A numerical study on the formation of a thermocline in shear-free turbulence

Y. Noh and H. J. S. Fernando
Department of Mechanical and Aerospace Engineering, Arizona State University, Tempe, Arizona 85287-6106

(Received 3 July 1990; accepted 5 November 1990)

The formation of a thermocline generated by the interaction between a stabilizing buoyancy flux and shear-free turbulence was studied using a numerical model. The time evolutions of the vertical distributions of the buoyancy and turbulent kinetic energy were calculated and were used to evaluate the depth of the thermocline and the time required for its formation. The numerical results are compared with the results of previous laboratory experiments. The mechanisms responsible for the formation of the thermocline are discussed in view of the numerical results.

I. INTRODUCTION

It has been observed that, when a stabilizing buoyancy flux is imposed on the surface of a turbulent fluid without mean shear, a horizontal front is formed at a certain depth where rapid changes of turbulent kinetic energy and density occur.1-3 A typical example of this phenomenon is the formation of a seasonal thermocline in the upper ocean during the summer (see, for example, Kitaigorodskii4). Although there has been extensive research on the response of a thermocline to the variability of sea-surface conditions,5-9 the physical mechanisms responsible for the formation of thermoclines in homogeneous fluids, subjected to stabilizing buoyancy fluxes, are not yet clearly understood.

A number of pertinent laboratory experiments, in which oscillating grids were used to generate turbulence, have been carried out.1-3 In these experiments, the depth at which the thermocline is formed has been found to be proportional to the modified Monin-Obukhov length scale \( L_q \) given by

\[
L_q = A_1 \left( \frac{K_0}{Q_0} \right)^{1/4},
\]

where \( K_0 \) and \( Q_0 \) are the eddy diffusivity and buoyancy flux at the surface of fluid, respectively, and \( A_1 \) is a constant. Theoretical arguments have been advanced by Hopfinger and Linden2 and Thompson10 to explain (1). Both formulations, however, employ special assumptions such as uniform density within the mixed layer2 and the existence of a local critical Richardson number \( R_i \) above which a thermocline is formed.10 In particular, the strong interaction between buoyancy flux and the structure of turbulence, which is critical to the formation of a thermocline, has not been considered explicitly.

In this paper, a one-dimensional numerical model is developed to investigate the formation of thermoclines in shear-free turbulence. So far, no attempts have been made to study this phenomenon numerically. The results are compared with those of the laboratory experiments, and the mechanisms for the thermocline formation are clarified.

II. FORMULATION OF MATHEMATICAL MODEL

The mean turbulent kinetic energy (TKE) equation, in the absence of a mean flow, can be written as11

\[
\frac{\partial E}{\partial t} = - \frac{\partial}{\partial z} \left( \frac{\rho'}{\rho_0} + \frac{u'_i u'_j}{2} \right) \frac{\rho'}{\rho_0} - \epsilon, \tag{2}
\]

where horizontal homogeneity is assumed. Here \( u'_i \) \((i=1,2,3)\) is the fluctuating velocity \((u'_i = w')\), \( E \) is the kinetic energy of turbulence \((= \frac{1}{2} \rho' u'_i u'_j)\), \( b' \) is the fluctuating buoyancy \((= g \rho' / \rho_0)\), \( \rho' \) and \( \rho' \) are the fluctuations of density and pressure, \( \rho_0 \) is the reference density, \( g \) is the gravitational acceleration, and \( \epsilon \) is the dissipation rate of TKE. The first and second terms on the rhs of (2) represent the turbulent diffusion of TKE and the buoyancy flux, respectively.

