EIGENSERIES SOLUTIONS TO OPTIMAL CONTROL PROBLEM AND CONTROLLABILITY PROBLEMS ON HYPERBOLIC PDES

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Abstract. A terminal-state tracking optimal control problem for linear hyperbolic equations with distributed control is studied in this paper. An analytic solution formula for the optimal control problem is derived in the form of eigenseries. We show that the optimal solution satisfies the approximate controllability property. An explicit solution formula for the exact controllability problem is also expressed by the eigenseries formula when the target state and the controlled state have matching boundary conditions. We demonstrate by numerical simulations that the optimal solutions expressed by the series formula approach the target functions.

1. Introduction. In this paper, we study both a terminal-state tracking optimal control problem and controllability problems for linear hyperbolic partial differential equations (PDEs) defined over the finite time interval $[0, T] \subset [0, \infty)$ and on a bounded, $C^2$ (or convex) spatial domain $\Omega \subset \mathbb{R}^d$ ($d=1$ or $2$ or $3$). Let target functions $W \in L^2(\Omega)$ and $Z \in L^2(\Omega)$ and initial conditions $w \in H^1_0(\Omega)$ and $z \in L^2(\Omega)$ be given. Let $f \in L^2(0,T;L^2(\Omega))$ denote the distributed control. We consider the following optimal control problem: minimize the terminal-state tracking functional

$$
\mathcal{J}(u,f) = \frac{T}{2} \int_{\Omega} |u(T,x) - W(x)|^2 \, dx + \frac{T}{2} \int_{\Omega} |u_t(T,x) - Z(x)|^2 \, dx + \frac{\gamma}{2} \int_0^T \int_{\Omega} |f(t,x)|^2 \, dx \, dt
$$

(1.1)

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(where $\gamma$ is a positive constant) subject to a hyperbolic PDE

$$\begin{cases}
  u_{tt} - \text{div}[A(x)\nabla u] = f & \text{in } Q \equiv (0,T) \times \Omega, \\
  u|_{\partial\Omega} = 0 & \text{in } (0,T), \\
  u|_{t=0} = w \quad \text{and} \quad u_t|_{t=0} = z & \text{in } \Omega.
\end{cases} \tag{1.2}$$

In (1.2), $A(x)$ is a symmetric matrix-valued, $C^1(\Omega)$ function that is uniformly positive definite.

Similar optimal control problems have been studied extensively during the last three decades. The existence and regularity of solutions of optimal control problems on partial differential equations were studied by Lions [16]. Specially, some studies have been conducted to explore the terminal-state tracking optimal control problems [11, 12, 14, 15]. Eigen series solutions to optimal control problems of linear parabolic equations were considered in [18, 22]. In [6], E. Fernandez-Cara and E. Zuazua estimated the cost of approximate controllability for the constant coefficient heat equation by employing the eigenvalues and eigenfunctions of $-\Delta$. Our main achievements in optimal control problems are the derivation and justification of analytic solution formula in the form of eigenseries even though the admissible state $u$ and the desired state $W$ have non-matching boundary conditions.

Terminal-state tracking problems are optimal control problems in their own right. However, terminal-state tracking optimal control problems are closely related to approximate and exact controllability problems [2, 18]. A number of researchers have developed many significant controllability theories for hyperbolic and parabolic equations. H. Fattorini and D. Russell [4, 5, 19, 20] considered a harmonic analysis method to derive sufficient conditions for the exact controllability. Lions [17] introduced the Hilbert Uniqueness Method for the exact controllability problem for hyperbolic equations. More recently, exact controllability problems for distributed parameter systems were studied by Fursikov and Imanuvilov [7, 8]. Some authors also explored a numerical approach to exact controllability problems for wave equations [9, 10, 13, 24]. The observability/controllability properties of a semi-discrete finite difference approximation for the 1D wave equation is discussed using eigenvalue problem [23]. In this paper, we wish to find a control $f$ that drives the state $u$ to $W$ and $u_t$ to $Z$ at the terminal time $T$ using the optimal control approach.

The works are organized as follows. In Section 2 the terminal-state tracking optimal control problem and controllability problems are formulated in an appropriate mathematical framework. In Section 3 we review some properties for the eigenvalues and eigenfunctions for the elliptic operator and establish certain results concerning eigenseries for some functions. In Section 4 we provide the derivation and justification of an explicit eigenseries solution formula for the terminal-state tracking optimal control problem. In Section 5 we show that the optimal solution $u$ approaches the target function $W$ and $u_t$ approaches $Z$ as the parameter $\gamma \to 0$, i.e. the optimal solution is a solution of approximate controllability problem. In Section 6 eigenseries solution formula for the exact controllability problem is derived and justified assuming matching boundary conditions for the controlled state and the target state. We present some computational results that illustrate the state tracking properties for the optimal solutions in Section 7. Finally, concluding remarks are provided in Section 8.
2. Formulation of optimal control and controllability problems. Throughout we freely make use of standard Sobolev space notations $H^m(\Omega)$ and $H^m_0(\Omega)$. We denote the norm for Sobolev space $H^m(\Omega)$ by $\| \cdot \|_m$. Note that $H^0(\Omega) = L^2(\Omega)$ so that $\| \cdot \|_0$ is the norm $L^2(\Omega)$ norm (see, e.g., [1]). Functional (1.1) can be written as

$$J(u, f) = \frac{T}{2} (\| u(T,x) - W(x) \|^2_0 + \| u_t(T,x) - Z(x) \|^2_0) + \frac{\gamma}{2} \int_0^T \| f(t,x) \|^2_0 dt. \quad (2.1)$$

**Definition 2.1.** Let $f \in L^2(0, T; L^2(\Omega))$, $w \in H^1_0(\Omega)$ and $z \in L^2(\Omega)$ be given. $u$ is said to be a weak solution of (1.2) if $u \in L^2(0, T; H^1_0(\Omega))$, with $u_t \in L^2(0, T; L^2(\Omega))$, $u_{tt} \in L^2(0, T; H^{-1}(\Omega))$ and $u$ satisfies

$$\begin{cases}
(\phi, u_t(t)) + \int_\Omega A(x) \nabla u(t) \cdot \nabla \phi dx = (f, \phi) \quad &\forall \phi \in H^1_0(\Omega), \\
u(0) = w, & u'(0) = z \text{ in } \Omega,
\end{cases} \quad (2.2)$$

where $(\cdot, \cdot)$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

An admissible element for the optimal control problem is a pair $(u, f)$ that satisfies the initial/boundary-value problem (2.2). The precise definition is given as follow.

**Definition 2.2.** Let $w \in H^1_0(\Omega)$ and $z \in L^2(\Omega)$. A pair $(u, f)$ is said to be an admissible element if $u \in L^2(0, T; H^1_0(\Omega))$, $u_t \in L^2(0, T; L^2(\Omega))$, $u_{tt} \in L^2(0, T; H^{-1}(\Omega))$, and $f \in L^2(0, T; L^2(\Omega))$ and $(u, f)$ satisfies the equation (2.2). The set of all admissible elements is denoted by $V_{ad}(0, T), w, z$ or simply $V_{ad}$.

For convenience, we define the temporal-spatial function spaces

$$V((0, T) \times \Omega) = \{ v \in L^2(0, T; H^2(\Omega)) : v_t \in L^2(0, T; H^1_0(\Omega)), v_{tt} \in L^2(0, T; L^2(\Omega)) \}.$$

The optimal control problem we study can be concisely stated as

$$\begin{align*}
\text{(OP)} \quad & \text{ seek a pair } (\hat{u}, \hat{f}) \in V_{ad} \text{ such that } J(\hat{u}, \hat{f}) = \inf_{(u, f) \in V_{ad}} J(u, f) \\
\text{ where the functional } J \text{ is defined by (1.1).}
\end{align*}$$

The existence and uniqueness of the optimal solution for (OP) follows from classical optimal control theories (see, e.g., [16]). The approximate and exact controllability problems are formulated as follows:

$$\begin{align*}
\text{(AP-CO}) \quad & \text{ seek a one-parameter set } \{ (u_\epsilon, f_\epsilon) : \epsilon > 0 \} \subset V_{ad} \text{ such that } \\
& \lim_{\epsilon \to 0} \| u_\epsilon(T) - W \|_0 = 0 \text{ and } \lim_{\epsilon \to 0} \| u'_\epsilon(T) - Z \|_0 = 0,
\end{align*}$$

and

$$\begin{align*}
\text{(EX-CO}) \quad & \text{ seek a pair } (u, f) \in V_{ad} \text{ such that } u(T) = W \text{ and } u'(T) = Z \text{ in } \Omega.
\end{align*}$$

The existence of a solution pair to exact controllability problems (EX-CO) is shown by the following theorem.

