HOMOLOGY GROUPS OF TYPES IN STABLE THEORIES AND THE HUREWICZ CORRESPONDENCE

JOHN GOODRICK, BYUNGHAN KIM, AND ALEXEI KOLESNIKOV

Abstract. We give an explicit description of the homology group $H_n(p)$ of a strong type p in any stable theory under the assumption that for every non-forking extension q of p the groups $H_i(q)$ are trivial for $2 \leq i < n$. The group $H_n(p)$ turns out to be isomorphic to the automorphism group of a certain part of the algebraic closure of n independent realizations of p; it follows from the authors' earlier work that such a group must be abelian. We call this the "Hurewicz correspondence" by analogy with the Hurewicz Theorem in algebraic topology.

The present paper is a part of the project to study type amalgamation properties in first-order theories by means of homology groups of types. Roughly speaking (more precise definitions are recalled below in Section 1), a strong type p is said to have *n-amalgamation* if commuting systems of elementary embeddings among algebraic closures of proper subsets of the set of n independent realizations of p can always be extended to the algebraic closure of all n realizations. The type p has *n-uniqueness* if this extension is essentially unique. Generalized amalgamation properties for systems of *models* were introduced by Shelah in [12] and played an important role in [13]. The type amalgamation properties were studied extensively by Hrushovski in [9] and applications were given. In fact, the type amalgamation properties have been used in model theory at least as far back as Hrushovski's classification of trivial totally categorical theories in [8].

In the previous paper [5], we introduced a notion of homology groups for a complete strong type in any stable, or even rosy, first-order theory. The idea was that these homology groups should measure information about the amalgamation properties of the type p. We proved that if p has n-amalgamation for all n, then $H_n(p) = 0$ for every n, and that the failure of 4-amalgamation (equivalently, the failure of 3-uniqueness)

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over dom(p) in a stable theory corresponds to non-triviality of the group $H_2(p)$. Furthermore, we established in that paper that $H_2(p)$ is isomorphic in stable theories to a certain automorphism group of closures of realizations of p. In [7], the theory is developed in a more general context of *amenable collection of functors*.

The current paper generalizes the main results of [5] in the stable context: if the type p does not have $(n+2)$-amalgamation, then for some i with $2 \leq i \leq n$ and some nonforking extension p' of p, the group $H_i(p')$ must be nonzero. Furthermore, at the first i for which such p' with $H_i(p') \neq 0$ exists, we show that $H_i(p')$ is isomorphic to a certain automorphism group $\Gamma_i(p')$, which immediately implies that $H_i(p')$ is a profinite group.

To structure the proof in a more transparent way, we state a technical lemma (Lemma 2.3) in Section 2 of this paper and prove the main result (Theorem 2.1) of the paper using the lemma. The proof of Lemma 2.3 uses certain algebraic structures (n-polygroupoids) that were linked to failure of $(n+2)$-amalgamation in the previous paper [6]; but neither the statement of the lemma, nor the proof of Theorem 2.1 from the lemma use these structures.

The proof of Lemma 2.3 is contained in Section 3 of the paper. It turns out that most of the results of [6] do not need global n-amalgamation assumptions, only the amalgamation properties for the type p and its non-forking extensions. The additional work to verify the results of [6] is contained in Section 4.

1. Notation and preliminaries

In this paper, we always work with a fixed complete stable theory $T = T^{eq}$ in a language $\mathcal{L}$, its saturated model $\mathcal{M} = \mathcal{M}^{\mathrm{eq}}$, and a complete type p over a small set $B = \mathrm{acl}(B)$. Throughout this paper, "independence" means nonforking independence. We use the usual notational conventions of stability theory, plus some further conventions from [5] and [6], some of which we will now recall.

Tuples of elements of the monster model or of variables will be denoted by lower-case letters (without a bar); the upper-case letters will denote sets. The base set B of the type p we have fixed is included in all algebraic closures. Algebraic closures will be denoted by a bar; for example, given a tuple c, the symbol $\bar{c}$ denotes $\mathrm{acl}(cB)$.

For sets A and C, the symbol $\mathrm{Aut}(A/C)$ denotes the group of elementary maps over C (i.e., fixing C pointwise) from $A \cup C$ onto $A \cup C$. For a type $q = \mathrm{tp}(a/C)$ with solution set A, $\mathrm{Aut}(q) := \mathrm{Aut}(A/C)$.

For the fixed type p, the symbol $p^{(k)}$ will denote the complete type of k independent realizations of the type p.

1.1. **Amalgamation properties and the homology groups.** We start by recalling from [4] and [5] the definition of a closed independent functor, which we then use to define both the type amalgamation properties and the homology groups of the type p.

A family of sets X ordered by inclusion can be endowed with a natural poset category structure: the objects are the elements of X and the morphisms are single inclusion maps $\iota_{u,v} : u \to v$ between any two sets $u, v \in X$ with $u \subseteq v$. The set X is called *downward-closed* if whenever $u \subseteq v \in X$, then $u \in X$.

We let $\mathcal{C}_B$ denote the category of all small algebraically closed subsets of $\mathcal{M}$ containing B, where morphisms are elementary maps over B (i.e., fixing B pointwise). For a downward closed X and a functor $f : X \to \mathcal{C}_B$ and $u \subseteq v \in X$, we write $f_v^u := f(\iota_{u,v})$ and $f_v^u(u) := f_v^u(f(u)) \subseteq f(v)$.

The simplicial structure on the sets of algebraic closures of independent sequences realizing a type is contained in Definitions 1.5 and 1.6 below. We begin by recalling an auxiliary definition.

Definition 1.1. A *(closed independent) p-functor* is a functor $f : X \to \mathcal{C}_B$ such that:

(1) The set $s := \bigcup X$ is a finite subset of ω, and X is a downward-closed subset of $\mathcal{P}(s)$;

(2) $f(\emptyset) \supseteq B$; and for $i \in s$, $f(\{i\})$ is of the form $\operatorname{acl}(Cb)$ where $b(\models p)$ is independent with $C = f_{\{i\}}^{\emptyset}(\emptyset)$ over B.

(3) For all non-empty $u \in X$, we have $f(u) = \operatorname{acl}(B \cup \bigcup_{i\in u} f_u^{\{i\}}(\{i\}))$ and the set $\{f_u^{\{i\}}(\{i\}) : i \in u\}$ is independent over $f_u^{\emptyset}(\emptyset)$.

If $f(\emptyset) = B$ (so for any $u \in X$, $f_u^{\emptyset}(\emptyset) = B$) then we say f is *over* B.

Next we define the amalgamation properties of the type p. As usual, $n = \{0, \ldots, n-1\}$.

Definition 1.2. Let $n \geq 1$.

(1) We say p has *n-amalgamation* (or *n-existence*) if for any p-functor $f : \mathcal{P}^-(n)(:= \mathcal{P}(n) \setminus \{n\}) \to \mathcal{C}_B$, there is an $(n-1)$-simplex g in p such that $g \supseteq f$. When f above ranges only over p-functors over B, we say p has *n-amalgamation over B.*

If p has k-amalgamation for every k with $1 \leq k \leq n$, then we say p has *n-complete amalgamation* (or *n-CA* for short).

(2) We say that $p \in S(B)$ has *n-uniqueness* if for any closed independent p-functor $f : \mathcal{P}^-(n) \to \mathcal{C}_B$ and any two $(n-1)$-simplices g_1 and

g_2 in p extending f, there is a natural isomorphism $F : g_1 \to g_2$ such that $F \restriction \mathrm{dom}(f)$ is the identity. Similarly we say p has *n-uniqueness over B* when f above ranges over p-functors over B.

If p has k-uniqueness for every k with $2 \leq k \leq n$, then we say p has *$(\leq n)$-uniqueness.*

At this point we should explain why it is necessary to consider closed independent p-functors f in which the maps f_t^s are anything but identity maps. Considered individually, all essential properties of a p-functor are preserved under natural isomorphisms, and (as pointed out in [4]) any p-functor $f : \mathcal{P}(n) \to \mathcal{C}_B$ is naturally isomorphic to a p-functor g in which every map g_t^s is the identity; call such a functor *untwisted*. However, the fact that *every* p-functor $f : \mathcal{P}^-(n) \to \mathcal{C}_B$ is naturally isomorphic to an untwisted functor is equivalent to the demand that p has n-amalgamation (and the latter is not always the case for $n \geq 4$). In particular, when we define the homology groups $H_n(p)$ below, we cannot automatically assume that all of the component maps in the p-functors involved are the identity. For similar reasons, it is important that the transition maps in Notation 2.2 are not identity maps.

The reader may assume that if f is a closed independent p-functor of the form $f : \mathcal{P}(s) \to \mathcal{C}_B$, where $|s| = k+1$, then there are $a_0, \ldots, a_k \models p^{(k+1)}$ such that $f(s) = \overline{\{a_i \mid i \in s\}}$ and the transition maps f_s^t are automorphisms of $\overline{\{a_i \mid i \in t\}}$. For the amalgamation properties and the definitions of the homology groups we will give below, assuming that all the constituent functors have this form does not cause any loss of generality.

It follows directly from the definitions above that the properties of n-uniqueness and n-existence of p are preserved under nonforking extensions. Note that 1-amalgamation and 2-amalgamation hold for any p by the existence of nonforking extensions, while 1-uniqueness and 2-uniqueness always hold by stationarity.

In general, the following holds:

Fact 1.3. [4] *Let $n \geq 1$. Then p has $(\leq n)$-uniqueness if and only if it has $(n+1)$-CA.*

The following fact is proved in [10]:

Fact 1.4. *For any $k, n \geq 1$, p has n-uniqueness if and only if $p^{(k)}$ has n-uniqueness.*

Finally, we come to the definition of the homology groups.

