ZEROS OF ZETA FUNCTIONS

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Dedicated to Professor Akio Fujii on the occasion of his retirement

1. Introduction

In this survey, we will focus on some results related to the explicit location of zeros of the Riemann zeta function.

We have few techniques to know the exact behavior of zeros of zeta functions. For the author's study on zeros of zeta functions, the Riemann-Siegel formula [33], [45] plays the central role. For this formula, see the next section. We have collected several examples whose structures look like the Riemann-Siegel formula and which satisfy the analogue of the Riemann hypothesis. The Riemann-Siegel formula can be simplified by

$$f(s) + \overline{f(1-\overline{s})},$$

where $f(s)$ satisfies certain nice conditions. Due to the symmetry, zeros of this formula tend to lie on $\mathrm{Re}(s) = 1/2$. Thus, our strategy to the Riemann hypothesis (RH) is to find a nice representation of the Riemann zeta function satisfying the above formula: if we are able to prove that complex zeros of $f(s)$ are in $\mathrm{Re}(s) \geqslant 1/2$ or in $\mathrm{Re}(s) \leqslant 1/2$, then we essentially derive RH.

For $T > 0$, we have

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T),$$

where $N(T)$ is the number of zeros of the Riemann zeta function in $0 < \mathrm{Im}(s) < T$. For this, see [52]. Thus, the average gap of consecutive zeros of the Riemann zeta function in $0 < \mathrm{Im}(s) < T$ is

$$\frac{2\pi}{\log T}.$$

What can we say about gaps of zeros of the Riemann zeta function? This question is one of the most important questions in studying the behavior of zeros of the Riemann zeta function. Together with the Riemann-Siegel formula, Montgomery's pair correlation conjecture [39] should be realized as the intrinsic property for the behavior of zeros of the Riemann zeta function.

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Montgomery's Pair correlation conjecture (PCC) *Assume the Riemann hypothesis. Then we have*

$$\sum_{\substack{0<\gamma,\gamma'<T \\ \frac{2\pi\alpha}{\log T}<\gamma'-\gamma<\frac{2\pi\beta}{\log T}}} 1 \sim N(T)\int_\alpha^\beta 1-\left(\frac{\sin \pi x}{\pi x}\right)^2 dx \qquad (0 \leqslant \alpha \leqslant \beta),$$

where $1/2+i\gamma$, $1/2+i\gamma'$ *are zeros of the Riemann zeta function.*

This conjecture says that there are arbitrarily small gaps of consecutive zeros and arbitrarily big gaps of consecutive zeros in comparison with the average gap of zeros of the Riemann zeta function. So called, locally zeros of the Riemann zeta function behave randomly, but globally there exists the beautiful harmony (PCC) of gaps of zeros. It should be noted that any true zeta function like the Riemann zeta function must fulfill PCC. Unfortunately, all examples in this survey that satisfy the analogue of RH do not have PCC, namely zeros of the examples behave regularly. Even we don't know any example satisfying PCC. In spite of the reconditeness, PCC causes a negative effect in resolving RH.

Recently, Weng [54] – [60] discovered new zeta functions. In fact, Weng's zeta functions are Arthur's periods for the Eisenstein series. Actually, Weng's zeta functions are linear combinations of the complete Riemann zeta function with coefficients of rational functions and satisfy functional equations. Thus, Weng's zeta functions do not have Euler products. Usually, without Euler product, we expect that zeta functions do not have RH, because we believe that the validity of RH comes from arithmetic properties (e.g., Euler products). Remarkably, several Weng's zeta functions satisfy the analogue of RH. Indeed, the author with Komori and Suzuki [30] shows that Weng's zeta functions for Chevalley groups satisfy the weak RH. Thus, as Weng [60] conjectured the truth of RH for Weng's zeta functions, we expect that any Weng's zeta function must have RH. We recall that we obtain Weng's zeta functions based on deep structures in Langlands' program. Hence it should be very important to investigate the behavior of zeros of Weng's zeta functions.

2. The Riemann-Siegel formula

It seems that without proof, Riemann [43] asserted that almost all zeros of the Riemann zeta function $\zeta(s)$ are on the critical line $\mathrm{Re}(s) = 1/2$. However, this statement still remains open, and we are puzzled what Riemann really said. In fact, Selberg [44] justified the presence of a positive proportion of zeros of $\zeta(s)$ on $\mathrm{Re}(s) = 1/2$. After 32 years, Levinson [37] showed that based on a wonderful property of the derivative of $\zeta(s)$, there are at least 1/3 'one third' on the critical line. Then, Conrey [7] followed Levinson's method to improve that there are at least 2/5 'two fifth' on the line, and recently Bui, Conrey and Young [5] did 41% on the line. Also, see [12]. Note that any of these results rely on arithmetic properties of $\zeta(s)$ and with current techniques, it looks unlikely to achieve a result like '45%'. We are not even halfway through what Riemann claimed. Thus, we are anxious to know Riemann's argument. One candidate is the Riemann-Siegel formula [33], [45]

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = F(s) + \overline{F(1-\overline{s})}, \tag{2.1}$$

where $\Gamma(s)$ is the Gamma function and

$$F(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\int_{0\swarrow 1}\frac{e^{i\pi x^2}x^{-s}}{e^{i\pi x}-e^{-i\pi x}}dx.$$

Here the symbol $0 \swarrow 1$ means that the path of integration is a line of slop 1 crossing the real axis between 0 and 1 and directed from upper right to lower left. Dividing by $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)$ and moving the path in the integral of $F(s)$ to the right, we obtain

$$\zeta(s) = \sum_{n\leqslant\sqrt{\frac{t}{2\pi}}}\frac{1}{n^s} + \chi(s)\sum_{n\leqslant\sqrt{\frac{t}{2\pi}}}\frac{1}{n^{1-s}} + O\left(t^{-\sigma/2}\right), \tag{2.2}$$

where $s = \sigma + it$ $(-1 \leqslant \sigma \leqslant 1,\ t > 10)$,

$$\chi(s) = \frac{\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)}{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}.$$

This Riemann-Siegel approximation formula has been very practical in investigating the behavior of $\zeta(s)$. In deed, the term $\sum_{n\leqslant\sqrt{\frac{t}{2\pi}}}\frac{1}{n^s}$ in (2.2) looks like $\zeta(s)$ and we can observe a strong symmetry from the formula. This strongly suggests the validity of RH.

For the explicit behavior of zeros of $\zeta(s)$, we need to understand the original form (2.1). With the Riemann-Siegel formula, can we justify RH (or Riemann's statement)? For instance, we would get RH if $F(s)$ in (2.1) has no zeros in $\mathrm{Re}(s) < 1/2$ or in $\mathrm{Re}(s) > 1/2$. Unfortunately, it seems that $F(s)$ has a positive proportion of zeros in $\mathrm{Re}(s) < 1/2$ and in $\mathrm{Re}(s) > 1/2$ both. This phenomenon for $F(s)$ is not weird, for the term $\sum_{n\leqslant\sqrt{\frac{t}{2\pi}}}\frac{1}{n^s}$ in (2.2) behaves like a zeta function: namely, zeros of a zeta function without an Euler product can be located arbitrary off the critical line. Under this circumstance, it is very difficult to prove RH or even Riemann's statement using (2.1). We note that even the behavior of zeros of $F(s)$ to the left half is obscure. We expect that for any real number σ, $F(s)$ in (2.1) has infinitely many zeros in $\mathrm{Re}(s) < \sigma$.

