ON THE SET OF ALL CONTINUOUS FUNCTIONS WITH UNIFORMLY CONVERGENT FOURIER SERIES

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(Communicated by Andreas R. Blass)

ABSTRACT. In this article we calculate the exact location in the Borel hierarchy of $UCF$, the set of all continuous functions on the unit circle with uniformly convergent Fourier series. It turns out to be complete $F_{\sigma \delta}$. Also we prove that any $G_{\delta \sigma}$ set that includes $UCF$ must contain a continuous function with divergent Fourier series.

INTRODUCTION

There are many criteria for uniform convergence of a Fourier series on the unit circle. One can find those tests in [Zy]. In the present paper, we study $UCF$ from the point of view of descriptive set theory. In [Ke], it was a conjecture that $UCF$ is complete $F_{\sigma \delta}$ (i.e., $F_{\sigma \delta}$ but not $G_{\delta \sigma}$). A lot of natural complete $F_{\sigma \delta}$ sets have been found. For example, the collection of reals that are normal or simply normal to base $n$ [KL]; $C^\infty(T)$, the class of infinitely differentiable functions (viewed as a $2\pi$-periodic function on $\mathbb{R}$); and $UC_X$, the class of convergent sequences in a separable Banach space $X$, are complete $F_{\sigma \delta}$ [Ke]. It turns out that $UCF$ is complete $F_{\sigma \delta}$. We give two different proofs for it. Ajtai and Kechris [AK] have shown that $ECF$, the set of all continuous functions with everywhere convergent Fourier series, is complete $CA$, i.e., coanalytic non-Borel. We show that there is no $G_{\delta \sigma}$ set $A$ such that $UCF \subset A \subset ECF$. Hence any $G_{\delta \sigma}$ set that includes $UCF$ must contain a continuous function with divergent Fourier series. From this point of view, although there are many natural complete $F_{\sigma \delta}$ sets, we can claim that $UCF$ is a very interesting set in analysis.

DEFINITIONS AND BACKGROUND

Let $\mathbb{N} = \{1, 2, 3, \cdots \}$ be the set of positive integers and $\mathbb{N}^\mathbb{N}$ the Polish space with the usual product topology and $\mathbb{N}$ discrete. Let $X$ be a Polish space. A subset $A$ of $X$ is $CA$ if there is a Borel function $f$ from $\mathbb{N}^\mathbb{N}$ to $X$ such that $f(\mathbb{N}^\mathbb{N}) = X - A$. A $CA$ ($F_{\sigma \delta}$) subset $A$ of $X$ is called complete $CA$ ($F_{\sigma \delta}$) if for any $CA$ ($F_{\sigma \delta}$) subset $B$ of $\mathbb{N}^\mathbb{N}$, there is a Borel (continuous) function $f$ from $\mathbb{N}^\mathbb{N}$ to $X$ such that the preimage of $A$ of $f$ is $B$, i.e., $B = f^{-1}(A)$. From the definition, it is easy to see that no
complete $CA (F_{\sigma\delta})$ set is Borel $(G_{\delta\sigma})$. In particular, if an $F_{\sigma\delta}$ subset $A$ of a Polish space is complete $F_{\sigma\delta}$ and the continuous preimage of an $F_{\sigma\delta}$ subset $B$ of a Polish space, then $B$ is also complete $F_{\sigma\delta}$.

Let $\mathbb{R}$ be the set of real numbers. Let $T$ denote the unit circle and $I$ the unit interval. Let $E$ be $T$ or $I$. We denote by $C(E)$ the Polish space of continuous functions on $E$ with the uniform metric

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in E\}.$$ 

$C(T)$ can also be considered as the space of all continuous $2\pi$-periodic functions on $\mathbb{R}$, viewing $T$ as $\mathbb{R}/2\pi\mathbb{Z}$. Let $UC$ denote the set of all sequences of continuous functions on $I$ that are uniformly convergent, i.e.,

$$UC = \{(f_n) \in C(I)^\mathbb{N} : (f_n) \text{ converges uniformly } \}.$$ 

To each $f \in C(T)$, we associate its Fourier series

$$S[f] \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx},$$

where $\hat{f}(n) = \frac{1}{2\pi} \int_{0}^{2\pi} f(t)e^{-int}dt$. Let

$$S_n(f, t) = \sum_{k=-n}^{n} \hat{f}(k)e^{ikt}$$

be the $n$th partial sum of the Fourier series of $f$. We say the Fourier series of $f$ converges at a point $t \in T$ if the sequence $(S_n(f, t))$ converges. Similarly, we define the uniform convergence of the Fourier series of $f$. Let $ECF$ denote the set of all continuous functions with everywhere convergent Fourier series. According to a standard theorem [Ka], the Fourier series of $f$ at $t$ converges to $f(t)$ if it converges. Hence we have

$$ECF = \{f \in C(T) : \forall t \in [0, 2\pi] \left((S_n(f, t)) \text{ converges}\right)\}\]

$$= \{f \in C(T) : \forall t \in [0, 2\pi] \left(f(t) = \lim_{n \to \infty} S_n(f, t)\right)\}.$$ 