Definition 1.5. Let $n \geq 0$ be a natural number. An *n-simplex in p* is a p-functor $f : \mathcal{P}(s) \to \mathcal{C}_B$ for some set $s \subseteq \omega$ with $|s| = n + 1$. The set s is called the *support of f*, or $\mathrm{supp}(f)$.

Let $\mathcal{S}_n(p)$ denote the collection of all n-simplices *over B* in p; and let $\mathcal{C}_n(p)$ denote the free abelian group generated by $\mathcal{S}_n(p)$; its elements are called *n-chains in p*. Similarly, we define $\mathcal{S}(p) := \bigcup_n \mathcal{S}_n(p)$, and $\mathcal{C}(p) := \bigcup_n \mathcal{C}_n(p)$. The *support of a chain c* is the union of the supports of all the simplices that appear in c with a nonzero coefficient.

The boundary maps are defined to be the usual simplicial boundary embeddings.

Definition 1.6. If $n \geq 1$ and $0 \leq i \leq n$, then the *i-th boundary operator* $\partial^i_n : \mathcal{C}_n(p) \to \mathcal{C}_{n-1}(p)$ is defined so that if f is an n-simplex in p with domain $\mathcal{P}(s)$, where $s = \{s_0, \ldots, s_n\}$ with $s_0 < \ldots < s_n$, then

$$\partial^i_n(f) = f \restriction \mathcal{P}(s \setminus \{s_i\}),$$

and we extend ∂^i_n linearly to a group map on all of $\mathcal{C}_n(p)$.

If $n \geq 1$ and $0 \leq i \leq n$, then the *boundary map* $\partial_n : \mathcal{C}_n(p) \to \mathcal{C}_{n-1}(p)$ is defined by the rule

$$\partial_n(c) = \sum_{0 \leq i \leq n} (-1)^i \partial^i_n(c).$$

We write ∂^i and ∂ for ∂^i_n and ∂_n, respectively, if n is clear from context.

The kernel of ∂_n is denoted $\mathcal{Z}_n(p)$, and its elements are called *n-cycles*. The image of ∂_{n+1} in $\mathcal{C}_n(p)$ is denoted $\mathcal{B}_n(p)$. The elements of $\mathcal{B}_n(p)$ are called *n-boundaries*.

It can be shown (by the usual combinatorial argument) that $\mathcal{B}_n(p) \subseteq \mathcal{Z}_n(p)$, or more briefly, "$\partial_n \circ \partial_{n+1} = 0$." Therefore we can define simplicial homology groups in the type p:

Definition 1.7. The *nth (simplicial) homology group of the type $p \in S(B)$* is

$$H_n(p) := \mathcal{Z}_n(p)/\mathcal{B}_n(p).$$

We now introduce a particular class of n-cycles representing all the members of $H_n(p)$ under the assumption of $(\leq n)$-uniqueness.

Definition 1.8. If $n \geq 1$, an *n-pocket* is an n-chain c of the form $\pm(f - g)$, where f and g are n-simplices that have the same boundary: $\partial f = \partial g$.

Notice that any n-pocket is an n-cycle.

Theorem 1.9. [5] *If p has $(\leq n)$-uniqueness for some $n \geq 1$, then*

$$H_n(p) = \{[c] : c \text{ is an } n\text{-pocket in } p \text{ with support } n+1\}.$$

If the functors $f, g \in \mathcal{S}_n(p)$ are naturally isomorphic and $f - g$ is an n-pocket, then $f - g$ is a boundary.

In any simple theory with elimination of hyperimaginaries, $H_1(p) = 0$.

1.2. **The group $\Gamma_n(p)$.** To the type p we will associate a certain automorphism group $\Gamma_n(p)$, which we will now define. Under the hypothesis of $(n+1)$-CA, $\Gamma_n(p)$ will turn out to be isomorphic to $H_n(p)$.

If n is a natural number, the symbol $[n]$ will denote the set $\{1, \ldots, n\}$. For a set X, the symbols $X^{(n)}$ and $X^{(<n)}$ denote, respectively, the set of n-element subsets of X and the set of subsets of X of size less than n.

If $c_1, \ldots, c_n$ realize $p^{(n)}$, then we let

$$\partial_A(c_1 ... c_n) := \operatorname{dcl}(\bigcup_{i=1}^{n} \overline{Ac_1 ... \hat{c}_i ... c_n}),$$

and we omit A when $A = \emptyset$.

A different part of the algebraic closure of a set of n independent realizations of the type p will be denoted by a tilde: if $c_1, ..., c_{n+1}$ realize $p^{(n+1)}$, we let

$$\widetilde{c_1 ... c_n} := \overline{c_1 ... c_n} \cap \partial_{c_{n+1}}(c_1 ... c_n).$$

Stationarity guarantees that the set $\widetilde{c_1 ... c_n}$ does not depend on the choice of the element c_{n+1}. If f is an $(n-1)$-simplex in p with the support $\{s_1, \ldots, s_n\}$ and $c_i = f_s^{\{s_i\}}(\{s_i\})$, then $\widetilde{f}$ will denote the set $\widetilde{c_1 ... c_n}$.

Finally, define

$$\Gamma_n(p) := \operatorname{Aut}(\widetilde{c_1 ... c_n} / \partial(c_1 ... c_n)),$$

where $c_1, ..., c_n \models p^{(n)}$. Since p is stationary, it is routine to check that this definition does not depend on the choice of the realizations c_i.

This group is closely linked to the amalgamation properties for p:

Fact 1.10. [4] *Let $n \geq 1$. Then if p has $(\leq n)$-uniqueness, all of the following are equivalent:*

(1) p has $(n+1)$-uniqueness over B;
(2) p has $(n+2)$-amalgamation over B;
(3) $\Gamma_n(p) = 0$.

We warn the reader that the boundary symbol ∂ will be used in three different contexts:

(1) to denote the boundary operation on n-simplices and n-chains, as described in Definition 1.6: if f is an n-simplex, then $\partial(f)$ is an $(n-1)$-chain;
(2) to denote a certain part of the algebraic closure of a set of n independent realizations of the type p, as in the definition of $\Gamma_n(p)$ just above; and
(3) to denote a part of the algebraic closure of the set $f(s)$ for a simplex f with the support s. Namely, if $f \in \mathcal{S}_{n-1}(p)$ for some $n \geq 1$ (and so $\operatorname{supp}(f) = \{s_1, \ldots, s_n\}$ for some n-element set $s \subset \omega$), then $\partial[f] := \partial(c_1 \ldots c_n)$ where $c_i := f_s^{\{s_i\}}(\{s_i\})$. We will use the square brackets $\partial[f]$ to separate this context from (1).

With the notation as in (3) above, note that $\Gamma_n(p) = \operatorname{Aut}(\widetilde{f}/\partial[f])$ for some (any) simplex $f \in \mathcal{S}_{n-1}(p)$.

2. Main result

The main result of the paper is the following.

Theorem 2.1. *Let $n \geq 1$. Let $T = T^{eq}$ be a stable theory and let p be a strong type in T. Assume that p has k-amalgamation for each $k \in \{2, \ldots, n+1\}$. Then $H_n(p) \cong \Gamma_n(p)$; the latter group is always a profinite abelian group.*

Note that Theorem 1.9 above and Fact 1.10 imply that Theorem 2.1 trivially holds for $n = 1$. In [5], the theorem for $n = 2$ was shown, and the above generalization was conjectured. We call this the Hurewicz correspondence since the result connects the homology groups of p to $\Gamma_n(p)$, which are similar to homotopy groups, as in algebraic topology (see [1]). In particular, it is shown in [4] that $\Gamma_2(p)$ is the profinite limit of the vertex groups of relatively definable groupoids obtained from the failure of 4-amalgamation of the type. Since the definable vertex groups consist of something like "homotopy equivalence of paths," as described in [3], the limit $\Gamma_2(p)$ of definable vertex groups is analogous to the fundamental groupoid π_1 of the type, as in singular homology theory (see, for example, [2]). Note that there is a mismatch in the numbering: our group Γ_n corresponds to the group π_{n-1} in algebraic topology.

To define a map from $H_n(p)$ to $\Gamma_n(p)$ and show that it is an isomorphism, we will use Lemma 2.3. The lemma shows that we can select, for each $(n-1)$-simplex f, a complete $*$-type Πf over the set $\partial[f]$ and a realization $\alpha(f)$ of Πf; the tuples $\alpha(f)$ control $\widetilde{f}$ in the sense that

$\operatorname{dcl}(\alpha(f)) = \widetilde{f}$; that the group $\operatorname{Aut}(\widetilde{f}/\partial[f])$ acts regularly and transitively on the set of realizations of Πf. These automorphism groups for different simplices turn out to be canonically isomorphic to a certain group G constructed in the proof of the lemma; this group turns out to be abelian.

The selector function α and the property in item (4) will allow us to assign an element of the group G to every n-simplex (and extend the assignment to chains by linearity). This assignment does depend on a particular choice of the function α. However, the key properties shown in items (5) and (6) allow us to establish that for any choice of the function α, the group element assigned to the boundary of an $(n+1)$-simplex has to be 0. These properties also allow us to characterize when the union of automorphisms of the $(n-1)$-faces of an n-simplex can be lifted to an automorphism of the entire n-simplex. The statement of the lemma does not use the language of polygroupoids, but these objects play a central role in its proof. We defer the proof of the lemma to Section 3; but show how Theorem 2.1 follows from it.