Concerning (2.2), we refer to [14] for an interesting study on zeros of $\zeta(s)$.

We have seen why the Riemann-Siegel formula is not suitable to resolve RH. However, we still ask if we can have alternative representations of $\zeta(s)$ like the Riemann-Sigel formula. Namely, we want to look for functions $F(s)$ such that it satisfies the similar formula as in (2.1) and $F(s)$ has no zeros in $\mathrm{Re}(s) < 1/2$ or in $\mathrm{Re}(s) > 1/2$. If we achieve this, the Riemann hypothesis will follow. Thus, heuristically, we can say that RH is valid if and only if there exists $F(s)$ such that (2.1) holds and $F(s)$ has no zeros in $\mathrm{Re}(s) < 1/2$ or in $\mathrm{Re}(s) > 1/2$. As a sufficient condition of this [26], we have

Proposition 2.1 *Let $W(z)$ be a function in $\mathbb{C}$. Suppose $W(z)$ satisfies*

$$W(z) = H(z)e^{\alpha z} \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{\rho_n}\right) \left(1 + \frac{z}{\overline{\rho_n}}\right) \right],$$

where $H(z)$ is a nonzero polynomial having N many (counted with multiplicity) in the lower half-plane, $\alpha \in \mathbb{R}$, $Im\rho_n \geqslant 0$ $(n = 1, 2, \ldots)$, and the infinite product converges uniformly in any compact subset of $\mathbb{C}$. Then, $W(z) + \overline{W(\overline{z})}$ (or $W(z) - \overline{W(\overline{z})}$) has at most N pair of conjugate complex zeros (counted with multiplicity).

Concerning $\zeta(s)$, we rotate the critical line to the real axis, i.e., $W(z) = F(1/2+iz)$. In this way, we often are able to have better understanding of the behavior of zeros of zeta functions.

In this proposition, one important ingredient is that α is real and often, this is related to some intrinsic properties of the function $W(z)$. We will see from some examples that $W(z)$ comes from zeta functions and we can say that the corresponding α characterizes the nature of $W(z) + \overline{W(\overline{z})}$ (or $W(z) - \overline{W(\overline{z})}$).

Proposition 2.1 can be justified by the following lemma [4].

Lemma 2.2 *Let $U(z)$ and $V(z)$ be real polynomials. Assume that $U \not\equiv 0$ and that $W(z) = U(z) + iV(z)$ has exactly n zeros (counted with multiplicity) in the lower half-plane. Then $U(z)$ can have at most n pairs of conjugate complex zeros (again counted with multiplicity).*

We note that $W(z) + \overline{W(\overline{z})} = 2U(z)$ and $W(z) - \overline{W(\overline{z})} = 2iV(z)$. This lemma immediately follows from checking $\arg(W(t))$ from $-\infty$ to ∞.

We have very limited techniques in knowing the exact location of zeros of zeta functions. Proposition 2.1 is one of them and it turns out that the proposition is quiet universally applicable to know the explicit behavior of zeros of zeta functions. We might develop other techniques rather than this, but essentially they are based on the sign change method or the argument principle.

3. The Riemann zeta function

In this section, we will introduce recent progresses for the behavior of zeros of functions related to $\zeta(s)$.

3.1. **The zeros of Fourier transforms.** For this section, we refer to [22]. We denote the Riemann Ξ-function $\Xi(z)$ by

$$\Xi(z) = \frac{s(s-1)}{2}\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

where $s = 1/2 + iz$. Then $\Xi(z)$ is an entire function with order 1. See [52] for this. The Riemann hypothesis is equivalent to the statement that the Riemann Ξ-function has real zeros only. The Riemann Ξ-function has the following Fourier transform [52]

$$\Xi(z) = \int_{-\infty}^{\infty} \Phi(t)e^{izt}dt$$

where

$$\Phi(t) = 2\sum_{n=1}^{\infty}(2n^4\pi^2 e^{9t/2} - 3n^2\pi e^{5t/2})e^{-n^2\pi e^{2t}}.$$

Here $\Phi(t) = \Phi(-t)$ and $\Phi(t) > 0$ for any $t \in \mathbb{R}$. Thus we get $\Xi(z) = \Xi(-z)$ and $\overline{\Xi(z)} = \Xi(\overline{z})$ for any complex number z.

Pólya considered the approximations of the Riemann Ξ-function $\Xi(z)$;

$$\Xi^*(z) = \int_{-\infty}^{\infty} 8\pi^2 \cosh(\frac{9}{2}t)e^{-2\pi\cosh(2t)}e^{izt}dt;$$

$$\Xi^{**}(z) = \int_{-\infty}^{\infty}\left(8\pi^2\cosh\frac{9}{2}t - 12\pi\cosh\frac{5}{2}t\right)e^{-2\pi\cosh 2t}e^{izt}dt.$$

Theorem 3.1 (Pólya, [42]) *$\Xi^*(z)$ and $\Xi^{**}(z)$ have real zeros only.*

Hejhal [15] considered general approximations to $\Xi(z)$. He took

$$\Xi_n(z) = \int_{-\infty}^{\infty}\Phi_n(t)e^{izt}dt,$$

where for each $n = 1, 2, \ldots,$

$$\Phi_n(t) = \sum_{k=1}^{n} e^{-2\pi k^2\cosh(2t)}\left[8\pi^2 k^4\cosh(\frac{9}{2}t) - 12\pi k^2\cosh(\frac{5}{2}t)\right].$$

Clearly, the approximation $\Xi_n(z)$ does not converge to $\Xi(z)$. Thus, it is meaningless to study these approximations concerning a possible proof of the Riemann hypothesis. However, the distribution of zeros of those approximations has their own interests.

We let f be a meromorphic function. Let $T > 0$. Define $N(T; f)$ and $N^s(T; f)$ by

$$N(T; f) = \text{ the number of zeros of } f(z) \text{ in the region } 0 < \mathrm{Re}(z) < T;$$

$$N^s(T; f) = \text{ the number of simple zeros of } f(z) \text{ in the region } 0 < \mathrm{Re}(z) < T.$$

From the standard argument [52], for $T > 0$ we have

$$N(T;\Xi) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T); \qquad N(T;\Xi_n) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T).$$

Theorem 3.2 (Hejhal, [15]) *Almost all zeros of $\Xi_n(z)$ are real and simple. More precisely,*

$$N(T;\Xi_n) - N^s(T;\Xi_n) = O\left(\frac{T\log T}{\log\log T}\right).$$

The author improved this theorem as follows.

Theorem 3.3 (Ki) $N(T;\Xi_n) - N^s(T;\Xi_n) = O(T)$.

Basically, the proof of this theorem follows from the sign change method. One cannot expect that $N(T;\Xi_n)$ has real zeros only, and so it is meaningless to approach to the Riemann hypothesis through this sort of approximations of the Riemann Ξ function.

We introduce a way to understand the exact behavior of zeros of Fourier transforms.