Introducing the eddy diffusivity \( K \) for the parametrization of turbulent diffusion of TKE and buoyancy, (2) can be rewritten as12,13

\[
\frac{\partial E}{\partial t} = \frac{\partial}{\partial z} \left( K \frac{\partial E}{\partial z} \right) + K \frac{\partial B}{\partial z} - \epsilon, \tag{3}
\]

where \( B \) is the mean buoyancy. The eddy diffusivity \( K \) and the dissipation rate can be modeled as

\[
K = c_p E^{1/2} \tag{4}
\]

and

\[
\epsilon = c_p E^{1/2} \frac{1}{\ell}, \tag{5}
\]

where \( \ell \) is the integral length scale, and \( c_p \) and \( c_p \) are constants. Similarly, it is possible to describe the variation of the mean buoyancy \( B \) using the turbulent diffusion equation as11

\[
\frac{\partial B}{\partial t} = - \frac{\partial}{\partial z} \frac{\rho'}{\rho_0} b' \tag{6}
\]

and

\[
\frac{\partial B}{\partial z} = \frac{\partial}{\partial z} \left( K \frac{\partial B}{\partial z} \right). \tag{7}
\]

Shear-free turbulence generated by oscillating grids, on which the current model is based, has been extensively studied in the laboratory.14,15 It has been shown that the
integral length scale and the rms velocity of turbulence in a homogeneous fluid are given by

\[ l_a = a_1(z + z_0), \]

\[ u = a_2 f S(z + z_0)^{-1}, \]

where \( f \) and \( S \) are the frequency and the stroke of the grid oscillations, respectively, \( z \) is the distance from the mid-plane of the grid oscillations located near the surface of fluid, and \( a_1, a_2, \) and \( z_0 \) are constants that depend on the grid geometry.

If the length scale is rescaled as \( \tau = l/a_1 \), (4) and (5) can be written as

\[ K = c_\mu E^{1/2}, \]

\[ \epsilon = c_\mu E^{1/2}, \]

where \( c_\mu = a_1 c_\mu \) and \( c_\mu = c_\mu/a_1 \). A similar rescaling of time by \( t = c_\mu t \) changes (3) and (7) to

\[ \frac{\partial E}{\partial \tau} = \frac{\partial}{\partial \tau} \left( K' \frac{\partial E}{\partial \tau} + \frac{\partial B}{\partial \tau} \right) - \left( c_\mu \right) \epsilon', \]

\[ \frac{\partial B}{\partial \tau} = \frac{\partial}{\partial \tau} \left( K' \frac{\partial B}{\partial \tau} \right), \]

where

\[ K' - E^{1/2}, \]

\[ \epsilon' = \epsilon^{1/2}. \]

Finally, the empirical constant \( c_\mu /c_\mu \) appearing in (12) can be obtained by substituting the experimental results (8) and (9) into the steady-state form of (12) with \( \partial B/\partial z = 0 \). This gives \( c_\mu /c_\mu = 6 \).

Experimental and theoretical studies\(^{16-18}\) have shown that, in stably stratified fluids, the growth of the vertical length scale of turbulence \( l_z \) is limited by the buoyancy length scale \( l_b = u/N \), where \( N \) is the Brunt–Väisälä frequency. On the basis of these results, it is assumed that \( l_v \) has the form

\[ l_v = \left[ 1 + \left( l_n/c_\mu \right)^2 \right]^{1/2}, \]

where \( c_1 \) is a constant of order 1, or

\[ l_v = \left[ 1 + c_R \left( l_n/l_v \right)^2 \right]^{1/2}, \]

where \( l_n = l_n/a_1 \) and \( c_R = (a_1/c_1)^2 \).

By defining \( U = [E(0,t')]^{1/2} \) and \( L = l_v(0,t') \), it is possible to define nondimensional parameters as

\[ u^* = \frac{u}{U}, \quad l_v^* = \frac{l_v}{L}, \quad z^* = \frac{z}{L}, \quad r^* = \frac{r}{(U/L)}, \]

\[ k^* = \frac{K'}{UL}, \quad \epsilon^* = \frac{\epsilon'}{(U^3/L)}. \]

The nondimensionalization for \( B \) was selected as

\[ B^* = \frac{B}{Q_0/(c_\mu U)}. \]