Before stating the lemma, let us make a number of notation agreements.

Notation 2.2. If $g \in \mathcal{S}_n(p)$ has the support $t = \{t_0, \dots, t_n\}$ listed in increasing order and $i \in n+1$, then $\Pi_i g$ will denote the image of the type $\Pi(\partial^i g)$ under the transition map $g_t^{t\setminus\{t_i\}}$. Thus if $c_i := g_t^{\{t_i\}}(\{t_i\})$ (so that $g(t) = \overline{c_0 \dots c_n}$), then $\Pi_i g$ is a complete $*$-type over $\partial(c_0, \dots, \widehat{c_i}, \dots, c_n)$.

The symbol $\alpha_i(g)$ will denote the image of the tuple $\alpha(\partial^i g)$ under the same transition map $g_t^{t\setminus\{t_i\}}$. Hence $\alpha_i(g)$ realizes the type $\Pi_i g$.

If $h \in \mathcal{S}_{n+1}(p)$, with the support $s = \{s_0, \dots, s_{n+1}\}$ (listed in increasing order) and $0 \le i < j \le n+1$, then $\Pi_{i,j} h$ will denote the image of the type $\Pi(\partial^i(\partial^j h))$ under the transition map $h_s^{s\setminus\{s_i,s_j\}}$. Similarly, $\alpha_{i,j}(h)$ will denote the image of the tuple $\alpha(\partial^i(\partial^j h))$ under $h_s^{s\setminus\{s_i,s_j\}}$.

For the remainder of this section, we will use the letters f, g, and h to denote simplices in $\mathcal{S}_{n-1}(p)$, $\mathcal{S}_n(p)$, and $\mathcal{S}_{n+1}(p)$, respectively. Greek letters will be used to denote the elements of the group G.

Lemma 2.3. *Suppose $T = T^{\mathrm{eq}}$ is a stable theory, B is an algebraically closed subset in the monster model $\mathcal{M}$ of T, and a complete type $p \in S(B)$ has $\le n$-uniqueness but fails $(n+1)$-uniqueness. There is a selector function (Π, α) on $\mathcal{S}_{n-1}(p)$ that produces, for every $(n-1)$-simplex $f \in \mathcal{S}_{n-1}(p)$ a $*$-type Πf over $\partial[f]$ and a tuple $\alpha(f) \models \Pi f$, as well as a complete $*$-type $q(x_0, \dots, x_n)$ over B and a profinite abelian group G such that:*

(1) for every $f \in \mathcal{S}_{n-1}(p)$, $\alpha(f)$ is a (possibly infinite) tuple of elements in $\widetilde{f}$;
(2) $\widetilde{f} = \mathrm{dcl}(w)$ for every $w \models \Pi f$;
(3) for each $f \in \mathcal{S}_{n-1}(p)$, there is a regular transitive action of G on the set of realizations of Πf. The action is $*$-type definable in the following sense: for every $\gamma \in G$, there is a unique $*$-type $r_\gamma(x, y)$ over B such that $\models r_\gamma(w, w')$ if and only if $w' = \gamma.w$;
(4) if $g \in \mathcal{S}_n(p)$, then for any realizations w_i of $\Pi_i g$ for $i = 0, \ldots, n$, there is a unique element $\gamma \in G$ such that $\models q(w_0, \ldots, w_{n-1}, \gamma.w_n)$;
(5) if $\models q(w_0, \ldots, w_n)$ and $\gamma_0, \ldots, \gamma_n$ are elements of G, then $\models q(\gamma_0.w_0, \ldots, \gamma_n.w_n)$ if and only if $\sum_{i=0}^{n}(-1)^i\gamma_i = 0$;
(6) if $h \in \mathcal{S}_{n+1}(p)$ and $\{w_{i,j} \models \Pi_{i,j}h \mid 0 \leq i < j \leq n+1\}$ are such that

$$\models q(w_{0,k}, \ldots, w_{k-1,k}, w_{k,k+1}, \ldots, w_{k,n+1}) \text{ for } 0 \leq k \leq n,$$

then $\models q(w_{0,n+1}, \ldots, w_{n,n+1})$.

Corollary 2.4. *The group G is isomorphic to the group Γ_n.*

Proof. By (2) and (3) in Lemma 2.3, the action by $g \in G$ on $\Pi(f)$ induces an automorphism in Γ_n. Since the action is regular and transitive, this correspondence is an isomorphism. □

Thus, we need to establish an isomorphism between $H_n(p)$ and G, and item (4) in Lemma 2.3 offers a way to do this.

Notation 2.5. For $g \in \mathcal{S}_n(p)$, let $\varepsilon(g) \in G$ be the unique element such that $\models q(\alpha_1(g), \ldots, \alpha_n(g), \varepsilon(g).\alpha_{n+1}(g))$. Thus we have a well-defined function $\varepsilon : \mathcal{S}_n(p) \to G$. Extending ε to $\mathcal{C}_n(p)$ by linearity, we get a function $\varepsilon : \mathcal{C}_n(p) \to G$.

It is clear that ε is a homomorphism from $\mathcal{C}_n(p)$ to G. In the three lemmas below, we establish that

- If $d \in \mathcal{B}_n(p)$, then $\varepsilon(d) = 0$. This will show that the function $[c] \in H_n(p) \mapsto \varepsilon(c) \in G$ is well-defined; we will use the symbol $\overline{\varepsilon}$ for this function.
- The homomorphism $\overline{\varepsilon}$ is injective, that is, if $d \in \mathcal{Z}_n(p)$ and $\varepsilon(d) = 0$, then $d \in \mathcal{B}_n(p)$.
- The homomorphism $\overline{\varepsilon}$ is surjective: for every $\gamma \in G$, there is $d \in \mathcal{Z}_n(p)$ such that $\varepsilon(d) = \gamma$.

Lemma 2.6. *If $d \in \mathcal{B}_n(p)$, then $\varepsilon(d) = 0$.*

Proof. Since ε is a linear function, it suffices to establish the claim of the lemma for the case when $d = \partial h$, where $h \in \mathcal{S}_{n+1}(p)$. In this case, $d = \sum_{j \le n+1} (-1)^j g_j$ where $g_j := \partial^j h$; and we need to show that $\sum_{j \le n+1} (-1)^j \varepsilon(g_j) = 0$.

Taking the elements $\alpha_{i,j}(h)$ (these are the elements described just before the statement of Lemma 2.3), we have:

$$\models q(\alpha_{0,k}(h), \ldots, \alpha_{k-1,k}(h), \alpha_{k,k+1}(h), \ldots, \varepsilon(g_k).\alpha_{k,n+1}(h)) \text{ for } 0 \le k \le n$$

by definition of the function ε. Lemma 2.3(6) now gives

$$\models q(\varepsilon(g_0).\alpha_{0,n+1}(h), \ldots, \varepsilon(g_n).\alpha_{n,n+1}(h))$$

On the other hand $\models q(\alpha_{0,n+1}(h), \ldots, \varepsilon(g_{n+1}).\alpha_{n,n+1}(h))$ and thus

$$\sum_{i=0}^{n} (-1)^i \varepsilon(g_i) = (-1)^n \varepsilon(g_{n+1}).$$

The latter equality gives the needed $\sum_{i \le n+1} (-1)^i \varepsilon(g_i) = 0$. □

Before we prove the injectivity of $\overline{\varepsilon}$, we need to establish the following proposition.

Proposition 2.7. *Suppose that $g, g' \in \mathcal{S}_n(p)$ are such that $\partial^i g = \partial^i g'$ for all $i = 0, \ldots, n$. The functors g and g' are naturally isomorphic if and only if $\varepsilon(g) = \varepsilon(g')$.*

Proof. If g and g' are naturally isomorphic, there is an elementary map $\eta : g(n) \to g'(n)$ such that $\eta(\alpha_i(g)) = \alpha_i(g')$ for all $i = 0, \ldots, n$. Since the map η is elementary, for all $\gamma \in G$

$$\models q(\alpha_0(g), \ldots, \alpha_{n-1}(g), \gamma.\alpha_n(g)) \\ \text{if and only if } \models q(\alpha_0(g'), \ldots, \alpha_{n-1}(g'), \gamma.\alpha_n(g')).$$

It follows that $\varepsilon(g) = \varepsilon(g')$.

