THE BOREL CLASSES
OF MAHLER'S $A$, $S$, $T$, AND $U$ NUMBERS

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ABSTRACT. In this article we examine the $A$, $S$, $T$, and $U$ sets of Mahler's
classification from a descriptive set theoretic point of view. We calculate the
possible locations of these sets in the Borel hierarchy. $A$ turns out to be $\Sigma^0_2$-
complete, while $U$ provides a rare example of a natural $\Sigma^0_2$-complete set. We
produce an upperbound of $\Sigma^0_4$ for $S$ and show that $T$ is $\Pi^0_3$ but not $\Sigma^0_3$. Our
main result is based on a deep theorem of Schmidt that allows us to guarantee
the existence of the $T$ numbers.

INTRODUCTION

Mahler [6] divided complex numbers into classes $A$, $S$, $T$, and $U$ according
to their properties of approximation by algebraic numbers. Some studies were
done on the structural properties of these sets. For example, Kasch and Volk-
mann [3] verified that the $T$ numbers have Hausdorff dimension zero. Also in
harmonic analysis, W. Morgan, C. E. M. Pearce, and A. D. Pollington [7] have
shown that the set of $T$ and $U$ numbers supports a measure whose Fourier
transform vanishes at infinity. In the present paper we study the $A$, $S$, $T$,
and $U$ sets from the point of view of Descriptive Set Theory. Among the
few sets whose exact Borel class is known, a large percentage turn out to be $\Pi^0_3$-
complete. For example, the collection of reals that are normal or simply normal
to base $n$ [4]; $C^\infty(\mathbb{T})$, the class of infinitely differentiable functions (viewed
as a $2\pi$-periodic function on $\mathbb{R}$); and $UC_X$, the class of convergent sequences
in a separable Banach space $X$ are $\Pi^0_3$-complete [2]. Apparently, there are few
known natural $\Sigma^0_2$-complete sets. Of course, the complement of a $\Pi^0_3$-complete
set is $\Sigma^0_3$-complete. But the complement of a natural set need not be natural!
Tom Linton [5] has shown that the family of $H$-sets, a class of thin sets from
harmonic analysis, is $\Sigma^0_3$-complete, and this is the only $\Sigma^0_3$-complete natural set
we know of (whose complement is not also natural). A. Kechris proposed to
find out what the Borel classes of the $A$, $S$, $T$, and $U$ sets are. It turns out
that $A$ is rather simple, being $\Sigma^0_2$-complete. On the other hand, $T$ is $\Pi^0_3$-hard,
while $U$ is $\Sigma^0_3$-complete. Our main results are based on a theorem of W. M.

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The exact Borel classes of the $S$ and $T$ sets are unknown to us.

**Definitions and Background**

For spaces $X$ and $Y$, $X^Y$ denotes the set of all functions $f$ from $Y$ to $X$, with the usual product topology, $X$ and $Y$ being endowed with their usual topologies ($2 = \{0, 1\}$ and $\mathbb{N} = \{1, 2, 3, \ldots\}$ being discrete). For sets $U$ and $V$, if $S$ is a function from $X^{n+1} \times Y^{n+1}$ to $U^{n+1} \times V^{n+1}$ and $n \in \mathbb{N}$, then $S|_n$ is the function from $X^{n+1} \times Y^{n+1}$ to $U^{n} \times V^{n}$ such that if $S((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})) = ((u_1, \ldots, u_{n+1}), (v_1, \ldots, v_{n+1}))$, then $S|_n((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})) = ((u_1, \ldots, u_{n}), (v_1, \ldots, v_{n}))$. $\mathbb{P} = \{x \in \mathbb{R} : x > 1\}$ and $A$ denotes the class of all nonzero real algebraic numbers in $\mathbb{C}$. We use the standard terminology of Addison to describe the Borel hierarchy. Thus the multiplicative sets of level $n$ are denoted by $\Pi^n_0$, while the additive class of level $n$ is denoted by $\Sigma^n_0$. In particular, $\Sigma^0_0 = \text{Open}$, $\Pi^0_0 = \text{Closed}$, $\Sigma^0_2 = F_\sigma$, $\Pi^0_2 = G_\delta$. In addition, the countable union of $\Pi^n_0$ sets is $\Sigma^{n+1}_0$; the countable intersection of $\Pi^n_0$ sets is a $\Sigma^{n+1}_0$ set; the complement of a $\Pi^n_0$ set is $\Sigma^n_0$; the $\Sigma^n_0$ sets are closed under finite intersection and countable union; while the $\Pi^n_0$ sets are closed under finite union and countable intersection. If the context demands it, we use $\Pi^n_0(X)$ to denote the $\Pi^n_0$ subsets of a space $X$.