In what follows, the asterisks denoting the nondimensional variables are dropped; instead they appear behind the equation numbers, whenever they are non-dimensional. Then the nondimensional forms of (12) and (13) are

\[ \frac{\partial E}{\partial \tau'} = \frac{\partial}{\partial \tau'} \left( K' \frac{\partial E}{\partial \tau'} + \frac{\partial B}{\partial \tau'} \right) + G \frac{\partial B}{\partial z'} - 6 \epsilon', \]

\[ \frac{\partial B}{\partial \tau'} = \frac{\partial}{\partial \tau'} \left( K' \frac{\partial B}{\partial \tau'} \right), \]

with

\[ G = \frac{Q_0}{\left( U^3/L \right)}. \]

Further, the nondimensionalized expressions for (14), (15), and (17) are given by

\[ K' = E^{1/2}, \]

\[ \epsilon' = \epsilon^{1/2}, \]

\[ l_n = l_n/\left( 1 + c_R \right), \]

where

\[ \text{Ri} = 3l_n^2 N^2 / 2E, \]

\[ = - \frac{3l_n^2}{2E} G \frac{\partial B}{\partial z}. \]

The initial conditions are taken as

\[ E(z,0) = (1 + z)^{-2}, \]

\[ B(z,0) = 0, \]

which represent the condition of oscillating-grid turbulence given by (8) before the start of buoyancy flux at the surface. The boundary conditions are given by

\[ E(0,t') = 1, \]

\[ \frac{\partial E(H,t')}{\partial z} = 0, \]

\[ \frac{\partial B(0,t')}{\partial z} = -1, \]

\[ \frac{\partial B(H,t')}{\partial z} = 0, \]

where \( z = H \) is the maximum height of the calculation. At the surface \( (z=0) \), the constant turbulent kinetic energy, maintained by the grid oscillation, and the constant buoyancy flux \( Q_0 = -K_0 \partial B/\partial z \) are given by (30) and (32), respectively. It is important to note that there are no empirical constants in (20)–(33) other than \( c_R \) which parametrizes the effect of stratification on the length scale of turbulence. The effect of the variation of \( c_R \) on the results will be discussed later. The calculations were performed using an implicit finite-difference method.
FIG. 1. The evolution of the buoyancy distribution with time at $G=10^{-5}$. The graphs correspond to $t'=n\Delta t'$ ($n=1,...,10$, $\Delta t'=200$).

III. NUMERICAL RESULTS

A. Evolution of turbulent kinetic energy and density distribution

The calculations were performed with $c_R = 0.1$; the reasons for this particular choice will be discussed in Sec. III B. The results of the time evolution of the vertical density distribution $B(z,t')$ are shown in Fig. 1. The graphs correspond to the time $t'=n\Delta t'$ ($n=1,2,...,10$, $\Delta t'=200$, and $G=10^{-5}$). At early times, the density distribution appears to be similar to the case of constant eddy diffusivity, in which $B$ decreases continuously with $z$ as

$$\frac{dB}{dz} = 1 - erf \left( \frac{z}{\sqrt{\Delta t'}} \right). \tag{34}$$

After some time ($n=3$), however, a weak thermocline appears at a certain depth, and then gradually becomes more pronounced as time increases. A relatively uniform density gradient is maintained within the mixed layer.

Figure 2 shows the corresponding evolution of the turbulent kinetic energy $E(z,t')$. The energy decreases rapidly with $z$ to the minimum at the depth of the thermocline $D$, as shown in Fig. 1. For $z > D$, the initial turbulence still persists, but dissipates slowly with time, because energy cannot propagate across the thermocline, once it is formed. It is also noted that the effect of the buoyancy flux on $E(z,t')$ is not important within the mixed layer except near the thermocline.

The features described above can be understood by considering the energy budget of (20); the individual terms are turbulent diffusion of kinetic energy, buoyancy flux, and viscous dissipation. A typical energy budget is shown in Fig. 3, which shows that the turbulent diffusion and viscous dissipation dominate within the mixed layer, and the buoyancy flux becomes important only as $z$ approaches $D$. It was also found that the energy budget, such as that shown in Fig. 3, remains unchanged with time.