To establish the other direction, suppose that for some $\gamma \in G$ we have

$$\models q(\alpha_0(g), \ldots, \alpha_{n-1}(g), \gamma.\alpha_n(g)) \text{ and } \models q(\alpha_0(g'), \ldots, \alpha_{n-1}(g'), \gamma.\alpha_n(g')).$$

For every proper subset $s \subsetneq n$, the sets $g(s)$ and $g'(s)$ are the same, so we can choose the component maps η_s, $s \in \mathcal{P}^-(n)$, to be the identity embeddings. It remains to construct the elementary embedding $\eta_n : g(n) \to g'(n)$ that commutes with the transition maps of g and g'. The latter requirement means that we are already given the elementary maps $h_i : \partial^i(g(n)) \to \partial^i(g'(n))$ for $i = 0, \ldots, n$. Namely,

$$h_i = (g')_n^{n \setminus \{i\}} \circ [g_n^{n \setminus \{i\}}]^{-1}.$$

The key point is that the union $\bigcup_{i \le n} h_i$ is an elementary embedding if $\varepsilon(g) = \varepsilon(g')$. To see this, first note that $h_i(\alpha_i(g)) = \alpha_i(g')$ for each

$i \leq n$, and in particular $h_n(\gamma.\alpha_n(g)) = \gamma.\alpha_n(g')$ since the action is $*$-type definable. Next, since the type q is complete, we have

$$\alpha_0(g), \ldots, \alpha_{n-1}(g), \gamma.\alpha_n(g)) \equiv \alpha_0(g'), \ldots, \alpha_{n-1}(g'), \gamma.\alpha_n(g')$$

and so the restriction of $\bigcup_{i \leq n} h_i$ to $\alpha_0(g), \ldots, \alpha_{n-1}(g), \gamma.\alpha_n(g)$ is an elementary embedding. And finally, since $\partial[g] = \mathrm{dcl}(\alpha_0(g), \ldots, \gamma.\alpha_n(g))$ and $\partial[g'] = \mathrm{dcl}(\alpha_0(g'), \ldots, \gamma.\alpha_n(g'))$, it follows that $\bigcup_{i \leq n} h_i : \partial[g] \to \partial[g']$ is elementary. So we can take η_n to be any elementary map extending $\bigcup_{i \leq n} h_i$. $\square$

Lemma 2.8. *The homomorphism $\overline{\varepsilon} : H_n(p) \to G$ is injective.*

Proof. By Fact 1.9, an element of $H_n(p)$ has the form $[g - g']$, where $\partial g = \partial g'$. Thus, it is enough to show that if $\partial g = \partial g'$ and $\varepsilon(g - g') = 0$, then $g - g'$ is a boundary. To show the latter, by Fact 1.9, it suffices to construct a natural isomorphism between the functors g and g'. But this is exactly what Proposition 2.7 does. $\square$

Lemma 2.9. *For every $\gamma \in G$, there is $d \in \mathcal{Z}_n(p)$ such that $\varepsilon(d) = \gamma$.*

Proof. Fix $c_0, \ldots, c_n \models p^{(n+1)}$ and let g be the n-simplex with constant transition maps such that $g(s) = \overline{c_s}$ for all $s \subset \{1, \ldots, n+1\}$. Let $\mu := \varepsilon(g)$. By the definition of ε, we have $\models q(\alpha_0(g), \ldots, \alpha_{n-1}(g), \mu.\alpha_n(g))$.

We now define another n-simplex g' that has the same boundary as the simplex g (which means that $g' - g$ is a member of $\mathcal{Z}_n(p)$) and such that $\varepsilon(g' - g) = \gamma$. Let $w := (-\gamma).\alpha_n(g)$.

By Lemma 2.3(3), there is $\sigma \in \mathrm{Aut}(\mathcal{M}/\partial(c_1 \ldots, c_n))$ taking $\alpha_n(g)$ to $(-\gamma).\alpha_n(g)$. For each $s \subset n$, we let $g'(s) := g(s)$; and all but one transition maps $(g')^s_t$ are the identity embeddings. The one exception is the map $(g')^{n-1}_n$: we define $(g')^{n-1}_n := \sigma \restriction g'(n-1)$. Since σ fixes $\partial(c_0 \ldots c_{n-1})$, it is easy to check that this transition map is compatible with the remaining transition maps.

We have $\alpha_i(g) = \alpha_i(g')$ for $i = 0, \ldots, n-1$ and $\alpha_n(g') = (-\gamma).g_{n+1}$. Then $\models q(\alpha_0(g'), \ldots, \alpha_{n-1}(g'), (\mu+\gamma).\alpha_n(g'))$, so $\varepsilon(g') = \mu+\gamma$. Finally, $\varepsilon(g' - g) = \gamma$. $\square$

As we mentioned above, the main result of the paper now follows easily from the three lemmas.

Proof of Theorem 2.1. By Lemma 2.6, the function $\overline{\varepsilon} : [c] \in H_n(p) \mapsto \varepsilon(c) \in G$ is a well-defined homomorphism. By Lemmas 2.8 and 2.9, the homomorphism $\overline{\varepsilon}$ is an isomorphism. $\square$

3. Proof of Lemma 2.3

Throughout this section, we will make the following assumptions: $T = T^{eq}$ is stable, $B = \operatorname{acl}(B)$ is a small subset of the monster model $\mathcal{M}$ (and without loss of generality, $B = \emptyset$), and the type $p \in S(B)$ has $(\leq n)$-uniqueness but fails $(n+1)$-uniqueness. In addition, we fix at the outset a tuple $c_1, \ldots, c_{n+1}$ realizing $p^{(n+1)}$. Ultimately the choice of the c_i's will not matter, but this will be convenient for some of our definitions.

3.1. Existence of symmetric systems.

For the proof of Lemma 2.3, it will be useful to have witnesses to the failure of $(n+1)$-uniqueness in p which are invariant under a certain family of automorphisms which we will now describe.

The next definition comes from [6].

Definition 3.1. An $(n-1)$-*symmetric system for* $\langle c_1, \ldots, c_{n+1}\rangle$ is a collection of (generally infinite) tuples $\{\bar{c}_s : s \in [n+1]^{(<n)}\}$ such that:

(1) $\bar{c}_s$ is a tuple enumerating $\operatorname{acl}(B\{c_i : i \in s\})$; and
(2) for any permutation σ of $[n+1]$, there is an automorphism $[\sigma] \in \operatorname{Aut}(\mathcal{M}/B)$ such that

$$[\sigma](\bar{c}_s) = \bar{c}_{\sigma(s)}$$

for every $s \in [n+1]^{(<n)}$.

Remark 3.2. Note that if $\{\bar{c}_s : s \in [n+1]^{(<n)}\}$ is an $(n-1)$-symmetric system as above, then it follows immediately from clause (2) that for any $s \in [n+1]^{(<n)}$ and any two permutations σ and τ of $[n+1]$,

$$[\sigma] \circ [\tau](\bar{c}_s) = [\sigma \circ \tau](\bar{c}_s).$$

Under the assumptions of this section, it is always possible to find an enumeration of the algebraic closures that forms a symmetric system. The following lemma is a strengthening of Lemma 1.20 of [6].

Lemma 3.3. *There is an* $(n-1)$*-symmetric system for* $\langle c_1, \ldots, c_{n+1}\rangle$.

Proof. This is similar to Lemma 1.20 of [6], which proved the existence of $(n-1)$-symmetric systems assuming that the theory T has $(\leq n)$-uniqueness, but we need a "local" version which uses only the hypothesis that p has $(\leq n)$-uniqueness. However, the proof of Lemma 1.20 of [6] applies verbatim once we have Lemma 1.18 of the cited paper. Lemma 1.18 in turn follows quickly from the property of relative $(n, n+1)$-uniqueness localized to p established in Lemma 4.1. The proof of the latter lemma is in Section 4 of this paper. □

We fix maps $\{[\sigma] : \sigma \in S_{n+1}\}$ witnessing an $(n-1)$-symmetric system for $\langle c_1, \dots, c_{n+1}\rangle$, as in Lemma 3.3.

3.2. **n-ary quasigroupoids and polygroupoids.** The proof of statements (4), (5), and (6) in Lemma 2.3 makes heavy use of some algebraic structures first defined in [6]: n-ary polygroupoids. These are n-sorted structures with sorts $P_1, \dots, P_n$; for the application here, the sort P_1 will be the set of realizations of a type p' of a finite tuple inside the algebraic closure of $c \models p$. The elements of sorts P_i will be contained in the algebraic closure of sets of i independent realizations of the type p'. There are projection maps between the sorts, and an $(n+1)$-ary relation on the elements of the "top" sort P_n – these will be ultimately used to form the type q in Lemma 2.3. The property of associativity of an n-ary polygroupoid (the difference between a quasi- and a polygroupoid) is essentially the property (6) in Lemma 2.3. To make the presentation self-contained, we include the definitions of these structures (quasigroupoid, polygroupoid) and their properties (associativity, local finiteness) in Section 4.

Definition 3.4. Let $p'(x) \in S(B)$ be a type of a finite tuple contained in the algebraic closure of a realization of p. We call a locally finite quasigroupoid $(P_1, P_2, \dots, P_n; Q; \pi^2, \dots, \pi^n)$ an *n-ary generic groupoid in p'* if all its sorts are B-type-definable and such that

(1) Q and π^i $(i = 2, \dots, n)$ are B-definable;
(2) P_1 is the solution set of $p'(x)$;
(3) for $b_1, ..., b_i \models p'$, $P_i(b_1, ..., b_i) \neq \emptyset$ if and only if $\{b_1, ..., b_i\}$ is B-independent; $P_i(b_1, ..., b_i) \subset \overline{b_1, ...b_i}$.
(4) for any any $i \in \{2, \dots, n\}$, any B-independent $b_1, ..., b_i$, and any $w \in P_i(b_1, ..., b_i)$ the type $\text{tp}(w/\pi^i(w))$ isolates $\text{tp}(w/\partial(b_1, ..., b_i))$, and if $w' \equiv_{\pi^i(w)} w$ then $w' \in P_i$;
(5) for any any $i \in \{2, \dots, n\}$, any B-independent $b_1, ..., b_i$, and any $w, w' \in P_i(b_1, ..., b_i)$ we have $\pi^i(w) = \pi^i(w')$ if and only if $w \equiv_{\overline{b_1, ...b_i}} w'$; and
(6) if $Q(v_1, \dots, v_{n+1})$ holds with $v_j \in P_n(c_1, \dots \widehat{c_j}, \dots, c_{n+1})$ then $\langle c_1, \dots, c_{n+1}\rangle$ is Morley in q, and
(7) associativity of Q holds on any Morley sequence of $(n+2)$-tuples in q (see Definition 4.6 below).