Let $\Gamma = \Sigma^0_0$ or $\Pi^0_0$. We call a set $C \subseteq X$ (a Polish space) $\Gamma$-hard if for any $B \in \Gamma(2^N)$, there is a continuous function $f$ from $2^N$ to $X$, such that $B = f^{-1}(C)$. If, moreover, $C \in \Gamma(X)$, we call $C$ $\Gamma$-complete. It is well known (see [2]) that a $\Pi^0_0$-complete set in an uncountable Polish space is $\Pi^0_0$ but not $\Sigma^0_0$, and if $A$ is $\Pi^0_0$-hard, then $A$ is not $\Sigma^0_0$. As well, in uncountable Polish spaces every $\Pi^0_0$ set and every $\Sigma^0_0$ set is $\Pi^{n+1}_0$ and $\Sigma^{n+1}_0$, so the Borel hierarchy is increasing in $n$.

For a given set $C \subseteq X$, in order to find the exact Borel class of $C$, one must first calculate an upperbound for $C$, by showing, for example, that $C$ is $\Pi^0_0$ and then prove a lowerbound for $C$'s Borel class, for example, by showing that $C$ is $\Pi^0_0$-hard. Usually, finding the upperbound is fairly easy. However, it can be difficult to prove the hardness of $C$. Since the Borel classes $\Pi^0_n$ and $\Sigma^0_n$ are closed under continuous preimages, if $B$ is $\Gamma$-hard ($\Gamma$-complete) and $B = f^{-1}(C)$, where $f$ is a continuous function, then $C$ is $\Gamma$-hard ($\Gamma$-complete, if also $C \in \Gamma$). This remark is the basis of a common method for showing that a given set $B$ is $\Gamma$-hard: Choose an already known $\Gamma$-hard set $B$ and show that there is a continuous function $f$ such that $B = f^{-1}(C)$.

Now we define the $A$, $S$, $T$, and $U$ sets, from Mahler's classification. For convenience we use Koksma's notation, which is equivalent to that of Mahler. Given algebraic $\alpha \in \mathbb{C}$, let $p(x) \in \mathbb{Z}[x]$ be its minimal polynomial. Fix $d, h \in \mathbb{N}$. Let $X_{d,h}$ be the finite collection of polynomials with degree $\leq d$ whose largest coefficient has absolute value $\leq h$. Let the height of a polynomial, $ht(p)$, be the maximum of the absolute values of the coefficients. Let $A_{d,h}$ be the finite collection of algebraic numbers $\alpha$ such that for some $p \in X_{d,h}$, $p(\alpha)$ is zero (recall that $0 \notin \mathbb{N}$). Thus, $A_{d,h}$ is the finite collection of algebraic (complex) numbers whose minimal polynomial has degree $\leq d$ and $ht \leq h$. Let $\xi$ be any complex number and let $\alpha$ belong to $A_{d,h}$ such that $|\xi - \alpha|$ takes
the smallest positive value; define \( \omega^{*}_{d}(\xi, h) \) by

\[
|\xi - \alpha| = \frac{1}{h^{d} \omega^{*}_{d}(\xi, h) + 1}.
\]

Set

\[
\omega^{*}_{d}(\xi) = \limsup_{h \to \infty} \omega^{*}_{d}(\xi, h) \quad \text{and} \quad \omega^{*}(\xi) = \limsup_{d \to \infty} \omega^{*}_{d}(\xi).
\]