Question 3.4 (de Bruijn, [4]) *Let $F(t)$ be a complex valued function defined on the real line; and suppose that $F(t)$ is integrable,*

$$F(t) = O\left(e^{-|t|^b}\right) \qquad (|t| \to \infty,\ t \in \mathbb{R})$$

for some constant $b > 2$, and

$$F(-t) = \overline{F(t)} \qquad (t \in \mathbb{R}).$$

Suppose also that for each $\epsilon > 0$ all but a finite number of the zeros of the Fourier transform of $F(t)$ lie in the strip $|Im(z)| \leqslant \epsilon$. Does it follow from these assumptions that for each $\lambda > 0$ the Fourier transform of $e^{\lambda t^2} F(t)$ has only a finite number of complex zeros?

Theorem 3.5 (Ki-Kim, [27]) *De Bruijn's question is true.*

This theorem is very useful to justify the exact location of zeros of Fourier transforms. Namely, we immediately meet some difficulties to show directly the explicit location of zeros of Fourier transforms, but due to Theorem 3.5, it suffices to prove a much weaker statement.

Theorem 3.6 (Ki-Kim, [28]) *We define the function $f(z)$ by*

$$f(z) = \int_{-\infty}^{\infty} Q(t) e^{P(t)} e^{izt} dt, \tag{2.1}$$

where the function $P(t)$ with even degree is a polynomial with the negative leading coefficient and $Q(t)$ an arbitrary polynomial. All but a finite number of the zeros of $f(z)$ are real and simple, if $P(-t) = \overline{P(t)}$ and $Q(-t) = \overline{Q(t)}$ for all $t \in \mathbb{R}$.

We recall that the Riemann Ξ-function is the Fourier transform of $\Phi(t)$. For a real number λ, we define $\Xi_\lambda(z)$ by

$$\Xi_\lambda(z) = \int_{-\infty}^{\infty} \Phi(t) e^{\lambda t^2} e^{izt} dt.$$

Thus, $\Xi_0(z) = \Xi(z)$.

Theorem 3.7 (Ki-Kim-Lee, [29]) *For any positive λ, all but finitely many zeros of $\Xi_\lambda(z)$ are real and simple.*

For proofs of Theorems 3.6 and 3.7, we need saddle point methods together with Theorem 3.5. Concerning Theorem 3.7, we note that RH is valid if and only if for any positive λ, $\Xi_\lambda(z)$ has real zeros only. It should be important to ask if we can prove an uniform version of Theorem 3.7, i.e., the number of zeros of $\Xi_\lambda(z)$ off the real axis is finite uniformly for any $\lambda > 0$. Then, this directly implies that all but finitely many zeros of $\zeta(s)$ are on the critical line. Unfortunately, using the same methods as in the proof of Theorem 3.7, the corresponding $\widetilde{\Xi}_\lambda$ for any suitable Dirichlet series with degree 1 or 2 and a functional equation satisfies the analogue of the theorem, for instance $\zeta(s)\zeta(s-2k+1)$ where k is any positive integer. Actually, zeros of $\Xi_\lambda(z)$ $(\lambda > 0)$ behave regularly, but zeros of $\zeta(s)$ or any genuine zeta function do irregularly according to PCC. This signs that our approach to RH by virtue of Fourier transforms will be unsuccessful.

We have a very interesting application of Theorem 3.7 as follows.

Theorem 3.8 (Ki-Kim-Lee, [29]) *We define Λ by the de Bruijn-Newman constant $\Lambda = 4\lambda^{(0)}$, where $\lambda^{(0)}$ denotes the infimum of the set of real numbers λ such that the entire function Ξ_λ has only real zeros. The de Bruijn-Newman constant Λ is less than* $1/2$.

Using the fact that $\zeta(s)$ has no zeros in $\mathrm{Re}(s) > 1$, de Bruijn [4] showed that for $\lambda \geqslant 1/8$, $\Xi_\lambda(z)$ has real zeros only. Thus, Theorem 3.8 says that we justify the analogue of the quasi-Riemann hypothesis, in terms of the weighted Riemann Ξ function $\Xi_\lambda(z)$! So far, any arithmetic applications of Theorem 3.8 have not been known. On the other hand, Newman [40] proved that for some $\lambda < 0$, $\Xi_\lambda(z)$ has nonreal zeros.

Conjecture 3.9 (Newman, [40]) *For any $\lambda < 0$, $\Xi_\lambda(z)$ has complex zeros.*

For some numerical results related to this conjecture, see [9], [41]. This conjecture means that as Newman [40] remarked, if the Riemann hypothesis is true, then it is only barely so. Based on Theorems 3.5 and 3.7, one might try to prove that for some $\lambda < 0$, all but a finite number of zeros of $\Xi_\lambda(z)$ lie in $|\mathrm{Im}(z)| < \delta$ for any $\delta > 0$. Then it follows from de Bruijn's question that RH is true, excluding finite many exceptions. However, one shouldn't take this seriously for RH because for $\lambda < 0$, there is no way to understand a precise behavior of zeros of $\Xi_\lambda(z)$ near the real axis as $z \to \infty$, and it is also very probable that $\Xi_\lambda(z)$ has infinitely many nonreal zeros.

3.2. **The derivative of the Riemann zeta function.** Speiser [47] proved that RH is equivalent to the nonexistence of nonreal zeros of $\zeta'(s)$ in $\mathrm{Re}(s) < 1/2$. It was reproved by Spira [48]. Levinson and Montgomery improved this result as follows.

Theorem 3.10 (Levinson-Montgomery, [38]) *Let $N^-(T)$ be the number of zeros of $\zeta(s)$ in $0 < t < T$, $0 < \sigma < 1/2$. Let $N_1^-(T)$ be the number of zeros of $\zeta'(s)$ in the same region. Then, we have*

$$N_1^-(T) = N^-(T) + O(\log T).$$

Unless $N^-(T) > T/2$ for all large T, there exists a sequence $\langle T_j \rangle$, $T_j \to \infty$ as $j \to \infty$ such that

$$N_1^-(T_j) = N^-(T_j)$$

Applying this theorem, Speiser's theorem was reproved in [38].

The most essential part of the proof of Theorem 3.10 is of the functional equation for $\zeta(s)$. With this theorem, we understand a beautiful relationship between zeros of $\zeta(s)$ and zeros of $\zeta'(s)$.