Note that in the definition above, since each π^i is B-definable, for $w \in P_i(\pi(f))$ and any B-automorphism σ permuting I, it follows $\sigma(\pi^i(w)) = \pi^i(\sigma(w))$ and $\sigma(w) \in P_i(\sigma(\pi(w)))$. Moreover since p has $(\leq n)$-uniqueness, then for any $u, v \in P_i$ $(i = 2, \dots, n)$, $u\pi^i(u) \equiv_B v\pi^i(v)$.

Notice that if we restrict a generic groupoid to the fibers over some infinite Morley sequence I_0, then the result will be a connected relatively I-definable polygroupoid that were studied in [6]. Conversely, any relatively I-definable polygroupoid in [6] can be uniquely expanded to a generic n-ary groupoid. This allows us to use the facts from [6]; they are collected in the following two propositions.

The first proposition establishes the existence of generic n-ary groupoid that are invariant, in the sense of the two definitions below, under the $(n-1)$-symmetric system of automorphisms constructed in Section 3.1.

Definition 3.5. We say a Morley sequence $\langle a_1, \ldots, a_{n+1}\rangle$ with $a_i \in \overline{c_i}$, is S_{n+1}-*compatible* if

$$[\sigma](a_1, \ldots, a_{n+1}) = (a_{\sigma(1)}, \ldots, a_{\sigma(n+1)})$$

for each $\sigma \in S_{n+1}$.

We say a Morley sequence $\langle a_1, \ldots, a_i\rangle$ with $a_i \in \overline{c_i}$ $(1 \leq i \leq n)$, is S_{n+1}-*compatible* if there are $a_{i+1}...a_{n+1}$ so that $\langle a_1, \ldots, a_{n+1}\rangle$ is S_{n+1}-compatible. Obviously if such a compatible $\langle a_1, \ldots, a_i\rangle$ is given, then $a_{i+1}...a_{n+1}$ is uniquely determined.

Let σ_j $(j \in [n])$ be the permutation of $[n+1]$ sending $(1, \ldots, n; n+1)$ to $(1, \ldots, \hat{j}, \ldots n+1; j)$. Note that the permutations $\sigma_1, \ldots, \sigma_n$ generate the symmetric group on the set $[n+1]$.

Definition 3.6. Given an S_{n+1}-compatible Morley sequence $\langle a_1, \ldots, a_{n+1}\rangle$, suppose that $(P_1, P_2, \ldots, P_n, Q, \pi^2, \ldots, \pi^n)$ is a generic n-ary groupoid in $\mathrm{tp}(a_1)$. Then we say $w \in P_n(a_1, \ldots, a_n)$ is *compatible with the* $[\sigma_j]$*'s* if $([\sigma_1](w), \ldots [\sigma_n](w), w)$ is a compatible $(n+1)$-tuple from P_n.

The following Proposition is adapted from a similar result from [6].

Proposition 3.7. *Let* $c_1, \ldots, c_n \models p^{(n)}$*, where* p *has* $(\leq n)$*-uniqueness, but not* $(n+1)$*-uniqueness.*

1. *The set* $\widetilde{c_1...c_n}$ *is not contained in* $\partial(c_1 \ldots c_n)$.
2. *For every finite tuple* $u \in \widetilde{c_1...c_n}$ *not contained in* $\partial(c_1 \ldots c_n)$*, there is a type* p_u*, an* S_{n+1}*-compatible Morley sequence* $\langle a_1, \ldots a_n\rangle$ *of finite tuples each realizing* p_u *with* $a_i \in \overline{c_i}$*, and a generic* n*-ary groupoid* $\mathcal{H}_u = (P_{u,1}, P_{u,2}, \ldots, P_{u,n}; Q_u; \pi^2_u, \ldots, \pi^n_u)$ *in* p_u *such that* $u \in \mathrm{dcl}(w)$ *for some* $w \in P_{u,n}(a_1, \ldots, a_n)$ *compatible with the* $[\sigma_j]$*'s.*

Proof. For part (1), now that we have established the existence of $(n-1)$-symmetric systems, the same construction as in the proof of Proposition 3.13 of [6] can be used to obtain a symmetric failure to

$(n+1)$-uniqueness, and for part (2), it is simple to modify the construction of the symmetric failure so that for some $w \in P_{u,n}(a_1, \ldots, a_n)$, $u \in \mathrm{dcl}(w)$. Then the rest of the proof of Theorem 3.3 of [6] can be carried out as before. □

Ultimately, we will construct a directed system of such generic n-ary groupoids, but before we do that, we describe a group that can be naturally attached to each such object. The next fact is a compilation of observations from [6]. The common setting for Proposition 3.8 and Lemma 3.9 is as follows. The type p' is the type of a finite tuple interalgebraic with a tuple realizing the type p. Since the algebraic closures of the realizations of p and p' are the same, the two types have the same uniqueness properties.

Proposition 3.8. *Assume that $p' \in S(B)$ has $(\leq n)$-uniqueness and $B = \mathrm{acl}(B)$. Let $\mathcal{H} = (I, P_2, \ldots, P_n; Q; \pi^2, \ldots, \pi^n)$ be a generic n-ary groupoid in p'. Then there is a B-definable finite abelian group $G_{\mathcal{H}}$ satisfying the following:*

(1) For any B-independent $b_1, \ldots, b_n \models p'$ and $w \in P_n(b_1, \ldots, b_n)$, $G_{\mathcal{H}}$ acts regularly (transitively and faithfully) on the solution set of $\Pi(x; b_1 ... b_n)$, where

$$\Pi(x; b_1 ... b_n) := \mathrm{tp}(w/\partial(b_1, \ldots, b_n)),$$

and so $|\Pi(\mathcal{M}; b_1 ... b_n)| = |G_{\mathcal{H}}|$.

For any $w, w' \models \Pi(x; b_1 ... b_n)$ and $\gamma \in G_{\mathcal{H}}$, we have $\mathrm{dcl}(wB) = \mathrm{dcl}(w'B)$ and $\mathrm{tp}(w, \gamma.w/B) = \mathrm{tp}(w', \gamma.w'/B)$.

Define the relation $\sim$ on the pairs in $(\Pi(\mathcal{M}; b_1 ... b_n))^2$ by letting $(w_0, w_0') \sim (w_1, w_1')$ if and only if there is $\gamma \in G_{\mathcal{H}}$ such that $w_j' = \gamma.w_j$ ($j = 0, 1$). Then $\sim$ is an equivalence relation on $(\Pi(\mathcal{M}; b_1 ... b_n))^2$ and the map $[\ ] : (\Pi(\mathcal{M}; b_1 ... b_n)^2/\sim) \to G_{\mathcal{H}}$ sending $[(w_j, w_j')]$ to γ is the unique bijection such that for $w, w', w'' \models \Pi(x; b_1 ... b_n)$,

$$[(w, w')] + [(w', w'')] = [(w, w'')].$$

The map $[\ ]$ induces the canonical isomorphism between $G_{\mathcal{H}}$ and $\mathrm{Aut}(\Pi(\mathcal{M}; b_1 ... b_n)/\partial(\bar{b}))$, sending $\gamma \in G_{\mathcal{H}}$ to the unique φ_γ such that $\gamma.w = \varphi_\gamma(w)$ for some (any) $w \models \Pi(x; b_1 ... b_n)$. In other words $G_{\mathcal{H}}$ uniformly and canonically binds all the automorphism groups $\mathrm{Aut}(\Pi(\mathcal{M}; b_1' ... b_n')$ with $b_1' ... b_n' \models (p')^{(n)}$.

(2) Assume $Q(w_1, ..., w_{n+1})$ holds. Then for any $(\gamma_1, \ldots, \gamma_{n+1}) \in G_{\mathcal{H}}^{n+1}$, we have

$$Q(\gamma_1.w_1, \gamma_2.w_2, \ldots, \gamma_{n+1}.w_{n+1}) \Leftrightarrow \sum_{i=1}^{n+1} (-1)^i \gamma_i = 0.$$

Proof. The construction of this group $G_{\mathcal{H}}$ and the proof of all of the properties listed above is given by the series of lemmas from Lemma 3.15 to Corollary 3.27 in [6] (there it was called simply "G"). Some steps in this proof use the property of relative (k,n)-uniqueness for values of $k \leq n$, but for these steps we can apply Lemma 4.1, since throughout we assume only $(\leq n)$-uniqueness for the type p' and not for T. $\square$

We state a new observation which will be crucial for the proof of Lemma 3.13 below.

Lemma 3.9. *Assume that $p' \in S(B)$ has $(\leq n)$-uniqueness, $B = \operatorname{acl}(B)$, and that a generic n-ary groupoid $(I, P_2, \ldots, P_n; Q; \pi^2, \ldots, \pi^n)$ in p' is given. Let $\Phi_{p'}(x_1, \ldots, x_{n+1})$ be the partial type over B saying that for some x_j^k ($k = 1, \ldots, n+2$, $j = 1, \ldots, n+1$) with $x_j = x_j^{n+2}$, and some Morley sequence $\langle a_1, \ldots, a_{n+2}\rangle$ in q,*

(1) $(x_1, \ldots, x_{n+1})$ is compatible from P_n, and $x_1 \ldots x_{n+1} \equiv_B x_1^k \ldots x_{n+1}^k$;
(2) $x_j^k = x_k^{j+1}$ for all $1 \leq k \leq j \leq n+1$; and
(3) $\pi(x_j^k) = (d_1, \ldots, \widehat{d_j}, \ldots, d_{n+1})$ where $d_1 \ldots d_{n+1} = a_1 \ldots \widehat{a_k} \ldots a_{n+2}$.

