A NOTE ON WEAK DIVIDING

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ABSTRACT. We study the notion of weak dividing introduced by S. Shelah. In particular we prove that T is stable iff weak dividing is symmetric.

In order to study simple theories Shelah originally defined *weak dividing* in [6] . This notion is overshadowed by that of *dividing*, as the first author proved that dividing is the right well-behaved notion for simple theories [2],[3],[5],and [4]. However Dolich's paper[1] reminded us that weak dividing is still an interesting notion. There he noted that weak dividing is symmetric and transitive in stable theories, and that simplicity is characterized by the property that dividing implies weak dividing.

Here we continue the investigation of the notion of weak dividing. Intriguingly, what weak dividing is to stability is analogous with what dividing is to simplicity. For example, we show that weak dividing is symmetric *only* in stable theories (2.5). Stability is also equivalent to left local character of weak dividing. However for the transitivity of weak dividing, a similar analogy does not exist. Namely, in a non-stable simple theory (e.g. the theory of the random graph), weak dividing can be transitive (2.7).

As usual, we work in a saturated model of an *arbitrary* complete theory T. Notation will be standard: a denotes a finite tuple, and M denotes a small elementary submodel. We assume that the reader has some familiarity with the basics of dividing/forking as in [3],[5] or [7].

1. Weak dividing

In this section we recall the definition of weak dividing from [6], state known facts from [6],[3],[4], [1], and some new observations. We sketch the proofs of the facts.

Definition 1.1. *We say $p(x) = tp(a/B)$* weakly divides *over $A(\subseteq B)$ if there is a formula $\psi(x_1, ..., x_n)$ over A such that $[p]^\psi := p(x_1) \cup ... \cup p(x_n) \cup \{\psi(x_1, ..., x_n)\}$ is inconsistent while $[q]^\psi$ is consistent where $q(x) = tp(a/A)$.*

The following easily come from the definition. (1.2.2 is also proved in 3.3.)

Fact 1.2. (1) *(Partial transitivity) Let $A \subseteq B \subseteq C$. If $tp(a/C)$ does not weakly divide over B and $tp(a/B)$ does not weakly divide over A, then $tp(a/C)$ does not weakly divide over A. Also if $tp(a/C)$ does not weakly divide over A, then $tp(a/B)$ does not weakly divide over A.*

(2) [6] *(Local character) For finite a and C, there is $C_0(\subseteq C)$ of size $\leq |T|$ such that $tp(a/C)$ does not weakly divide over C_0.*

But in general, $tp(a/C)$ may weakly divide over B, while $tp(a/C)$ does not weakly divide over A, see (2.6).

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Fact 1.3. [6] *If $tp(a/Bc)$ divides over B, then $tp(c/Ba)$ weakly divides over B.*

Proof. Since $tp(a/Bc)$ divides over B, there are $c = c_0, c_1, ..., c_k$ such that, $tp(c_i/B) = tp(c/B)$ and $\varphi(x, c_0) \wedge ... \wedge \varphi(x, c_k)$ is inconsistent for a formula $\varphi(x, y)$ over B, while $a \models \varphi(x, c_0)$. Hence $tp(c_0...c_n/B) \cup p(x_0) \cup ... \cup p(x_n)$ is inconsistent for $p(x) = tp(c/Ba)$. Hence by compactness, p weakly divides over B. □

Fact 1.4. [3],[4] *The following are equivalent.*

(1) *T is simple, i.e. dividing satisfies the local character property.*
(2) *Dividing is symmetric.*
(3) *Dividing is transitive.*

Fact 1.5. [6],[1] *T is simple iff dividing implies weak dividing.*

Proof. (⇒) If $tp(a/Bc)$ divides over B in simple T, then by symmetry $tp(c/Ba)$ divides over B. Then by 1.3, $tp(a/Bc)$ weakly divides over B.

(⇐) By 1.2.2. □

Remark 1.6. *The following are equivalent.*

(1) *$p(x) = tp(a/B)$ does not weakly divide over $A(\subseteq B)$.*
(2) *For any set of tuples $\{a_i | \ i \in I\}$ such that $tp(a/A) = tp(a_i/A)$, there is $B'(\models tp(B/A))$ such that, for all $i \in I$, $tp(a_iB'/A) = tp(aB/A)$ (i.e. $\bigcup\{r(\bar{x}/a_iA) | \ i \in I\}$ is consistent, where $r(\bar{x}/aA) = tp(B/aA)$).*

Proof. (1)⇒(2). See [6].

