevent \( h \) from being reached.
(ii) even when his own action prevents \( h \) from being reached.

- A pure strategy, \( s_i : H_i \to A \), specifies an action \( s_i(h) \in C_h \) for player \( i \) at each \( h \in H_i \). \( s_i(\cdot) \in S_i \).

- There are two ways of randomization.
  - A mixed strategy \( \sigma_i \in \Delta(S_i) \) randomizes over pure strategies. \( \sigma_i \in \Sigma_i \).
  - A behavior strategy \( b_i : h \in H_i \mapsto b_i(h) \in \Delta(C_h) \) randomizes over available actions at each information set of player \( i \). \( b_i \in \mathcal{B}_i \).
  - When studying extensive form games directly, it is more convenient to use behavior strategies.

**Example 4.2. (Strategies in Extensive Form Game)** Consider the following extensive form game:

![Game Diagram](image)

The set of pure strategies for player 2 is \( S_2 = \{wy, wz, xy, wz\} \). A mixed strategy can be written as \( \sigma_2 = (\sigma_2(wy), \sigma_2(wz), \sigma_2(xy), \sigma_2(xz)) \) with \( \sum_{s_2 \in S_2} \sigma_2(s_2) = 1 \) while a behavior strategy can be written as \( b_2 = ((b_2(w), b_2(x)), (b_2(y), b_2(z))) \) with \( b_2(w) + b_2(x) = 1 \) and \( b_2(y) + b_2(z) = 1 \).

Each extensive form game \( \Gamma_E \) has a normal form representation.

**Example 4.3. (Normal Form Representation)** Consider the following extensive form games:

![Game Diagram](image)
Normal form representations of these games are respectively given by

\[
\begin{array}{cccc}
1 & 3,3 & 3,3 & 2,1 & 2,1 \\
D & 1,2 & 4,4 & 1,2 & 4,4 \\
\end{array}
\]

Consider the following extensive form game:

\[
\begin{array}{c}
L \\
U \quad 3,1 \\
D \quad 2,2 \\
\end{array}
\begin{array}{c}
R \\
U \quad 4,1 \\
D \quad 3,0 \\
\end{array}
\begin{array}{c}
1 \\
2 \\
\end{array}
\]

Normal form representation of this game is given by

\[
\begin{array}{cccc}
1 & 4,1 & 2,2 & 3,1 & 3,1 \\
D & 3,0 & 3,0 & 3,1 & 3,1 \\
\end{array}
\]

By merging redundant pure strategies \( Rl \) and \( Rr \) into a single strategy \( R \), we can obtain a reduced normal form as follows:

\[
\begin{array}{ccc}
1 & 4,1 & 2,2 & 3,1 \\
D & 3,0 & 3,0 & 3,1 \\
\end{array}
\]
We now study the problem of Nash equilibrium that arises when applied to the extensive form games.

**Example 4.4. (Entry Deterrence II)** Consider the following *predation* game. Firm E is considering entering a market that currently has a single incumbent (firm I). If it enters, the incumbent can either ‘accommodate’ or ‘fight’. This extensive form game can be depicted as

![Game Tree](image)

Its normal form representation is

<table>
<thead>
<tr>
<th></th>
<th>Fight</th>
<th>Accommodate</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>E</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Out</td>
<td>0, 2</td>
<td>0, 2</td>
</tr>
<tr>
<td>In</td>
<td>-3, -1</td>
<td>2, 1</td>
</tr>
</tbody>
</table>

There are two pure strategy NE: (Out, Fight) and (In, Accommodate). But the former NE is not a sensible prediction for this game since if E chooses to enter, the only reasonable response for I is to accommodate. In other word, the incumbent’s strategy “fight if E is in” is not credible.

In a NE of an extensive form game, there may be information sets which cannot be reached. We say that those information sets are “off the equilibrium path”. We will refine the concept of Nash equilibrium by requiring that an equilibrium strategy specifies optimal behavior at *every* information set, whether or not it is on the equilibrium path. This is called the principle of *sequential rationality.*
5 Subgame Perfect Equilibrium

- A subgame \( \Gamma'_E \) of an extensive form game \( \Gamma_E \) is a subset of \( \Gamma_E \) which
  
  (i) Begins at a single decision node and contains this node, all of its successors, and no other nodes.
  
  (ii) Is closed under information sets: if \( x \in \Gamma'_E \), \( x \in h \), and \( y \in h \), then \( y \in \Gamma'_E \).

**Example 5.1. (Subgames)** Consider the following extensive form game:

```
  1
  /|
  / \
 2b 2c
  |
  1c 1d
  /   |
  3g 3h
  /   |
  1i
```

Only \( a, b, \) and \( i \) are the initial nodes of subgames while the others are not. \( \square \)

- A profile \( \sigma \) is a subgame perfect equilibrium (SPE) of \( \Gamma_E \) if it induces a NE in every subgame of \( \Gamma_E \).
  