Then $\models \Phi_{p'}(v_1, \ldots, v_{n+1})$ if and only if there is some $\mathcal{L}(B)$-formula $Q'(x_1, \ldots, x_{n+1})$ realized by compatible $(v_1, \ldots, v_{n+1})$ from P_n, such that $(I, \ldots, P_n; Q'; \pi^i)$ forms a generic n-ary groupoid in p' (where we have just replaced Q by Q').

Proof. ($\Leftarrow$): This direction is clear due to the associativity of Q' and the $(\leq n)$-uniqueness property for p'.

($\Rightarrow$): Assume $\Phi_{p'}(v_1, \ldots, v_{n+1})$ holds. So there is some Morley sequence $\langle a_1, \ldots a_{n+1}\rangle$ in p' such that $\pi(v_j) = (a_1, \ldots, \widehat{a_j}, \ldots, a_{n+1})$ is witnessed by a formula, say $\theta(x_j; y_1, \ldots, \widehat{y_j}, \ldots, y_{n+1}) \in \mathcal{L}(B)$. Moreover notice that $v_j \in \operatorname{dcl}(v_1 \ldots \widehat{v_j} \ldots v_{n+1}; B)$ for each $j \in [n{+}1]$, since due to $(\leq n)$-uniqueness there is v'_j such that $v_1 \ldots v'_j \ldots v_{n+1} \equiv_B w_1 \ldots w_{n+1}$ where $(w_1, \ldots, w_{n+1}) \models Q$, so $v'_j \in \operatorname{dcl}(v_1 \ldots \widehat{v_j} \ldots v_{n+1}; B)$, and by Fact 3.8(1), v_j and v'_j are interdefinable over B. Hence the fact that $v_j \in \operatorname{dcl}(v_1, \ldots \widehat{v_j} \ldots v_{n+1}; B)$ is witnessed by some $\mathcal{L}(B)$-formula

$$\alpha_j(x_j; x_1, \ldots, \widehat{x_j}, \ldots, x_{n+1}).$$

We also choose a formula $\beta(x_1, \ldots, x_{n+1}) \in \mathcal{L}(B)$ which says "the $(n{+}1)$-tuple $(x_1, \ldots, x_{n+1})$ is compatible." Now we let $Q'(x_1, \ldots, x_{n+1})$ be the formula

$$\beta(x_1, \ldots, x_{n+1}) \wedge$$
$$\exists y_1 \ldots y_{n+1} \bigwedge_{j=1}^{n+1} (\theta(x_j; y_1, \ldots, \widehat{y_j}, \ldots, y_{n+1}) \wedge \alpha_j(x_j; x_1, \ldots, \widehat{x_j}, \ldots, x_{n+1})),$$

realized by $(v_1, \ldots, v_{n+1})$. (In fact, $Q'(v_1, \ldots, v_{n+1})$ together with a type over B saying that for some Morley $\langle a_1, \ldots a_{n+1}\rangle$ in p', $\pi(v_j) = (a_1, \ldots, \widehat{a_j}, \ldots, a_{n+1})$ for each $j \in [n+1]$, determines $\mathrm{tp}(v_1, \ldots, v_{n+1}/B)$.) Now it suffices to check that Q' satisfies the associativity on a Morley sequence of $(n+2)$-tuple in p'.

Since $(v_1, \ldots, v_{n+1})$ satisfies (1),(2), and (3), there exist elements v_j^k and a B-Morley sequence $\langle a_1, \ldots a_{n+2}\rangle$ satisfying these conditions (where v_j^k plays the role of x_j^k). In particular, $\models Q'(v_1^k, \ldots, v_{n+1}^k)$ for all $k \in [n+2]$. On the other hand, assume that there exist u_j^k ($k = 1, \ldots, n+2$, $j = 1, \ldots, n+1$) such that for each $k \in [n+2]$, $(u_1^k, \ldots, u_{n+1}^k)$ is compatible from P_n; $u_j^k = u_k^{j+1}$ for all $1 \le k \le j \le n+1$; and $\pi(x_j^k) = (d_1, \ldots, \widehat{d_j}, \ldots, d_{n+1})$ where $d_1 \ldots d_{n+1} = a_1 \ldots \widehat{a_k} \ldots a_{n+2}$. Moreover assume $Q'(u_1^k, \ldots, u_{n+1}^k)$ holds for each $k \in [n+1]$. We want to verify that it holds as well when $k = n+2$ (the other cases being similar).

Note first that due to $(\le n)$-uniqueness for p', it follows that $v_1 \ldots v_n \equiv_B u_1^k \ldots u_n^k$ for all $k \in [n+2]$. Since Q' isolates $\mathrm{tp}(v_{n+1}/v_1 \ldots v_n B)$, this implies

$$v_1 \ldots v_{n+1} \equiv_B u_1^k \ldots u_{n+1}^k$$

for all $k \in [n+1]$. Furthermore, again by $(\le n)$-uniqueness, we have

$$\langle v_1^k \ldots v_{n+1}^k |\ k = 1, \ldots, n\rangle \equiv_B \langle u_1^k \ldots u_{n+1}^k |\ k = 1, \ldots, n\rangle.$$

Then due to the Q'-relation on $(v_1^{n+1}, \ldots, v_{n+1}^{n+1}) = (v_n^1, \ldots, v_n^{n+1})$, and on $(u_1^{n+1}, \ldots, u_{n+1}^{n+1}) = (u_n^1, \ldots, u_n^{n+1})$, it clearly follows that

$$\langle v_1^k \ldots v_{n+1}^k |\ k = 1, \ldots, n+1\rangle \equiv_B \langle u_1^k \ldots u_{n+1}^k |\ k = 1, \ldots, n+1\rangle.$$

Then since $Q'(v_1^{n+2}, \ldots, v_{n+1}^{n+2})$ holds, so does $Q'(u_1^{n+2}, \ldots, u_{n+1}^{n+2})$. $\square$

3.3. **Constructing a directed system of generic n-ary groupoids.** The first key observation, stated in Proposition 3.7, is that since p fails $(n+1)$-uniqueness, there is some

$$w \in \widetilde{c_1 \ldots c_n} \setminus \partial(c_1 \ldots c_n).$$

If we could pick a single $w \in \mathcal{M}$ which was interdefinable with all of $\widetilde{c_1 \dots c_n}$, then letting $q = \mathrm{tp}(w/\partial(c_1 \dots c_n))$, we would have

$$G \cong G_q := \mathrm{Aut}(q(\mathcal{M})/\partial(c_1 \dots c_n)),$$

and G would be finite since q is algebraic.

However, in general there is no such w which is maximal with respect to definable closure, and the group G will be constructed as an inverse limit of finite groups G as above. It will take some work to construct an appropriate index set for the inverse limit and check that this has all the properties we need.

The next Lemma is a generalization and adaptation of Claim 2.9 from [5], and it can be proved in the same manner (using Proposition 3.7 above).

Lemma 3.10. *There is a directed partially ordered set $(J, \leq)$, with $|J| \leq |T|$, and families $c_{1,u}, \dots, c_{n+1,u}$, p_u, $\mathcal{H}_u$, and w_u indexed by $u \in J$, satisfying all of the following properties for every $u \in J$:*

(1) $c_{1,u}, \dots, c_{n+1,u}$ is a Morley sequence over B such that each $c_{i,u}$ is interalgebraic with c_i over B;
(2) $p_u = \mathrm{tp}(c_{1,u}/B)$;
(3) $\mathcal{H}_u = (I_u, P_{2,u}, \dots, P_{n,u}; Q_u; \pi^2_u, \dots, \pi^n_u)$ is an n-ary generic groupoid in p_u;
(4) $w_u \in P_{n,u}(c_{1,u}, \dots, c_{n,u})$;
(5) w_u is compatible with the $[\sigma_j]$'s;
(6) If $u \leq v$, then $c_{1,u} \in \mathrm{dcl}\, c_{1,v}$, $w_u \in \mathrm{dcl}(w_v)$, and

$$c_{1,u}c_{1,v} \equiv c_{2,u}c_{2,v} \equiv \dots \equiv c_{n+1,u}c_{n+1,v};$$

and

(7)

$$\widetilde{c_1 \dots c_n} = \mathrm{dcl}_B\left(\bigcup_{u \in J} w_u\right).$$

The following notation is designed to match the both the notation in Proposition 3.8 and the notation for the selector function Π in Lemma 2.3. It is also convenient to have symbols for the sets of realizations of certain types. These elements explicitly witness the failure of $(n+1)$-uniqueness for the type p and have a lot of built-in symmetry with respect to the automorphisms $[\sigma_j]$; so we chose the notation "SW" for "symmetric witness".

Notation 3.11. If $u \in J$,

$$\Pi_u(x_u; y_u) = \mathrm{tp}(w_u; \pi^n_u(w_u))$$

and if $d \equiv_B \pi_u^n(w_u)$, then

$$SW_u(d) = \Pi_u(\mathcal{M}; d);$$

and

$$SW_u = \bigcup_{d \equiv_B \pi_u^n(w_u)} SW_u(d).$$

Then, as in Lemma 2.10 of [5], we obtain:

Lemma 3.12. *For any $u, v \in J$ such that $u \leq v$, there are surjective functions*

$$\tau_{v,u} : p_v(\mathcal{M}) \to p_u(\mathcal{M})$$

and

$$\rho_{v,u} : SW_v \to SW_u$$

which are relatively B-definable (that is, their graphs are the intersections of B-definable relations with the product of the domain and the range) and satisfy all of the following:

(1) $\tau_{v,u}(c_{i,v}) = c_{i,v}$*,*
(2) $\rho_{v,u}(w_v) = w_u$*,*

and whenever $u \leq v \leq w$,

(3) $\tau_{v,u} \circ \tau_{w,v} = \tau_{w,u}$*, and*
(4) $\rho_{v,u} \circ \rho_{w,v} = \rho_{w,u}$*.*

The next Lemma says that we can make slight changes in the definable relations Q_u in order to make them coherent with the projection maps $\rho_{v,u}$. This will be crucial below to ensure that we can take an inverse limit of the groups G_u.