We discuss the importance of $\zeta'(s)$ for RH. From the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s),$$

we readily deduce

$$\frac{\chi'(s)}{\chi(s)}H(s)\zeta(s) = H(s)\zeta'(s) + H(1-s)\zeta'(1-s), \tag{3.1}$$

where $H(s) = \pi^{-s/2}\Gamma(s/2)$. Thus, this fulfills the Riemann-Siegel formula (2.1). We should recall that RH if and only if $\zeta'(s)$ does not have complex zeros in $\mathrm{Re}(s) < 1/2$. Thus it is expected that $\zeta'(s)$ does not have complex zeros in $\mathrm{Re}(s) < 1/2$. We notice that $H(s)\zeta'(s)$ has simple poles at $s = 0, -2, -4, \ldots$ and one simple real zero in each interval $(-2n-2, -2n)$ $(n = 1, 2, 3, \ldots)$. These poles play a positive role and these real zeros do the other way that zeros of $\zeta(s)$ tend to be located on $\mathrm{Re}(s) = 1/2$. The important point is that they are equally distributed on the negative real axis and so they can be ignored. Thus, it suffices to know the behavior of complex zeros of $\zeta'(s)$. We must realize that after we differentiate $\zeta(s)$, zeros of $\zeta'(s)$ move to the right of the critical line $\mathrm{Re}(s) = 1/2$. This is an extremely important phenomenon in studying zeros of $\zeta(s)$. Applying this wonderful property, Levinson achieved his landmark theorem [37], that is, at least one third of zeros of $\zeta(s)$ are on $\mathrm{Re}(s) = 1/2$. Probably, (3.1) is the representation what we have been looking for. One can ask what if we consider higher derivatives of $\zeta(s)$. Namely, we can definitely enjoy the phenomena that zeros of higher derivatives move to the right further and further. However, for higher derivatives of $\zeta(s)$, we do not have functional equations and so it seems very difficult to recover zeros of $\zeta(s)$ from zeros of higher derivatives. If we can do so, for instance we immediately prove that almost all zeros of $\zeta(s)$ are on $\mathrm{Re}(s) = 1/2$. For this, we refer to [31].

We have a heuristic argument for the simple zeros of $\zeta(s)$. We assume RH. Suppose $\zeta(s)$ has a zero on $\mathrm{Re}(s) = 1/2$ with multiplicity $n > 1$. Then, certainly, this multiple zero remains the same zero for $\zeta'(s)$ on $\mathrm{Re}(s) = 1/2$ with multiplicity $n-1$ and the other one more zero moves to the right or disappears. Based on this, we rearrange (2.1) as follows:

$$\frac{\chi'(s)}{\chi(s)}H(s)\zeta(s) = f(s)\left(H(s)\widetilde{\zeta}'(s) + H(1-s)\widetilde{\zeta}'(1-s)\right),$$

where $f(s)$ is the obvious product from zeros of $\zeta'(s)$ on $\mathrm{Re}(s) = 1/2$. Clearly, we have

$$f(s) = f(1-s).$$

Then, one can justify the following.

Theorem 3.11 (Ki) *Assuming RH, all complex zeros of* $H(s)\widetilde{\zeta}'(s)+H(1-s)\widetilde{\zeta}'(1-s)$ *are simple and on* $\mathrm{Re}(s) = 1/2$.

Through the function in this theorem, we reproduce zeros of $\zeta(s)$. It is unreasonable that these reproduced zeros overlap with the zeros of $f(s)$. Under this hypothesis, we can deduce that all zeros of $\zeta(s)$ are simple.

3.3. **Gaps of zeros of the Riemann zeta function.** In order to understand the explicit behavior of zeros of $\zeta(s)$, we need to understand gaps of zeros of the function. We should know deep arithmetic properties of $\zeta(s)$ in acquiring that gaps are arbitrary small in comparison with the average gap of zeros of $\zeta(s)$. The property of small gaps is heavily related to the Landau-Siegel zero problem. Indeed, this is the origin of PCC [39]. For more detailed accounts, we refer to [8], and also see [18]. In [11], using the derivative of $\zeta(s)$, the authors introduced a new point of view for the Landau-Siegel zero problems. Thus, especially, the explicit behavior of zeros of the derivative of $\zeta(s)$ becomes very important. For this, we will briefly introduce a theorem [25].

$\rho' = \beta' + i\gamma'$ denotes a zero of $\zeta'(s)$. Under RH, K. Soundararajan [46] demonstrated the presence of zeros of $\zeta'(s)$ in the region $\sigma < 1/2 + \nu/\log T$ for all $\nu \geqslant 2.6$. Assuming the truth of RH, K. Soundararajan in the same paper conjectured that the following two statements are equivalent:

(i) $\liminf(\beta' - 1/2)\log\gamma' = 0$;

(ii) $\liminf(\gamma^+ - \gamma)\log\gamma = 0$ where $1/2 + i\gamma$, $1/2 + i\gamma^+$ are zeros of $\zeta(s)$ and γ^+ is the least ordinate with $\gamma^+ > \gamma$.

Y. Zhang [61] has shown that (ii) implies (i) as follows.

Theorem 3.12 (Zhang, [61]) *Assume RH. Let α_1 and α_2 be positive constants satisfying $\alpha_1 < 2\pi$ and*

$$\alpha_2 > \alpha_1 \left(1 - \sqrt{\frac{\alpha_1}{2\pi}}\right)^{-1}.$$

If $\rho = 1/2 + i\gamma$ is a zero of $\zeta(s)$ such that γ is sufficiently large and $\gamma^+ - \gamma < \alpha_1(\log\gamma)^{-1}$, then there exists a zero ρ' of $\zeta'(s)$ such that

$$|\rho' - \rho| < \alpha_2(\log\gamma)^{-1}.$$

Is it true that (i) implies (ii)? M. Z. Garaev and C. Y. Yildirim [13] proved a weaker form of the converse; if we assume RH and $\liminf(\beta' - 1/2)\log\gamma'(\log\log\gamma')^2 = 0$, then we have $\liminf_{n\to\infty}(\gamma_{n+1} - \gamma_n)\log\gamma_n = 0$.

We have the following.

Theorem 3.13 (Ki, [25]) *Assume RH. Then, the following are equivalent:*

(1) $\liminf(\beta' - 1/2)\log\gamma' \neq 0$;

(2) *For any $c > 0$ and $s = \sigma + it$ with $0 < |\sigma - 1/2| < c/\log t$ $(t > t_0(c))$, we have*

$$\frac{\zeta'}{\zeta}(s) = \frac{1}{s - \widehat{\rho}} + O(\log t),$$

where $\widehat{\rho} = 1/2 + i\gamma$ is the zero of ζ closest to s (and to the origin, if there are two such).

By Theorem 3.13 we have

Corollary 3.14 *Assume RH and* $\liminf(\beta' - 1/2)\log\gamma' \neq 0$. *Then, for any* $c > 0$ *and* $s = \sigma + it$ $(t > t_1(c))$, *we have*

$$\frac{\zeta'}{\zeta}(s) = O\left((\log t)^{2-2\sigma}\right)$$

uniformly for $1/2 + c/\log t \leqslant \sigma \leqslant \sigma_1 < 1$.

Assuming RH, we have

$$\frac{\zeta'}{\zeta}(s) = O\left((\log t)^{2-2\sigma}\right)$$

holds uniformly for $1/2 + c/\log\log t \leqslant \sigma \leqslant \sigma_1 < 1$, where $t \geqslant 20$, $c > 0$ and 'O' depends upon c and σ_1. For the proof of this, see [25]. We immediately find a huge difference between the last two formulas. Assuming RH, with current techniques it appears difficult to disprove or prove the formula in Corollary 3.14.

Based on Theorems 3.12, 3.13 and Soundararajan's conjecture, we speculate as follows.