  - In the Example 4.4, the only SPE is (In,Accommodate).
  
  - We say that a game \( \Gamma_E \) has perfect information if every information set is a singleton.
  
  - Zermelo's theorem: *Every finite game of perfect information has a pure strategy SPE.* Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique SPE. \( \leftarrow \) Easy to prove using ‘backward induction procedure’.

**Example 5.2. (Entry Deterrence III)** Suppose that there are two entrants (player 1 and 2) and one incumbent (player 3) in an industry. Entrants decide sequentially whether to enter (\( E \)) or not (\( N \)). After entry, the incumbent can choose to fight (\( F \)) or accommodate (\( A \)). Each firm’s payoff is given as 5, 2, or \(-1\) if it is a monopolist, duopolist, or triopolist, respectively. It costs 1 (3) for the incumbent (entrant) to fight.
In this game, there are 64 pure strategy profiles and 20 Nash equilibria, which yield 3 outcomes, (0,0,5), (0,2,2), and (2,0,2). For instance, one NE which yields (0,2,2) is 
\((N, E'E''', AA' A''')\). Among all NE, only \((E, E''N''', AA' A''')\) is an SPE whose outcome is \((2,0,2)\). □

**Example 5.3. (Centipede Game)** There are 2 players who move alternatingly. Each player begins with $1 in his pile. When moving, a player can stop the game or continue. If a player continues, his pile is reduced by $1 while his opponent’s pile is increased by $2. The game ends when a player stops or both piles have $100; players get their piles.

\[
\begin{array}{cccccccccccc}
1 & C & 2 & C & 1 & C & 2 & C & \ldots & 1 & C & 2 & C \\
1,1 & 0,3 & 2,2 & 1,4 & 98,98 & 97,100 & 99,99 & 98,101
\end{array}
\]

In the unique SPE, everyone always stops, yielding the outcome \((1,1)\). (Note: Player 1 also stops immediately in all NE) □

Subgame perfection requires Nash behavior in every subgame. This isn’t always enough to guarantee sequential rationality. Consider the following example.

**Example 5.4. (Entry Deterrence IV)** There are one incumbent and one entrant. Now the entrant (player 1) can enter \((E)\) or enter aggressively \((E')\). The incumbent (player 2)
cannot observe what kind of entry has occurred. Aggressive entry is more profitable for 1 if 2 accommodates while it is more costly if 2 fights.

There are two SPE: $(\hat{E}, A)$ and $(N, F)$. Note that against either entry, 1 is worse off if he fights, which implies that $\hat{E}, A$ is the only reasonable SPE. Still, $(N, F)$ is an SPE since the whole game is the only subgame so subgame perfection has no bite here.
6 Refinements of SPE

We require that each player plays optimally at every information set, not just every subgame, where he moves. In case a player reaches an information set containing multiple nodes, he forms a belief over those nodes and should play an optimal response to his belief.

- Given an extensive form game $\Gamma_E$, beliefs are a map $\mu : D \rightarrow [0, 1]$ satisfying $\sum_{x \in h} \mu(x) = 1$ for all information sets $h$.
  - If $x \in h \in H_i$, then $\mu(x)$ is the probability player $i$ assigns to being at node $x$ given that he is at $h$.
- A strategy profile $b$ is sequentially rational given beliefs $\mu$ if for each player $i$ and each information set $h \in H_i$, player $i$’s behavior conditional on $h$ being reached, maximizes his expected utility, given $b_{-i}$ and $\mu$.
  - All of our refinements combine sequential rationality with some restriction on beliefs.
  - One essential restriction is that a system of beliefs is Bayesian: Letting $P(x|b)$ denote the probability that node $x$ is reached given the behavioral strategy $b$, beliefs are Bayesian if, for every $h \in H$ and $x \in h$,
    $$\mu(x) = \frac{P(x|b)}{\sum_{x' \in h} P(x'|b)} \text{ whenever } \sum_{x' \in h} P(x'|b) > 0.$$  

Example 6.1. (Example 5.4 Continued) Suppose that player 1 plays $b_1 = \frac{1}{4}N + \frac{1}{2}E + \frac{1}{4}\hat{E}$. Then, $P(x|b) = 1/4$, $P(y|b) = 1/2$, and $P(z|b) = 1/4$. Thus,
  $$\mu(x) = \frac{1/4}{3/4} = \frac{1}{3} \text{ and } \mu(y) = \frac{2}{3}.$$  

- The strategy-beliefs pair $(b, \mu)$ is a (weak) perfect Bayesian equilibrium (PBE) if
  (i) $b$ is sequentially rational given $\mu$.
  (ii) $\mu$ is Bayesian given $b$.
Note that no restriction is imposed on beliefs at information sets that are never reached.