Lemma 3.13. *There is a family $\langle Q'_u : u \in J\rangle$ of formulas over B such that, after replacing $\mathcal{H}_u$ by $\mathcal{H}'_u := (I_u, \ldots; Q'_u; \pi_u^2, \ldots, \pi_u^n)$ (only changing Q_u to Q'_u), we have:*

(1) The family $\langle \mathcal{H}'_u : u \in J\rangle$ is still an n-ary generic groupoid; and
(2) Whenever $u \leq v$, $(d_1, \ldots, d_{n+1}) \models (p_v)^{(n+1)}$, and $w_1, \ldots, w_{n+1} \in P_v$ are elements satisfying

$$\pi_v(w_i) = (d_1, \ldots, \widehat{d_i}, \ldots, d_{n+1})$$

and

$$Q'_v(w_1, \ldots, w_{n+1}),$$

then

$$Q'_u(\rho_{v,u}(w_1), \ldots, \rho_{v,u}(w_{n+1})).$$

Proof. For each $u \in J$, since

$$([\sigma_1](w_u), [\sigma_2](w_u), \ldots, [\sigma_n](w_u), w_u)$$

is a compatible tuple from a generic n-ary groupoid, by Lemma 3.9, there is a relation Q'_u which is satisfied by this tuple such that $\mathcal{H}'_u = (I_u, \ldots; Q'_u; \pi^2_u, \ldots, \pi^n_u)$ is still a generic n-ary groupoid.

To check the second condition in the Lemma, suppose that $u, v \in J$ and $u \leq v$ and that $w_1, \ldots, w_{n+1} \in P_v$ satisfy $Q'_v(w_1, \ldots, w_{n+1})$ and $\pi_v(w_i) = (d_1, \ldots, \widehat{d_i}, \ldots, d_{n+1})$ for some $d_1, \ldots, d_{n+1}$.

Claim 3.14. *There is an elementary map $\varphi \in \operatorname{Aut}(\mathcal{M}/B)$ such that $\varphi(d_i) = c_{i,v}$ for $i \in \{1, \ldots, n+1\}$ and $\varphi(w_i) = [\sigma_i](w_v)$ for $i \in \{1, \ldots, n\}$.*

Proof. As in Definition 3.1 of [4], we say that a type p has *relative n-uniqueness* just in case p has relative (n, n)-uniqueness. Now by the proofs of Theorem 3.10 and Proposition 3.14 of [4], the fact that p has $(\leq n)$-uniqueness implies that p has $B(n)$ and thus relative n-uniqueness (in fact, just the n-uniqueness and $(n-1)$-uniqueness of p would be sufficient for this step).

Since $\{c_{1,v}, \ldots, c_{n+1,v}\}$ and $\{d_1, \cdots, d_{n+1}\}$ are Morley sequences over B in the same type, there is an elementary map $\varphi_0 \in \operatorname{Aut}(\mathcal{M}/B)$ such that $\varphi_0(d_i) = c_{i,v}$ for each $i \in \{1, \ldots, n+1\}$. By relative n-uniqueness of p over $\operatorname{acl}(Bd_1)$, there is an elementary map $\varphi_1 \in \operatorname{Aut}(\mathcal{M}/\operatorname{acl}(Bc_{1,v}))$ such that φ_1 fixes $\operatorname{acl}_B(c_{1,v}, \ldots, \widehat{c_{i,v}}, \ldots, c_{n+1,v}\}$ pointwise for every $i \in \{2, \ldots, n+1\}$ and such that

$$\varphi_1(\varphi_0(w_i)) = [\sigma_i](w_v)$$

for every $i \in \{1, \ldots, n\}$. Now $\varphi := \varphi_1 \circ \varphi_0$ is the map we seek.

□

Now since φ is elementary, it preserves the Q'_v relation, and therefore $\varphi(w_{n+1}) = w_v$.

Likewise, since the elementary maps $[\sigma_i]$ commute with the definable map $\rho_{v,u}$,

$$\begin{aligned} &(\rho_{v,u}([\sigma_1](w_v)), \ldots, \rho_{v,u}([\sigma_n](w_v), \rho_{v,u}(w_v)) \\ &= ([\sigma_1](\rho_{v,u}(w_v)), \ldots, [\sigma_n](\rho_{v,u}(w_v)), \rho_{v,u}(w_v)) \\ &= ([\sigma_1](w_u), \ldots, [\sigma_n](w_u), w_u), \end{aligned}$$

and Q'_u holds of the final tuple by definition. That is, the $(n+1)$-tuple $([\sigma_1](w_v), \ldots, [\sigma_n](w_v), w_v)$ satisfies the formula

$$Q'_u(\rho_{v,u}(x_1), \ldots, \rho_{v,u}(x_{n+1}))$$

(assigning $[\sigma_1](w_v)$ to $x_1, \ldots$, $[\sigma_n](w_v)$ to x_n, and w_v to x_{n+1}). Taking preimages via the elementary map φ, we see that $(w_1, \ldots, w_{n+1})$ also satisfies this formula, which is what we wanted to prove.

□

3.4. **Proof of Lemma 2.3.**

Proof. First, let $\overline{w}$ be a (possibly infinite) tuple listing $\{w_u : u \in J\}$, and let $q(x_0, \ldots, x_n)$ be the complete $*$-type over B of

$$([\sigma_1](\overline{w}), [\sigma_2](\overline{w}), \ldots, [\sigma_n](\overline{w}), \overline{w}).$$

Let r be the complete $*$-type

$$r(x; y) := \text{tp}(\overline{w}; \partial(c_1, \ldots, c_n)/B).$$

Given any $f \in \mathcal{S}_{n-1}(p)$, note that the infinite tuple $\partial[f]$ can be ordered so that $\partial[f] \equiv \partial(c_1, \ldots, c_n)$: this follows from $(\leq n)$-uniqueness of the type p by the same argument as in the proof of Claim 3.14 above. Pick any such ordering of $\partial[f]$, and let Πf be $r(x; \partial[f])$.

Given $f \in \mathcal{S}_{n-1}(p)$, let $\alpha(f)$ be *any* realization of Πf.

By Proposition 3.8 above, for each $u \in J$, the generic n-ary polygroupoid $\mathcal{H}'_u$ has a finite abelian "binding group" $G_{\mathcal{H}'_u}$, which we will denote by simply G_u. Each G_u acts as described in Proposition 3.8 on the set of all realizations of $SW_u(d)$ for any $d \equiv_B \pi^n_u(w_u)$.

Now for $u, v \in J$ such that $u \leq v$, we can define a surjective group homomorphism $\chi_{v,u} : G_v \to G_u$ as follows: if $\gamma \in G_v$, then for any $w \in SW_v(\pi^n_v(w_v))$, we have that $\gamma.w$ is also an element of $SW_v(\pi^n_v(w_v))$. Now we define $\chi_{v,u}(\gamma)$ to be the unique element of G_u such that

$$\chi_{v,u}(\gamma).\rho_{v,u}(w) = \rho_{v,u}(\gamma.w).$$

This definition of $\chi_{v,u}$ does not depend on the choice of w since for any other $w' \in SW_v(\pi^n_v(w_v))$, we have $(w, \gamma.w) \equiv_B (w', \gamma.w')$, and so the B-definability of $\rho_{v,u}$ implies that

$$(\rho_{v,u}(w), \rho_{v,u}(\gamma.w)) \equiv_B (\rho_{v,u}(w'), \rho_{v,u}(\gamma.w')),$$

whence $\chi_{v,u}(\gamma).\rho_{v,u}(w') = \rho_{v,u}(\gamma.w')$.

The fact that $\chi_{v,u}$ is a group homomorphism now follows directly from its definition, and it is surjective since G_v acts transitively on $SW_v(\pi^n_v(w_v))$.

Finally, our group G will be the inverse limit of $\langle G_u : u \in J\rangle$. If we think of each $\alpha \in G$ concretely as a tuple $\langle \gamma_u : u \in J\rangle$ such that whenever $u \leq v$, we have $\chi_{v,u}(\gamma_v) = \gamma_u$, then we can define the action of γ as follows: if $\overline{w}' \models \Pi f$, then $\overline{w}'$ enumerates some elements $\{w'_u : u \in J\}$ in such a way that $(\overline{w}', \partial[f]) \equiv_B (\overline{w}, \partial(c_1, \ldots, c_n))$, and we define $\alpha \cdot \overline{w}'$ as the enumeration (in the same order) of the elements

$\{\gamma_u \cdot w'_u : u \in J\}$. The remaining desired properties of the action (Lemma 2.3 parts (3)-(6)) now follow from Proposition 3.8. □

4. Symmetric systems and polygroupoids under local uniqueness assumptions

This section verifies the statements that are necessary to strengthen the results of [6] to the form needed for this paper.

4.1. **Symmetric system.**

Lemma 4.1. *Suppose that T is stable, $p \in S(B)$ (with $B = \mathrm{acl}^{eq}(B)$) has n-uniqueness, and $m \geq n$. Then p has the following property:*

For any Morley sequence $a_1, \ldots, a_m$ over B such that each a_i is interalgebraic (over B) with a realization of p, and for any set of maps $\{\varphi_u : u \subset_{n-1} [m]\}$ such that

(1) φ_u is an automorphism of $\mathrm{acl}(\{a_i : i \in u\} \cup B)$, and

(2) For all $v \subsetneq u$, φ_u acts as the identity on $\mathrm{acl}(\{a_i : i \in v\} \cup B)$,

we have that $\bigcup_u \varphi_u$ is an elementary map.