So the values of \( \omega^{*}_{d}(\xi) \) and \( \omega^{*}(\xi) \) measure how fast \( \xi \) is approximated by algebraic numbers. We define, according to the values of \( \omega^{*}_{d}(\xi) \) and \( \omega^{*}(\xi) \), the \( A, S, T, \) and \( U \) sets as follows:

\[
A = \{ \xi \in \mathbb{C} : \omega^{*}(\xi) = 0 \},
\]

\[
S = \{ \xi \in \mathbb{C} : 0 < \omega^{*}(\xi) < \infty \},
\]

\[
T = \{ \xi \in \mathbb{C} : \omega^{*}(\xi) = \infty \quad \text{and} \quad \forall d \in \mathbb{N} \quad (\omega^{*}_{d}(\xi) < \infty) \},
\]

\[
U = \{ \xi \in \mathbb{C} : \omega^{*}(\xi) = \infty \quad \text{and} \quad \exists d \in \mathbb{N} \quad (\omega^{*}_{d}(\xi) = \infty) \}.
\]

Thus, the \( A \) numbers are slowly approximated by algebraic numbers. The \( S \) numbers are approximated a bit more quickly than \( A \) numbers. On the other hand, the \( T \) numbers and the \( U \) numbers are very rapidly approximated, i.e., the value of \( \omega^{*}(\xi) \) is infinite. In particular, the approximation of the \( U \) numbers is so quick that for some \( d \in \mathbb{N}, \omega^{*}_{d}(\xi) \) diverges. For these reasons, we claim that the set of complex numbers is naturally partitioned by the \( A, S, T, \) and \( U \) numbers.

**RESULTS**

**Lemma 1.** \( \xi \in A \Leftrightarrow \xi \) is an algebraic number.

**Proof.** See [1, pp. 85–94].

**Proposition 2.** (i) The \( A \) numbers are \( \Sigma^{0}_{2} \)-complete, and the \( U \) numbers are \( \Sigma^{0}_{3} \).

(ii) The \( S \) numbers are \( \Sigma^{0}_{4} \), while the collection of \( T \) numbers are \( \Pi^{0}_{3} \).

**Proof of Proposition 2.** (i) For each \( d \in \mathbb{N}, \) let \( U_{d} \) be the collection of \( \xi \in \mathbb{C} \) such that \( \omega^{*}_{d}(\xi) = \infty \). Then \( U_{d} \) is \( \Pi^{0}_{d} \), since

\[
\xi \in U_{d} \Leftrightarrow \omega^{*}_{d}(\xi) = \infty
\]

\[
\Leftrightarrow \forall a \in \mathbb{N} \forall b \in \mathbb{N} \exists c \in \mathbb{N} \ (\omega^{*}_{d}(\xi, b + c) > a)
\]

\[
\Leftrightarrow \forall a \in \mathbb{N} \forall b \in \mathbb{N} \exists c \in \mathbb{N} \exists \alpha \in A_{d,b+c} \left( 0 < |\xi - \alpha| < \frac{1}{(b + c)^{ad+1}} \right)
\]

\[
\Leftrightarrow \xi \in \bigcap_{a \in \mathbb{N}} \bigcap_{b \in \mathbb{N}} \bigcap_{c \in \mathbb{N}} \bigcup_{\alpha \in A_{d,b+c}} V(a, b, c, \alpha),
\]

where \( V(a, b, c, \alpha) \) is the collection of \( \xi \in \mathbb{C} \) such that \( 0 < |\xi - \alpha| < 1/(b + c)^{ad+1} \), which is open. Since it is easy to see that for each \( d, \omega^{*}_{d}(\xi) = \infty \) implies \( \omega^{*}_{d+1}(\xi) = \infty \), we have \( U = \bigcup_{d=1}^{\infty} U_{d} \) and \( U \) is \( \Sigma^{0}_{3} \). It is well known that if \( D \) is a countable dense set in a perfect Polish space, then \( D \) is \( \Sigma^{0}_{2} \)-complete. Thus, by Lemma 1, \( A \) is \( \Sigma^{0}_{2} \)-complete.

(ii) By definition, \( T \) is the collection of \( \xi \in \mathbb{C} \) such that \( \omega^{*}(\xi) = \infty \) and \( \forall a \in \mathbb{N} \ (\omega^{*}_{a}(\xi) < \infty) \). Thus, \( T = M \cap N \), where \( M = \{ \xi \in \mathbb{C} : \omega^{*}(\xi) = \infty \} \)

and
and $N = \{ \xi \in \mathbb{C} : \forall \alpha \in \mathbb{N} \; (\omega^*_\alpha(\xi) < \infty) \}$. Now $M$ is $\Pi^0_3$, since