**Example 6.2. (Example 5.4 Continued)** Player 2 must play a BR to \((\mu(x), \mu(y))\), which is \(A\) since it is dominant at this information set. Thus, \(((\tilde{E}, A), \mu(y) = 1)\) is the unique PBE. □

In some games, PBE is not strong enough.

**Example 6.3. (Entry Deterrence V)** Now, the entrant (player 1) first decides whether to enter or not and then both entrant and incumbent have to decide whether to fight or accommodate.

This game has a unique SPE of \(((E, A), A')\): In the subgame, \(A\) is dominant for 1, which induces 2 to play \(A'\). And then 1 plays \(E\). This is also a PBE with \(\mu(y) = 1\). There is another PBE, \(((N, A), F', \mu(x) = 1)\), which is not SPE. Note that \(\mu(x)\) is not inconsistent with the Bayes’ rule since the node \(x\) is not reached. □

The above example demonstrates that we need to impose some restriction on beliefs at off-the-equilibrium information sets.

- A system of beliefs \(\mu\) is **consistent given** \(b\) if there exists a sequence of strategy-beliefs pairs \(b^k, \mu^k\) such that
  1. each \(b^k\) is completely mixed, that is \(b^k \gg 0\),
  2. each \(\mu^k\) is Bayesian given \(b^k\), and
  3. \((b^k, \mu^k)\) converges to \((b, \mu)\).
− Note that given (a), every information set is reached so the Bayes’ rule uniquely determines \( \mu^k \) at every information set.
− Note that \( b^k \) need not be BR to \( \mu^k \).
− Idea here is that \( \mu \) should be close to the beliefs that would arise if players had small probabilities of making mistakes.

• A strategy-beliefs pair \((b, \mu)\) is a *sequential equilibrium* \((SE)\) if
  
  (i) \( b \) is sequentially rational given \( \mu \)
  
  (ii') \( \mu \) is consistent given \( b \).

− Note that going from PBE to SE, only (ii) has been replaced by (ii').

**Example 6.4.** (Example 6.3 Continued) Can \(((N, A), F', \mu(x) = 1)\) be an SE? No.
Consistency requires that for some sequence \((b^k, \mu^k) \gg 0\) converging to \(((E, A), A')\),

\[
\mu(x) = \lim_{k \to \infty} \mu^k(x) = \lim_{k \to \infty} \frac{b^k(E)b^k_1(F)}{b^k_1(F)b^k(F) + b^k(E)b^k_1(A)} = \lim_{k \to \infty} \frac{b^k(F)}{b^k_1(F) + b^k(A)} = \frac{b_1(F)}{b_1(F) + b_1(A)} = 0.
\]

• While the definition of consistency is not very intuitive, it implies the followings:

  − \( \mu \) is Bayesian given \( \sigma \)
  − Parsimony: Let \( D_x \) be the set of deviations from \( \sigma \) required to reach \( x \). If \( x, y \in h \) and \( D_y \) is a strict subset of \( D_x \), then \( \mu(x) = 0 \).
  − Cross-player consistency: Two player with the same information must have the same beliefs about opponents’ deviations.

**Theorem 6.1.** *Sequential equilibrium exists and are subgame perfect.*

We have the following relationship among equilibrium concepts:

\[
\text{SE} \quad \Longrightarrow \quad \text{SPE} \quad \Longrightarrow \quad \text{NE}
\]

\[
\text{PBE} \quad \Downarrow
\]

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Example 6.5. (Centipede with a Possibly Generous Player) Continuing costs you 1 but benefit your opponent 5. Player 2 has a 1/20 chance of being crazy and always choosing to continue. Player 2’s type is not observed by player 1.

\[
\begin{array}{c}
\text{Normal (19/20)} \\
S^1 \\
0,0 \\
2 \\
S^2 \\
4,4 \\
3,9 \\
8,8 \\
\text{Crazy (1/20)} \\
C^1 \\
1_y \\
2 \\
C^2 \\
1_z \\
8,8 \\
\end{array}
\]

In order to compute SE, \((b, \mu)\), we first take care of easy parts of strategies and beliefs and then solve backward for the remaining parts. First of all, \(\mu(w) = 19/20\) by Bayes’ rule and \(b_2(s^2) = 1\). At the second information set of player 1, consistency requires that for some sequence \(b^k \gg 0\) converging to \(b\),

\[
\mu(y) = \lim_{k \to \infty} \mu^k(y) = \lim_{k \to \infty} \frac{19}{20} b_1^k(C^2) b_2^k(c^1) = \lim_{k \to \infty} \frac{19 b_2^k(c^1)}{19 b_2^k(c^1) + 1} = \frac{19 b_2(c^1)}{19 b_2(c^1) + 1}.
\]