**Theorem 2.3.** Assume that $w, W \in H^1_0(\Omega) \cap H^2(\Omega)$ and $z, Z \in L^2(\Omega) \cap H^1_0(\Omega)$. Then the hyperbolic PDE (1.2) is exactly controllable. (i.e. (EX-CO) has a solution.)
Proof. Let \( \tilde{u} \) be a function satisfying
\[
\tilde{u} \in W((0, T) \times \Omega), \quad \tilde{u} = 0 \text{ on } (0, T) \times \partial \Omega,
\]
\[
\tilde{u}|_{t=0} = w \in H_0^1(\Omega) \cap H^2(\Omega),
\]
\[
\tilde{u}_t|_{t=0} = z \in L^2(\Omega) \cap H^1_0(\Omega).
\]
The existence of such a \( \tilde{u} \) is guaranteed by the existence and regularity results (see [3]) for the hyperbolic problem
\[
\begin{cases}
  u_{tt} - \Delta u = 0 \text{ in } (0, T) \times \Omega, \\
  u = 0 \text{ in } (0, T) \times \partial \Omega, \\
  u|_{t=0} = w, \\
  u_t|_{t=0} = z.
\end{cases}
\]
Likewise, there exists a \( \tilde{\tilde{u}} \) satisfying
\[
\tilde{\tilde{u}} \in W((0, T) \times \Omega), \quad \tilde{\tilde{u}} = 0 \text{ on } (0, T) \times \partial \Omega,
\]
\[
\tilde{\tilde{u}}|_{t=T} = W \in H_0^1(\Omega) \cap H^2(\Omega),
\]
\[
\tilde{\tilde{u}}_t|_{t=T} = Z \in L^2(\Omega) \cap H^1_0(\Omega).
\]
We now choose a function \( \theta = \theta(t) \in C^\infty\{0, T\} \) such that
\[
\theta(t) = 1 \forall t \in [0, T/3], \quad 0 \leq \theta(t) \leq 1 \forall t \in [T/3, 2T/3], \quad \theta(t) = 0 \forall t \in [2T/3, T]
\]
and set
\[
u = \theta(t)\tilde{u} + [1 - \theta(t)]\tilde{\tilde{u}} \text{ on } (0, T) \times \Omega.
\]
Then we have
\[
\begin{cases}
  u \in W((0, T) \times \Omega), \quad u = 0 \text{ on } (0, T) \times \partial \Omega, \\
  u|_{t=0} = w, \quad u_t|_{t=0} = z, \\
  u|_{t=T} = W, \quad u_t|_{t=T} = Z.
\end{cases}
\]
By defining
\[
f \equiv u_{tt} - \text{div}(A(x)\nabla u) \in L^2(0, T; L^2(\Omega)),
\]
we see that \((u, f)\) solves the exact controllability problem (EX-CON).

3. Results concerning eigenfunction expansions. The main objective of this paper is to find an explicit solution formula, expressed in terms of eigenfunctions, for the optimal control problem and for the exact controllability problem. To do that, we first review some properties for the eigenvalues and eigenfunctions for the elliptic operator. We recall the following lemma (see [3], p.335, Theorem 1):

**Lemma 3.1.** The set \( \Lambda \) of all eigenvalues for the elliptic operator \(-\text{div}(A(x)\nabla)\) where \(A(x)\) is defined by (1.2) may be written \( \Lambda = \{\lambda_i\}_{i=1}^\infty \subset \mathbb{R} \) where
\[
0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots \quad \text{and} \quad \lambda_i \to \infty \quad \text{as} \quad i \to \infty.
\]
Furthermore there exists a set of corresponding eigenfunctions \( \{e_i\}_{i=1}^\infty \subset H^2(\Omega) \cap H_0^1(\Omega) \) which form an orthonormal basis of \(L^2(\Omega)\) (with respect to the \(L^2(\Omega)\) inner product).

In the sequel we let \( \{(\lambda_i, e_i)\}_{i=1}^\infty \) denote a set of eigenpairs as stated in Lemma 3.1. We quote several lemmas (see [18]).
Lemma 3.2. The set \( \{e_i/\sqrt{\lambda_i}\}_{i=1}^{\infty} \) forms an orthonormal basis of \( H^1_0(\Omega) \) with respect to the inner product
\[
(u, v) \mapsto B[u, v] = \int_{\Omega} A(x)\nabla u \cdot \nabla v \, dx, \quad \forall u, v \in H^1_0(\Omega).
\] (3.1)
The set \( \{e_i/\lambda_i\}_{i=1}^{\infty} \) forms an orthonormal basis of \( H^3(\Omega) \cap H^1_0(\Omega) \) with respect to the inner product
\[
(u, v) \mapsto B[u, v] = \int_{\Omega} \text{div}[A(x)\nabla u]\text{div}[A(x)\nabla v] \, dx, \quad \forall u, v \in H^2(\Omega) \cap H^1_0(\Omega).
\] (3.2)

Lemma 3.3. Assume that \( y \in L^2(\Omega) \) and \( y = \sum_{i=1}^{\infty} y_i e_i \in L^2(\Omega) \). Then the following statements are equivalent:
\begin{enumerate}
  \item[i)] \( y \in H^1_0(\Omega) \);
  \item[ii)] \( \sum_{i=1}^{\infty} y_i e_i \in H^1_0(\Omega) \);
  \item[iii)] \( \sum_{i=1}^{\infty} \lambda_i |y_i|^2 < \infty \).
\end{enumerate}

Lemma 3.4. Assume that \( y \in L^2(\Omega) \) and \( y = \sum_{i=1}^{\infty} y_i e_i \in L^2(\Omega) \). Then the following statements are equivalent:
\begin{enumerate}
  \item[i)] \( y \in H^2(\Omega) \cap H^1_0(\Omega) \);
  \item[ii)] \( \sum_{i=1}^{\infty} y_i e_i \in H^2(\Omega) \cap H^1_0(\Omega) \);
  \item[iii)] \( \sum_{i=1}^{\infty} \lambda_i |y_i|^2 < \infty \).
\end{enumerate}

The main results of this section are the two theorems below concerning term-by-term differentiations of eigenseries for functions in \( W((0, T) \times \Omega) \). We first quote a lemma (see [21], p169, Lemma 1.1 and [3], p286, Theorem 2).

Lemma 3.5. Assume that \( u \in L^2(0, T; L^2(\Omega)), u_t \in L^2(0, T; L^2(\Omega)) \). Then
\[
-\int_0^T \phi'(t) \int_{\Omega} u(t)v dx dt = \int_0^T \phi(t) \int_{\Omega} u_t(t)v dx dt \quad \forall \phi \in C_0^\infty(0, T), \forall v \in L^2(\Omega).
\]

Theorem 3.6. Assume that \( u \in W((0, T) \times \Omega), u = 0 \) on \((0, T) \times \partial \Omega\) and
\[
\begin{align*}
u(t) &= \sum_{i=1}^{\infty} u_i(t)e_i \quad \text{in } L^2(\Omega) \quad \text{a.e. } t \in (0, T) \\
\end{align*}
\]
Then
\[
\begin{align*}
&\sum_{i=1}^{\infty} \int_0^T \left( |u_i''(t)|^2 + \lambda_i |u_i(t)|^2 \right) dt = \|u_t(t)\|^2_{L^2((0, T) \times \Omega)} + \int_0^T B[u, v] dt < \infty, \\
&\sum_{i=1}^{\infty} \lambda_i |u_i(0)|^2 < \infty, \quad \sum_{i=1}^{\infty} |u_i'(0)|^2 < \infty, \\
&u_{tt}(t) = \sum_{i=1}^{\infty} u_i''(t)e_i(\mathbf{x}) \quad \text{in } L^2(\Omega), \text{ a.e. } t \in (0, T),
\end{align*}
\]
and
\[
-\text{div}[A(x)\nabla u] = \sum_{i=1}^{\infty} \lambda_i u_i e_i \quad \text{in } L^2(\Omega), \text{ a.e. } t.
\]

Proof. We first note the continuous embedding \( W((0, T) \times \Omega) \hookrightarrow C([0, T]; H^1(\Omega)) \) and the boundary condition \( u = 0 \) on \((0, T) \times \partial \Omega\) imply that \( u(t) \in H^1_0(\Omega) \) for every \( t \in [0, T] \). By Lemma 3.3 we have
\[
u(t) = \sum_{i=1}^{\infty} u_i(t)e_i \text{ in } H^1_0(\Omega), \forall t \in [0, T].
\]
In particular, since \( u(0) \in H^1_0(\Omega) \), Lemma 3.3 yields \( \sum_{i=1}^{\infty} |\lambda_i||u_i(0)|^2 < \infty \). Similarly since \( u \in W((0,T) \times \Omega) \) and we use the same continuous embedding theorem, we arrive at \( u_t \in C([0,T];L^2(\Omega)) \). Thus \( u'(0) \in L^2(\Omega) \) and so \( \sum_{i=1}^{\infty} |u'_i(0)|^2 < \infty \).