We denote by $NCF$ the complement of $ECF$. Let $UCF$ denote the set all continuous functions with uniformly convergent Fourier series, i.e.,

$$UCF = \{f \in C(T) : \text{the Fourier series of } f \text{ converges uniformly}\}.$$ 

**Results**

**Theorem** ([AK]). $ECF$ is complete $CA$.

(See [AK].)

**Proposition 1.** $UCF$ and $UC$ are $F_{\sigma\delta}$. 

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Proof. Let $\mathbb{Q}$ be the set of all rational numbers. We consider $T$ as $[0, 2\pi]$ with 0 and $2\pi$ identified. By the definition of $\text{UCF}$,

$$f \in \text{UCF} \iff S_N(f) \text{ converges uniformly}$$

$$\iff \forall \ a \in \mathbb{N} \ \exists \ b \in \mathbb{N} \ \forall \ c, d \in \mathbb{N} \ \forall \ e \in \mathbb{Q} \left( |S_{b+c}(f, e) - S_{b+d}(f, e)| \leq \frac{1}{a} \right)$$

$$\iff f \in \bigcap_{a \in \mathbb{N}} \bigcup_{b \in \mathbb{N}} \bigcap_{c, d \in \mathbb{N}} \bigcap_{e \in \mathbb{Q} \cap [0, 2\pi]} V(a, b, c, d, e),$$

where $V(a, b, c, d, e)$ is the collection of $f \in C(T)$ such that $|S_{b+c}(f, e) - S_{b+d}(f, e)| \leq \frac{1}{a}$, which is closed, since the function $f \mapsto -\hat{f}(n)$ is continuous. Hence $\text{UCF}$ is $F_{\sigma \delta}$. Similarly, so is $\text{UC}$. We are done.

Lemma 2. The set $C_3 = \{ \alpha \in \mathbb{N}^\mathbb{N} : \lim_{n \to \infty} \alpha(n) = \infty \}$ is complete $F_{\sigma \delta}$.

(See [Ke, p. 180].) This set will be used to prove our main theorem.

Proposition 3. $\text{UC}$ is complete $F_{\sigma \delta}$.

Proof. We define the function $F$ from $\mathbb{N}^\mathbb{N}$ to $C(I)\mathbb{N}$ as follows: for each $\beta \in \mathbb{N}^\mathbb{N}$,

$$F(\beta) = \left( \frac{1}{\beta(n)} \right).$$

Then it is easy to see that

$$\beta \in C_3 \iff F(\beta) \text{ converges to } 0 \iff F(\beta) \text{ converges uniformly},$$

since $F(\beta)$ is a sequence of constant functions. Clearly, $F$ is continuous. Hence $\text{UC}$ is the continuous preimage of $C_3$. By Proposition 1 and Lemma 2, $\text{UC}$ is complete $F_{\sigma \delta}$.

Theorem 4. There is a continuous function $H$ from $\mathbb{N}^\mathbb{N}$ to $C(T)$ such that for all $A$ with $\text{UCF} \subseteq A \subseteq \text{ECF}$,

$$\beta \in C_3 \iff H(\beta) \in A,$$

and

$$\beta \notin C_3 \iff H(\beta) \in \text{NCF}.$$ 

In particular, $\text{UCF}$ is complete $F_{\sigma \delta}$.

By this theorem, we have the following corollary.

Corollary 5. There is no $G_{\delta \sigma}$ set $A$ such that

$$\text{UCF} \subseteq A \subseteq \text{ECF},$$

i.e., any $G_{\delta \sigma}$ set that includes $\text{UCF}$ must contain a continuous function with divergent Fourier series.

Proof. Suppose a $G_{\delta \sigma}$ set $A$ satisfies $\text{UCF} \subseteq A \subseteq \text{ECF}$. Then by Theorem 4, we obtain $H^{-1}(A) = C_3$. Since $A$ is $G_{\delta \sigma}$, so is $C_3$. By Lemma 2, this contradicts our assumption. \qed
It is a basic fact of descriptive set theory [Ke] that any Borel set is coanalytic. Since $ECF$ is complete $CA$ by Theorem [AK], it is a very natural guess that the complement of $C_3$ can be reducible to $ECF - UCF$. In fact, we have the following theorem.