We consider three cases for the player 1’s strategy at his second information set: (i) If \(b_1(S^2) = 1\), then we must have \(b_2(s^1) = 1\) and thus \(\mu(y) = 0\), to which \(b_1(C^2) = 1\) is optimal, contradiction. (ii) If \(b_1(C^2) = 1\), then we must have \(b_2(c^1) = 1\) and thus \(\mu(y) = 19/20\), to which \(b_1(S^2) = 1\) is optimal (since \(4 > \frac{1}{20} 3 + \frac{19}{20} 4\)), contradiction. (iii) Player 1 mixes between
$S^2$ and $C^2$, which requires $\mu(y)3 + (1 - \mu(y))8 = 4$ or $\mu(y) = 4/5$. This in turn requires $b_2(c^1) = 4/19$. For 2 to be willing to mix between $c^1$ and $s^1$, $5 = (1 - b_1(C^2))4 + b_1(C^2)9$ or $b_1(C^2) = 1/5$. At his first information set, 1 gets 0 from $S^1$ and

\[
\frac{19}{20} \left( \frac{15}{19}(-1) + \frac{4}{19} \left( \frac{4}{5} + \frac{1}{5}3 \right) \right) + \frac{1}{20} \left( \frac{4}{5} + \frac{1}{5}8 \right) = \frac{1}{4}
\]

from $C^1$, which means $b_1(C^1) = 1$. Thus, a unique SE is

\[
((C^1, \frac{4}{5}S^2 + \frac{1}{5}C^2), (\frac{15}{19}s^1 + \frac{4}{19}c^1, s^2), \mu(w) = \frac{19}{20}, \mu(y) = \frac{4}{5}).
\]

There are other PBE'a. For instance,

\[
((S^1, S^2), (s^1, s^2), \mu(w) = \frac{19}{20}, \mu(y) \geq \frac{4}{5})
\]

is a PBE but not SE. □
7 Signaling Games

These are games in which player 1 (sender) receives a private signal and then chooses an action (a message) while player 2 (receiver), observing only the message, chooses an action (a response). These games have many applications. But even sequential equilibrium can fail to produce relatively precise prediction of play. This motivates us to further refine SE.

• A typical signaling games consists of
  
  − $T$: Set of player 1’s types.
  − $\pi$: Prior distribution on $T$ with $\pi(t)$ for all $t \in T$.
  − $M$: Set of player 1’s actions (messages).
  − $M(t)$: Messages available to type $t$.
  − $T(m)$: Set of types who can send message $m$.
  − $R(m)$: Set of actions (responses) available to 2 after receiving message $m$.
  − $u_i(t, m, r)$: Player $i$’s utility function.

Example 7.1. (Signaling Game I) A signaling game is often described as follow:
We denote by $\mu(t|m)$ the (posterior) probability that 2 assigns to 1 being of type $t$ after receiving message $m$. For refining SE, we will ask what is a reasonable restriction on $\mu$.

- Players’ strategies are given by

  - $b_1(m|t)$: Probability that the sender of type $t$ chooses $m$.
  - $b_2(r|m)$: Probability that the receiver chooses $r$ after $m$.
  - $B_1(b_2|t)$: Player 1’s best responses to $b_2$, if his type is $t$.
  - $B_2(m, \mu)$: Player 2’s best responses to $m$, given his belief is $\mu$.
  - $B_2(m, I) = \bigcup_{\mu: \mu(I|m) = 1} BR(m, \mu)$: Set of player 2’s best responses to his beliefs placing all weights on $I \subset T$.

- In signaling games, NE and SE can be stated as

  - $b$ is NE if there exists a belief $\mu$ such that
    (i) For each $t$, $b_1(m|t) > 0 \implies m \in B_1(b_2|t)$
    (ii) For each $m$ sent with positive probability, $b_2(r|m) > 0 \implies r \in B_2(\mu, m)$
    (iii) $\mu$ is Bayesian given $b$.
  - $(b, \mu)$ is SE if
    (i) For each $t$, $b_1(m|t) > 0 \implies m \in B_1(b_2|t)$
    (ii') For each $m$, $b_2(r|m) > 0 \implies r \in B_2(\mu, m)$
    (iii) $\mu$ is Bayesian given $b$.
  - Consistency does not restrict beliefs at unreached information sets any further than PBE does. Thus, in signaling games,

$$SE \iff PBE \implies NE.$$ 

**Example 7.2. (Signaling Game II)** Consider the following signaling game.
Consider SE'a:

\[(b_1(m_1|t_1) = b_1(m_1|t_2) = 1, b_2(r_1) = 1, \mu(t_2|m_2) \geq 1/2)\]

and

\[(b_1(m_1|t_1) = b_1(m_1|t_2) = 1, b_2(r_1) \geq 1/2, \mu(t_2|m_2) = 1/2).\]

Here, \(m_2\) is strictly dominated by \(m_1\) for \(t_2\). Thus, after observing \(m_2\), the receiver should believe that the sender is of type \(t_1\) or \(\mu(t_1|m_2) = 1\), to which \(b_2(r_1) \geq 1/2\) is not optimal.

\[\Box\]

- An SE \(b, \mu\) fails the \textit{dominance criterion} if there exist some unsent message \(m\) and and some types \(t\) and \(t'\),

  (i) \(\max_{r \in R(m)} u_1(t, m, r) < \min_{r \in R(\hat{m})} u_1(t, \hat{m}, r)\) for some \(\hat{m}\).