B. The depth of the thermocline

The depth of the thermocline $D$ can be defined as the depth where the minimum of $E$ occurs (for example, see Fig. 2), and its equilibrium depth $D_e$ is obtained from the limiting value of $D$ as $t'$ goes to infinity. In practice, as is

FIG. 2. The time evolution of turbulent kinetic energy distribution at $G=10^{-5}$. The graphs correspond to $t'=n\Delta t'$ ($n=1,...,10$, $\Delta t'=200$).

FIG. 3. Comparison of the magnitude of each term appearing in the rhs of (20); ---, turbulent transfer (TT); ---, buoyancy flux (BF); --, viscous dissipation (VD). The graphs for turbulent transfer and viscous dissipation are approximately overlapped.
shown in Fig. 4, the calculations clearly show the convergence of $D$ to an asymptotic value corresponding to $D_p$ soon after the first appearance of the minimum $E$. The calculations were made until $E_{\text{min}} < 10^{-24}$. In order to ensure that the effects of the upper boundary are negligible, the value of $H$ was selected so that $D/H < 0.2$.

Figure 5 shows the variation of $D_Q$ with $G$ for several values of $c_R$ (0.0, 0.1, 1.0, 10.0). It is important to note that all graphs show the relation

$$D_Q = A_2 G^{\alpha}, \quad \alpha = 0.25 \pm 0.005,$$

in accordance with the experimental result (1), regardless of $c_R$. On the other hand, the proportionally constant $A_2$ is found to decrease significantly with $c_R$. This means that the parameter $c_R$ is important in the determination of $D_Q$, although the relation $D_Q \propto G^{-1/4}$ is valid regardless of $c_R$.

The dimensional form of (35)

$$D_Q = A_2 (U^3 L^3 / Q_0) \frac{1}{4},$$

(36)

can be compared with the experimental results suggested by Hopfinger and Linden, 2

$$D_Q = (20r)^{1/4} (U^3 L^3 / Q_0) \frac{1}{4},$$

(37)

where $r = 2.2$. A good agreement between (36) and (37) is found with $c_R = 0.1$.

C. Time scale of thermocline formation

The time scale of the thermocline formation $t_Q$ which is defined as the time of the first appearance of the minimum $E$, was evaluated and is shown in Fig. 6. The results suggest

$$t_Q \propto G^{-\beta}, \quad \beta = 0.5 \pm 0.01,$$

(38)

which is in agreement with the relation expected from the simple dimensional analysis, viz., $t_Q \sim (K/Q_0)^{1/2}$. It is also found that $t_Q$ decreased with $c_R$, as is expected from the fact that reduction of the vertical length scale of turbulence due to stratification facilitates the formation of a thermocline.

IV. CONCLUSIONS AND DISCUSSION

In the previous sections, a numerical model was presented to describe the formation of a thermocline when a stabilizing buoyancy flux $Q_0$ is imposed on shear-free turbulence. The model predictions for the equilibrium depth of the thermocline $D_p$ which is found to vary as $D_p \propto G^{-1/4}$, where $G = (Q_0 L / U^2)$ and $A_2 = \text{const}$, is in good agreement with previous experimental data. The time scale for the onset of a thermocline is predicted as $t_Q \propto G^{-1/2}$. Unlike previous models, 2,10 the present model does not employ a priori assumptions such as constant eddy diffusivity and a uniform density distribution within the mixed layer.

In explaining the Hopfinger and Linden data, 2 Thompson 10 conjectured the existence of a local critical Richardson number above which the growth of the mixed layer is not possible. In the present work, the Hopfinger and Linden data are reproduced without introducing this Richardson number constraint. In Thompson's model, the interaction between the turbulence and the buoyancy flux was not taken into consideration explicitly, and the eddy diffusivity was assumed to be unaffected by the buoyancy flux. Nevertheless, due to the stabilizing buoyancy flux, the turbulence is suppressed, and the eddy diffusivity is reduced. The
local reduction of the eddy diffusivity is capable of inducing a strong density gradient at a certain depth, where the turbulence is suppressed even further. This mechanism leads to the formation of a thermocline across which both the propagation of turbulent kinetic energy and the buoyancy flux are prohibited.