Conjecture 3.15 *Assume RH. Then the following are equivalent:*

(i)$'$ *For any* $c > 0$ *and* $s = \sigma + it$ *with* $t > t_2(c)$, *we have*

$$\frac{\zeta'}{\zeta}(s) = O\left((\log t)^{2-2\sigma}\right)$$

uniformly for $1/2 + c/\log t \leqslant \sigma \leqslant \sigma_1 < 1$;

(ii)$'$ *The negation of* (ii), *i.e.,* $\liminf(\gamma^+ - \gamma)\log\gamma \neq 0$.

We expect the negation of (ii)$'$. It is believed that even after we achieve the validity of RH, we will not be capable to disprove this.

4. The Eisenstein series

We define the Eisenstein series by

$$E_0(z;s) = \frac{1}{2}\sum_{(m,n)\neq(0,0)} \frac{y^s}{|mz+n|^{2s}} \qquad (\mathrm{Re}(s) > 1),$$

where $z = x + yi$, $x \in \mathbb{R}$, $y > 0$. The Eisenstein series is analytically continued to the whole complex plane except for the simple pole at $s = 1$. It has the Fourier expansion [6] as follows:

$$E_0(z;s) = \zeta(2s)y^s + \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)y^{1-s} + 4\pi^s\sqrt{y}\sum_{n=1}^{\infty} n^{1/2-s}\sum_{d|n} d^{2s-1}\frac{K_{s-1/2}(2\pi ny)}{\Gamma(s)}\cos(2\pi nx).$$

Also we have the functional equation:

$$\pi^{-s}\Gamma(s)E_0(z;s) = \pi^{-1+s}\Gamma(1-s)E_0(z;1-s). \tag{4.1}$$

The Eisenstein series with class-number 1 has an Euler product, for instance

$$E_0(i;s) = 4\zeta(s)L(s,\chi_{-4}).$$

Unfortunately, the Eisenstein series with class number > 1 doesn't have an Euler product. Thus, we expect that in general, the Eisenstein series doesn't satisfy the analogue of RH, for example, the Eisenstein series with class-number 2

$$E_0(\sqrt{5}i;s), \qquad E_0(i\sqrt{6};s), \qquad E_0(i\sqrt{10};s).$$

Namely, applying methods [10], we can prove that the number of zeros of the Eisenstein series with class-number > 1 is $\gg T$ in $\mathrm{Re}(s) > 1$ and $0 < \mathrm{Im}(s) < T$. Voronin [53] showed the analogue of this result for any strip inside the interval $(1/2, 1)$. Recently, Lee [36] established that the number of zeros of any given Eisenstein series with class-number > 1 in $\mathrm{Re}(s) > 1$ and $0 < \mathrm{Im}(s) < T$ is

$$cT + o(T) \qquad (T \to \infty),$$

for any interval (σ_1, σ_2), where $\sigma_1 > 1/2$, $\sigma_1 < \sigma_2$ and c is depending on σ_1, σ_2. Can we still expect that almost all zeros of the Eisenstein series are on $\mathrm{Re}(s) = 1/2$? For this question, some results are known. We refer to Hejhal's theorem [16]: for any L-functions $L_1(s), L_2(s)$ satisfying some (rather complicated) conditions, the number of zeros of a cetain linear combination of L-functions

$$(\cos\alpha)e^{i\omega_1}L_1(s) + (\sin\alpha)e^{i\omega_2}L_2(s)$$

in $\mathrm{Re}(s) > 1/2 + G/\log T$ and $0 < \mathrm{Im}(s) < T$ is between

$$C_1\frac{T\log T}{G\sqrt{\log\log T}} \qquad \text{and} \qquad C_2\frac{T\log T}{G\sqrt{\log\log T}},$$

where $(\log\log T)^\kappa \leqslant G \leqslant (\log T)^\delta$, $\kappa > 1, \delta > 0$, and C_1, C_2 $(C_1 < C_2)$ are depending on $L_1, L_2, \omega_1, \omega_1, \kappa, \delta$. Hejhal [17] proposed a generalization of this result. We know that the Eisenstein series with class-number 2 can be represented by a linear combination of L-functions. Thus, it might be very interesting if one can prove the analogue of Hejhal's theorem for the Eisenstein series. It can be shown that the Eisenstein series has at least $\gg T$ many zeros on $\mathrm{Re}(s) = 1/2$ and in $0 < \mathrm{Im}(s) < T$. On the other hand, we refer to [3]: namely, it was shown that with several assumptions, almost all zeros of a linear combination of L-functions are on $\mathrm{Re}(s) = 1/2$.

We recall

$$E_0(i;s) = 4\zeta(s)L(s,\chi_{-4}).$$

Thus, the Eisenstein series contains the Riemann zeta function $\zeta(s)$. Hence, it should be meaningful to investigate zeros of the Eisenstein series. We consider truncations of the Eisenstein series as follows:

$$E_{0,0}(z;s) = \zeta(2s)y^s + \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)y^{1-s};$$

$$E_{0,N}(z;s) = \zeta(2s)y^s + \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)y^{1-s} + 4\pi^s\sqrt{y}\sum_{n=1}^{N} n^{1/2-s}\sum_{d|n} d^{2s-1}\frac{K_{s-1/2}(2\pi ny)}{\Gamma(s)}\cos(2\pi nx).$$

Interestingly, these truncations fulfill the same functional equation as in (4.1)

$$\pi^{-s}\Gamma(s)E_{0,N}(z;s) = \pi^{-1+s}\Gamma(1-s)E_{0,N}(z;1-s)$$

for $N = 0, 1, 2, \ldots$. Thus, we expect that zeros of truncations $E_{0,N}(z;s)$ tend to be located on $\mathrm{Re}(s) = 1/2$.

We have several methods to show that all complex zeros of the constant term of the Eisenstein series in its Fourier series with $\mathrm{Im}(z) \geqslant 1$ lie on $\mathrm{Re}(s) = 1/2$. Hejhal [15] justified this as follows. For his proof, we briefly introduce the Maass-Selberg formula for the Eisenstein series. $\mathcal{F}$ denotes the standard fundamental domain for the full modular group $\mathrm{SL}(2,\mathbb{Z})$ by

$$\mathcal{F} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0,\ |z| > 1,\ |\mathrm{Re}(z)| \leqslant 1/2\}.$$

Applying Green's Theorem, we derive the Maass-Selberg formula as follows:

$$\begin{aligned}
&[s(1-s) - w(1-w)] \iint_{\mathcal{F}} \widetilde{E}_0(z;s)\widetilde{E}_0(z;w)\frac{dxdy}{y^2} \\
&\quad = (s-w)\left[\psi(s)\psi(w)Y^{1-s-w} - \phi(s)\phi(w)Y^{s+w-1}\right] \\
&\qquad + (1-s-w)\left[\phi(s)\psi(w)Y^{s-w} - \psi(s)\phi(w)Y^{w-s}\right],
\end{aligned}$$

where

$$\phi(s) = \zeta(2s), \qquad \psi(s) = \pi^{\frac{1}{2}}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\zeta(2s-1),$$

$$\widetilde{E}_0(z;s) = \begin{cases} E_0(z;s) & \text{if } y \leqslant Y \\ E_0(z;s) - \zeta(2s) - \pi^{\frac{1}{2}}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\zeta(2s-1) & \text{if } y > Y. \end{cases}$$

Using this formula, we readily prove that all complex zeros of the constant term of the Eisenstein series for $\mathrm{Im}(z) \geqslant 1$ lie on $\mathrm{Re}(s) = 1/2$. Also, this can be readily obtained by Pólya's method [42]. There is still another way by the author.