  (ii) \(\max_{r \in R(m)} u_1(t', m, r) \geq \min_{r \in R(\hat{m})} u_1(t', \hat{m}, r)\) for all \(\hat{m}\).

  (iii) \(\mu(t|m) > 0\).

- (i) says \(m\) is dominated for \(t\) while (ii) says it is not dominated for some other type \(t'\).

- According to (iii), receiver assigns a positive probability to \(t\) after observing \(m\).
Example 7.3. (The Beer-Quiche Game (Cho and Kreps (1987))) The sender can be wimpy or surly. Surly type likes beer for breakfast; wimpy type likes quiche. Getting the preferred breakfast yields 1 unit of payoff. The receiver likes to duel with wimpy type but walk away from surly type: To duel with wimpy (surly) type yields the receiver 1 (-1) while to walk away yields 0. Avoiding duel is worth 2 units of payoffs to all sender types.

Consider the following SE’a:

(†) \((b_1(B|t_w) = b_1(B|t_s) = 1, (b_2(W|B) = 1, b_2(D|Q) = 1(\geq 1/2)), \mu(t_w|Q) \geq 1/2(= 1/2))\)

and

(‡) \((b_1(Q|t_w) = b_1(Q|t_s) = 1, (b_2(W|Q) = 1, b_2(D|B) = 1(\geq 1/2)), \mu(t_w|B) \geq 1/2(= 1/2))\)

We ask if (‡) is reasonable. The wimpy type is getting his highest possible payoff by choosing quiche; he can only get hurt by switching to beer. Therefore, if the receiver sees beer, he should expect a surly type and walk away. Expecting this, surly types should deviate to beer. \(\Box\)
Let \((b, \mu)\) be an SE of the signaling game. Letting \(u^*_1(t)\) denote the equilibrium payoff of type \(t\), \((b, \mu)\) fails the \textit{intuitive criterion} if there exist some unsent message \(m\) and some type \(t'\),

\[
\tag{\S} \quad u^*_1(t') < \min_{r \in B_2(T(m) - D(m), m)} u_1(t', m, r),
\]

where

\[
D(m) := \{ t \in T(m) : u^*_1(t) > \max_{B_2(T(m), m)} u_1(t, m, r) \}.
\]

- In words, \(D(m)\) is the set of types who get less than their equilibrium payoff by choosing \(m\), provided that the receiver does not play a never-a-best-response strategy.
- The equilibrium fails the intuitive criterion if there exists a type \(t'\) who would necessarily do better by choosing \(m\) than in equilibrium as long as the receiver’s beliefs assign probability 0 to types in \(D(m)\).

**Example 7.4.** \textbf{(Example 7.3 Continued)} In the equilibrium \((\dagger)\), \(B\) is unused. We have \(B_2(T(B), B) = \{D, W\}\) and

\[
\begin{align*}
u_1^*(t_w) &= 3 > 2 = u_1(t_w, B, W) \\
&> 0 = u_1(t_w, B, D).
\end{align*}
\]

But \(u_1^*(t_s) = 2 < 3 = u_1(t_s, B, W)\). Thus, \(D(B) = \{t_w\}\) and \(T(B) - D(B) = \{t_s\}\), which implies

\[
u_1^*(t_s) < \min_{r \in B_2(t_s, B)} u_1(t_s, B, r) = u_1(t_s, B, W) = 3,
\]
satisfying \((\S)\). In the equilibrium \((\ddagger)\), \(Q\) is unused. \(t_s\) should never deviate, so only \(t_w\) might play \(Q\). If \(Q\) is played, then the receiver plays \(D\). Since this gives \(t_w\) a payoff of \(1 < u_1^*(t_w) = 2\), he does not deviate. Thus, \((\ddagger)\) does not violate the intuitive criterion if \(\mu(t_w | Q) = 1\). \(\square\)
8 Repeated Games

In many applications, players face the same interaction repeatedly. How does this affect our predictions of play?