Using the \( L^2(\Omega) \) orthogonality of \( \{e_i\} \) we have

\[
|u|^2_{L^2(0,T;L^2(\Omega))} = \int_0^T |u(t)|^2 dt = \int_0^T \sum_{i=1}^{\infty} |u_i(t)|^2 dt \geq \int_0^T |u_j(t)|^2 dt, \forall j
\]

so that each \( u_j \in L^2(0,T) \). Since \( u_{tt} \in L^2(0,T;L^2(\Omega)) \), we may write

\[
u_{tt}(t) = \sum_{i=1}^{\infty} v_i(t)e_i \quad \text{in} \quad L^2(\Omega), \text{a.e.} \ t,
\]

and

\[
|u_{tt}|^2_{L^2(0,T;L^2(\Omega))} = \int_0^T |u_{tt}(t)|^2 dt = \int_0^T \sum_{i=1}^{\infty} |v_i(t)|^2 dt \geq \int_0^T |v_j(t)|^2 dt, \forall j,
\]

so that each \( v_j \in L^2(0,T) \). Using Lemma 3.5 we have that

\[
- \int_0^T \phi''(t) \int_{\Omega} u(t)e_j dx dt = \int_0^T \phi'(t) \int_{\Omega} u_t(t)e_j dx dt
\]

\[
\quad = - \int_0^T \phi(t) \int_{\Omega} u_{tt}(t)e_j dx dt \quad \forall \phi \in C_0^\infty(0,T), j = 1, 2, \ldots.
\]

Substituting series expressions for \( u \) and \( u_{tt} \) into the last equation and using the \( L^2(\Omega) \) orthogonality of \( \{e_i\} \) we obtain

\[
\int_0^T \phi''(t)u_j dt = \int_0^T \phi(t)v_j dt \quad \forall \phi \in C_0^\infty(0,T), j = 1, 2, \ldots,
\]

so that \( v_j = u_j'' \) for \( j = 1, 2, \ldots \). This provides \( u_{tt}(t) = \sum_{i=1}^{\infty} u''_i(t)e_i(x) \) in \( L^2(\Omega), \text{a.e.} \ t \).

Since \( u(t) \in H^2(\Omega) \cap H^1_0(\Omega) \) for almost every \( t \), Lemma 3.4 implies that

\[
u(t) = \sum_{i=1}^{\infty} u_i(t)e_i \quad \text{in} \quad H^2(\Omega) \cap H^1_0(\Omega), \text{a.e.} \ t,
\]

so that

\[
-\text{div}[A(x)\nabla u(t)] = \sum_{i=1}^{\infty} -\text{div}[A(x)\nabla u_i(t)e_i] = \sum_{i=1}^{\infty} \lambda_i u_i e_i, \quad \text{in} \quad L^2(\Omega), \text{a.e.} \ t.
\]

From the above (3.4) and Lemma 3.4 we obtain

\[
\int_0^T \bar{B}[u,u] dt = \int_0^T ||\text{div}[A(x)\nabla u(t)]||^2 dt = \int_0^T \sum_{i=1}^{\infty} |\lambda_i|^2 |u_i|^2 dt < \infty.
\]

Adding up (3.3) and (3.4) and applying the Monotone Convergence Theorem we have that

\[
\sum_{i=1}^{\infty} \int_0^T (|u''_i(t)|^2 + |\lambda_i|^2 |u_i(t)|^2) dt = ||u_{tt}||^2_{L^2(0,T;L^2(\Omega))} + \int_0^T \bar{B}[u,u] dt < \infty.
\]

It completes the proof.
Theorem 3.7. Assume that the set of functions \( \{ u_i(t) \}_{i=1}^{\infty} \subset H^2(0,T) \) satisfies
\[
\sum_{i=1}^{\infty} \int_0^T (|u''_i(t)|^2 + \lambda_i |u'_i(t)|^2 + |\lambda_i|^2 |u_i(t)|^2) dt < \infty, \tag{3.6}
\]
and
\[
\sum_{i=1}^{\infty} |\lambda_i| |u_i(0)|^2 dt < \infty, \quad \sum_{i=1}^{\infty} |u'_i(0)|^2 dt < \infty. \tag{3.7}
\]
Then the function \( u(t) = \sum_{i=1}^{\infty} u_i(t) e_i \) satisfies \( u \in W((0,T) \times \Omega) \), \( u = 0 \) on \( (0,T) \times \partial \Omega \),
\[
u_{tt}(t) = \sum_{i=1}^{\infty} u''_i(t) e_i(x) \text{ in } L^2(\Omega), \text{ a.e. } t, \tag{3.8}
\]
and
\[-\text{div}[A(x) \nabla u(t)] = \sum_{i=1}^{\infty} \lambda_i u_i(t) e_i(x) \text{ in } L^2(\Omega), \text{ a.e. } t. \tag{3.9}\]

Proof. We note that
\[
\sum_{i=1}^{\infty} \int_0^T |u_i(t)|^2 dt \leq \frac{1}{|\lambda_1|^2} \sum_{i=1}^{\infty} \int_0^T |\lambda_i|^2 |u_i(t)|^2 dt < \infty,
\]
so that \( u(t) = \sum_{i=1}^{\infty} u_i(t) e_i \) in \( L^2(\Omega) \) for almost every \( t \in (0,T) \). Moreover we have
\[
\sum_{i=1}^{\infty} \int_0^T |u'_i(t)|^2 dt \leq \frac{1}{|\lambda_1|} \sum_{i=1}^{\infty} \int_0^T \lambda_i |u'_i(t)|^2 dt < \infty.
\]
Therefore \( \{ u_i(t) \}_{i=1}^{\infty} \) satisfies the assumptions of Theorem 3.7 in [18], and thus we conclude the function \( u(t) = \sum_{i=1}^{\infty} u_i(t) e_i \) satisfies \( u \in L^2(0,T;H^2(\Omega)) \), \( u_t \in L^2(0,T;H^2(\Omega)) \), \( u = 0 \) on \( (0,T) \times \partial \Omega \),
\[
u_i(t) = \sum_{i=1}^{\infty} u_i(t) e_i(x) \text{ in } L^2(\Omega), \text{ a.e. } t,
\]
and
\[-\text{div}[A(x) \nabla u(t)] = \sum_{i=1}^{\infty} \lambda_i u_i(t) e_i(x) \text{ in } L^2(\Omega), \text{ a.e. } t.
\]
Moreover assumption (3.6) and Lemma 3.3 imply that \( u_t \in L^2(0,T;H^1_0(\Omega)) \).

By assumption (3.6) we are justified to define \( f \in L^2(0,T;L^2(\Omega)) \) as the series function
\[
f = \sum_{i=1}^{\infty} f_i(t) e_i = \sum_{i=1}^{\infty} u''_i(t) e_i \text{ in } L^2(\Omega), \text{ a.e. } t \in (0,T).
\]
Since \( H^2(0,T) \) is continuously embedded into \( C^1[0,T] \), \( u_i(0) \) and \( u'_i(0) \) are well defined for each \( i \). Assumption (3.7) and Lemma 3.3 imply that \( u_{t|t=0} \in H^1_0(\Omega) \) and \( u_{t|t=0} \in L^2(\Omega) \) where \( u_{t|t=0} = \sum_{i=1}^{\infty} u_{i|t=0} e_i \) and \( u_{i|t=0} = \sum_{i=1}^{\infty} u_{i|t=0} e_i \).