(2)⇒(1). Assume (2). Now suppose that for a formula $\psi(x_1, ..., x_n)$ over A, $[q]^\psi$ is realized by $(a_1, ..., a_n)$, where $q(x) = tp(a/A)$. By (2), there is $B'(\models tp(B/A))$ such that $tp(a_iB'/A) = tp(aB/A)$. Now by moving B' to B over A, we can find $(a'_1, ..., a'_n)$ realizing $[p]^\psi$. Hence p does not weakly divide over A. □

Fact 1.7. [1] *Let T be stable and let $p \in S(B)$. Then p does not weakly divide over $A(\subseteq B)$ iff p is the unique nondividing extension of $q = p\lceil A$ over B.*

Proof. (⇒) By 1.5, p can not be a dividing extension of q. By choosing in 1.6, $\{a_i | i \in I\}$ to be a set of realizations of the distinct strong types over A extending q (and using the stationarity of strong types), we see that p is the unique nondividing extension of q over B.

(⇐) Easily comes from 1.6 and the extension axiom. □

Hence in a stable theory the notions of weak dividing and dividing coincide over a model (In 2.5, we shall see this property characterizes stability), although the two notions are different in general over a set, even in a stable theory (e.g. an equivalence relation only having two infinite equivalence classes).

Fact 1.8. [1] *If T is stable, then weak dividing is symmetric.*

Proof. Easy to see from 1.7. □

We shall prove the converse (2.5). Also from 1.7, we notice that stability implies the *full* transitivity of weak dividing. We see this again in 1.9, and 1.10. (But the converse does not hold for transitivity (2.7).)

Lemma 1.9. *Let $A \subseteq B$. Suppose that $tp(c/Bd)$ weakly divides over B. Then for some $b \in B$, $tp(cb/Ad)$ weakly divides over A.*

Proof. Since $p = tp(c/Bd)$ weakly divides over B, there is a formula $\psi(x_1, ..., x_n; b_0; a)$ ($b_0 \in B \setminus A$, $a \in A$) such that $[p]^\psi$ is inconsistent (*) while $[p\lceil B]^\psi$ is consistent (**). By (*), there is $b_1 \in B$ such that $[p\lceil b_1 d]^\psi$ is inconsistent (***). Let $\psi'(x_1 y_0 y_1, ..., x_n y_0 y_1; a) = \psi(x_1, ..., x_n; y_0; a) \wedge y_1 = y_1$. Now, by (**), $[tp(cb_0 b_1/A)]^{\psi'}$ is consistent. On the other hand if $[tp(cb_0 b_1/Ad)]^{\psi'}$ is consistent, then there are $c_1 ... c_n$ and $b_0' b_1'$ such that $c_i b_0' b_1' \models tp(cb_0 b_1/Ad)$ and $\psi'(c_1 b_0' b_1', ..., c_n b_0' b_1'; a)$. By moving $b_0' b_1'$ to $b_0 b_1$ over Ad, we can find a tuple $(c_1' ... c_n')$ realizing $[p\lceil b_1 d]^\psi$. This contradicts (***). □

Theorem 1.10. *For any T, if weak dividing is symmetric then it is transitive.*

Proof. Due to 1.2.1, it suffices to show that if $tp(a/C) = p$ does not weakly divide over A (*), then p does not weakly divide over B ($A \subseteq B \subseteq C$). Suppose not, say p weakly divides over B. Then by symmetry, there is $c \in C$ such that $tp(c/Ba)$ weakly divides over B. Then by 1.9, there is $b \in B$ such that $tp(cb/Aa)$ weakly divides over A. Again by symmetry, $tp(a/Acb)$ weakly divides over A. This contradicts (*), and hence symmetry (of weak dividing) implies transitivity. □