We call this property relative (n,m)-uniqueness localized to p.

Proof. This follows from the next two Claims, which are localized versions of two lemmas from [4].

Claim 4.2. *If p has n-uniqueness, then p has the property $B(n)$ (see Definition 3.1 of [4] – to say that p has this property means that it is true when the tuples a_i are interalgebraic over B with realizations of p).*

Proof. The same argument as in the proof of Lemma 3.3 of [4] applies without changes. □

Claim 4.3. *If p has $B(n)$, then relative (n,m)-uniqueness localized to $p^{(k)}$ holds for any k.*

Proof. By the main result of [10], since p has n-uniqueness, so does $p^{(k)}$ for any k. So by Claim 4.2, every $p^{(k)}$ also has $B(n)$. Now we can carry out the same argument as in the proof of Lemma 4.4 of [4], which goes by induction on m. This argument proves $(n, m+1)$-uniqueness of $p^{(k)}$ by applying relative (n,m)-uniqueness to systems of elementary maps defined over independent sets of the form $\{b_1, \ldots, b_m\}$ where b_i realizes $p^{(k)}$ or $p^{(2k)}$, which holds by the induction hypothesis. □

□

4.2. **Polygroupoids.**

Definition 4.4. Let $M = (I, P_2, \dots, P_n, \pi^2, \dots, \pi^n)$ be a structure with sorts P_i, $i = 2, \dots, n$ and functions $\pi^k : P_k \to (P_{k-1})^k$, $k = 2, \dots, n$; sometimes we may use "P_1" to refer to the sort I. We use the symbol $\pi^k_i(w)$ to refer to the ith element of the tuple $\pi^k(w)$.

(1) We say that a tuple $(w_1, \dots, w_{k+1}) \in (P_k)^{k+1}$ is *compatible* if
  (a) $k = 1$ and $w_1 \neq w_2$, or
  (b) $k \geq 2$ and $\pi^k_i(w_j) = \pi^k_{j-1}(w_i)$ for all $1 \leq i < j \leq k+1$.

  We say that a k-tuple

$$(w_1, \dots, w_{\ell-1}, \widehat{\cdot}, w_{\ell+1}, \dots, w_{k+1}) \in (P_k)^k$$

  with the deleted term number $\ell (\leq k+1)$, is *partially compatible* if either $k = 1$, or $k \geq 2$ and $\pi^k_i(w_j) = \pi^k_{j-1}(w_i)$ for all $1 \leq i (\neq \ell) < j (\neq \ell) \leq k+1$.

(2) Assume that for every $w \in P_k$, the image $\pi^k(w)$ forms a compatible tuple: Then for any $i \in \{2, \dots, n\}$ and any $w \in P_i$, we iteratively say that w *is over* $(a_1, \dots, a_i) \in I^{(i)}$ if,
  (a) $i = 2$ and $(a_2, a_1) = \pi^2(w)$, or
  (b) $i > 2$ and for every $j \in \{1, \dots, i\}$, $\pi^i_j(w)$ is over $(a_1, \dots, \widehat{a}_j, \dots, a_i)$;

  For any $(a_1, \dots, a_i) \in I^{(i)}$, we denote by $P_i(a_1, \dots, a_i)$ the set of all $w \in P_i$ which are over $(a_1, \dots, a_i)$. If $w \in P_i(a_1, \dots, a_i)$, we also write "$\pi(w) = (a_1, \dots, a_i)$"; Note that then the sort P_i is the disjoint union of all the "fibers" $P_i(a_1, \dots, a_i)$ where $(a_1, \dots, a_i) \in I^{(i)}$.

  Given $w \in P_i(a_1, \dots, a_i)$, and a j-tuple s $(0 < j \leq i)$ of increasing numbers from $[i]$, we write $\pi_s(w)$ to denote the unique element in $P_{|s|}(a_j |\, j \in s)$ which is an image of w under a composition of $\pi^2, \dots, \pi^i$. For example, $\pi_{\langle 1, \dots, i\rangle}(w) = w$, $\pi_{\langle 1 \rangle}(w) = a_1$, and $\pi_{\langle 1, \widehat{2}, \widehat{3}, \dots, i\rangle}(w) = \pi^{i-1}_2(\pi^i_2(w))$.

Definition 4.5. If $n \geq 2$, an *n-ary quasigroupoid* is a structure $\mathcal{H} = (I, P_2, \dots, P_{n-1}, P, Q)$ with n disjoint sorts $I = P_1, P_2, \dots, P_n = P$ equipped with an $(n+1)$-ary relation $Q \subseteq P^{n+1}$ and a system of maps $\langle \pi^k : 2 \leq k \leq n \rangle$ satisfying the following axioms:

(1) (Coherence) For each $k \in \{2, \dots, n\}$, the function π^k maps an element $w \in P_k$ to a compatible k-tuple $(\pi^k_1(w), \dots, \pi^k_k(w)) \in (P_{k-1})^k$.
(2) (Compatibility and Q) If $Q(w_1, \dots, w_{n+1})$ holds, then $(w_1, \dots, w_{n+1})$ is a compatible $(n+1)$-tuple of elements of P_n.
(3) (Uniqueness of horn-filling) Whenever $Q(w_1, \dots, w_{n+1})$ holds, then for any $i \in \{1, \dots, n+1\}$, w_i is the unique element $x \in P$

which satisfies

$$Q(w_1, \ldots, w_{i-1}, x, w_{i+1}, \ldots, w_{n+1}).$$

We say the n-ary quasigroupoid $\mathcal{H} = (I, \ldots, P, Q)$ is *locally finite* if for every $z \in I^{(n)}$, the set $P(z)$ is finite.

Definition 4.6. If $\mathcal{H} = (I, P_2, \ldots, P_n, Q)$ is an n-ary quasigroupoid, we say that $\mathcal{H}$ is an *n-ary polygroupoid* if it satisfies the following condition:

(Associativity) Suppose that $\{w^i_j \mid 1 \leq i \leq n+2,\ 1 \leq j \leq n+1\}$ is a collection of elements in P such that for each $i = 1, \ldots, n+2$ the elements $\{w^i_j \mid 1 \leq j \leq n+1\}$ are compatible and such that $w^i_j = w^{j+1}_i$ for all $1 \leq i \leq j \leq n+1$.

For each $\ell = 1, \ldots, n+2$, if $Q(w^i_1, \ldots, w^i_{n+1})$ hold for all $i \in \{1, \ldots, n+2\} \setminus \{\ell\}$, then $Q(w^\ell_1, \ldots, w^\ell_{n+1})$ holds too.

If associativity holds for any compatible such tuples w^i_j with $\pi(w^i_j) = (d^i_1, \ldots, \widehat{d^i_j}, \ldots, d^i_{n+1}) \in I^{(n)}$ where $d^i_1 \ldots d^i_{n+1} = c_1 \ldots \widehat{c_i} \ldots c_{n+2}$, then we say *the associativity of Q holds on* $(c_1, \ldots, c_{n+2})$.

References

[1] Glen E. Bredon. *Topology and Geometry.* Springer-Verlag, NewYork, 1993.

[2] Ronald Brown. *Topology and Groupoids.* www.groupoids.org, 2006.

[3] John Goodrick and Alexei Kolesnikov. Groupoids, covers, and 3-uniqueness in stable theories. *Journal of Symbolic Logic*, 75 (2010), 905-929.

[4] John Goodrick, Byunghan Kim, and Alexei Kolesnikov. Amalgamation functors and boundary properties in simple theories. *Israel Journal of Mathematics*, 193 (2013) 169-207.

[5] John Goodrick, Byunghan Kim, and Alexei Kolesnikov. Homology groups of types in model theory and the computation of $H_2(p)$. *Journal of Symbolic Logic*, 78 (2013), 1086–1114.

[6] John Goodrick, Byunghan Kim, and Alexei Kolesnikov. Type-amalgamation properties and polygroupoids in stable theories. To appear in *Journal of Mathematical Logic.*

[7] John Goodrick, Byunghan Kim, and Alexei Kolesnikov. Amalgamation functors and homology groups in model theory. *Proceedings of ICM 2014*, II (2014), 41–58.

[8] Ehud Hrushovski. Totally categorical structures. *Transactions of the American Mathematical Society* 313 (1989), 131–159.

[9] Ehud Hrushovski. Groupoids, imaginaries and internal covers. *Turkish Journal of Mathematics* 36 (2012), 173–198.

[10] Byunghan Kim. On the n-uniqueness of types in rosy theories. Submitted.

[11] Jacob Lurie. *Higher Topos Theory.* Princeton University Press, 2009.

[12] S. Shelah. Classification theory for nonelementary classes. The number of uncountable models of $\psi \in L_{\omega_1 \omega}$. Parts A, B. *Israel Journal of Mathematics*, **46** (1983), 212–271.

[13] Saharon Shelah. *Classification theory and the number of non-isomorphic models.* North Holland, 1990.

Department of Mathematics, Universidad de los Andes, Bogotá, Colombia

Department of Mathematics, Yonsei University, Seoul, Korea

Department of Mathematics, Towson University, MD, USA
E-mail address: `jr.goodrick427@uniandes.edu.co`
E-mail address: `bkim@yonsei.ac.kr`
E-mail address: `akolesnikov@towson.edu`