Let’s assume that \( \bar{u}(t) = \sum_{i=1}^{\infty} \bar{u}_i(t) e_i \) is a solution of the following problem;
\[
\begin{align*}
\bar{u}_{tt}(t) &= f \quad \text{in } (0,T) \times \Omega, \\
\bar{u}_{t|t=0} &= u_{t|t=0}, \\
\bar{u}_{|t=0} &= u_{t|t=0}.
\end{align*}
\tag{3.10}
\]
It is clear that \( \bar{u}_{tt} \in L^2(0, T; L^2(\Omega)) \). As in the proof of Theorem 3.6 we have

\[
\bar{u}_{tt}(t) = \sum_{i=1}^{\infty} \tilde{u}_i''(t)e_i(x) \in L^2(\Omega), \text{ a.e. } t. \tag{3.11}
\]

Thus we may write (3.10) in the series form

\[
\begin{cases}
\sum_{i=1}^{\infty} \tilde{u}_i''(t)e_i = \sum_{i=1}^{\infty} f_i(t)e_i & \text{ in } (0, T) \times \Omega, \\
\sum_{i=1}^{\infty} \tilde{u}_i(0)e_i = \sum_{i=1}^{\infty} u_i(0)e_i & \text{ in } L^2(\Omega), \\
\sum_{i=1}^{\infty} \tilde{u}_i'(0)e_i = \sum_{i=1}^{\infty} u_i'(0)e_i & \text{ in } L^2(\Omega),
\end{cases}
\]

so that for each \( i \),

\[
\begin{cases}
\tilde{u}_i''(t) = f_i(t) & \text{ in } (0, T), \\
\tilde{u}_i(0) = u_i(0), \\
\tilde{u}_i'(0) = u_i'(0).
\end{cases} \tag{3.12}
\]

From the definition of \( f_i \) and the uniqueness of the solution for the initial value problem (3.12), we obtain \( u_i \equiv \tilde{u}_i \) in \( (0, T) \) for each \( i \) so that \( u(t) = \bar{u}(t) \) in \( L^2(\Omega) \) for every \( t \). Hence \( u_{tt} = \bar{u}_{tt} \in L^2(0, T; L^2(\Omega)) \). Also, the equations (3.11) complete the proof.

\[ \square \]

4. Solution of the optimal control problem. In this section, we will derive an explicit formula for the optimal solution expressed as a series of eigenfunctions \( \{e_i\} \). We consider all functions as \( L^2(\Omega) \)-convergent series of eigenfunctions \( \{e_i\} \) for the elliptic operator:

\[
\begin{align*}
&u(t, x) = \sum_{i=1}^{\infty} u_i(t)e_i(x), \quad w(x) = \sum_{i=1}^{\infty} w_i e_i(x), \quad z(x) = \sum_{i=1}^{\infty} z_i e_i(x), \\
&f(t, x) = \sum_{i=1}^{\infty} f_i(t)e_i(x), \quad W(x) = \sum_{i=1}^{\infty} W_i e_i(x), \quad Z(x) = \sum_{i=1}^{\infty} Z_i e_i(x).
\end{align*}
\]

**Theorem 4.1.** Assume that \( w \in H^1_0(\Omega) \cap H^2(\Omega) \), \( z \in L^2(\Omega) \cap H^1_0(\Omega) \) and \( W, Z \in L^2(\Omega) \). Let \((\bar{u}, \bar{f})\) be the solution pair of optimal control problem in \( \mathcal{W}((0, T) \times \Omega) \times H^1(0, T; L^2(\Omega)) \). Then

\[
\bar{u}(t, x) = \sum_{i=1}^{\infty} \bar{u}_i(t)e_i(x) \tag{4.1}
\]

where

\[
\bar{u}_i(t) = (C_1 + tC_3) \cos \sqrt{\lambda_i}t + (C_2 + tC_4) \sin \sqrt{\lambda_i}t \tag{4.2}
\]

with the coefficients \( C_1, C_2, C_3 \), and \( C_4 : \)

\[
C_1 = w_i, \quad C_2 = \frac{1}{\sqrt{\lambda_i}}(z_i - C_3), \quad C_3 = \text{constant}, \quad C_4 = \text{constant}.
\]
We also obtain by Theorem 3.6

\[ C_3 = \left( \frac{(2\gamma + T^2)\sqrt{\lambda_i}A + BT}{\lambda_i(2\gamma\lambda_i + T^2)(2\gamma + T^2) + 2\gamma(\lambda_i - 1)\sqrt{\lambda_i}ABT - B^2T^2} \right), \]

\[ C_4 = \left( \frac{(2\gamma + T^2)\sqrt{\lambda_i}A - BT}{\lambda_i(2\gamma\lambda_i + T^2)(2\gamma + T^2) + 2\gamma(\lambda_i - 1)\sqrt{\lambda_i}ABT - B^2T^2} \right), \]

where \( A = \cos \sqrt{\lambda_i}T \) and \( B = \sin \sqrt{\lambda_i}T. \)

Proof. Let \((u, f)\) be an arbitrary element in \( W((0, T) \times \Omega) \times H^1(0, T; L^2(\Omega)). \) Then we may write

\[ u(t, x) = \sum_{i=1}^{\infty} u_i(t)e_i(x) \quad \text{and} \quad f(t, x) = \sum_{i=1}^{\infty} f_i(t)e_i(x) \quad \text{in} \quad L^2(\Omega). \]

We also obtain by Theorem 3.6

\[ u_{tt}(t) = \sum_{i=1}^{\infty} u''_i(t)e_i(x) \quad \text{in} \quad L^2(\Omega), \quad \text{a.e. } t \]

and

\[ -\text{div}[A(x)\nabla u(t)] = \sum_{i=1}^{\infty} \lambda_i u_i(t)e_i(x) \quad \text{in} \quad L^2(\Omega), \quad \text{a.e. } t. \]

Thus we may rewrite the constraint equations (2.2) as

\[
\begin{align*}
\int_{\Omega} \left( \sum_{j=1}^{\infty} [u''_j(t) + \lambda_j u_j(t)] e_j \right) e_i dx &= \int_{\Omega} \left( \sum_{j=1}^{\infty} f_j e_j \right) e_i dx, \quad i = 1, 2, \cdots, \\
\int_{\Omega} \sum_{j=1}^{\infty} u_j(0)e_j dx &= \int_{\Omega} \sum_{j=1}^{\infty} u_j(0)e_j dx, \quad i = 1, 2, \cdots, \\
\int_{\Omega} \sum_{j=1}^{\infty} u'_j(0)e_j dx &= \int_{\Omega} \sum_{j=1}^{\infty} z_j e_j dx, \quad i = 1, 2, \cdots,
\end{align*}
\]

so that for each \( i, \)

\[
\begin{align*}
u''_i(t) + \lambda_i u_i(t) &= f_i(t) \quad \text{in} \quad (0, T), \\
u_i(0) &= w_i, \\
u'_i(0) &= z_i.
\end{align*}
\]

The functional \( \mathcal{J} \) also can be written in the series form

\[ \mathcal{J}(u, f) = \frac{T}{2} \sum_{i=1}^{\infty} \left( |u_i - W_i|^2 + |u'_i - Z_i|^2 \right) + \frac{\gamma}{2} \sum_{i=1}^{\infty} \int_0^T |f_i|^2 dt. \]

The optimal control problem is recast into:

\((\tilde{OP})\) minimize functional (4.4) subject to the constraints (4.3) for all \( i = 1, 2, \cdots. \)
Since the constraint equations are fully uncoupled for each \( i \), the optimal control problem \((\tilde{OP}_i)\) is equivalent to

\[
\begin{align*}
\text{(\tilde{OP}_i)} & \quad \text{for each } i = 1, 2, \ldots, \text{minimize functional } J_i(u_i, \hat{f}_i) \text{ subject to the constraints } (4.3) \text{ where the functional } J_i(u_i, \hat{f}_i) \text{ is defined by}
\end{align*}
\]

\[
J_i(u_i, \hat{f}_i) = \frac{T}{2} \left( |u_i - W_i|^2 + |u_i' - Z_i|^2 \right) + \frac{\gamma}{2} \int_0^T |\hat{f}_i|^2 dt.
\]

Therefore the pair

\[
(\tilde{u}, \hat{f}) = \left( \sum_{i=1}^{\infty} \tilde{u}_i e_i(x), \sum_{i=1}^{\infty} \hat{f}_i e_i(x) \right)
\]

is the solution pair of the optimal control problem if and only if \((\tilde{u}_i, \hat{f}_i)\) is the solution for \((\tilde{OP}_i)\) for each \( i \). Through the use of Lagrange multiplier rules for the constrained minimization problem \((\tilde{OP}_i)\), we can derive an optimality system which consists of the state equation (4.3) with initial conditions, the costate equation with terminal conditions,

\[
\left\{ \begin{array}{l}
\xi''(t) + \lambda_i \xi(t) = 0 \quad \text{in } (0, T), \\
\xi_i(T) = T(u_i'(T) - Z_i), \\
\xi_i''(T) = -T(u_i(T) - W_i),
\end{array} \right.
\]

and the optimality condition,

\[
\xi_i(t) = -\gamma f_i(t).
\]

By eliminating \( \xi_i \) from (4.6)-(4.7), we have

\[
\left\{ \begin{array}{l}
f''_i(t) + \lambda_i f_i(t) = 0 \quad \text{in } (0, T), \\
f_i(T) = -\frac{T}{\gamma}(u_i'(T) - Z_i), \\
f_i'(T) = \frac{T}{\gamma}(u_i(T) - W_i),
\end{array} \right.
\]