$$
\xi \in M \iff \forall a \in \mathbb{N} \; \forall b \in \mathbb{N} \; \exists c \in \mathbb{N} \; (\omega^*_\alpha(\xi) > a)
\iff \forall a \in \mathbb{N} \; \forall b \in \mathbb{N} \; \exists c \in \mathbb{N} \; \exists d \in \mathbb{N} \; \forall e \in \mathbb{N} \; \exists f \in \mathbb{N}
\left( \omega^*_\alpha(\xi, e + f) > a + \frac{1}{d + 1} \right)
\iff \xi \in \bigcap_{a \in \mathbb{N}} \bigcap_{b \in \mathbb{N}} \bigcap_{c \in \mathbb{N}} \bigcap_{d \in \mathbb{N}} \bigcap_{e \in \mathbb{N}} \bigcap_{f \in \mathbb{N}} W(a, b, c, d, e, f),
$$

where $W(a, b, c, d, e, f)$ is the collection of $\xi \in \mathbb{C}$ such that $\omega^*_\alpha(\xi, e + f) > a + 1/(d + 1)$, which is open by the argument above. So $N$ is $\Pi^0_3$, since by (i) $U$ is $\Sigma^0_3$ and

$$
\xi \in N \iff \forall a \in \mathbb{N} \; (\omega^*_\alpha(\xi) < \infty)
\iff \xi \in C - U.
$$

Hence $T$ is $\Pi^0_4$, being the intersection of two $\Pi^0_4$ sets. Since $\xi \in S \iff \xi \notin T$, $\xi \notin U$, and $\xi \notin A$, $S$ is $\Sigma^0_4$. $\square$

In $2^\mathbb{N}$, $Q$ is the collection of sequences which end in zeros.

**Lemma 3.** There exists a continuous function $\nu$ from $2^\mathbb{N}$ to $\mathbb{N}^\mathbb{N}$ such that

(i) for each $d \in \mathbb{N}$, $\alpha \in 2^\mathbb{N}$, $\nu(\alpha)(d) \leq \nu(\alpha)(d + 1)$;

(ii) $\alpha \in Q \iff \lim_{d \to \infty} \nu(\alpha)(d) < \infty$.

**Proof of Lemma 3.** Let $\alpha \in 2^\mathbb{N}$. We produce $\beta = \nu(\alpha)$ recursively. First $\beta(1) = \alpha(1)$. Suppose that we have defined $\beta(i)$ for all $i \leq k$. Put $\beta(k + 1) = \beta(k)$ if $\alpha(k + 1) = 0$ and $\beta(k + 1) = \beta(k) + 1$ otherwise. It is easy to see that the function $\nu$ satisfies (i). As long as $\alpha$ ends in zeros, so does $\nu(\alpha)$ in constants. Otherwise, $\nu(\alpha)(d)$ goes to infinity as $d \to \infty$, because for infinitely many $d$’s, $\nu(\alpha)(d + 1) = \nu(\alpha)(d) + 1$. So (ii) is valid. For given $d \in \mathbb{N}$, $\alpha_1, \alpha_2 \in 2^\mathbb{N}$, such that $\alpha_1(i) = \alpha_2(i)$ for all $i \leq d$, $\nu(\alpha_1)(i) = \nu(\alpha_2)(i)$ for all $i \leq d$. So $\nu$ is continuous. This completes Lemma 3. $\square$

From Lemma 3, $\alpha \notin Q \iff \lim_{d \to \infty} \nu(\alpha)(d) = \infty$. To prove our main theorem, we need a standard example of the $\Pi^0_3$-complete set.

**Lemma 4.** The set $P_3 = \{ \alpha = (\alpha_d) \in (2^\mathbb{N})^\mathbb{N} : \forall d \in \mathbb{N} \; (\alpha_d \in Q) \}$ is $\Pi^0_3$-complete.

**Proof.** See [2].

The following theorem is the main result of the paper.

**Theorem 5.** There is a continuous function $f$ from $(2^\mathbb{N})^\mathbb{N}$ to $\mathbb{C}$ such that

$$
\alpha \in P_3 \iff f(\alpha) \in T \; \text{ and } \; \alpha \notin P_3 \iff f(\alpha) \in U.
$$

In particular, $T$ is $\Pi^0_3$-hard and $U$ is $\Sigma^0_3$-complete.