Theorem 4.1 (Ki, [24]) *For $y \geqslant 1$, all complex zeros of the constant term are simple and on $Re(s) = 1/2$.*

By Proposition 2.1, we can prove this theorem for the second part. It is quite amazing that this proof relies on an easy classical method. We get the simplicity of zeros of the constant term by checking the argument change of the constant term on $\mathrm{Re}(s) = 1/2$. With a careful computation, we have

Theorem 4.2 (Lagarias-Suzuki, [34]) *There exists $y^*(= 4\pi e^{-\gamma})$ such that all zeros of $a_0(z,s)$ lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$ for $1 \leqslant y \leqslant y^*$ and for $y > y^*$ there are exactly two zeros off the critical line. These exceptional zeros are real simple zeros ρ_y, $1-\rho_y$ with $\frac{1}{2} < \rho_y < 1$ and $\rho_y \to 1$ as $y \to \infty$.*

From the behavior of zeros of the constant term of the Eisenstein series, we need to investigate the location of zeros of truncations of the Eisenstein series. We already knew the failure of the analogue of RH. However, Hejhal [15] proved the similar result as in Theorem 3.2, namely almost all zeros of truncations of the Eisenstein series are on $\mathrm{Re}(s) = 1/2$. Interestingly, the author proves the following.

Theorem 4.3 (Ki, [21]) *All but finitely many zeros of truncations of the Eisenstein series in any strip containing the line $Re(s) = 1/2$ are simple and on $Re(s) = 1/2$. If $Im(z) \geqslant 1$, then all but finitely many zeros of truncations of the Eisenstein series are simple and on $Re(s) = 1/2$.*

Concerning this theorem, we also refer to [20], [23].

Based on Theorem 4.3, we can approach RH due to

$$E_0(i; s) = 4\zeta(s)L(s, \chi_{-4}).$$

The constant term and the first several truncations of this Eisenstein series satisfy the analogue of RH. Thus, we ask if we can find a sequence $\langle E_{0,N_k}(i; s)\rangle$ such that all complex zeros of $E_{0,N_k}(i; s)$ are on $\mathrm{Re}(s) = 1/2$ and $N_k \to \infty$ as $k \to \infty$. This immediately implies the validity of the Riemann hypothesis. Unfortunately, in general, the truncations have off-line zeros! For instance, it turns out that for the case $N = 100$, $E_{0,100}(i; s)$ has off-line zeros around $2\pi N = 200\pi$ and even some of these off-line zeros are very close to the line $\mathrm{Re}(s) = 1$. This suggests that in this way, it seems very difficult to demonstrate the validity of RH. However, we still propose the following.

Conjecture 4.4 *For any $T > 0$, there is a positive integer n_0 such that for any $n \geqslant n_0$, all zeros of $E_{0,n}(i; s)$ in $0 < Im(s) < T$ are on $Re(s) = 1/2$.*

This implies the validity of RH. Numerical computations show that Conjecture 4.4 tends to be correct up to $1 \leqslant N \leqslant 120$. Thus, we can still pursue RH in this direction.

5. Weng's Zeta functions

The constant term for the Eisenstein series belongs to periods. The concept of periods has widely been studied and applied in mathematics. For instance, this is one of core concepts in Langlands' program. We shall introduce this in the theory of Langlands' program and shall discuss its relation to our study.

Recently, Weng [60] discovered new zeta functions defined by periods. In order to introduce Weng's zeta functions, we need some backgrounds in Automorphic Forms. For this purpose, we mostly follow [60]. Also, see [1], [19], [51] and [56]. We let F be a number field with $\mathbb{A} = \mathbb{A}_F$ its ring of adèles. Let G be a quasi-split connected reductive algebraic group over F. Let Z be the central subgroup of G. Fix a Borel subgroup P_0 of G over F. Write $P_0 = M_0U_0$ where M_0 is a maximal torus and U_0 is the unipotent radical of P_0. Let $P \supset P_0$ be a parabolic subgroup of G over F. Write $P = MU$ with $M_0 \subset M$ the standard Levi and U the unipotent radical. Let W be the Weyl group of the maximal F-split subtorus of M_0 in G. Let Δ_0 be the set of simple roots. ρ_P denotes half the sum of roots in U. We fix a maximal compact subgroup $\mathbf{K}$ of $G(\mathbb{A})$ such that $P(\mathbb{A}) \cap \mathbf{K} = (M(\mathbb{A}) \cap \mathbf{K})(U(\mathbb{A}) \cap \mathbf{K})$.

We write $X(G)_F$ for the additive group homomorphisms from G to $GL(1)$ over F. We form the real vector space $\mathfrak{a}_G = \mathrm{Hom}_F(X(G)_F, \mathbb{R})$. We set $\mathfrak{a}_P = \mathfrak{a}_M$ $(P = MU)$ and $\mathfrak{a}_0 = \mathfrak{a}_{P_0}$. We write Δ_0^P for the set of simple roots in P. Let Δ_P be the set of linear forms on $\mathfrak{a}_P$ obtained by restriction of elements in the complement $\Delta_0 - \Delta_0^P$. We write $\widehat{\Delta}_0 = \{\varpi_\alpha : \alpha \in \Delta_0\}$ for the set of simple weights. Set $\widehat{\Delta}_P = \{\varpi_\alpha : \alpha \in \Delta_0 - \Delta_0^P\}$. We write $\widehat{\tau}_p$ for the characteristic function of the subset

$$\{t \in \mathfrak{a}_P : \varpi(t) > 0,\ \varpi \in \widehat{\Delta}_P\}.$$

Fix a sufficiently regular $T \in \mathfrak{a}_0$ ($\alpha(T) \gg 0$ for any simple root α). For a continuous function ϕ on $G(F) \setminus G(\mathbb{A})^1$, define Arthur's analytic truncation $\left(\Lambda^T \phi\right)(x)$ by

$$\left(\Lambda^T \phi\right)(x) = \sum_P (-1)^{\dim(A_P/Z)} \sum_{\delta \in P(F)\setminus G(F)} \phi_P(\delta x) \widehat{\tau}_P(H(\delta x) - T),$$

where A_P is the central subgroup of M $(P = MU)$, $\phi_P(x) = \int_{U(F)\setminus U(\mathbb{A})} \phi(nx)dn$ and the sum is over all parabolic subgroups containing P_0 (see [1], [2]).

