COMMENTS BY THE EDITOR

d like to take this opportunity to welcome to the Editorial Board, I. Richard Savage f The Florida State University. At the same time that Professor Savage is joining e Editorial Board two others are leaving. On behalf of the Editors, I want to extend ecial thanks to William McPhee and Roland Robertson for their help during the st two years of the Journal's existence. Finally, the Journal has made use of a large mber of reviewers and to them I wish to extend a special thank you. No journal is y better than the quality of the reviewing that is done. My special thanks go there-re to the following list of people who have given graciously of their time and talent reviewing manuscripts for the *Journal of Mathematical Sociology* this last year:

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SYSTEMS OF SOCIAL EXCHANGE

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This paper develops a formal model for exchange within a system of action. The system of action is defined by *actors*, *events*, *control* of actors over events, and *interests* of actors in the outcomes of events. The system is designed to deal with divisible events (best exemplified by private goods) or indivisible events (such as a bill on which a vote is taken), and events with or without externalities. This paper considers mostly divisible events, but some steps are taken toward analysis of indivisible events on which there are opposing interests.

In this paper, I want to present a theoretical framework appropriate for behavior in informal groups, but with extensions to more formalized structures of collective action. The basic structure will turn out to be equally appropriate for representing the structure of action in an economic market of private goods, while the extensions involve action structures that violate the simplicity of private goods markets. I will first outline the conceptual framework, then show its application to simple informal situations, and finally shows its extension to structures that are less simple, involving collective decisions and collective action.

EVENTS, ACTORS, CONTROL, INTEREST

A useful point from which to initiate any examination of behavior in informal groups is the idea of 'power.' Power is a very ambiguous term in social organization, ambiguous in several respects. First, it is sometimes used to refer to a relation between two individuals ('A has power over B'), and sometimes used to refer to a relation between an individual and a certain activity ('A has power over activity X'). For example, in the Encyclopedia of the Social Sciences, Dahl defines power as a relation between individuals, while most of the work on community power structures (by Hunter, Freeman, and others) implicitly or explicitly defines power as a relation between an individual and an activity.

Second, power sometimes refers to a dimension or ordering created by transitivity of the power relation, so that if A has power over B, he also has power over those persons $C, D, E, \ldots$, which B has power over. In such a case, we could speak of more and less power, and know that if A has more power than B, he can always get his way in any struggle with B. In other work, however, power is not treated as necessarily transitive, but is a private relation between A and B that does not have any implication for the power of A over any other person.

These confusions arise, I believe, from the absence of an internally consistent conceptual framework within which power is embedded. In the framework I will present, the fundamental relation is a relation between individuals and activities (or 'events' as I shall usually call them), with the result that power between individuals is a derivative quantity. It will turn out to have the following properties:

1. In a system with perfect social exchange, power is transitive, and also contains a metric, such that if the power of A plus the power of C exceeds that of B, then A and C can determine the outcome of the event if they both favor one outcome, and B favors the other—assuming that A, B, and C all have equal interests in the event's outcome.

2. In a system without perfect exchange, power is not always transitive, the amount of intransitivity depending on the imperfections in exchange. (Demonstrations of this must be reserved to a later paper, for this paper considers only perfect exchange systems.)

3. When control over an event is divided, as a result of the physical or constitutional constraints under which action takes place, the outcome of the event depends not only upon the power of the actors and their interests in the outcome, but also upon the particular decision rule that the physical environment or constitution imposes.

These points give some idea of the kind of conceptual system to be developed. It will outline it in more detail below. A complete exposition may be found in two forthcoming publications (Coleman, 1973a, 1973b).

We first consider a system of action composed of two kinds of elements, actors and events. Actors have an acting self and an object self, the acting self taking actions to affect the outcomes of events, and the object self being affected by the outcomes of events. There are, then, two properties linking actors and events: control of events by actors (i.e., by the acting self), and consequences of events for actors, or as I will describe it, interests of actors in events (i.e., consequences of events for the object self). There is, in addition, one behavioral postulate: each actor will act so as to best satisfy his interests, given his resources.