- Consider a stage game \( G = \{ N, \{ A_i \}_{i \in N}, \{ u_i \}_{i \in N} \} \) with \( n \) players
  - \( a_i \in A_i \): Pure action. \( a = (a_1, \cdots, a_n) \in A \)
  - \( \alpha_i \in \Delta A_i \): Mixed action. \( \alpha = (\alpha_1, \cdots, \alpha_n) \)

- Consider now the infinite repetition of \( G \), denoted \( G^\infty = \{ N, \{ S_i \}_{i \in N}, \{ \pi_i \}_{i \in N}, \delta \} \)
  - \( a^t \in A \): Action profile in period \( t = 0, 1, \cdots \)
  - \( h^t = (a^0, a^1, \cdots, a^{t-1}) \): History as of period \( t = 0, 1, \cdots \). \( h^t \in H^t \). \( H^0 = \{ \emptyset \} \).
  - \( H = \bigcup_{t=0}^\infty H^t \): All finite histories
  - \( H^\infty = A^\infty \): All infinite histories. \( h^\infty = (a^0, a^1, \cdots) \in H^\infty \)
  - \( s_i : H \to A_i \): Pure strategy. \( s_i \in S_i, s = (s_1, \cdots, s_n) \in S \)
  - \( \sigma_i : H \to \Delta A_i \): Behavior strategy. \( \sigma_i \in \Sigma_i, \sigma = (\sigma_1, \cdots, \sigma_n) \in \Sigma \)
  - \( a^t(s) \): Action profile in period \( t \) induced by a pure strategy profile \( s \).
    \[
    \begin{align*}
    a^0(s) &\equiv (s_1(\emptyset), \cdots, s_n(\emptyset)) \\
    a^1(s) &\equiv (s_1(a^0(s)), \cdots, s_n(a^0(s))) \\
    a^2(s) &\equiv (s_1(a^0(s), a^1(s)), \cdots, s_n(a^0(s), a^1(s))) \\
    &\ddots
    \end{align*}
    \]
  - Given a strategy profile \( s \), player \( i \)'s payoff is given by
    \[
    \pi(s) = (1 - \delta) \sum_{t=0}^{\infty} \delta^t u_i(a^t(s))
    \]
  - Note that the subgame starting after any history has the same structure as the original repeated game.
Example 8.1. (Infinitely Repeated Prisoners’ Dilemma) Let $G$ be given as

<table>
<thead>
<tr>
<th></th>
<th>Cooperate</th>
<th>Defect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooperate</td>
<td>$-1, -1$</td>
<td>$-3, 0$</td>
</tr>
<tr>
<td>Defect</td>
<td>$0, -3$</td>
<td>$-2, -2$</td>
</tr>
</tbody>
</table>

One easy NE (and also SPE) for $G^\infty$ is

$$s_i(h_t) = D, \forall h_t \in H.$$ 

If $\delta \geq \frac{1}{2}$, then there is another NE (and also SPE) in which players use ‘grim trigger strategy’:

$$s_i(h_t) = \begin{cases} 
C & \text{if } t = 0 \text{ or } a^s = (C, C), \forall s \leq t - 1 \\
D & \text{otherwise}
\end{cases}$$

that is to say ‘I will defect forever if you have ever defected’. To check this is an SPE, consider two kinds of history, one where no one has deviated yet and the other where someone has deviated before. In the latter history, no one wants to deviate from $D$, given that the other always plays $D$. In the former, if agent $i$ deviates to play $C$, then from that time on, he will

$$(1 - \delta)(0 + \sum_{t=1}^{\infty} \delta^t \times (-2)) = -2\delta$$

while if he does not deviate, he obtains $-1$, which is (weakly) greater than $-2\delta$ if $\delta \geq \frac{1}{2}$.

\[\square\]

Observation: It is an SPE of $G^\infty$ for players to play an NE action profile of $G$ in every period.

There is a useful principle for checking whether a strategy profile in repeated games constitutes an SPE.

**Proposition 8.1 (One-Shot Deviation Principle).** A strategy profile $\sigma$ is an SPE of $G^\infty$ if and only for all $i \in N$ and all $h_t \in H$, there is no $\hat{\sigma}_i \in S_i$ which agrees with $\sigma_i$ at all histories except $h_t$ and which yields player $i$ a higher payoff than $\sigma_i$ does in the subgame following $h_t$. 

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Example 8.2. (Tit-for-tat Strategy in the Repeated Prisoners’ Dilemma) Consider the ‘tit-for-tat’ strategy as follows:

\[
s_i(h^t) = \begin{cases} 
C & \text{if } t = 0 \text{ or } a_{j}^{t-1} = C \text{ for } t \geq 2 \\
D & \text{if } a_{j}^{t-1} = D \text{ for } t \geq 2
\end{cases}
\]

that is to say ‘I do to you whatever you did to me in the last period’. First of all, the tit-for-tat constitutes an NE if and only if \(\delta \geq \frac{1}{2}\). The best deviation strategy is to defect once and go back to cooperate (check this), which yields

\[
(1 - \delta)(0 - 3\delta - 1(\delta^2 + \delta^3 + \cdots)) = -3\delta + 2\delta^2.
\]

This is not greater than the equilibrium payoff \(-1\) if and only if \(\delta \geq \frac{1}{2}\).

Claim The tit-for-tat constitutes an SPE if and only if \(\delta = \frac{1}{2}\).

Proof. We need to consider four subgames starting after \(a^{t-1} = (D, D), (C, C), (D, C),\) and \((C, D),\) respectively.

