Geometric Simplicity Theory

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ALC 10
September 1-6, 2008
Outline

1. $n$-amalgamation
2. $\mathcal{M}^{\geq}$
3. $n$-simplicity
4. Fields in simple theories
Geometric Simplicity Theory

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Let $\mathcal{C}_T$ be a category of the algebraically closed substructures of $\mathcal{M}$. Recall that any poset is a category. For $n \in \omega$, write $\mathcal{P}(n)^- := \mathcal{P}(n) - \{n\}$.
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**Definition**

- A functor $a : \mathcal{W}(\subseteq \mathcal{P}(n)) \to \mathcal{C}_T$ is said to be independence preserving (i.p.) if
  1. for any $w_0, w_1 \subseteq w \in \mathcal{W}$, $a_{w_0 \downarrow a_{w_0 \cap w_1}} a_{w_1}$ holds within $a_w$;
  2. for $w \in \mathcal{W}$, $a_w = acl(\bigcup \{a_{\{i\}} \mid i \in w\})$.

- We say $T$ has $n$-amalgamation if any i.p. functor $a : \mathcal{P}(n)^- \to \mathcal{C}_T$ can be extended to i.p. $\hat{a} : \mathcal{P}(n) \to \mathcal{C}_T$.

3-amalgamation is type-amalgamation (the Independence Theorem). Hence any simple $T$ has 3-amalgamation.
3-amalgamation

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Geometric Simplicity Theory
4-amalgamation
4-amalgamation over acl base set
$T$ stable.

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**Example**

Consider $[A]^2 = \{\{a, b\}|a \neq b \in A\}$ where $A$ infinite. Let $B = [A]^2 \times \{0, 1\}$ where $\{0, 1\} = \mathbb{Z}_2$. Also let $E \subseteq A \times [A]^2$ be a membership relation, and let $P$ be a subset of $B^3$ such that $((w_1, \delta_1)(w_2, \delta_2)(w_3, \delta_3)) \in P$ iff there are distinct $a_1, a_2, a_3 \in A$ such that for $\{i, j, k\} = \{1, 2, 3\}$, $w_i = \{a_j, a_k\}$, and $\delta_1 + \delta_2 + \delta_3 = 0$. Let $M = (A, [A]^2, B; E, P; \text{Pr}_1 : B \to [A]^2)$. Then $M$ is stable.
The stable example does not have 4-amalgamation over $\emptyset = \text{acl}(\emptyset)$.

Why

Note first that $\text{dcl}(\emptyset) = \text{acl}(\emptyset)$, and for $a \in A$, $\text{dcl}(a) = \text{acl}(a)$. Now choose distinct $a_1, a_2, a_3, a_4 \in A$. For $\{i, j, k\} \subseteq \{1, 2, 3, 4\}$, fix an enumeration $\overline{a_i a_j} = (b_{ij}, \ldots)$ of $\text{acl}(a_i a_j)$ where $b_{ij} = (\{a_i, a_j\}, \delta) \in B = [A]^2 \times \{0, 1\}$. Let $r_{ij}(x_{ij}) = \text{tp}(\overline{a_i a_j})$, and let $x_{ij}^1$ be the variable for $b_{ij}$. Note that $b_{ij} = (\{a_i, a_j\}, \delta)$ and $b_{ij}' = (\{a_i, a_j\}, \delta + 1)$ have the same type over $a_i a_j$. Hence there is $\overline{(a_i a_j)'} = (b_{ij}', \ldots)$ also realizing $r_{ij}(x_{ij})$. Therefore we have complete types $r_{ijk}(x_{ijk})$, $r_{ijk}'(x_{ijk}')$ both extending $r_{ij}(x_{ij}) \cup r_{ik}(x_{ik}) \cup r_{jk}(x_{jk})$ realized by some enumerations of $\text{acl}(a_i a_j a_k)$ such that $P(x_{ij}^1 x_{ik}^1 x_{jk}^1) \in r_{ijk}$ whereas $\neg P(x_{ij}^1 x_{ik}^1 x_{jk}^1) \in r_{ijk}'$. Then it is easy to see that $r_{123} \cup r_{124} \cup r_{134} \cup r_{234}'$ is inconsistent.
Resolution

In his recent preprint [http://arxiv.org/abs/math/0603413v1], Hrushovski showed that if $\mathcal{M} \models T$ is stable, then there is $CM^* \models T^*$ in $\mathcal{L}^*(\supseteq \mathcal{L})$ such that $\mathcal{M}$ is stably embedded into $\mathcal{M}^*$, and $\mathcal{M}^*$ has $n$-amalgamation over any acl bases. We may write $\mathcal{M}^*$ as $\mathcal{M}^{geq}$.

In short, like $\mathcal{M}^{eq}$, wlog, we can assume $\mathcal{M} = \mathcal{M}^{geq}$ when $T$ is stable.
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Open Problem

Can we construct such $\mathcal{M}^*$ for simple $T$? If yes, then possibly we can remove the assumption of 4-amalgamation in the group configuration theorem.
In the following notions of $n$-simplicity and $K(n)$-simplicity, for convenience, we use imprecise definitions good enough however representing the essence of notions. Also for convenience, we describe notions only over $\emptyset = acl(\emptyset)$. 
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**Fact**

Let $I = \langle a_i \mid i \in \omega \rangle$ be Morley, and let $b \downarrow a_0$. Then there is $b' \equiv_{a_0} b$ such that $I$ is Morley over $b'$ and $b' \downarrow I$.