The simplest system of action to be considered is one in which the events have two properties that I will call *divisibility* and *internality*. A divisible event is one in which a fraction of control over the event represents full control of a fraction of the consequences; and an internal event is one in which exercise of control gives consequences only to that actor who exercises control, and no others. The best examples of divisible and internal events are finely divisible private goods whose consumption creates no externalities for others. Fractions of control over a quantity of a private good can be realized through division of that quantity into appropriate fractions fully controlled by each actor. Consumption of that quantity of the good has consequences only for the consumer, if the good has no externalities. An event which is divisible but not internal is exemplified by what economists term a private good with consumption externalities. It can be divided into quantities (i.e., sub-events), with independent outcomes, but consumption of each of these quantities produces some consequences for persons other than the consumer. An event which is not divisible is an event which cannot be divided into sub-events with independent outcomes, but has a single outcome. A fraction of control can mean only a fraction of the power to determine the event's outcome (through voting or some other means). An example of an indivisible but internal event is a private good which cannot be divided, but whose use has consequences only for the user. A house or other large indivisible commodity is an

example. Often, such indivisible goods are jointly owned, with rights of usage divided among the owners. In poor countries, an automobile may be owned by several persons; in all countries, many goods are in effect owned by a household consisting of several members, all of whom have rights of usage. Figure 1 shows the types of events deriving from the two dimensions of divisibility and internality.

		Internal: Yes	Internal: No
Divisible	Yes	private goods or actions that affect no one else	events or actions with externalities
	No	goods or events too large to be individually controlled. collective decisions required	events requiring collective decisions

FIGURE 1 Types of events distinguished by divisibility and internality

Although systems of action involving divisible and internal events are exemplified by economic markets of private goods, there are also non-economic activities that fit these definitions. In a social group in which each person is interested in gaining the attention or time of particular others, each person's attention or time can be conceived as a divisible event to be distributed among other persons. So long as the group is not engaged in any collective action to be participated in jointly by all or by some subset, but all activities involve pairwise interactions, then a system of divisible internal events can mirror this action.† Thus we can think of a model for divisible internal events as appropriate for a group in which there is interaction but no joint or common action, no joint or collective decisions which entail joint or collective action.

In a system with divisible internal events, there is one state of the system, one distribution of control over events, in which each actor has no reason for interaction with others, because his distribution of control over events is the one most satisfactory to him, given his resources. But if the system is in any state other than this, some actors will find that they can best realize their interests by exchanging control over one event for control over another.

Such a system of activities can be modelled by a linear system with the following characteristics:

1. There are m events and n actors

2. The amount of initial control over event i by actor j is represented by c_{ij}, where $0 \le c_{ij} \le 1$, and $\sum_j c_{ij} = 1$. Since events are divisible, c_{ij} represents the fraction of i that actor j has full control over.

† A complication is produced when outcomes of the events are not independent. For example, if j spends time with k, this means that k must spend that same amount of time with j. For this paper, I will ignore such complications.

3. Each actor has an interest in event i, x_{ji}, with the following properties:

(a) $0 \le x_{ji} \le 1$

(b) $\sum_i x_{ji} = 1$

(c) x_{ji} represents the fraction of his resources that he allocates in a perfect exchange system toward control of event i, independent of the total size of his resources and independent of the cost of gaining control of event i. This independence of allocation from cost and resources represent two behavioral assumptions. They can also be expressed in economists' terms as assumptions that the price elasticity and resource (or income) elasticity for all events equal -1 and $+1$ respectively.† In the appendix, these two properties (independence of cost and independence of total resources) are derived from the Weber-Fechner law, in conjunction with the behavioral postulate of maximization of satisfaction.

The quantities x_{ji} and c_{ij} are the fundamental properties relating actors and events.

If we let v_i be the value of full control over event i, and c_{ij}^* be the amount of control over event i that actor j controls at equilibrium, then $v_i c_{ij}^*$ is the amount of resources he must devote to event i to control the amount c_{ij}^*. If we let r_j be defined as the actor's total amount of resources, then according to (c) above, the amount of resources he devotes to i also equals $x_{ji} r_j$. Thus by definition of x_{ji}, we have in a perfect exchange system the equation

$$v_i c_{ij}^* = x_{ji} r_j, \qquad (1)$$

where, following the two assumptions under (c) above, x_{ji} is a constant for actor j and event i, independent of r_j and v_i.

The derived quantities v_i and r_j can be defined in terms of the matrix of x_{ji}'s and the matrix of c_{ij}'s by summing equation (1) over i and over j. First summing over i gives:

$$\sum_{i=1}^{n} v_i c_{ij}^* = r_j \sum_{i=1}^{m} x_{ji},$$

and since $\sum_i x_{ji} = 1$,

$$r_j = \sum_{i=1}^{m} v_i c_{ij}^*. \qquad (2)$$

Because in a perfect exchange system the value of an actor's total resources does not change, r_j is also equal to the value of his initial control.