Now we recall some properties of heirs and coheirs, which will be used in the next sections. Let $M \subseteq A \subseteq B$ and let $p = tp(c/A)$ and $q = p\lceil M$. We say p is a *coheir* of q if any $\varphi(x) \in p$ is realized by a tuple from M. p is called an *heir* of q if for any $a \in A$, $tp(a/Mc)$ is a coheir of $tp(a/M)$. Now suppose that p is a coheir of q. Then p does not divide over M and, moreover p has an extension r in $S(B)$ which is a coheir of p. In particular, if $b_1,\ b_2 \subseteq B$ and $tp(b_1/A) = tp(b_2/A)$, then for any $c \models r$, $tp(cb_1/A) = tp(cb_2/A)$. Hence if $I = \langle c_i | i < \omega \rangle$ ($c_i \models p$) is a *coheir sequence* of p (i.e. for each $i < \omega$, $tp(c_{i+1}/Ac_0 ... c_i)$ is a coheir of q and an extension of $tp(c_i/Ac_0 ... c_{i-1})$), then I is A-indiscernible. Note that T is stable iff for any $q \in S(M)$ and $M \subseteq A$ there is a unique coheir in $S(A)$ of q iff the same uniqueness over a model holds for heirs.

Remark We can obviously extend the notion of heir to an arbitrary set. If $p_1 = tp(d/B)$ is an heir of $p_2 = tp(d/A)(A \subseteq B)$, then p_1 does not weakly divide over A [1]: By 1.6, for given $\{d_i | i \in I\}$ with $d_i \models p_2$ (*), it suffices to see $\Gamma = \bigcup\{r(\bar{x}/d_i A) | i \in I\}$ is consistent, where $r(\bar{x}/dA) = tp(B/dA)$. For a finite subset of Γ, say $\Gamma_0 = \{\varphi(x, d_{i_0} a'), ..., \varphi(x, d_{i_k} a')\}$ ($x \subseteq \bar{x}, a' \in A$), since p_1 is an heir of p_2 there is $a \in A$ such that $\varphi(a, d_{i_0} a')$. Then from (*), $a \models \Gamma_0$. Hence by compactness, Γ is consistent.

2. Weak dividing left-chain and symmetry

Recall that $\{tp(b/A\{a_\beta | \beta < \alpha\}) | \alpha < |T|^+\}$ is called a *dividing(forking) chain* if for each $\alpha < |T|^+$, $tp(b/A\{a_\alpha | \beta \leq \alpha\})$ is a dividing(forking) extension of $tp(b/A\{a_\alpha | \beta < \alpha\})$. (This notion of a chain is different from the chain notion in Dolich's paper.) As is well-known the non-existence of the dividing/forking chain is a criterion of a theory being simple. Namely, T is not simple iff there is a dividing chain iff there is a forking chain. But we shall see that in *any* T, there *does not exist* a weak dividing chain (which of course is defined in the same manner.) The correct chain notion for weak dividing is as follows.

Definition 2.1. *A collection of types $\{tp(a_\alpha/Ab\{a_\beta|\ \beta < \alpha\}) | \alpha < |T|^+\}$ is said to be a* weak dividing left-chain *if for each $\alpha < |T|^+$, $tp(a_\alpha/Ab\{a_\beta|\ \beta < \alpha\})$ weakly divides over $A \cup \{a_\beta|\ \beta < \alpha\}$.*

Note that, by 1.3, if T has a dividing chain, then it produces a weak dividing left-chain. We call this a *left-chain* since it is convenient to imagine the sequence $\langle a_\alpha | \alpha < |T|^+ \}$ in 2.1 goes to the left hand side, while the sequence goes to the right hand side in the definition of the dividing chain. Hence we also call the usual chain, a dividing/forking *right*-chain. In the same vein, one may be tempted to define a dividing/forking *left*-chain, but there is *no* such chain (3.3). We shall observe an interesting connection between the notions of the weak dividing left-chain and stability. Namely, as the non-existence of the dividing chain characterizes simplicity, that of the weak dividing left-chain is equivalent to stability (2.5).

Lemma 2.2. *The following are equivalent.*

(1) *T has a weak dividing left-chain.*
(2) *There are an $\mathcal{L}$-formula $\psi(x_1, ..., x_n)$, an indiscernible sequence $I = \langle a_i | i \in \omega \rangle$ over c, and $\bar{d}^i = d^i_1..d^i_n (\models tp(\bar{d}^0))$ $(i \in \omega)$ such that $\bar{d}^i \models [tp(a_0)]^\psi$ whereas $[tp(a_0/c)]^\psi$ is inconsistent, and moreover $tp(d^i_j/a_0...a_{i-1}) = tp(a_i/a_0...a_{i-1})$ for $j = 1, .., n$.*
(3) *There are a finite tuple c and a set A such that $tp(A/A_0c)$ weakly divides over $A_0 (\subseteq A)$ for any subset A_0 of cardinality $\leq |T|$.*