Now combining (4.8) and (4.3), we arrive at a fourth order ordinary differential equation with initial and terminal conditions:

\[
\left\{ \begin{array}{l}
u^{(4)}_i(t) + 2\lambda_i u''_i(t) + \lambda_i^2 u_i(t) = 0 \quad \text{in } (0, T), \\
u_i(0) = w_i, \\
u'_i(0) = z_i, \\
u''_i(T) + \lambda_i u_i(T) = -\frac{T}{\gamma}(u_i'(T) - Z_i), \\
u'''_i(T) + \lambda_i u'_i(T) = \frac{T}{\gamma}(u_i(T) - W_i),
\end{array} \right.
\]

The general solution to this differential equation is

\[
u_i(t) = C_1 \cos \sqrt{\lambda_i} t + C_2 \sin \sqrt{\lambda_i} t + t(C_3 \cos \sqrt{\lambda_i} t + C_4 \sin \sqrt{\lambda_i} t)
\]

\[
\quad = (C_1 + tC_3) \cos \sqrt{\lambda_i} t + (C_2 + tC_4) \sin \sqrt{\lambda_i} t.
\]
In order to determine the coefficients, $C_1, C_2, C_3$, and $C_4$ using the given initial and terminal conditions we evaluate the followings:

$$u_1'(t) = (C_2 \sqrt{\lambda_i} + C_3 + t C_4 \sqrt{\lambda_i}) \cos \sqrt{\lambda_i} t + (-C_1 \sqrt{\lambda_i} - t C_3 \sqrt{\lambda_i} + C_4) \sin \sqrt{\lambda_i} t,$$

$$u_1''(t) = (-C_1 \lambda_i - t C_3 \lambda_i + 2 C_4 \sqrt{\lambda_i}) \cos \sqrt{\lambda_i} t - (C_2 \lambda_i + 2 C_3 \sqrt{\lambda_i} + t C_4 \lambda_i) \sin \sqrt{\lambda_i} t,$$

$$u_1'''(t) = (-C_2 \lambda_i \sqrt{\lambda_i} - 3 C_3 \lambda_i - t C_4 \lambda_i \sqrt{\lambda_i}) \cos \sqrt{\lambda_i} t + (C_1 \lambda_i \sqrt{\lambda_i} + t C_3 \lambda_i \sqrt{\lambda_i} - 3 C_4 \lambda_i) \sin \sqrt{\lambda_i} t.$$

By the given initial and terminal conditions in (4.9) we finally obtain the system for the coefficients, $C_1, C_2, C_3$, and $C_4$.

$$C_1 = w_i,$$

$$\sqrt{\lambda_i} C_2 + C_3 = z_i,$$

$$-\frac{T}{\gamma} \sqrt{\lambda_i} \sin \sqrt{\lambda_i} T C_1 + \frac{T}{\gamma} \sqrt{\lambda_i} \cos \sqrt{\lambda_i} T C_2$$

$$+ (-2 \sqrt{\lambda_i} \sin \sqrt{\lambda_i} T + \frac{T}{\gamma} \cos \sqrt{\lambda_i} T - \frac{T^2}{\gamma} \sqrt{\lambda_i} \sin \sqrt{\lambda_i} T) C_3$$

$$+ (2 \sqrt{\lambda_i} \cos \sqrt{\lambda_i} T + \frac{T^2}{\gamma} \sqrt{\lambda_i} \cos \sqrt{\lambda_i} T) \cos \sqrt{\lambda_i} T + \frac{T}{\gamma} \sin \sqrt{\lambda_i} T) C_4 = \frac{T}{\gamma} Z_i,$$

$$\frac{T}{\gamma} \cos \sqrt{\lambda_i} T C_1 + \frac{T}{\gamma} \sin \sqrt{\lambda_i} T C_2 + (2 \lambda_i \sin \sqrt{\lambda_i} T + \frac{T^2}{\gamma} \cos \sqrt{\lambda_i} T) C_3$$

$$+ (2 \lambda_i \sin \sqrt{\lambda_i} T + \frac{T^2}{\gamma} \sin \sqrt{\lambda_i} T) C_4 = \frac{T}{\gamma} W_i.$$  

(4.11)

Solving for $C_1, C_2, C_3,$ and $C_4$ and then plugging them into the general solution (4.10) we find the solution formula (4.1)-(4.2).  

\[\square\]

5. Dynamics of the optimal control solution. In this section we will show the optimal solution is a solution of the approximate controllability problem using the formula (4.1)-(4.2).

Remark 5.1. We recall that a set of eigenfunctions \(\{\psi_i\}\) is an orthonormal basis of of \(L^2(\Omega)\) so that for any function \(\phi(x) = \sum_{i=1}^{\infty} \psi_i(x)\) in \(L^2(\Omega)\) we have \(\|\phi\|^2 = \sum_{i=1}^{\infty} |\psi_i|^2\).

Theorem 5.2. Assume that \(w \in H^1_0(\Omega) \cap H^2(\Omega), z \in L^2(\Omega) \cap H^3(\Omega)\) and \(W, Z \in L^2(\Omega)\). Let \((\hat{u}, \hat{f})\) be the solution pair of optimal control problem in \(W((0, T) \times \Omega) \times H^1(0, T; L^2(\Omega)).\) Then the optimal solution \(u\) as a function of the parameter \(\gamma\) satisfies the approximate controllability property, i.e.,

$$\lim_{\gamma \to 0} \|u(T) - W\|_0 = 0, \quad \lim_{\gamma \to 0} \|u(T) - Z\|_0 = 0.$$

(5.1)

Proof. Using the inequality \((a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2)\) and the formula (4.1)-(4.2), we have the following inequality:
\[ \|u(T) - W\|_0^2 = \sum_{i=1}^{\infty} |u_i(T) - W_i|^2 \]
\[ \leq 4 \sum_{i=1}^{\infty} (I)^2|w_i|^2 + 4 \sum_{i=1}^{\infty} (II)^2|W_i|^2 + 4 \sum_{i=1}^{\infty} (III)^2|z_i|^2 + 4 \sum_{i=1}^{\infty} (IV)^2|Z_i|^2 \]

where

\[ I \leq \frac{4\lambda_i^2 A\gamma^2 + 2\sqrt{\lambda_i}T(\lambda_i\sqrt{\lambda_i}AT + \lambda_iAB - AB + A^2B + \lambda_iB^3T)\gamma}{4\lambda_i^2\gamma^2 + 2\sqrt{\lambda_i}T(\lambda_i\sqrt{\lambda_i}T + \lambda_iAB - AB + \sqrt{\lambda_i}T)\gamma + T^2(\lambda_iT^2 - B^2)}, \]

\[ II \leq \frac{-4\lambda_i^2\gamma^2 - 2\sqrt{\lambda_i}T(\lambda_i\sqrt{\lambda_i}T + \lambda_iAB)\gamma}{4\lambda_i^2\gamma^2 + 2\sqrt{\lambda_i}T(\lambda_i\sqrt{\lambda_i}T + \lambda_iAB - AB + \sqrt{\lambda_i}T)\gamma + T^2(\lambda_iT^2 - B^2)}, \]

\[ III \leq \frac{4\lambda_i\sqrt{\lambda_i}BT^2\gamma}{4\lambda_i^2\gamma^2 + 2\sqrt{\lambda_i}T(\lambda_i\sqrt{\lambda_i}T + \lambda_iAB - AB + \sqrt{\lambda_i}T)\gamma + T^2(\lambda_iT^2 - B^2)} \]

and

\[ IV \leq \frac{2\lambda_i BT^2\gamma}{4\lambda_i^2\gamma^2 + 2\sqrt{\lambda_i}T(\lambda_i\sqrt{\lambda_i}T + \lambda_iAB - AB + \sqrt{\lambda_i}T)\gamma + T^2(\lambda_iT^2 - B^2)}. \]

Now consider the first term of the second line in (5.2). Note that \( \lambda_iT^2 - B^2 > 0 \) for all \( i \in \mathbb{N} \), because \( x - \sin x > 0 \) for \( x > 0 \). We will prove that

\[ \sum_{i=1}^{\infty} (I)^2|w_i|^2 \to 0 \quad \text{as} \quad \gamma \to 0. \]

Let \( \epsilon > 0 \) be given. There exists an \( n \in \mathbb{N} \) such that

\[ \sum_{i=n}^{\infty} |w_i|^2 < \frac{\epsilon^2}{8}. \] (5.3)