For ϕ an automorphic form of G, we define Arthur's period $A(\phi; T)$ by

$$A(\phi; T) = \int_{G(F)\setminus G(\mathbb{A})} \Lambda^T \phi(g) dg.$$

For ϕ an M-level automorphic form, we form the associated Eisenstein series $E(\phi, \lambda)(g)$ by

$$E(\phi, \lambda)(g) = \sum_{\delta \in P(F)\setminus G(F)} m_P(\delta g)^{\lambda + \rho_P} \phi(\delta g) \qquad (\mathrm{Re}\lambda \in \mathcal{C}_P^+),$$

where m_P is the Harish-Chandra homomorphism and $\mathcal{C}_P^+$ denotes a certain positive chamber in $\mathfrak{a}_P$ and $\lambda = (\lambda_1, \ldots, \lambda_r)$ with r the rank of the group. For a cusp form ϕ, then we obtain that the Eisenstein period $A(E(\phi; \lambda); T)$ is

(1) 0 if $P \neq P_0$;

(2) $$v \sum_{w \in W} \frac{e^{\langle w\lambda - \rho_{P_0}, T\rangle}}{\prod_{\alpha \in \Delta_0} \langle w\lambda - \rho_{P_0}, \alpha^\vee \rangle} \times \int_{M_0(F)\setminus M_0(\mathbb{A})^1 \times \mathbf{K}} (M(w, \lambda)\phi)(mk) dm\, dk$$

if $P = P_0$, where $v = \mathrm{vol}\left(\{\sum_{\alpha \in \Delta_0} a_\alpha \alpha^\vee : a_\alpha \in [0, 1)\}\right)$, $\alpha^\vee$ is the coroot associated to α and for $g \in G(\mathbb{A})$

$$(M(w, \lambda)\phi)(g) = m_{P'}(g)^{\omega\lambda + \rho_{P'}} \int_{U'(F)\cap wU(F)w^{-1}\setminus U'(\mathbb{A})} m_P(w^{-1}n'g)^{\lambda + \rho_P} dn'$$

with $M' = wMw^{-1}$ and $P' = M'U'$ (see [19]). Following Weng [60], define the period $\omega_F^G(\lambda)$ of G over F by

$$\omega_F^G(\lambda) = \sum_{w \in W} \frac{1}{\prod_{\alpha \in \Delta_0} \langle w\lambda - \rho_{P_0}, \alpha^\vee \rangle} \times M(w, \lambda),$$

where

$$M(w,\lambda)=m_{P'}(e)^{\omega\lambda+\rho_{P'}}\int_{U'(F)\cap wU(F)w^{-1}\backslash U'(\mathbb{A})}m_P(w^{-1}n')^{\lambda+\rho_P}dn'.$$

A precise formula for G a classical semisimple algebraic group over $\mathbb{Q}$, can be found in [35], and also see [57] and [59].

Let P be a fixed maximal parabolic subgroup of G. Then, P corresponds to a simple root $\alpha_P\in\Delta_0$. We write $\Delta_0\setminus\{\alpha_P\}=\{\beta_1,\ldots,\beta_{r-1}\}$, where $r=r(G)$ denotes the rank of G. Following Weng [60], define the period $\omega_{\mathbb{Q}}^{G/P}$ for (G,P) over $\mathbb{Q}$ by

$$\omega_{\mathbb{Q}}^{G/P}(\lambda_P)=\mathrm{Res}_{\langle\lambda-\rho,\beta^\vee_{r(G)-1}\rangle=0}\cdots\mathrm{Res}_{\langle\lambda-\rho,\beta_1^\vee\rangle=0}\left(\omega_{\mathbb{Q}}^G(\lambda)\right),\qquad\lambda_P\gg 0,$$

where with the constraint of taking residues along with $(r-1)$ singular hyperplanes

$$\langle\lambda-\rho,\beta_1^\vee\rangle=0,\ldots,\langle\lambda-\rho,\beta^\vee_{r(G)-1}\rangle=0,$$

there is only one variable λ_{i_0} left among z_i's, re-scale it when necessary and rename it λ_P. After suitable normalizations, we define the Weng zeta function $\xi_{\mathbb{Q}}^{G/P}(s)$ from $\omega_{\mathbb{Q}}^{G/P}(\lambda_P)$. Komori [32] demonstrated that $\xi_{\mathbb{Q}}^{G/P}(s)$ has the usual functional equation $\xi_{\mathbb{Q}}^{G/P}(s)=\xi_{\mathbb{Q}}^{G/P}(1-s)$. Weng [60] conjectured the following.

Conjecture 5.1 *The analogue of RH for the Weng zeta function $\xi_{\mathbb{Q}}^{G/P}(s)$ holds.*

Weng [60, pp. 38–43] provides the following ten zeta functions:

$\xi_{\mathbb{Q}}^{SL(2)/P_0}$ for $SL(2)$; $\xi_{\mathbb{Q}}^{SL(3)/P_{1,2}}$ for $SL(3)$; $\xi_{\mathbb{Q}}^{SL(4)/P_{1,3}}$, $\xi_{\mathbb{Q}}^{SL(4)/P_{2,2}}$ for $SL(4)$;
$\xi_{\mathbb{Q}}^{SL(5)/P_{1,4}}$, $\xi_{\mathbb{Q}}^{SL(5)/P_{2,3}}$ for $SL(5)$; $\xi_{\mathbb{Q}}^{Sp(4)/P_1}$, $\xi_{\mathbb{Q}}^{Sp(4)/P_2}$ for $Sp(4)$;
$\xi_{\mathbb{Q}}^{G_2/P_{\mathrm{long}}}$, $\xi_{\mathbb{Q}}^{G_2/P_{\mathrm{short}}}$ for G_2,

where each zeta function is associated with a maximal parabolic subgroup for each group. It is shown in [34], [49], [50], [51], [57], [58] that all zeros of five zeta functions $\xi_{\mathbb{Q}}^{SL(2)/P_0}$, $\xi_{\mathbb{Q}}^{SL(3)/P_{1,2}}$, $\xi_{\mathbb{Q}}^{Sp(4)/P_1}$, $\xi_{\mathbb{Q}}^{G_2/P_{\mathrm{long}}}$, $\xi_{\mathbb{Q}}^{G_2/P_{\mathrm{short}}}$ are on $\mathrm{Re}(s)=1/2$. On the other hand, the author proved

Theorem 5.2 (Ki, [26]) *All zeros of ten Weng's zeta functions $\xi_{\mathbb{Q}}^{SL(2)/P_0}$, $\xi_{\mathbb{Q}}^{SL(3)/P_{1,2}}$, $\xi_{\mathbb{Q}}^{SL(4)/P_{1,3}}$, $\xi_{\mathbb{Q}}^{SL(4)/P_{2,2}}$, $\xi_{\mathbb{Q}}^{SL(5)/P_{1,4}}$, $\xi_{\mathbb{Q}}^{SL(5)/P_{2,3}}$, $\xi_{\mathbb{Q}}^{Sp(4)/P_1}$, $\xi_{\mathbb{Q}}^{Sp(4)/P_2}$, $\xi_{\mathbb{Q}}^{G_2/P_{long}}$, $\xi_{\mathbb{Q}}^{G_2/P_{short}}$ are on $Re(s)=1/2$ and simple.*

The most complicated one among 10 examples in Theorem 5.2 is $\xi_{\mathbb{Q}}^{SL(5)/P_{2,3}}(s)$ as follows.