Proof. (1)⇒(2). Suppose that there are a tuple b, and a set $\{u_i | \ i < |T|^+\}$ such that for each $\alpha < |T|^+$, $tp(u_\alpha/Ab\{u_\beta | \ \beta < \alpha\})$ weakly divides over $A \cup \{u_\beta | \ \beta < \alpha\}$. Since there are only $|T|$-many $\mathcal{L}$-formulas, we can find a fixed $\mathcal{L}$-formula $\phi(w_1, ..., w_n; y)$ and $|T|^+$-subset τ of $|T|^+$ such that, for all $i \in \tau$, there is $s_i \in A\{u_j | j < i\}$ with $\phi_{s_i} = \phi(w_1, ..., w_n; s_i)$ such that $[tp(u_i/A\{u_j | j < i\})]^{\phi_{s_i}}$ is consistently realized by saying $u^i_1...u^i_n$ (†), whereas $[tp(u_i/A\{u_j | j < i\}b)]^{\phi_{s_i}}$ is inconsistent. Again we can assume that inconsistency for each i is witnessed by a single $\mathcal{L}$-formula. Namely, there is $\varphi(w; yy'z)$ so that, for $e_i \in A\{u_j | j < i\}$, $\phi(w_1, ..., w_n; s_i) \wedge \varphi(w_1; s_ie_ib) \wedge ... \wedge \varphi(w_n; s_ie_ib)$ is inconsistent ($\varphi(w; s_ie_ib) \in tp(u_i/A\{u_j | j < i\}b)$ (*)). Then again we can assume that there is a single $\sigma(yy'z)$ realized by s_ie_ib such that $\phi(w_1, ..., w_n; y) \wedge \varphi(w_1; yy'z) \wedge ... \wedge \varphi(w_n, yy'z) \wedge \sigma(yy'z)$ is inconsistent. We now let $x_k \equiv w_kyy'$ and let $\psi(x_1, ..., x_n) \equiv \phi(w_1, ..., w_n; y) \wedge y' = y'$ (‡). Also let $\gamma(x; z) \equiv \varphi(wyy'; z) \wedge \sigma(yy'z)$ (**) Then $\psi(x_1, ..., x_n) \wedge \gamma(x_1; z) \wedge ... \wedge \gamma(x_n; z)$ is inconsistent (***).

Since ω is embedded into τ, we suppose $\omega \subseteq \tau$. Now consider the sequence $\langle u_is_ie_i | i \in \omega \rangle$, and $u^i_1...u^i_n$: By (†) and (‡), $tp(u^i_js_ie_i/u_0s_0e_0...u_{i-1}s_{i-1}e_{i-1}) = tp(u_is_ie_i/u_0s_0e_0...u_{i-1}s_{i-1}e_{i-1})$ for $j = 1, .., n$, and $u^i_1s_ie_i...u^i_ns_ie_i \models \psi(x_1, ..., x_n)$. On the other hand, by (*) and (**), $u_is_ie_i \models \gamma(x, b)$. Then by (***), $\psi(x_1, ..., x_n) \wedge \gamma(x_1; b) \wedge ... \wedge \gamma(x_n; b)$ is inconsistent, (and so $tp(u_is_ie_i/u_0s_0e_0...u_{i-1}s_{i-1}e_{i-1}b)$ weakly divides over $u_0s_0e_0...u_{i-1}s_{i-1}e_{i-1}$.)

Now the previous paragraph is a type-definable condition on $u_is_ie_ibu^i_1...u^i_n$. Hence by compactness and a Ramsey style argument, we can further assume that $\langle u_is_ie_i | i \in \omega \rangle$ is b-indiscernible and $tp(u^i_1s_ie_i...u^i_ns_ie_i) = tp(u^0_1s_0e_0...u^0_ns_0e_0)$. Then the condition of (2) is satisfied by letting $a_i = u_is_ie_i$, $c = b$ and $\bar{d}^i = u^i_1s_ie_i...u^i_ns_ie_i$.