Since \( A = \cos \sqrt{\lambda_i}T \leq 1 \), we obtain

\[ I \leq \frac{4A\gamma^2 + 2AT^2\gamma + \frac{2ABT\gamma}{\sqrt{\lambda_i}}}{4\gamma^2 + 2T^2\gamma + \frac{2ABT\gamma}{\lambda_i\sqrt{\lambda_i}} - \frac{2ABT\gamma}{\lambda_i\sqrt{\lambda_i}} + \frac{2A^2BT\gamma}{\lambda_i\sqrt{\lambda_i}} + \frac{2B^3T^2\gamma}{\lambda_i\sqrt{\lambda_i}} \sqrt{\lambda_i}} \]

\[ \leq \frac{4\gamma^2 + 2T^2\gamma + \frac{2ABT\gamma}{\lambda_i}}{4\gamma^2 + 2T^2\gamma + \frac{2ABT\gamma}{\lambda_i}} - \frac{2ABT\gamma}{\lambda_i} + \frac{2A^2BT\gamma}{\lambda_i\sqrt{\lambda_i}} + \frac{2B^3T^2\gamma}{\lambda_i\sqrt{\lambda_i}} \sqrt{\lambda_i} \]

Let \( K_i \) be the right hand side of the last inequality. Since \( \frac{1}{\lambda_i} \geq \frac{1}{\lambda_2} \geq \frac{1}{\lambda_3} \geq \cdots \) and \( \frac{1}{x_i} \to 0 \) as \( i \to \infty \), we have

\[ \lim_{i \to \infty} K_i = 1. \]

So there exists an \( m \in \mathbb{N} \) such that

\[ I \leq K_i \leq 2 \quad \text{for} \quad i \geq m. \] (5.4)

Letting \( N = \max(n, m) \) and then holding this \( N \) fixed, we may choose a \( \gamma_0 \) such that

\[ I \leq K_i \leq 2 \quad \text{for} \quad i \geq m. \]

Letting \( N = \max(n, m) \) and then holding this \( N \) fixed, we may choose a \( \gamma_0 \) such that
where.

Then the functions

Theorem 6.1. Assume that $w, W \in H^1_0(\Omega) \cap H^2(\Omega)$ and $z, Z \in L^2(\Omega) \cap H^1_0(\Omega)$. Then the functions

$$u(t, x) = \sum_{i=1}^{\infty} u_i(t) e_i(x), \quad f(t, x) = \sum_{i=1}^{\infty} f_i(t) e_i(x),$$

where

$$u_i(t) = (C_1 + t C_3) \cos \sqrt{\lambda_i} t + (C_2 + t C_4) \sin \sqrt{\lambda_i} t,$$

and

$$f_i(t) = 2 \sqrt{\lambda_i} C_4 \cos \sqrt{\lambda_i} t - 2 \sqrt{\lambda_i} C_3 \sin \sqrt{\lambda_i} t,$$

form a solution pair to the exact controllability problem (EX-CON), provided

$$A = \cos \sqrt{\lambda_i} T, \quad B = \sin \sqrt{\lambda_i} T, \quad C_1 = w_i, \quad C_2 = \frac{1}{\sqrt{\lambda_i}} (z_i - C_3),$$

$$C_3 = \frac{1}{\lambda_i T^2 - B^2} \left( T \sqrt{\lambda_i} A + B \right) \left( \sqrt{\lambda_i} W_i - \sqrt{\lambda_i} A w_i - B z_i \right) - (Z_i + \sqrt{\lambda_i} B w_i - A z_i) \sqrt{\lambda_i} BT,$$

for each $\gamma \in [0, \gamma_0]$. Thus we have from (5.3), (5.4) and (5.5) that

$$\sum_{i=1}^{\infty} (I)^2 |w_i|^2 = \sum_{i=1}^{\infty} (I)^2 |w_i|^2 + \sum_{i=N}^{\infty} (I)^2 |w_i|^2 < \frac{\epsilon^2}{2} + 4 \frac{\epsilon^2}{8} = \epsilon^2$$

for each $\gamma \in [0, \gamma_0]$. Therefore, we obtain from (5.2) that

$$\lim_{\gamma \to 0} ||u(T) - W||_0 = 0.$$

Moreover, we can similarly show that $\lim_{\gamma \to 0} ||u(t) - Z||_0 = 0$. This completes the proof.

6. Solution of the exact controllability problem. Recall that the exact controllability problem (EX-CON) has a solution pair $(u, f)$ if $w, W \in H^1_0(\Omega) \cap H^2(\Omega)$ and $z, Z \in L^2(\Omega) \cap H^1_0(\Omega)$. In this section, a solution formula for the exact controllability problem (EX-CON) will be expressed by the eigenseries formula. Formally setting $\gamma = 0$ in (4.2) we expect to obtain the solution formula. But this formula needs justification as infinite series functions are involved.

Theorem 6.1. Assume that $w, W \in H^1_0(\Omega) \cap H^2(\Omega)$ and $z, Z \in L^2(\Omega) \cap H^1_0(\Omega)$. Then the functions

Theorem 6.1. Assume that $w, W \in H^1_0(\Omega) \cap H^2(\Omega)$ and $z, Z \in L^2(\Omega) \cap H^1_0(\Omega)$. Then the functions

$$u(t, x) = \sum_{i=1}^{\infty} u_i(t) e_i(x), \quad f(t, x) = \sum_{i=1}^{\infty} f_i(t) e_i(x),$$

where

$$u_i(t) = (C_1 + t C_3) \cos \sqrt{\lambda_i} t + (C_2 + t C_4) \sin \sqrt{\lambda_i} t,$$

and

$$f_i(t) = 2 \sqrt{\lambda_i} C_4 \cos \sqrt{\lambda_i} t - 2 \sqrt{\lambda_i} C_3 \sin \sqrt{\lambda_i} t,$$

form a solution pair to the exact controllability problem (EX-CON), provided

$$A = \cos \sqrt{\lambda_i} T, \quad B = \sin \sqrt{\lambda_i} T, \quad C_1 = w_i, \quad C_2 = \frac{1}{\sqrt{\lambda_i}} (z_i - C_3),$$

$$C_3 = \frac{1}{\lambda_i T^2 - B^2} \left( T \sqrt{\lambda_i} A + B \right) \left( \sqrt{\lambda_i} W_i - \sqrt{\lambda_i} A w_i - B z_i \right) - (Z_i + \sqrt{\lambda_i} B w_i - A z_i) \sqrt{\lambda_i} BT,$$
\[ C_4 = \frac{\left( T \sqrt{\lambda_i} A - B \right) \left( Z_i + \sqrt{\lambda_i} B w_i - A z_i \right) + (\sqrt{\lambda_i} W_i - \sqrt{\lambda_i} A w_i - B z_i) \sqrt{\lambda_i} B T}{\lambda_i T^2 - B^2}. \]

**Proof.** It is easy to verify that \( u_i(0) = w_i, u_i(T) = W_i, u_i'(0) = z_i \) and \( u_i'(T) = Z_i \). Thus we have that \( u(0) = w, u(T) = W, u'(0) = z \) and \( u'(T) = Z \). To show that the pair \((u, f)\) is a solution to (EX-CON) we need to show that

\[
\begin{cases}
    u_{tt} - div[A(x) \nabla u] = f & \text{in } (0, T) \times \Omega, \\
    u = 0 & \text{on } (0, T) \times \partial \Omega,
\end{cases}
\]

and we will do so by verifying the assumptions of Theorem 3.7.