**Theorem 6.** There is a continuous function $\tilde{H}$ from $\mathbb{N}^\mathbb{N}$ to $C(\mathbb{T})$ such that

$$\beta \in C_3 \iff \tilde{H}(\beta) \in UCF,$$

and

$$\beta \notin C_3 \iff \tilde{H}(\beta) \in ECF - UCF.$$

In particular, $UCF$ is complete $F_{\sigma\delta}$.

In order to prove Theorem 4 and Theorem 6, we need the following criterion due to Dini and Lipschitz [Zy, p. 63]. Let $f$ be defined in a closed interval $J$, and let

$$\omega(\delta) = \omega(\delta; f) = \sup\{|f(x) - f(y)| : x, y \in J \text{ and } |x - y| \leq \delta\}.$$

The function $\omega(\delta)$ is called the *modulus of continuity* of $f$.

**The Dini-Lipschitz test.** If $f$ is continuous and its modulus of continuity $\omega(\delta)$ satisfies the condition $\omega(\delta) \log \delta \to 0$, then the Fourier series of $f$ converges uniformly.

We introduce the Féjer polynomials, for given $0 < n < N \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$Q(x, N, n) = 2 \sin Nx \sum_{k=1}^{n} \frac{\sin kx}{k},$$

$$R(x, N, n) = 2 \cos Nx \sum_{k=1}^{n} \frac{\sin kx}{k}.$$

These two polynomials were used in [Zy] to prove that there exists a continuous function whose Fourier series diverges at a point.

**Lemma 7.** There are positive numbers $C_1, C_2 > 0$ such that

$$|Q| < C_1 \text{ and } |R| < C_2,$$

i.e., these polynomials are uniformly bounded in $x, N, n$.

Since

$$\sum_{k=1}^{n} \frac{\sin kx}{k}$$

is uniformly bounded in $n$ and $x$, Lemma 7 follows. By Lemma 7, we immediately have the following.

**Proposition 8.** Let $(N_k)$ and $(n_k)$ be any two sequences of positive integers, with $n_k < N_k$, and let $(\alpha_k)$ be a sequence of real numbers such that $\alpha_1 + \alpha_2 + \alpha_3 + \cdots < \infty$. Then the series $\sum \alpha_k Q(x, N_k, n_k)$, $\sum \alpha_k R(x, N_k, n_k)$ converge to continuous functions.
Proof of Theorem 4. We fix $A$ with $UCF \subseteq A \subseteq ECF$. Let $\alpha_k = 2^{-k}$, $n_k = N_k/2 = 2^{2^k}$ ($k = 1, 2, 3, \ldots$). We define $H$ from $\mathbb{N}^\mathbb{N}$ to $C(\mathbb{T})$ as follows: for all $\beta \in \mathbb{N}^\mathbb{N}$,

$$H(\beta) = \sum \alpha_k \frac{1}{\beta(k)} Q(x, N_k, n_k).$$

Claim 1. $H$ is continuous and well-defined.

Proof. By Proposition 8, $H$ is well-defined. By Lemma 7, it is easy to see that $H$ is continuous.

We divide the rest of the proof into two parts so that we have more intuition.

Case 1. $\lim_{n \to \infty} \beta(k) \neq \infty$.

We want to show that $H(\beta) \in NCF$. For each $k \in \mathbb{N}$ the inequality

$$|S_{N_k+n_k}(H(\beta), 0) - S_{N_k}(H(\beta), 0)| = \left| \sum_{|l| \leq N_k+n_k} \hat{H}(\beta)(l) - \sum_{|l| \leq N_k} \hat{H}(\beta)(l) \right|$$

$$= \alpha_k \frac{1}{\beta(k)} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n_k} \right) > \alpha_k \frac{1}{\beta(k)} \log n_k = 2^{-k} \frac{1}{\beta(k)} \log 2^{2^k}$$

holds. Since $\lim_{n \to \infty} \beta(k) \neq \infty$, there exists a $p \in \mathbb{N}$ such that for infinitely many $k$'s, $\beta(k) = p$. Hence the Fourier series of $H(\beta)$ does not converge, since in (1) we have $1/p \log 2$ for infinitely many $k$'s. So we derive $H(\beta) \in NCF$.

Case 2. $\lim_{n \to \infty} \beta(k) = \infty$.