(2)⇒(3). Assume (2). By compactness, we can further assume that the length of I is $|T|^+$, say $I = \langle a_i | \ i < |T|^+ \rangle$. Let $A = \{a_i | \ i < |T|^+\}$. Then for any set $A_0 (\subseteq A)$ of cardinality $\leq |T|$, there is $a_\alpha \in A$ such that $\alpha > i$ for any $a_i \in A_0$. Hence by (2), there is $d^\alpha_1...d^\alpha_n (\models tp(\bar{d}))$ such that $tp(d^\alpha_j/A_0) = tp(a_\alpha/A_0)$ for $j = 1, .., n$. Then $tp(a_\alpha/A_0c)$ weakly divide over A_0. Thus $tp(A/A_0c)$ weakly divides over A_0 too.

(3)⇒(1). Assume (3). Then in particular $tp(A/c)$ weakly divides over $\emptyset$. Hence there is finite $a_0 \in A$ such that $tp(a_0/c)$ weakly divides over $\emptyset$. Now to use induction, suppose that we have found $\{a_i | \ i < \alpha\} \subseteq A$ $(\alpha < |T|^+)$ such that for $\beta < \alpha$, $tp(a_\beta/c\{a_i | i < \beta\})$ weakly divide over $\{a_i|i<\beta\}$. Now by (3), $tp(A/c\{a_i|i<\alpha\})$ weakly divides over $\{a_i|i<\alpha\}$. Hence for some finite $a_\alpha \in A$, $tp(a_\alpha/c\{a_i|i<\alpha\})$ weakly divides over $\{a_i|i<\alpha\}$. Then $\{a_\alpha|\ \alpha<|T|^+\}$ forms a weak dividing left-chain. $\square$

Remark 2.3.

1. We call the negation of 2.2.3, *left local character* of weak dividing: For any finite tuple c and a set A, there is $A_0(\subseteq A)$ of cardinality $\leq |T|$ such that $tp(A/A_0c)$ does not weakly divide over A_0. Then the lemma says that T does not have a weak dividing left-chain if and only if left local character of weak dividing holds.
2. We state a fact (folklore) which will be used later. Let $\langle a_i|i\leq\omega\rangle$ be an indiscernible sequence. Then $p:=tp(a_\omega/I)=Lstp(a_\omega/I)$ where $I=\langle a_i|i\in\omega\rangle$: Let $b\models p$. Then the set of formulas saying that there are $x_i(i\in\omega)$ such that $a_\omega x_0x_1...$ and $bx_0x_1...$ both indiscernible over I, is consistent, since a finite subset is realized by some tuple from I itself. Hence $Lstp(b/I)=Lstp(a_\omega/I)=p$.

Lemma 2.4. *Let T be unstable. Then there are a_1,a_2,b and a model M such that $tp(a_1/M)=tp(a_2/M)$ and $tp(a_1a_2/Mb)$ is a coheir of $tp(a_1a_2/M)$, while $tp(a_1a_2/Mb)$ weakly divide over M, and $tp(b/Ma_1a_2)$ does not weakly divide over M.*

Proof. Since T is unstable, there must exist a model M, and a complete type $p(x)\in S(M)$ such that $p(x)$ has two distinct coheirs $q_1(x),q_2(x)$ over Mb. Hence some formula $\varphi(x,m,b)\in q_1(x)$, while $\neg\varphi(x,m,b)\in q_2(x)$ for $m\in M$ (*). Now, let $a_1\models q_1(x)$. There are a coheir extension $q_2'(x)$ of $q_2(x)$ over Mba_1, and a realization $a_2\models q_2'(x)$. Then, $r(xy)=tp(a_1a_2/Mb)$ is a coheir of $tp(a_1a_2/M)$. (†).

Claim 1) $tp(a_1a_2/Mb)$ weakly divides over M: Since a_1,a_2 both realize $p(x)\in S(M)$, there is a_3 such that $tp(a_1a_2/M)=tp(a_2a_3/M)$. Now, if $tp(a_1a_2/Mb)$ does not weakly divides over M, then by 1.6, there is $b'\models tp(b/M)$ such that $tp(a_1a_2b'/M)=tp(a_2a_3b'/M)=tp(a_1a_2b/M)$. Hence $tp(a_2b'/M)=tp(a_2b/M)$ (**) and $tp(a_2b'/M)=tp(a_1b/M)$ (***). Then by (*) and (**), $\neg\varphi(a_2,m,b')$. On the other hand by (*) and (***), $\varphi(a_2,m,b')$, a contradiction.