In explaining their results on thermocline formation, Hopfinger and Linden\(^2\) assumed that the density distribution within the mixed layer was uniform. Therefore the lhs of (6) becomes independent of \(z\) and buoyancy flux \(Q\) \((= -\bar{w}b')\) was obtained as

\[
Q = Q_0(1 - z/D), \tag{39}
\]

using \(Q=0\) at \(z=D\). Both the numerical results shown in Fig. 1 and their own experimental data, however, show a significant density gradient within the mixed layer proportional to the buoyancy flux. On the other hand, it is interesting to find that (39) describes the distributions of the buoyancy flux shown in Fig. 3 reasonably well. This can be easily explained from the fact that the rate of increase of buoyancy \(\partial B/\partial t\), rather than \(B\), is independent of \(z\), when \(z<D\), as is evident from Fig. 1.

An important feature of Eqs. (20)–(25) is that the actual size of the length scale of turbulence \(l\) does not affect the depth of the thermocline, but only the values of the buoyancy in the mixed layer. In other words, the empirical constant \(a_1\) (or \(c'_{\mu} = a_1c_{\mu}\)), which gives the actual size of \(l\) by (8), does not appear in the equations, and is used only to convert the variation of buoyancy to the actual value by \(t' = c'_{\mu}t\) and \(B^* = B/(Q_0/c'_{\mu}U)\). The value of \(c'_{\mu}\) can be obtained as \(c'_{\mu} ~ 1\) from a comparison with the experimental data of Hopfinger and Linden\(^2\) where \(\partial B/\partial z ~ Q_0/\gamma T\). This suggests that the relatively large size eddies \((l~z)\) are important for the transfer of \(B\) and \(E\).

Although the effect of stratification on the integral length scale of turbulence \(l\) has been neglected in both of these models,\(^2,10\) the present work shows that it is important in the determination of the proportionally constant \(A_2\) of (36). The calculations performed with \(c'_{\mu} = 0\), i.e., without consideration of stratification effects on \(l\), overestimate the value of \(D_Q\) as shown in Fig. 4. Further, note that \(A_2\) has a weak dependence on \(a_1\) in view of \(c_{\mu} \approx (a_1/c_{\mu})^2\). Hopfinger and Linden\(^2\) suggested a constant value for \(A_2\) \(= (20r)^{1/4}, r = 2.2\), but it is important to note that, in their own model, \(r\) is related to \(a_1\) by \(c_{\mu} = a_1r = 3\). In their experiment \(a_1\) varies between 0.1 and 0.25. If the median value of \(a_1\) is taken as \(a_1 = 0.175, c_{\mu} = 0.56\) is obtained. The variation of \(A_2\) with \(a_1\) in their experiment is then given as \(2.4 < A_2 < 2.9\), which is within the error margins of their data.

As pointed out by Hopfinger and Linden,\(^2\) the relation (37) [or (1)] depends on the decay law of turbulent kinetic energy \((u \sim z^{-3/2}\) used here). If, following Thompson and Turner,\(^{16}\) the decay law \(u \sim z^{-3/2}\) is employed, the corresponding result shows that \(D_Q = G/2^{1/4}\). Finally, it should be noted that, in the present model, the turbulent diffusivity is assumed to be the same for \(E\) and \(B\). This appears to be consistent with previous results including oscillating-grid-generated turbulence.\(^{20,21}\) A small deviation of the ratio of turbulent diffusivities for \(E\) and \(B\), \(\alpha\), from unity does not affect the main results (36), because \(G\) can be modified as \(G' = \alpha G\) and this changes (36) to \(D_Q = A_2G'^{1/4}G^{-1/4} \equiv A_2G'^{-1/4}\).

**ACKNOWLEDGMENTS**

The computations were carried out in double precision on the VAX 8650 and CRAY X-MP/14se at Arizona State University. The authors wish to thank Professor D. F. Janowski for careful comments.

During the period of the preparation of this paper, the authors were supported by the Physical Oceanography and Arctic Science Program of the Office of the Naval Research and the National Science Foundation.

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