1. \(a^{t-1} = (D, D)\) : One-shot deviation by player 1 to \(C\) will result in the play path \((C, D), (D, C), (C, D), (D, C), \cdots\), for a payoff of

\[
(1 - \delta)(-3 \times (1 + \delta^2 + \delta^4 + \cdots) + 0 \times (\delta + \delta^3 + \cdots)) = \frac{-3}{1 + \delta}
\]

while ‘no deviation’ yields \(-2\). It is necessary to have

\[
\frac{-3}{1 + \delta} \leq -2 \text{ or } \delta \leq \frac{1}{2}
\]

2. \(a^{t-1} = (C, C)\) : One-shot deviation to \(D\) by player 1 will result in the play path \((D, C), (C, D), (D, C), (C, D), \cdots\), for a payoff of

\[
(1 - \delta)(0 \times (1 + \delta^2 + \delta^4 + \cdots) - 3 \times (\delta + \delta^3 + \cdots)) = \frac{-3\delta}{1 + \delta}
\]

while ‘no deviation’ yields \(-1\). It is necessary to have

\[
\frac{-3\delta}{1 + \delta} \leq -1 \text{ or } \delta \geq \frac{1}{2}.
\]
(3) $a^{t-1} = (D, C)$ : One-shot deviation to $D$ by player 1 results in the play path $(D, D), (D, D), (D, D), \ldots$, for a payoff of $-2$ while ‘no deviation’ results in $(C, D), (D, C), (C, D), \ldots$, for a payoff of $\frac{-3}{1+\delta}$ while ‘no deviation’ yields $-1$. It is necessary to have
\[
\frac{-3}{1+\delta} \geq -2 \text{ or } \delta \geq \frac{1}{2}.
\]

(4) $a^{t-1} = (C, D)$ : One-shot deviation to $C$ by player 1 results in the play path $(C, C), (C, C), (C, C), \ldots$, for a payoff of $-1$ while ‘no deviation’ results in $(D, C), (C, D), (D, C), \ldots$, for a payoff of $\frac{-3\delta}{1+\delta}$. It is necessary to have
\[
\frac{-3\delta}{1+\delta} \geq -1 \text{ or } \delta \leq \frac{1}{2}.
\]

There are a plethora of equilibria in repeated games. In fact, any feasible payoff can be supported as an equilibrium outcome as long as it is not too low. What does then ‘too low’ mean?

- Let us first introduce a few definitions as follows:

  - The set of feasible payoffs in $G$ is
    \[ F \equiv \text{Convex Hull of } \{ v \in \mathbb{R}^n | \exists a \in A \text{ s.t. } v = u(a) \}. \]
  
  - Player $i$’s minmax payoff in $G$ is given as
    \[ v_i = \min_{\alpha_{-i} \in \prod_{j \neq i} \Delta A_j} \left[ \max_{a_i \in A_i} u_i(a_i, \alpha_{-i}) \right]. \]

    \[ \rightarrow \text{ Let } \hat{\alpha}_i = (\hat{\alpha}_i^i, \hat{\alpha}_i^{-i}) \text{ denote a (mixed) minmax action profile for player } i \text{ such that } \hat{\alpha}_i^i \in B_i(\alpha_i^i) \text{ and } v_i = u_i(\hat{\alpha}_i^i, \hat{\alpha}_i^{-i}). \]

  - A payoff vector $v \in \mathbb{R}^n$ is (strictly) individually rational if $v_i > v_i$ for all $i \in N$.
  
  - Thus, the set of feasible and individually rational payoffs is given by
    \[ F^{IR} \equiv \{ v = (v_1, \ldots, v_n) \in F | v_i > v_i, \forall i \in N \}. \]
Example 8.3. (Feasible and Individually Rational Payoffs in the Battle of Sexes) Let $G$ be given as

<table>
<thead>
<tr>
<th></th>
<th>Woman</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>(2,1)</td>
<td>D</td>
</tr>
<tr>
<td>D</td>
<td>(0,0)</td>
<td>(1,2)</td>
</tr>
</tbody>
</table>

Here, $v_1 = \min_{\alpha_2(S) \in [0,1]} \left[ \max\{2\alpha_2(S), (1 - \alpha_2(S))\} \right]$, which yields $v_1 = \frac{2}{3}$. Also, $\hat{\alpha}_2(S) = 1$ and $\hat{\alpha}_1(S) \in [0,1]$. By symmetry, $v_2 = v_1 = \frac{2}{3}$.

Thus, $F^{IR}$ is illustrated as follows

Proposition 8.2. Suppose that $\sigma$ is an NE of $G^\infty$. Then, $\pi(\sigma) \geq v_i$.

Proof. Given $\sigma_{-i}$, player $i$ can always play the myopic best response in each period. □

Proposition 8.3 (NE Folk Theorem). Consider $v \in F^{IR}$. Then, there is $\delta < 1$ such that for each $\delta > \delta$, there exists an NE of $G^\infty$ with payoffs $v$.