Roughly speaking, the original statement of a theorem of Schmidt is the following: Let $\alpha_1, \alpha_2, \ldots$ be any nonzero algebraic numbers and let $\nu_1, \nu_2, \ldots$ be any real numbers exceeding 1. Then we may find $\xi \in \mathbb{C}$ such that according to $\alpha_1, \alpha_2, \ldots$ and $\nu_1, \nu_2, \ldots, \xi$ is a $U$ number or $T$ number.

By using $\nu$, which is constructed in Lemma 3, we shall effectively control $\nu_i$’s so that we are able to prove Theorem 5. In order to make it work, we need to state the reformulated version of a theorem of Schmidt which will play a crucial role in the proof of Theorem 5.
Theorem S (Schmidt). There exists a sequence \( (S_n) \) such that for each \( n \in \mathbb{N} \),

(i) \( S_n \) is a function from \( \mathbb{A}^n \times \mathbb{P}^n \) to \( \mathbb{A}^n \times (0, 1)^n \) and \( S_{n+1}|_n = S_n \).

(ii) Suppose that \( S_n((\theta_1, \theta_2, \ldots, \theta_n), (\nu_1, \nu_2, \ldots, \nu_n)) = ((\gamma_1, \gamma_2, \ldots, \gamma_n), (\lambda_1, \lambda_2, \ldots, \lambda_n)) \).

Then for each \( j < n \), \( \gamma_j/\theta_j \) is rational, \( H_{j+1} > 2H_j \) and \( \frac{1}{4} H_j^{-1} < \gamma_{j+1} - \gamma_j < \frac{1}{2} H_j^{-1} \), where \( H_j = h'_{\gamma_j} \) and \( h_j = \text{ht}(\gamma_j) \), and furthermore, we have \( |\gamma_j - \beta| > B^{-1} \) for all algebraic numbers \( \beta \) with degree \( d \leq j \) distinct from \( \gamma_1, \ldots, \gamma_j \), where \( B = \lambda_d^{-1} b(3d)^4 \) and \( b \) denotes the height of \( \beta \).

Proof. See [1, pp. 85-94].

Using Theorem S we define the function \( S^* \) from \( \mathbb{A}^N \times \mathbb{P}^N \) to \( \mathbb{A}^N \times (0, 1)^N \) as follows: \( S^*((\theta_1, \theta_2, \ldots), (\nu_1, \nu_2, \ldots)) = ((\gamma_1, \gamma_2, \ldots, \gamma_n), (\lambda_1, \lambda_2, \ldots, \lambda_n)) \), where for each \( n \), \( S_n((\theta_1, \ldots, \theta_n), (\nu_1, \ldots, \nu_n)) = ((\gamma_1, \ldots, \gamma_n), (\lambda_1, \ldots, \lambda_n)) \). \( S^* \) is well defined by Theorem S(i).

Proof of Theorem 5. Let \( \alpha \in (2^N)^N \). Fix a bijection \( \langle \, \, \rangle \) from \( \mathbb{N} \times \mathbb{N} \) to \( \mathbb{N} \). For each \( d, k \in \mathbb{N} \), define

\[
\nu(d,k) = (\nu(\alpha_d)(k) + 1)(3d)^5 \quad \text{and} \quad \theta(d,k) = \theta_{d,k},
\]

where the function \( \nu \) is constructed in Lemma 3. Put \( A = \{\theta(d,k)\} \) and \( \deg(\theta(d,k)) = d \). Say

\[
S^*((\theta_1, \theta_2, \ldots), (\nu_1, \nu_2, \ldots)) = ((\gamma_1, \gamma_2, \ldots), (\lambda_1, \lambda_2, \ldots)).
\]

Then by Theorem S(ii), \( \gamma_1, \gamma_2, \ldots \) tends to a limit \( \xi \) which is a real number and satisfies

1. \( |\xi - \beta| > B^{-1} \) for all algebraic numbers \( \beta \) distinct from \( \gamma_1, \gamma_2, \ldots \),

and also

2. \( \frac{1}{4} H_j^{-1} \leq \xi - \gamma_j \leq H_j^{-1} \) for all \( j \).

Define

\[
f(\alpha) = \lim_{j \to \infty} \gamma_j = \xi.
\]

Claim. \( f \) is continuous from \( (2^N)^N \) to \( \mathbb{C} \).