We show that $H(\beta) \in UCF$. We will demonstrate that $\omega(\delta; H(\beta)) \log \delta \to 0$ as $\delta \to 0$. Then by the Dini-Lipschitz test, this shows that the Fourier series of $H(\beta)$ converges uniformly. We take any $0 < \delta \leq 1/2$ and define $\nu = \nu(\delta)$ as the largest integer $k$ satisfying $2^k \leq 1/\delta$. By Lemma 7, we have the following inequality:

$$\left| \sum_{k=\nu+1}^{\infty} \alpha_k \frac{1}{\beta(k)} Q(x+\delta, N_k, n_k) - \sum_{k=\nu+1}^{\infty} \alpha_k \frac{1}{\beta(k)} Q(x, N_k, n_k) \right|$$

$$\leq 2C \sum_{k=\nu+1}^{\infty} \alpha_k \frac{1}{\beta(k)} \leq 2C \sup \{ \frac{1}{\beta(k)} : k > \nu \} \sum_{k=\nu+1}^{\infty} \alpha_k$$

$$= 4C \sup \{ \frac{1}{\beta(k)} : k > \nu \} 2^{\nu-1} \leq 4C \sup \{ \frac{1}{\beta(k)} : k > \nu \} \frac{\log 2}{\log \delta}.$$

Now we calculate the rest of $H(\beta)$. We clearly have

$$Q'(x, N, n) = N R(x, N, n) + 2 \sin N x \sum_{k=1}^{n} \cos kx, \quad |Q'| \leq NC + 2n = nC_1,$$
for $N = 2n$ and $C_1 = 2C + 2$. By the mean value theorem, we have the following inequality:

$$\left| \sum_{k \leq \nu} \alpha_k \frac{1}{\beta(k)} Q(x + \delta, N_k, n_k) - \sum_{k \leq \nu} \alpha_k \frac{1}{\beta(k)} Q(x, N_k, n_k) \right|$$

$$\leq C_1 \delta \left( 2^{-1} 2^{2^1} \frac{1}{\beta(1)} + \cdots + 2^{-\nu} 2^{2^\nu} \frac{1}{\beta(\nu)} \right)$$

$$\leq C_1 2^{2^-} \sum_{k \leq \nu} 2^{-k} 2^{k} \frac{1}{\beta(k)} \leq C_1 \frac{1}{\log \delta} 2^{2^-} \sum_{k \leq \nu} 2^{2^k - k} \frac{1}{\beta(k)}.$$  

By (2) and (3), we have the following:

(4) \[ |\nu| \leq \frac{2^\nu}{\log \delta}. \]

Now if $\delta \to 0$, then $\nu \to \infty$. So it suffices to show that the right part of (4) goes to 0 as $\nu \to \infty$. Since $\beta(\nu) \to +\infty$ as $\nu \to \infty$, $\sup\{1/\beta(k) : k > \nu\}$ goes to 0. We need to show that the rest goes to zero as $\nu$ diverges to infinity. This requires the following small claim.

**Claim 2.** $\sum_{k \leq \nu} 2^{2^k - k} \leq 2^{2^\nu + 1}.$

**Proof.** Use induction on $\nu$. For $\nu = 1$, $2^{2^1} = 2 \leq 2^{2^0} = 2$. Suppose it is true for $\nu$. By the induction assumption, $\sum_{k \leq \nu} 2^{2^k - k} + 2^{2^\nu + 1} \leq 2^{2^\nu + 1} + 2^{2^\nu + 1}$. It is enough to show that $2^{2^\nu + 1} + 2^{2^\nu + 1} \leq 2^{2^\nu + 1}$. Letting $\theta = 2^\nu$, one can verify this inequality.

Fix $\epsilon$. Take $N_0$ such that $1/\beta(k) < \epsilon$ for all $k \geq N_0$. For this $N_0$, we choose $N > N_0$ so that $2^{-2^\nu} \sum_{k \leq N_0} 2^{2^k - k} < \epsilon$ for all $\nu \geq N$. Then for all $\nu \geq N$, by claim 2, the following inequality is valid:

$$2C_1 2^{-2^\nu} \sum_{k \leq \nu} 2^{2^k - k}$$

$$< 2C_1 \left( 2^{-2^\nu} \sum_{k \leq N_0} 2^{2^k - k} \frac{1}{\beta(k)} + 2^{-2^\nu} \sum_{N_0 < k \leq \nu} 2^{2^k - k} \frac{1}{\beta(k)} \right)$$

$$< 2C_1 \left( \epsilon + 2^{-2^\nu} \epsilon \sum_{N_0 < k \leq \nu} 2^{2^k - k} \right) < 2\epsilon C_1 \left( 1 + \frac{2^{2^\nu - 2^\nu + 4}}{2^{2^\nu + 1}} \right)$$

$$= 3\epsilon C_1.$$  

Hence the right side of (4) converges to zero as $\nu$ goes to the infinity, i.e., as $\delta \to 0$. So we derive $H(\beta) \in UCF$.