By Lemma 3.3 and Lemma 3.4, the assumptions \( w, W \in H^1_0(\Omega) \cap H^2(\Omega) \) and \( z, Z \in L^2(\Omega) \cap H^1_0(\Omega) \) imply

\[
\sum_{i=1}^{\infty} |\lambda_i| |w_i| \leq 2 \quad \text{and} \quad \sum_{i=1}^{\infty} |\lambda_i||W_i| \leq 4,
\]

\[
\sum_{i=1}^{\infty} |\lambda_i|^2 |w_i| \leq 2 \quad \text{and} \quad \sum_{i=1}^{\infty} |\lambda_i|^2 |W_i| \leq 2, \quad (6.3)
\]

\[
\sum_{i=1}^{\infty} |\lambda_i| |z_i| \leq 2 \quad \text{and} \quad \sum_{i=1}^{\infty} |\lambda_i||Z_i| \leq 2.
\]

Since \( u_i(0) = w_i \) and \( u_i'(0) = z_i \), we obviously obtain

\[
\sum_{i=1}^{\infty} \lambda_i |u_i(0)| \leq 2 \quad \text{and} \quad \sum_{i=1}^{\infty} \lambda_i |w_i| \leq 2,
\]

\[
\sum_{i=1}^{\infty} |u_i'(0)| \leq 2 \quad \text{and} \quad \sum_{i=1}^{\infty} |z_i| \leq 2.
\]

Note that

\[
C_1 = w_i,
\]

\[
C_2 = \frac{1}{\lambda_i T^2 - B^2} \left[ (T \sqrt{\lambda_i} + AB) w_i + \sqrt{\lambda_i} T^2 z_i - (T \sqrt{\lambda_i} A + B) W_i + TBZ_i \right],
\]

\[
C_3 = \frac{1}{\lambda_i T^2 - B^2} \left[ -(T \sqrt{\lambda_i} A B) w_i - B^2 z_i + (T \lambda_i A + \sqrt{\lambda_i} B) W_i - T \sqrt{\lambda_i} B Z_i \right],
\]

\[
C_4 = \frac{1}{\lambda_i T^2 - B^2} \left[ -\sqrt{\lambda_i} B^2 w_i + (AB - \sqrt{\lambda_i} T) z_i + T \lambda_i BW_i + (T \sqrt{\lambda_i} A - B) Z_i \right].
\]

Using the inequalities \((a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2), B \leq \sqrt{\lambda_i} T, A \leq 1, \text{ and } B \leq 1\), we have the following estimates:

\[
|C_1|^2 = |w_i|^2,
\]

\[
|C_2|^2 \leq \frac{4}{(\lambda_i T^2 - B^2)^2} \left[ 4T^2 \lambda_i |w_i|^2 + T^4 \lambda_i |z_i|^2 + 4T^2 \lambda_i |W_i|^2 + T^4 \lambda_i |Z_i|^2 \right],
\]

\[
|C_3|^2 \leq \frac{4}{(\lambda_i T^2 - B^2)^2} \left[ 4T^2 \lambda_i^2 |w_i|^2 + T^2 \lambda_i |z_i|^2 + 4T^2 \lambda_i^2 |W_i|^2 + T^2 \lambda_i |Z_i|^2 \right],
\]

\[
|C_4|^2 \leq \frac{4}{(\lambda_i T^2 - B^2)^2} \left[ \lambda_i |w_i|^2 + 4T^2 \lambda_i |z_i|^2 + T^2 \lambda_i |W_i|^2 + 4T^2 \lambda_i |Z_i|^2 \right].
\]
We know that \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \) and \( \lambda_i \to \infty \) as \( i \to \infty \). Thus there exists an \( N \in \mathbb{N} \) such that

\[
\frac{4B^2}{T^2} \leq \lambda_i \quad \text{for} \quad i \geq N,
\]

which implies \( \lambda_i T^2 \geq 4B^2 \) for \( i \geq N \). Now we observe that \( (\lambda_i T^2 - B^2)^2 - \frac{1}{2} \lambda_i^2 T^4 = \frac{1}{2} \lambda_i^2 T^4 - 2 \lambda_i B^2 T^2 + B^4 = \frac{1}{2} \lambda_i T^2 (\lambda_i T^2 - 4B^2) + B^4 \geq 0 \), and so

\[
(\lambda_i T^2 - B^2)^2 \geq \frac{1}{2} \lambda_i^2 T^4 \quad \text{for} \quad i \geq N.
\]

Therefore for each \( i \geq N \),

\[
|C_1|^2 = |w_i|^2,
\]

\[
|C_2|^2 \leq \frac{8}{\lambda_i^2 T^4} \left[ 4T^2 \lambda_i |w_i|^2 + T^4 \lambda_i |z_i|^2 + 4T^2 \lambda_i |W_i|^2 + T^4 \lambda_i |Z_i|^2 \right],
\]

\[
|C_3|^2 \leq \frac{8}{\lambda_i^2 T^4} \left[ 4T^2 \lambda_i^2 |w_i|^2 + T^2 \lambda_i |z_i|^2 + 4T^2 \lambda_i^2 |W_i|^2 + T^2 \lambda_i |Z_i|^2 \right],
\]

\[
|C_4|^2 \leq \frac{8}{\lambda_i^2 T^4} \left[ \lambda_i |w_i|^2 + 4T^2 \lambda_i |z_i|^2 + T^2 \lambda_i^2 |W_i|^2 + 4T^2 \lambda_i |Z_i|^2 \right].
\]

From (6.1) and the above inequalities, for each \( i \geq N \) we have

\[
\int_0^T \lambda_i^2 |u_i(t)|^2 dt = \int_0^T |\lambda_i^2|(C_1 + tC_3) \cos \sqrt{\lambda_i} t + (C_2 + tC_4) \sin \sqrt{\lambda_i} t|^2 dt,
\]

\[
\leq \int_0^T 4|\lambda_i|^2 (|C_1|^2 + |C_2|^2 + T^2|C_3|^2 + T^2|C_4|^2) dt,
\]

\[
\leq \frac{4}{T} \left[ 40|\lambda_i| + 33T^2|\lambda_i|^2 \right] |w_i|^2 + 192T|\lambda_i||z_i|^2,
\]

\[
+ \frac{32}{T} \left[ 4|\lambda_i|^2 + 5T^2|\lambda_i|^2 \right] |W_i|^2 + 192T|\lambda_i||Z_i|^2.
\]

Combining (6.3) and (6.4) we arrive at

\[
\sum_{i=1}^\infty \int_0^T |\lambda_i|^2 |u_i(t)|^2 dt = \sum_{i=1}^{N-1} \int_0^T |\lambda_i|^2 |u_i(t)|^2 dt + \sum_{i=N}^{\infty} \int_0^T |\lambda_i|^2 |u_i(t)|^2 dt < \infty.
\]

Differentiating (6.1) and then we have

\[
u_i'(t) = (\sqrt{\lambda_i}C_2 + C_4 + t\sqrt{\lambda_i}C_4) \cos \sqrt{\lambda_i} t + (C_4 - \sqrt{\lambda_i}C_1 - t\sqrt{\lambda_i}C_3) \sin \sqrt{\lambda_i} t,
\]

\[
u_i''(t) = (2\sqrt{\lambda_i}C_4 - \lambda_i C_1 - t\lambda_i C_3) \cos \sqrt{\lambda_i} t - (\lambda_i C_2 + 2\sqrt{\lambda_i} C_3 + t\lambda_i C_4) \sin \sqrt{\lambda_i} t.
\]

According to the above we have similar estimations as follows:

\[
|\nu_i'(t)|^2 \leq 4 \left[ |\lambda_i||C_1|^2 + |\lambda_i||C_2|^2 + 2(1 + T^2|\lambda_i|)(|C_3|^2 + |C_4|^2) \right],
\]

\[
|\nu_i''(t)|^2 \leq 4 \left[ |\lambda_i|^2|C_1|^2 + |\lambda_i|^2|C_2|^2 + 2(4|\lambda_i| + T^2|\lambda_i|^2)(|C_3|^2 + |C_4|^2) \right].
\]
Thus
\[
\int_0^T \lambda_i |u_i'(t)|^2 dt \\
\leq 4T \int_0^T \left[ |\lambda_i||C_1|^2 + |\lambda_i||C_2|^2 + 2(1 + T^2)|\lambda_i||C_3|^2 + 2(1 + T^2)|\lambda_i||C_4|^2 \right] \\
\leq 4T \left( (65|\lambda_i|^2 + \frac{112}{T^2}|\lambda_i| + \frac{16}{T^2}) |w_i|^2 + 4T \left( 88|\lambda_i| + \frac{80}{T^2} \right) |z_i|^2 \right) \\
+ 4T \left( (80|\lambda_i|^2 + \frac{112}{T^2}|\lambda_i|) |W_i|^2 + 4T \left( 88|\lambda_i| + \frac{80}{T^2} \right) |Z_i|^2, \right.
\]
and
\[
\int_0^T |u''_i(t)|^2 dt \\
\leq 4T \left[ |\lambda_i|^2|C_1|^2 + |\lambda_i|^2|C_2|^2 + 2(4|\lambda_i| + T^2)|\lambda_i|^2|C_3|^2 \right. \\
\left. + 2(4|\lambda_i| + T^2)|\lambda_i|^2|C_4|^2 \right] \\
\leq 4T \left( (65|\lambda_i|^2 + \frac{304}{T^2}|\lambda_i| + \frac{64}{T^4}) |w_i|^2 + 4T \left( 88|\lambda_i| + \frac{320}{T^2} \right) |z_i|^2 \right) \\
+ 4T \left( (80|\lambda_i|^2 + \frac{352}{T^2}|\lambda_i|) |W_i|^2 + 4T \left( 88|\lambda_i| + \frac{320}{T^2} \right) |Z_i|^2. \right.
\]
By the similar method that we used before, we obtain
\[
\sum_{i=1}^{\infty} \int_0^T \lambda_i |u_i'(t)|^2 dt < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \int_0^T |u_i''(t)|^2 dt < \infty.
\]
Therefore we have verified the assumptions of Theorem 3.7, and thus we use the theorem to conclude that \( u \in W((0, T) \times \Omega) \), \( u = 0 \) on \((0, T) \times \partial \Omega\),
\[
u(t) - \text{div}[A(x) \nabla u(t)] = \sum_{i=1}^{\infty} u_i''(t)e_i(x) + \sum_{i=1}^{\infty} \lambda_i u_i(t)e_i(x)
\]
\[
= \sum_{i=1}^{\infty} [u_i''(t) + \lambda_i u_i(t)] e_i(x) \text{ in } L^2(\Omega), \text{ a.e. } t,
\]
where \( u_i''(t) \) and \( u_i(t) \) are given in (6.5) and (6.1).