Proof of the claim. Suppose \( (\alpha_d^{(m)}) \to (\alpha_d) \) as \( m \to \infty \), where for each \( m \), \( (\alpha_d^{(m)}) \in (2^N)^N \) and \( (\alpha_d) \in (2^N)^N \). Say for each \( m \),

\[
f((\alpha_d^{(m)})) = \xi_m = \lim_{k \to \infty} \gamma_k^{(m)} \quad \text{and} \quad f((\alpha_d)) = \xi = \lim_{k \to \infty} \gamma_k,
\]

where for each \( k \in \mathbb{N} \), \( \gamma_k^{(m)} \) and \( \gamma_k \) are defined by \( S^* \), according to \( (\alpha_d^{(m)}) \) and \( (\alpha_d) \). Let \( \varepsilon > 0 \). Choose \( a_0 \) such that \( 1/2^{a_0-2} < \varepsilon \). Since \( (\alpha_d^{(m)}) \) goes to \( (\alpha_d) \) as \( m \to \infty \), by the definition of \( \gamma_k^{(m)} \) and \( \gamma_k \) we may find \( N_0 \in \mathbb{N} \) such that \( |\gamma_k^{(m)} - \gamma_k| = 0 \) for all \( m \geq N_0 \). Then for all \( m \geq N_0 \), we have the following inequality:

\[
|\xi_m - \xi| \leq |\xi_m - \gamma_k^{(m)}| + |\gamma_k^{(m)} - \gamma_k| + |\gamma_k - \xi| < \frac{1}{2^{a_0-2}} < \varepsilon.
\]
since from (2) and Theorem S(ii),
\[|\xi_m - \gamma_a^{(m)}| \leq (H_a^{(m)})^{-1} \leq \frac{1}{2a-1} (H_1^{(m)})^{-1} \leq \frac{1}{2a-1}\]
and
\[|\xi - \gamma_a| \leq H_a^{-1} \leq \frac{1}{2a-1} H_1^{-1} \leq \frac{1}{2a-1}\]
for all \( a \geq 1 \). So \( f \) is a continuous function. \( \Box \)

Now we show the main part of the theorem. Depending on the properties of \( \nu \), Theorem S guarantees that we produce a \( T \) number or \( U \) number. So we divide the following two cases so that one can have more intuitive ideas.

**Case 1.** \( \alpha = (\alpha_d) \notin P_3 \), i.e., \( \exists d \in \mathbb{N} \) \((\alpha_d \notin \mathbb{Q})\).

Fix such \( d \), i.e., \( \alpha_d \notin \mathbb{Q} \). Then by Lemma 3, we have
\[\lim_{k \to \infty} (\nu(\alpha_d)(k) + 1) = \infty.\]

It is clear that for all \( k, h = h(d,k), \)
\[h^{-d\omega^*_d(\xi,h)-1} \leq |\xi - \gamma(d,k)| \leq h^{-\nu(d,k)} \]
from (2) and the definition of \( \omega^*_d(\xi,h), \)
where \( f(\alpha) = \xi \). So \( d\omega^*_d(\xi,h(d,h)) \geq \nu(d,k) - 1 \), i.e.,
\[\omega^*_d(\xi,h(d,h)) \geq \frac{\nu(d,k) - 1}{d} \geq (\nu(\alpha_d)(k) + 1)3^5d^4 - \frac{1}{d} \text{ for all } k.\]

It is easy to see that \( \limsup_{k \to \infty} h(d,k) = \infty \), since the right side of (3) goes to infinity as \( k \to \infty \). This shows that we may choose \( \{k_m\} \) such that \( k_m \to \infty \) and \( h(d,k_m) \to \infty \) as \( m \to \infty \). From (3) we get the following inequality:
\[\omega^*_d(\xi,h) = \limsup_{m \to \infty} \omega^*_d(\xi,h) \geq \limsup_{m \to \infty} \omega^*_d(\xi,h(d,k_m)) \geq \lim_{m \to \infty} (\nu(\alpha_d)(k_m) + 1)3^5d^4 - \frac{1}{d} = \infty.\]

Therefore, \( \omega^*_d(\xi) = \infty \) and \( f(\alpha) = \xi \in U \). So we derive \( \alpha \notin P_3 \Rightarrow f(\alpha) = \xi \in U.\)