Claim 2) $tp(b/Ma_1a_2)$ does not weakly divide over M: This follows from (†) and the remark at the end of section 1. $\square$

Theorem 2.5. *The following are equivalent.*

1. *T is stable.*
2. *In T, weak dividing is symmetric.*
3. *Weak dividing is symmetric over a model.*
4. *The notions of weak dividing and dividing are equal over any model.*
5. *T does not have a weak dividing left-chain.*
6. *Weak dividing has left local character.*

Proof. (1)⇒(2). By 1.8.

(2)⇒(3). Obvious

(3)⇒(1). By lemma 2.4.

(1)⇒(4). See 1.7.

(4)⇒(1). Again by 2.4.

(1)⇒(5). We prove the contrapositive. Assume that T has a weak dividing left-chain. Then by 2.2, there is a c-indiscernible sequence $\langle a_i|i\leq\omega\rangle$ such that $tp(a_\omega/Ic)$ weakly divides over $I = \langle a_i | i < \omega \rangle$. But since any $\psi(x) \in tp(a_\omega/Ic)$ is realized in $I = \langle a_i | i < \omega \rangle$, $tp(a_\omega/Ic)$ can not divide over I. Moreover since $tp(a_\omega/I) = Lstp(a_\omega/I)$ (2.3.2), if T is stable then by 1.7, any non-dividing extension of $tp(a_\omega/I)$ must be a non weak dividing extension. Hence T is not stable.

(5)⇒(1). Again we prove the contrapositive. If T is unstable, then by 2.4, there are d_0 and b such that $p := tp(d_0/Mb)$ is a coheir of $q := tp(d_0/M)$ and p weakly divides over M. Hence for some ψ over M, $[q]^\psi$ is consistent realized by say $\bar{d} = d_0...d_n$, while $[p]^\psi$ is inconsistent (*). Now let $\langle a_i/i \in \omega \rangle$ be a coheir sequence of p. Then I is an indiscernible sequence over Mb. Let $J = \langle c_i/i \in \omega \rangle$ be an Mb-indiscernible sequence (**) such that $tp(c_0...c_i/Mb) = tp(a_i...a_0/Mb)$.

Claim) For each i, there is $\bar{d}^i = d^i_0...d^i_n (\models tp(\bar{d}/M))$ such that $tp(d^i_j/Mc_0...c_{i-1}) = tp(c_i/Mc_0...c_{i-1})$ for $j = 0, .., n$: Since $r(x) = tp(c_0...c_{i-1}/Mc_i)$ is a coheir of $r'(x) = tp(c_0...c_{i-1}/M)$, there is an extension r_1 of r over $Mc^i_0...c^i_n$, where $c^i_0...c^i_n \models tp(\bar{d}/M)$ and $c_i = c^i_0$, such that r_1 is a coheir of r'. By moving a realization of r_1 to $c_0...c_{i-1}$ over Mc_i, we can find the desired $\bar{d}^i$ which is an Mc_i-automorphic image of $c^i_0...c^i_n$.

Hence by the claim, $[tp(c_i/Mc_0...c_{i-1})]^\psi$ is realized by $\bar{d}^i$, while by (*)(**), $[tp(c_i/Mc_0...c_{i-1}b)]^\psi$ ($\supseteq [tp(c_i/Mb)]^\psi = [p]^\psi$) is inconsistent. Then by indiscernibility, we obtain a weak dividing left-chain.

(5)⇔(6). See 2.2. □

As mentioned in section 1, for stable T, weak dividing satisfies transitivity. One may then expect, as above, that weak dividing being transitive is equivalent to the theory being stable. But in the random graph, weak dividing also satisfies transitivity. First, we examine a simple theory in which the transitivity of weak dividing fails.

Example 2.6. Let $(V, \langle , \rangle)$ be a vector space V over a finite field equipped with an inner product giving orthogonality between two independent vectors. Let a, b, c be independent vectors in V such that a, b are orthogonal (*) while b, c and a, c are not (**). One can see that $tp(a/cb)$ does not weakly divide over $\emptyset$. But $tp(a/cb)$ weakly divides over c: Let a' be a vector in the plane generated by a,and c such that a, a' are independent and $tp(ac) = tp(a'c)$. Then there is no b' such that $tp(acb') = tp(a'cb')$, since otherwise c is a linear combination of a and a', and thus by (*), c must be orthogonal with b', which contradicts to (**).