Above fact is a particular case of 3-amalgamation (IT), and used crucially in showing IT for simple $T$. 
Definition

$T$ is $K(n)$-simple if for $k \leq n$ and any Morley $l = \langle a_i \mid i \in \omega \rangle$, whenever $l_k = \langle a_i \mid i < k \rangle$ is Morley over $b$ with $b \downarrow l_k$, there is $b' \equiv_{l_k} b$ such that $l$ is Morley over $b'$ and $b' \downarrow l$. 

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- We say $T$ has $n$-CA if $T$ has $k$-amalgamation for all $k \leq n$. 
Definition

- $T$ is $K(n)$-simple if for $k \leq n$ and any Morley $I = \langle a_i | i \in \omega \rangle$, whenever $I_k = \langle a_i | i < k \rangle$ is Morley over $b$ with $b \downarrow I_k$, there is $b' \equiv_{I_k} b$ such that $I$ is Morley over $b'$ and $b' \downarrow I$.
- $T$ is $n$-simple if for $k \leq n$ and any Morley $I = \langle a_i | i \leq k \rangle$, whenever $I_k = \langle a_i | i < k \rangle$ is Morley over $b$ with $b \downarrow I_k$, there is $b' \equiv_{I_k} b$ such that $I$ is Morley over $b'$ and $b' \downarrow I$.
- We say $T$ has $n$-CA if $T$ has $k$-amalgamation for all $k \leq n$.

Both $n$-simplicity, $K(n)$-simplicity are particular cases of $(n + 2)$-CA.

Question

Are those 3 notions equivalent?

simple = 1-simple = $K(1)$-simple = 3-amalgamation = 3-CA
Yes

(Kolesnikov) 2-simple = K(2)-simple = 4-amalgamation = 4-CA
<table>
<thead>
<tr>
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Yes

(Kolesnikov) 2-simple = K(2)-simple = 4-amalgamation = 4-CA

Yes and No

(K, Kolesnikov, Tsuboi) Yes:
\( n \)-simple = \((n + 2)\)-CA

No: For each \( n \geq 3 \), there is an example of \( K(n) \)-simple but not having \((n + 2)\)-CA.
\( \mathcal{L} = \{ R \} \), \( R \) is a \( n \)-ary relation.

\[ \mathcal{K} := \{ A \mid A \text{ is a finite } R\text{-structure; } R \text{ is symmetric and irreflexive; for any } A_0 \subseteq A \text{ with } |A_0| = n + 1, \text{ the no. of } n\text{-element subsets of } A_0 \text{ holding } R \text{ is even } \} \]

The Fraïssé limit of \( \mathcal{K} \) is the simple \( \omega \)-categorical example.
Stable
Stable | Simple

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Geometric Simplicity Theory

- $n$-amalgamation
- $\mathcal{M}^{\text{eq}}$
- $n$-simplicity

Fields in simple theories

Diagram:

- Stable
- Superstable
- Simple
Fields in simple theories

- $n$-amalgamation
- $\mathcal{M}^{\text{eq}}$
- $n$-simplicity

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Fields in simple theories

\[ n\text{-amalgamation} \quad M_{\text{eq}} \quad n\text{-simplicity} \]

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Geometric Simplicity Theory
Fields in simple theories

$n$-amalgamation

$M \mathrel{\models} n$-simplicity

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Geometric Simplicity Theory
Fields in simple theories

$n$-amalgamation

$\mathcal{M}^\text{geq}$

$n$-simplicity

Geometric Simplicity Theory

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Stable
SCF
Superstable
ACF, DCF
V.Sp

Simple
PAC
Supersimple
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**Geometric Simplicity Theory**

- **Stable**
  - SCF
  - ACF, DCF
  - V.Sp

- **Superstable**
  - ACF, DCF
  - V.Sp

- **Simple**
  - PAC
  - Psf, V.Sp with forms
Definition

$F$ a field.

- $F$ is said to be PAC if any absolutely irreducible algebraic set $V$ defined over $F$ has $F$-rational point.
- $F$ is *pseudo-finite* if it is an infinite field of the theory of all finite fields.
- An extension field $E$ of $F$ is *separable* if it is algebraic over $F$, and for each $e \in E$, $\text{irr}_F(e)$ has no multiple roots.
- $F$ *perfect* if every algebraic extension is separable; equivalently either $\text{Char}(F) = 0$ or $x \mapsto x^p$ where $p = \text{Char}(F) > 0$ is onto.
- $F$ is *bounded* for each $n > 1$, there are only finitely many separably algebraic extensions of degree $n$.
- $F$ is *separably closed* if it has no proper separable extension.
Fact

\( F \) PAC.

- (Chatzidakis) \( F \) is simple iff \( F \) is bounded.
- (Hrushovski, Pillay, Poizat) \( F \) is supersimple iff \( F \) is perfect and bounded.
- (Duret, Wood) \( F \) is stable iff \( F \) is separably closed.
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Fact

(Ax)

A field \( F \) is pseudo-finite iff
1. it is perfect
2. it is PAC, and
3. for each \( n \) it has exactly one algebraic extension of degree \( n \)

Hence pseudo-finite fields are unstable supersimple (of SU-rank 1).
Fact

• (Macintyre; Cherlin, Shelah) Superstable division ring is an algebraically closed field.
• (Pillay, Scanlon, Wagner) Supersimple division ring is a field.
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- (Macintyre; Cherlin, Shelah) Superstable division ring is an algebraically closed field.
- (Pillay, Scanlon, Wagner) Supersimple division ring is a field.

Open Problem

- Is any stable field separably closed?
- Is any supersimple field PAC?