Proof. Suppose for simplicity that there is a pure action profile $\bar{a}$ such that $u_i(\bar{a}) = v_i, \forall i \in N$. Consider the following ‘grim trigger strategy’:

$$
\sigma_i(h^t) = \begin{cases} 
\bar{a}_i & \text{if } t = 0 \text{ or } a^s = \bar{a}, \forall s \leq t - 1 \\
\hat{\alpha}_i^j & \text{if } a_{-j}^s = \bar{a}_{-j}, \forall s \leq t - 1 \text{ and } a_{-j}^{s'} \neq \bar{a}_j \text{ for some } s' \leq t - 1 \\
\text{any } a_i & \text{otherwise}
\end{cases}
$$

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If every player follows this equilibrium strategy, then the resulting path will be \( \bar{a}, \bar{a}, \bar{a}, \cdots \) with payoff \( v_i \) for player \( i \). If player \( i \) unilaterally deviates, he can get at most
\[
(1 - \delta) \max_a u_i(a) + \delta v_i,
\]
which is less than \( v_i \) if \( \delta \) is sufficiently close to 1.

What if \( v \) above cannot be obtained using a pure action profile?

Example 8.4. (Example 8.3 Continued) For instance, \( v = (3/2, 3/2) \in F^{IR} \) but no pure action profile can generate it. In such case, replace \( \bar{a} \) above with a correlated strategy \( \left( \frac{1}{2}(S, S) + \frac{1}{2}(D, D) \right) \). That is to say, players see the result of a coin toss; they play \((S, S)\) if head comes up and \((D, D)\) if tail. If say player 1 does not follow the instruction, then player 2 plays \( \hat{\alpha}^1_2(S) = \frac{1}{3} \) thereafter. \( \square \)

However, the grim trigger strategy may not constitute an SPE since once someone deviates, to punish him using the minmax action profile can hurt the punishers themselves. One way to fix this problem is to punish a deviator by reverting to an NE strategy of the stage game.

Proposition 8.4 (Nash Reversion Perfect Folk Theorem). Let \( \alpha^* \) be an NE of \( G \) and let \( v \in F \) satisfy \( v_i > u_i(\alpha^*) \). Then, there is \( \delta > 1 \) such that for each \( \delta > \delta \), there exists an SPE of \( G^\infty \) with payoff \( v \).

Proof. Suppose for simplicity that there exists \( \bar{a} \in A \) such that \( u(\bar{a}) = v \). Consider the following ‘Nash reversion strategy’:
\[
\sigma_i(h^t) = \begin{cases} 
\bar{a}_i & \text{if } t = 0 \text{ or } a^* = \bar{a}, \forall s \leq t - 1 \\
\alpha_i^* & \text{otherwise}
\end{cases}
\]

We only need to consider two kinds of history, one where no one has deviated yet and the other where someone has deviate before. Clearly, no one wants to deviate in the latter history. In the former, the equilibrium path will be \( \bar{a}, \bar{a}, \bar{a}, \cdots \). If player \( i \) one-shot deviates, then he will get at most
\[
(1 - \delta) \max_a u_i(a) + \delta u_i(\alpha^*),
\]
which is less than \( v_i \) if \( \delta \) is sufficiently close to 1. \( \square \)
Example 8.5. (Repeated ‘Game of Chicken’) Let $G$ be given as

$$
\begin{array}{c|cc}
S & N \\
\hline
(2,2) & (1,5) \\
(5,1) & (0,0)
\end{array}
$$

Here, $v_1 = \min_{\alpha_2(S) \in [0,1]} \max\{2\alpha_2(S) + (1 - \alpha_2(S)), 5\alpha_2(S)\}$, which yields $v_1 = 1 = v_2$. There exists one mixed NE strategy profile $(\frac{1}{4}S + \frac{3}{4}N, \frac{1}{4}S + \frac{3}{4}N)$. Calling it $\alpha^*$, we have $u_1(\alpha^*) = u_2(\alpha^*) = \frac{5}{4}$.

Can we expand the set of SPE payoffs to $F^{IR}$? The answer is yes.

Proposition 8.5 (Perfect Folk Theorem). Suppose that

(1) $n = 2$ or

(2) $n \geq 3$ and there are no two player $i$ and $j$ such that $u_j(\cdot) = a + bu_i(\cdot)$ for $a, b > 0$.

Let $v \in F^{IR}$. Then, there is $\delta < 1$ such that for each $\delta > \delta$, there exists an SPE of $G^\infty$ with payoff $v$.

Idea of Proof: Use ‘stick and carrot’ strategy. Stick: Punish a deviating player for a sufficient amount of time. Carrot: Provide each punisher with an incentive to punish by promising him a carrot once the punishment phase elapses.