**Case 2.** \( \alpha = (\alpha_d) \in P_3 \), i.e., \( \forall d \in \mathbb{N} \) \((\alpha_d \in \mathbb{Q})\).

Fix \( d \in \mathbb{N} \). Then for all \( h, k, m \), we have
\[\xi - \gamma(m,k) \geq \frac{1}{4} h^{-\nu(\alpha_d)(k)+1}(3m)^3, \]
\[|\xi - \beta| \geq \lambda_{\deg(\beta)}(ht(\beta))^{-3\deg(\beta)} \]
for all algebraic numbers \( \beta \) distinct from \( \gamma_1, \gamma_2, \ldots \) from (1) and (2), where \( f(\alpha) = \xi \). In fact, all nonzero algebraic numbers appear in these two inequalities. Let \( h \) be a given natural number. Then from (4) and the definition of \( \omega^*_d(\xi,h) \), we have the following inequality:
\[h^{-d\omega^*_d(\xi,h)} \geq \min\{\left\{\frac{1}{4} h^{-M_0(3d)^3}, \lambda(d)h^{-3(3d)^4}\right\}, \]
where \( M_0 = \sup\{\nu(\alpha_s)(k)+1 : s \leq d \text{ and } k < \infty\} \) and \( \lambda(d) = \min\{\lambda_s : s \leq d\} \). Even if for \( s \leq d \), there is no \( k \) such that \( h(s,k) = h \), this inequality can be applied. The value \( \lambda(d) \) is positive and \( 1 \leq M_0 < \infty \), since \( \{\lambda_s : s \leq d\} \)
is the finite set of positive values and by assumption and Lemma 3, \( \forall d \in \mathbb{N} \) \( \lim_{k \to \infty} \nu(\alpha_d)(k) < \infty \). So from (5) we get

\[
\omega_d^*(\xi, h) \leq \max \left\{ \frac{\log 4}{\log h} + 3^5 M_0 d^4, \frac{\log 1/\lambda(d)}{d \log h} + 3^5 d^4 \right\} < \infty
\]

and

\[
\omega_d^*(\xi) = \limsup_{h \to \infty} \omega_d^*(\xi, h) \leq \max\{3^5 M_0 d^4, 3^5 d^4\} = 3^5 M_0 d^4 < \infty.
\]

Hence we can see that the inequality

\[ (6) \quad \omega_d^*(\xi) = \limsup_{h \to \infty} \omega_d^*(\xi, h) < \infty \]

holds for all \( d \). But for all \( d, k \), we obtain

\[
\omega_d^*(\xi, h(d, k)) \geq \frac{\nu(d, k) - 1}{d} \geq (\nu(\alpha_d)(k) + 1)3^5 d^4 - \frac{1}{d}.
\]

As in Case 1, \( \omega_d^*(\xi) \geq 3^5 d^4 M_1 - \frac{1}{d} \), where \( M_1 = \lim_{k \to \infty} \nu(\alpha_s)(k) + 1 \geq 1 \). Therefore,

\[ (7) \quad \omega_d^*(\xi) \geq (3d)^4 \quad \text{and} \quad \omega^*(\xi) = \limsup_{d \to \infty} \omega_d^*(\xi) = \infty. \]

From (6) and (7), for all \( d \in \mathbb{N} \), \( \omega_d^*(\xi) < \infty \) and \( \omega^*(\xi) = \infty \), i.e., \( f(\alpha) = \xi \in T \). So we derive \( \alpha \in P_3 \Rightarrow f(\alpha) = \xi \in T \).

By Case 1 and Case 2, we obtain \( \alpha \in P_3 \Rightarrow f(\alpha) \in T \) and \( \alpha \notin P_3 \Rightarrow f(\alpha) \in U \). By definition of \( T \), \( U \), it is easy to see that they are disjoint. So the continuous function \( f \) satisfies \( P_3 = f^{-1}(T) \) and \( C - P_3 = f^{-1}(U) \). This fact implies that \( T \), \( U \) are \( \Pi^0_3 \)-hard, \( \Sigma^0_3 \)-complete respectively, since by Lemma 4, \( P_3 \) is \( \Pi^0_3 \)-complete. We complete the proof of Theorem 5. \( \Box \)

Remark. We conjecture that \( S \), \( T \) are \( \Sigma^0_4 \)-complete, \( \Pi^0_4 \)-complete, respectively.

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