$$r_j = \sum_{i=1}^{n} v_i c_{ij}. \qquad (3)$$

Summing equation (1) over j gives

$$v_i \sum_j c_{ij}^* = \sum_{j=1}^{n} r_j x_{ji},$$

† Price and income elasticities of -1 and $+1$ are often considered by economists to be the 'normal' lasticities in private goods consumption, from which certain goods may deviate. They imply declining marginal utility of a good, or more particularly, that the marginal utility of a good is inversely proportional to the amount of the good already held.

and since $\sum_j c_{ij}^* = 1$,

$$v_i = \sum_{j=1}^{n} r_j x_{ji}. \qquad (4)$$

Equations (2) and (4) provide intuitively appealing definitions for the resources held by each actor and the value of each event in a system of perfect exchange. Stated in words, these definitions are, from equation (3) and equation (4):

...The resources held by actor j consist of the sum of the initial control he has over all events, each event weighted by its value; and

...The value of an event is the sum of interests in the event, each actor's interests weighted by his total resources.

These definitions constitute the framework of the simple system of divisible events. The initially given quantities are the matrix of control, C, with elements c_{ij}, and the matrix of interests X, with elements x_{ji}. From these may be calculated the value of each event, and the resources of each actor in a system of perfect exchange, through equation (2), equation (4), and the final, equilibrium control that will be held by each actor, through equation (1). The solutions for resources and value may be found from joint use of equations (3) and (4). Substituting for v_i in equation (3) its value from equation (4) gives

$$r_j = \sum_{i=1}^{m} \sum_{k=1}^{n} r_k x_{ki} c_{ij}. \qquad (5)$$

Solution of this set of simultaneous equations for $r_1, \ldots, r_n$ (using also the fact that $\sum_j r_j = 1$) allows calculation of resources. Similarly, substitution for r_j in equation (4) its value from equation (3) gives

$$v_i = \sum_{j=1}^{n} \sum_{k=1}^{m} v_k c_{kj} x_{ji}. \qquad (6)$$

Solution of this set of simultaneous equations, using the fact that $\sum_i v_i = 1$, allows calculation of values of events.

This framework, though it contains no elements of conflict (since events are divisible with internal consequences), begins to give an idea of the way power will be treated in this system. Power in this system is merely another name for what I have called resources. It is derived from control over events, and is a quantity with a metric, showing just how much of the value of the system is held by actor j. Like money in an economic system of private goods, it is not a relation between two actors, but something which can be used in exchange to increase satisfaction, subject to the quantity that one begins with. It is not, however, limited to economic systems, as the example of its application in informal groups in the next section will show.

EXAMPLE OF APPLICABILITY TO INFORMAL GROUPS

Because data are ordinarily not collected in ways that allow the applicability of this theory to real groups, the examples in this and other sections are hypothetical. Assume there are three people together in a ski resort. There are two men and one girl. One man knows how to ski, and the other two people, have some interest in learning (which can only be from him). The man who does not ski has the most money, the girl has only half as much, and the skier only a sixth as much. The two men each have an interest in money equal to their interest in the girl's attention. She has an equal interest in attention from each of them, and an interest in money equal to her interest in attention from the men. The structure of interests and control is given by the following matrices:

$X =$	1 skier's attention	2 m.m. attention	3 girl's attention	4 money	5 learning to ski
1 skier	0	0	.5	.5	0
2 moneyed man	0	0	.4	.4	.2
3 girl	.2	.2	0	.4	.2

$C =$	1	2	3
1	1	0	0
2	0	1	0
3	0	0	1
4	.1	.6	.3
5	1	0	0

These matrices of interest and control can be used, with equation (5), to calculate the resources of the skier, the moneyed man, and the girl. Use of equation (5) and $\sum r_j = 1$ gives

$$r_1 = .05r_1 + .24r_2 + .44r_3,$$
$$r_2 = .3\ r_1 + .24r_2 + .44r_3,$$
$$r_3 = .65r_1 + .52r_2 + .12r_3,$$

and $1 = r_1 + r_2 + r_3$.

Solutions are:

$$r_1 = .268$$
$$r_2 = .335$$
$$r_3 = .396$$

Thus the three persons have unequal resources in the situation with which to realize their interests. The skier has least, the girl has most, and the moneyed man is between the two. Use of equation (4) and equation (1) allow calculation of the distribution of control after exchange, at equilibrium:

$C^* =$	1	2	3
1	0	0	1
2	0	0	1
3	.5	.5	0
4	.314	.314	.371
5	0	.459	.541

The girl has all the attention of both men, the skier has half and the moneyed man half of the girl's attention, the skier now has 31.5% of the money and the girl somewhat more than she started with, and both the moneyed man and the girl get ski instruction, though the girl gets more.