$$
\begin{aligned}
&\frac{\xi(2)^2\xi(3)}{5s-5}\cdot\xi(5s-1)\xi(5s)+\frac{\xi(2)}{4(5s-3)}\cdot\xi(5s-1)\xi(5s)\\
&+\frac{\xi(2)}{(5s-1)^2(5s-5)}\cdot\xi(5s-3)\xi(5s-2)-\frac{\xi(2)\xi(3)}{2(5s-4)}\cdot\xi(5s-1)\xi(5s)\\
&-\frac{\xi(2)^2}{3(5s-4)}\cdot\xi(5s-1)\xi(5s)-\frac{\xi(2)^2}{3(5s-3)}\cdot\xi(5s-1)\xi(5s)\\
&-\frac{1}{4(5s-3)(5s-1)}\cdot\xi(5s-2)\xi(5s-1)+\frac{\xi(2)^2}{3(5s-2)}\cdot\xi(5s-4)\xi(5s-3)\\
&-\frac{1}{2(5s-4)(5s-2)(5s-1)}\cdot\xi(5s-3)\xi(5s-2)+\frac{1}{8(5s-3)}\cdot\xi(5s-4)\xi(5s-3)\\
&-\frac{\xi(2)}{(5s-4)^2(5s)}\cdot\xi(5s-2)\xi(5s-1)-\frac{\xi(2)}{6(5s-2)}\cdot\xi(5s-4)\xi(5s-3)\\
&-\frac{1}{2(5s-5)(5s-2)^2}\cdot\xi(5s-3)^2-\frac{1}{4(5s-2)(5s-3)}\cdot\xi(5s-3)\xi(5s-1)\\
&+\frac{\xi(2)\xi(3)}{2(5s-1)}\cdot\xi(5s-4)\xi(5s-3)+\frac{\xi(2)}{2(5s-5)(5s-1)}\cdot\xi(5s-3)\xi(5s-2)\\
&+\frac{\xi(2)}{2(5s-4)(5s-1)}\cdot\xi(5s-3)\xi(5s-2)+\frac{1}{6(5s-3)}\xi(2)\cdot\xi(5s-1)\xi(5s)\\
&+\frac{\xi(2)}{6(5s-2)}\cdot\xi(5s-1)\xi(5s)+\frac{\xi(2)}{2(5s-4)(5s-1)}\cdot\xi(5s-2)\xi(5s-1)\\
&+\frac{1}{(5s-4)^2(5s-1)^2}\cdot\xi(5s-2)^2+\frac{1}{3(5s-3)(5s-1)}\xi(2)\cdot\xi(5s-3)\xi(5s-1)\\
&-\frac{1}{8(5s-2)}\cdot\xi(5s-1)\xi(5s)-\frac{\xi(2)}{6(5s-3)}\cdot\xi(5s-4)\xi(5s-3)\\
&-\frac{\xi(2)}{4(5s-2)}\cdot\xi(5s-4)\xi(5s-3)+\frac{\xi(2)}{2(5s-4)(5s)}\cdot\xi(5s-2)\xi(5s-1)\\
&+\frac{1}{2(5s-3)^2(5s)}\cdot\xi(5s-1)^2-\frac{\xi(2)\xi(3)}{(5s-5)(5s)}\cdot\xi(5s-3)\xi(5s-1)\\
&-\frac{\xi(2)^2}{(5s-5)(5s)}\cdot\xi(5s-2)\xi(5s-1)-\frac{\xi(2)}{(5s-4)(5s-3)(5s)}\cdot\xi(5s-1)^2\\
&+\frac{\xi(2)\cdot\xi(5s-3)\xi(5s-1)}{3(5s-4)(5s-2)}+\frac{\xi(5s-2)\xi(5s-1)}{2(5s-4)(5s-3)(5s-1)}\\
&-\frac{1}{4(5s-4)(5s-2)}\cdot\xi(5s-3)\xi(5s-2)-\frac{1}{(5s)}\xi(2)^2\xi(3)\cdot\xi(5s-4)\xi(5s-3)\\
&-\frac{\xi(2)^2}{(5s-5)(5s)}\cdot\xi(5s-3)\xi(5s-2)+\frac{\xi(2)}{(5s-5)(5s-2)(5s-1)}\cdot\xi(5s-3)^2\\
&+\frac{\xi(2)^2}{3(5s-1)}\cdot\xi(5s-4)\xi(5s-3);
\end{aligned}
$$

As we see, this complicated Weng zeta function does not have an Euler product, but it is remarkable that this Weng zeta function satisfies the analogue of RH! We have to recall that any Eisenstein series with class number > 1 that has the similar structure as Weng's zeta functions do not satisfy RH. Based on Theorem 5.2, we strongly believe the truth of RH for Weng's zeta funtions in general. We have speculated that the Euler product for a zeta function is intrinsic and RH mainly comes from this. However, Theorem 5.2 with the Weng zeta function above is against to this belief. The functional equation for the Weng zeta function has an important role for the validity of RH, but the author believes that there must exist more profound structures to be realized in justifying RH of general cases.

Theorem 5.3 (Ki-Komori-Suzuki, [30]) *Let G be a Chevalley group defined over $\mathbb{Q}$, in other words, G is a connected semisimple algebraic group defined over $\mathbb{Q}$ endowed with a maximal ($\mathbb{Q}$-)split torus. Let P_0 be a Borel subgroup of G containing the maximal torus. Let P be a maximal parabolic subgroup of G defined over $\mathbb{Q}$ containing P_0.*

Then all but finitely many zeros of $\hat{\zeta}^{(G,T)}_{\mathbb{Q},P/B}(s)\ \xi^{G/P}_{\mathbb{Q}}$ are simple and on the critical line of its functional equation.

This theorem supports RH for general Weng's zeta functions. We give a sketch of the proof of this theorem. The Weng's zeta function in Theorem 5.3 can be represented as

$$f(s) + f(1-s),$$

for a suitable function in such a way that $f(s)$ has only finitely many zeros to the right of the line $\mathrm{Re}(s) = 1/2$. To get these, we heavily introduce properties of the root system for the Chevalley group. Also, we need to know that the coefficient in the dominant term of $f(s)$ should not be vanished. Amazingly, the coefficient is related to the volume of a certain fundamental domain and so we acquire the nonvanishing property of the coefficient in the dominant term. We then are ready to use Proposition 2.1. In order to apply Proposition 2.1, we have to understand analytic behaviors of $f(s)$ which are very complicated. It is quite amazing that the constant α of the Hadamard factorization of $W(z)$ ($W(z) = f(1/2 + iz)$) in Proposition 2.1 is real. In fact, the realness of α relies on some intrinsic properties of the Weng's zeta function. Even if the constant α is not real, we can still justify Theorem 5.3 using some methods in [21]. However, we have much better understanding for Weng's zeta functions if we can follow Proposition 2.1.

Can we improve Theorem 5.3? We note that general Weng's zeta functions are extremely complicated, for the size of the Weyl group grows rapidly as the rank becomes large. Thus, we immediately see how difficult RH of general Weng's zeta functions is. So called, is it provable to justify the validity of Weng's conjecture or RH for Weng's zeta functions in general? We note that we don't know yet applications of RH for Weng's zeta functions to Number Theory. However, we have to recall that we know very little examples that behave like zeta functions and that fulfill RH, and RH of Weng's zeta functions might be as hard as the Riemann hypothesis. Thus, we will understand deeply the behavior of zeros of zeta functions if we achieve the validity of RH of Weng's zeta functions.

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