By case 1 and case 2, we obtain

- $\beta \notin C_3 \Rightarrow H(\beta) \notin NCF$ and
- $\beta \in C_3 \Rightarrow H(\beta) \in UCF$, respectively. Since $NCF$ and $A$ are disjoint, we have the following:

\[ \beta \notin C_3 \iff H(\beta) \notin NCF \quad \text{and} \]

\[ \beta \in C_3 \iff H(\beta) \in A. \]
We have shown the first assertion of the theorem. In particular, \( C_3 \) is the preimage of \( UCF \). Hence by Lemma 2, the second assertion follows. We have completed the proof of Theorem 4.

**Proof of Theorem 6.** Instead of \( Q \), we use \( R \). With \( N_k, n_k \), and \( \alpha_k \) as in the proof of Theorem 4, we define \( \tilde{H} \) from \( N^N \) to \( C(\mathbb{T}) \) as follows: for each \( \beta \in N^N \),

\[
\tilde{H}(\beta) = \sum \alpha_k \frac{1}{\beta(k)} R(x, N_k, n_k).
\]

The same proof as before will demonstrate that this function is well-defined and continuous and that if \( \lim_{n \to \infty} \beta(n) = \infty \), then the Fourier series of \( \tilde{H}(\beta) \) converges uniformly. So it suffices to show that if \( \lim_{n \to \infty} \beta(n) \neq \infty \), then \( \tilde{H}(\beta) \in ECF - UCF \). Suppose \( \lim_{n \to \infty} \beta(n) \neq \infty \). The representation of \( \tilde{H}(\beta) \) as Fourier series is \( \sum a_v \sin vx \). We see that \( \sum a_v \sin vx \) converges uniformly for any \( \delta > 0 \), since the partial sums of \( R(x, N_k, n_k) \) are uniformly bounded in \( k \) and \( x \), for \( \delta \leq |x| \leq \pi \). The series \( \sum a_v \sin vx \) contains sines only, and hence it converges for \( x = 0 \), and so everywhere. Now we will show that \( \sum a_v \sin vx \) does not converge uniformly. It is easy to see that

\[
\sum_{v=1}^{3n_k} a_v \sin vx - \sum_{v=1}^{2n_k} a_v \sin vx = \sum_{v=2n_k+1}^{3n_k} a_v \sin vx = 2^{-k} \frac{1}{\beta(k)} \sum_{v=1}^{n_k} \frac{\sin(2n_k + v)x}{v}.
\]

So if we let \( x = \pi/4n_k \), then we have

\[
\left| \sum_{v=1}^{3n_k} a_v \sin vx - \sum_{v=1}^{2n_k} a_v \sin vx \right| = 2^{-k} \frac{1}{\beta(k)} \sum_{v=1}^{n_k} \frac{\sin(2n_k + v)x}{v} \geq 2^{-k} \frac{1}{\beta(k)} \sin \frac{\pi}{4} \sum_{v=1}^{n_k} \frac{1}{v},
\]

since, for all \( v \) in the interval \( 1 \leq v \leq n_k \),

\[
\frac{3}{4} \pi \geq \frac{\pi}{4n_k} (2n_k + v) \geq \frac{\pi}{2}.
\]

So finally,

\[
\left| \sum_{v=1}^{3n_k} a_v \sin vx - \sum_{v=1}^{2n_k} a_v \sin vx \right| \geq 2^{-k} \frac{1}{\beta(k)} \sin \frac{\pi}{4} \sum_{v=1}^{n_k} \frac{1}{v} \geq 2^{-k} \frac{1}{\beta(k)} \log n_k \sin \frac{\pi}{4} = \frac{\log 2}{\sqrt{2}} \frac{1}{\beta(k)}.
\]

Hence \( \sum a_v \sin vx \) does not converge uniformly, since in (5) the same value appears for infinitely many \( k \)'s. Hence, as in the proof of Theorem 4, we finish the proof of Theorem 6.

\( \square \)

**Acknowledgment**

I wish to thank Professor A. Kechris for his constant guidance. The author is indebted to the referee and Professor A. R. Blass for many valuable comments.
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