Example 2.7. Let (M, R) be a universal model of the random graph. To show the transitivity of weak dividing, it suffices to show that for finite $\bar{a}$ and $A \subseteq B \subseteq C$, $tp(\bar{a}/C)$ does not weakly divide over B (*), if $tp(\bar{a}/C)$ does not weakly divide over A (**): Hence we suppose (**). To show (*), let us suppose that $\{\bar{a}_i | i \in I\}$ is given so that $\bar{a}_i \models tp(\bar{a}/B)$ (†). By (**), there is $C'(\supseteq A)$ such that $tp(\bar{a}C/A) = tp(\bar{a}_iC'/A)$ for $i \in I$ (‡). Then there is $(A \subseteq) B' \subseteq C'$ such that $tp(B'/A) = tp(B/A)$. Thus by 1.6, it suffices to see that $tp(B'/\{\bar{a}_i | i \in I\}A) = tp(B/\{\bar{a}_i | i \in I\}A)$. Now by (†) and (‡), for $i \in I$, $tp(\bar{a}_iB) = tp(\bar{a}_iB')$. Then as the two types are determined by the relation R between an element in $\{\bar{a}_i | i \in I\}$ and an element in B (or B'), the two types are the same.

3. Miscellaneous

Now let us study a notion which looks to stability as the tree property is to simplicity. Recall from [2] that $\varphi(x, y) \in L$ has the *n-tree property* $(2 \leq n < \omega)$ if there is a set of tuples

$\{a_{\bar{v}} | \bar{v} \in n^{<\omega}\}$ such that, for each $\bar{v} \in n^{<\omega}$, $\{\varphi(x, a_{\bar{v}\frown i}) | i < n\}$ is pairwise inconsistent, and for every $\bar{u} \in n^{\omega}$, $\{\varphi(x, a_{\bar{u}\restriction i}) | i < \omega\}$ is consistent.

Lemma 3.1. (1) *If $\varphi(x; y)$ has the 2-tree property, then for each n, some finite conjunction $\varphi(x; y_1) \wedge ... \wedge \varphi(x; y_k)$ has the n-tree property.*
(2) *If $\varphi(x; y)$ is unstable, then for each $n \geq 2$, a finite conjunction of $\varphi(x; y_1) \wedge \neg\varphi(x; y_2)$ has the n-tree property.*
(3) *If $\varphi(x; y)$ has the n-tree property, then $\varphi(x, y)$ is unstable.*

Proof. 1. Suppose that $\varphi(x; y)$ has the 2-tree property witnessed by $\{a_{\bar{v}} | \bar{v} \in 2^{<\omega}\}$. Now for example for $n = 3$, $\varphi(x; y_1) \wedge \varphi(x; y_2)$ has the 3-tree property witnessed by $\{a_{\bar{u}'} a_{\bar{u}} | \ \bar{u} \in \{00, 01, 10\}^{<\omega}, \bar{u}' = \bar{u}\lceil(|\bar{u}| - 1)\}$ ($a_{\bar{u}'}$ for y_1, $a_{\bar{u}}$ for y_2). Similarly one can see for each n, a finite conjunction of the formula has the n-tree property.

2. [2] Assume $\varphi(x; y)$ is unstable. Now by compactness, there are $\varphi(x, y)$ and a sequence $\langle b_s a_s | s \in \mathbb{Q}\rangle$ such that $\models \varphi(b_s, a_t)$ if and only if $s < t$. Now for given $n \geq 2$, choose $r_1 < r_2 < ... < r_{2n-1} \in \mathbb{Q}$ and consider the following n formulas.

$$\psi_m \equiv \neg\varphi(x, a_{r_m}) \wedge \bigwedge\nolimits_{1 \leq i \leq n-1} \varphi(x, a_{r_{m+i}}). \quad (m = 1, ..., n)$$