From (6.2) and the last equality, we deduce
\[
f(t) = u(t) - \text{div}[A(x) \nabla u(t)] \text{ in } L^2(\Omega), \text{ a.e. } t,
\]
so that
\[
f(t) = u(t) - \text{div}[A(x) \nabla u(t)] \in L^2((0, T); L^2(\Omega)).
\]
Therefore the pair \((u, f)\) is indeed a solution to (EX-CON) which completes the proof.

\[
7. \text{ One-dimensional numerical experiments.} \text{ In one-dimensional space, it is well known that the eigenpairs} \\{ \{\lambda_i, e_i\}\}_{i=1}^{\infty} \text{ of the elliptic operator can be easily obtained by solving the following ordinary differential equations:}
\]
\[
\begin{cases}
-\epsilon''(x) = \lambda e(x) \quad \text{in } 0 \leq x \leq 1, \\
e(0) = e(1) = 0.
\end{cases}
\]
The solutions of the equation are
\[ \lambda_i = (i\pi)^2 \text{ and } e_i(x) = \sqrt{2}\sin(i\pi x), \quad i = 1, 2, \ldots. \]

Using the eigenpairs of the elliptic operator the optimal solution can be computed from the series solution formulae \((4.1)-(4.2)\) with the following constrain equations on the spatial interval \(\Omega = (0, 1)\):
\[
\begin{cases}
u_{tt} - u_{xx} = f \quad \text{in } (0, T) \times (0, 1), \\
u(t, 0) = u(t, 1) = 0 \quad \text{in } (0, T), \\
u|_{t=0} = w \quad \text{and} \quad u_t|_{t=0} = z \quad \text{in } (0, 1).
\end{cases}
\]

with given target functions \(W(x), Z(x)\).

Consider two sets of data:

**Data I.**
\[
w_i = \sum_{i=1}^{10} \frac{i}{\sqrt{2}} e_i(x), \quad z_i = 0,
\]
\[
W_i = \int_0^1 W(x)e_i(x) \, dx \quad \text{and} \quad Z_i = \int_0^1 Z(x)e_i(x) \, dx.
\]

**Data II.**
\[
w_i = \sin i\pi x, \quad z_i = \pi \sin i\pi x,
\]
\[
W_i = 1, \quad Z_i = 0.
\]

We choose \(T = 1\) for Data I and \(T = 2\) for Data II.

Note that \(w_i, W_i, z_i, \) and \(Z_i\) are calculated by
\[
w_i = \int_0^1 w(x)e_i(x) \, dx, \quad W_i = \int_0^1 W(x)e_i(x) \, dx \quad \text{and} \quad Z_i = \int_0^1 Z(x)e_i(x) \, dx.
\]

From the given initial conditions and target functions in Data I and II, we have

**Data I.**
\[
w_i = \begin{cases}
\frac{i}{\sqrt{2}}, & i \leq 10, \\
0, & i > 10,
\end{cases} \quad W_i = \begin{cases}
\frac{1}{\sqrt{2}}, & i = 2, \\
0, & i \neq 2,
\end{cases}
\]
\[
z_i = 0 \quad \text{for all } i, \quad Z_i = \begin{cases}
\frac{1}{\sqrt{2}}, & i = 1, \\
\frac{1}{2\sqrt{2}}, & i = 2, \\
0, & \text{otherwise},
\end{cases}
\]

**Data II.**
\[
w_i = \begin{cases}
\frac{1}{\sqrt{2}}, & i = 1, \\
0, & i \neq 1,
\end{cases} \quad W_i = \sqrt{2} \frac{1 - (-1)^i}{i\pi} \quad \text{for all } i,
\]
\[
z_i = \begin{cases}
\frac{\pi}{\sqrt{2}}, & i = 1, \\
0, & i \neq 1,
\end{cases} \quad Z_i = 0 \quad \text{for all } i.
\]

For each data set we solve the optimal control problem by the series solution formulae \((4.1)-(4.2)\). In each case we vary the parameter \(\gamma\) and calculate the difference between optimal solutions and target functions with \(L^2(\Omega)\) norm and \(H^1(\Omega)\) norm (see Table 1 and 2). Moreover, we plot the optimal solutions \(\hat{u}(t)\) and \(\hat{u}_t(t)\) at the terminal time \(T\) (the “*” curve) versus the target functions \(W\) and \(Z\) (the “−” curve). See Figures 1 and 2 for Data I, and Figures 3 and 4 for Data II.

For Data I the admissible state \((u)\) and the target state \((W)\) have matching boundary conditions (both have homogeneous boundary conditions). For Data II the admissible state and the target function have nonmatching boundary conditions.
optimal control problem. The convergence of the solutions of the optimal control controllability problem can also be naturally obtained based on the results for the approximate controllability problem. Thus, an eigenseries solution for the exact results in separate papers.

$$\hat{u}(t)$$ and $$\hat{u}_k(t)$$ at the terminal time $$T$$ approach to the target functions $$W$$ and $$Z$$ in $$L^2$$-sense as the parameter $$\gamma$$ is getting smaller. This good job of tracking is predicted by Theorem 5.2. However, we cannot guarantee the convergence of the optimal solution $$\hat{u}(T)$$ to the target functions $$W$$ in $$H^1$$-sense specially for Data II because of their nonmatching boundary condition (See Table 2).

8. Conclusions. In the paper, we study the eigenseries solutions of the terminal-state tracking optimal distributed control problem governed by linear hyperbolic equations. An explicit analytic formula using eigenvalues/eigenfunctions of the elliptic operator and infinite series is successfully derived for the solution of the optimal control problem. It is also proved that this optimal control solution as a function of the parameter $$\gamma$$ (a parameter in the objective functional) is a solution of the approximate controllability problem. Thus, an eigenseries solution for the exact controllability problem can also be naturally obtained based on the results for the optimal control problem. The convergence of the solutions of the optimal control problem to the target functions at the terminal time was numerically demonstrated with varying the parameter $$\gamma$$. We do think that the techniques and methodologies presented here will be applied to other types of constrained equations, for instance, Stoke’s equation. We are currently working on these issues and will present the results in separate papers.

<table>
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<tr>
<th>$$\gamma$$</th>
<th>$$|u(T) - W|_0$$</th>
<th>$$|\hat{u}_k(T) - Z|_0$$</th>
<th>$$|u(T) - W|_1$$</th>
<th>$$|\hat{u}_k(T) - Z|_1$$</th>
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<td>$$0.2370 \times 10^0$$</td>
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<td>$$0.2465 \times 10^3$$</td>
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<tr>
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<td>$$0.3991 \times 10^{-2}$$</td>
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<td>$$10^{-8}$$</td>
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<td>$$0.4020 \times 10^{-6}$$</td>
<td>$$0.7389 \times 10^{-2}$$</td>
<td>$$0.8697 \times 10^{-6}$$</td>
</tr>
</tbody>
</table>

Table 2. Difference between optimal solutions and target functions with Data II ($$T = 2$$)
Acknowledgments. The authors would like to thank the anonymous reviewers for constructive comments to improve this paper.

REFERENCES

Figure 1. Optimal solution $\hat{u}(T)$ and target $W$ with Data I ($T = 1$)
*: optimal solution $\hat{u}(T)$  -: target function $W$

Figure 2. Optimal solution $\hat{u}_t(T)$ and target $Z$ with Data I ($T = 1$)
*: optimal solution $\hat{u}_t(T)$  -: target function $Z$

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Figure 3. Optimal solution $\hat{u}(T)$ and target $W$ with Data II ($T = 2$)
*: optimal solution $\hat{u}(T)$   -: target function $W$

Figure 4. Optimal solution $\hat{u}_t(T)$ and target $Z$ with Data II ($T = 2$)
*: optimal solution $\hat{u}_t(T)$   -: target function $Z$