All ψ_m are obtained from a single formula $\psi \in L$ by adding different parameters. It is easy to check $\{\psi_m | 1 \leq m \leq n\}$ is pairwise inconsistent and each ψ_m is consistent. It consists of the first level of the n-tree. For the second level, let us look at ψ_1 for example. Select points $r_1 < r_{1,1} < r_{1,2} < ... < r_{1,2n-1} < r_2$ so that we can define

$$\psi_{1,m} \equiv \neg\varphi(x, a_{r_{1,m}}) \wedge \bigwedge\nolimits_{1 \leq i \leq n-1} \varphi(x, a_{r_{1,m+i}}). \quad (m = 1, ..., n)$$

Again each $\psi_{1,m}$ is obtained from ψ and $\{\psi_{1,m} | 1 \leq m \leq n\}$ is pairwise inconsistent. Moreover $\psi_1 \wedge \psi_{1,m}$ is consistent for every m. Since $\mathbb{Q}$ is dense, we can construct the n-tree of ψ.

3. This follows from the fact that there are continuum many $\varphi(x; y)$-types over some countable set. □

Note that a formula $\varphi(x; y)$ being unstable does not imply that some conjunction of the formula has the n-tree property, e.g. $\varphi(x; y) \equiv x_1 < y < x_2$ in $(\mathbb{Q}, <)$.

Corollary 3.2. *The following are equivalent.*
(1) *T is not stable.*
(2) *For each $n \geq 2$, there is an $\mathcal{L}$-formula having the n-tree property.*
(3) *For some $n \geq 2$, a formula satisfies the n-tree property.*

As we have seen there are analogies between the weak dividing left-chain/left local character of weak dividing and the weak dividing right-chain/(right) local character of dividing. It is natural to ask when right local character of weak dividing or left local character of dividing holds. We can make the following rather interesting observation. (3.3.2 was noticed in [6].)

Proposition 3.3. *For an arbitrary theory T, the following hold.*
(1) *There is no weak dividing right-chain.*
(2) *(Local character of weak dividing) For any finite tuple c and a set A, there is $A_0 (\subseteq A)$ of cardinality $\leq |T|$ such that $tp(c/A)$ dos not weakly divide over A_0.*
(3) *There is no dividing/forking left-chain.*

(4) *(Left local character of dividing/forking) For any finite tuple c and a set A, there exists $A_0(\subseteq A)$ of cardinality $\leq |T|$ such that $tp(A/A_0c)$ does not divide/fork over A_0.*

Proof. 1. Suppose that there is a weak dividing right-chain. Then by the similar reasons in the proof of 2.2, it is not difficult to see that there is a c-indiscernible sequence $I = \langle a_i |\ i \leq \omega \rangle$ such that $tp(c/I) := p$ weakly divides over $I' = \langle a_i |\ i < \omega \rangle$. But clearly since p is an heir of $p\lceil I'$, p can not weakly divides over I', a contradiction.

2. Again if a type does not satisfy local character of weak dividing, then a weak dividing chain is produced.

3. It is not difficult to see that a dividing/forking left-chain produces a c-indiscernible sequence $I = \langle a_i |\ i \leq \omega \rangle$ such that $tp(a_\omega/I'c)$ divides/forks over $I' = \langle a_i |\ i < \omega \rangle$, which is again impossible.

4. Similar. □

From above 3.3.4, one may attempt to extend notions of ranks in simplicity theory into the context of arbitrary first-order theories. For example, define LU-rank in such a way that for a complete type $p = tp(c/A)$, $LU(p) \geq \alpha + 1$ if there is b such that $tp(b/Ac)$ divides over A and $LU(tp(c/Ab)) \geq \alpha$. Then clearly $c \in acl(A)$ iff $LU(c/A) = 0$.

Lemma 3.4. *The following are equivalent.*

(1) *The LU-rank of any complete type is ordinal valued.*
(2) *For any finite tuple c and a set A, there is finite $A_0(\subseteq A)$ such that $tp(A/A_0c)$ does not divide over A_0.*
(3) *There do not exist a tuple c and a sequence of sets $A_i (i \in \omega)$ such that $A_i \subseteq A_j$ for $i \leq j$ and $tp(A_{i+1}/A_ic)$ divides over A_i.*

In spite of this lemma, these extensions of notions do not seem to blend well with existing theories. For instance, there is no reason to see that in general $LU(p) \geq LU(q)$ for q an extension of p. Also different from expectation, an o-minimal structure need not be LU-rank 1: (S. Starchenko) Let $M = (\mathbb{Q}, <, S)$ where $S(x) = x + 2$. Then $0 < LU(1/0) < LU(1)$.

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