This is a trivial example, but it illustrates how the conceptual framework operates to characterize a system of action in a group. If the moneyed man had had a greater

proportion of the money at the start, he would have ended both with more of the girl's attention and with more of the ski instruction. If the girl and the moneyed man had had more interest in learning to ski, the skier would have gained both more of the girl's attention and more of the money; if the skier were uninterested in money he would have had more of the girl's attention, and she more instruction; and so on for other variations.

EXTENSIONS TO SYSTEMS WITH INDIVISIBLE EVENTS

In a system of action with divisible events, a fraction of final control over event i by actor j, c^*_{ij}, is well-defined. It means full control of a fraction of the event or good. But with indivisible events, the quantity c^*_{ij} is not well-defined unless it is 0 or 1, or unless it has been given a definition by introduction of a decision rule. For example, if the decision rule is a majority rule with a coin flip when control is equally divided, then $c^*_{ij} > .5$ represents full control, while $c^*_{ij} < .5$ represents no control, and $c^*_{ij} = 0.5$ represents full control with probability 1/2 and no control with probability 1/2.

In addition to the problem of divided control that is posed by indivisible events, there is also the question of what is meant by 'interest' when there is no continuously-divisible quantity. For with an indivisible event, interest cannot be expressed in terms of the increment of satisfaction per increment of control. If it is a good, and truly indivisible, one experiences (or consumes) it either completely or not at all. If it is some more general state of the world, it either comes to pass or does not. Thus with truly indivisible events, we can think of only two outcomes, each giving some level of satisfaction or utility. It is possible, with such events, to design a procedure which could assign quantitative measures to these utility levels, which are specified up to an arbitrary scale coefficient and an arbitrary zero point.† The utility difference between outcomes is then specified up to a scale constant (since the constants for the zero point cancel). This arbitrary constant is specified by the criterion that $\sum_i x_{ji} = 1$.

Thus it is possible to think of the interests in indivisible events as the *relative utility differences* between the positive and negative outcomes of an event. Mathematically, if u^+_{ji} and u^-_{ji} are respectively the utilities of positive and negative outcomes of event i to actor j, then

$$x_{ji} = \frac{|u^+_{ji} - u^-_{ji}|}{\sum_{k=1}^{m} |u^+_{jk} - u^-_{jk}|}. \qquad (7)$$

As indicated above, any other measures of utility w_{ji} which are related to u_{ji} such

† One procedure for doing this has been described by Von Neumann and Morgenstern (1947). This procedure involves choice between risky situations involving event outcomes, and involves the assumption that subjectively perceived probabilities correspond to objective probabilities. Another procedure, involving partial control of the event through vote trading in a collective decision, is described in the Appendix. It is the latter procedure that should be taken as defining interests in the present model, because the procedure involves observations of behavior that is intrinsic to the theory—an important criterion in arriving at measures of concepts in a theory.

that $w_{ji} = a + bu_{ji}$ are valid. This can be seen by substituting $a+bu_{ji}$ for u_{ji} in equation (7), giving an equation which reduces back to equation (1). (For another reason which can be seen from discussion in the appendix, the sign of b must be positive.)

Thus interest must be defined differently for indivisible events than for divisible events, since they cannot be experienced or consumed in partial quantities. The definition of interest, however, as relative utility difference between positive and negative outcomes, is both compatible with the definition for divisible events and intuitively appealing.

It appears reasonable to apply the theory to indivisible internal events, for certain circumstances in which there is no conflict over the desired outcome. Examples of this are as indicated earlier, private goods that cannot be finely divided, such as a country club, or a pleasure boat. The fineness of division necessary is relative to the resources of the actors; a good example is automobiles, which are indivisible, but individually owned in rich countries, while they are often jointly owned because of their indivisibility in poor countries.

The formal analysis for indivisible internal events is like that for divisible ones, up to a certain point. But once the analysis is carried out, showing the final control, then further questions arise, if the matrix of final control shows divided control of indivisible events. For example, in the ski resort example, suppose a constraint existed, that ski instruction must be given to the two learners together, rather than individually. Then the matter becomes more complicated, because any of several conditions might exist. First, the ski instruction may have the nonconservative property of a public good, so that the total possible amount of instruction available now is twice what it was before, with the constraint that either learner can come to control a maximum of 1. That would imply one kind of analysis, in which ski instruction is split into two events which are tied together, in the sense that control of them either passes to the two learners or remains in the hands of the skier. If control does pass to the two learners, then there arises the problem between them of how costs are allocated between them. This is the usual problem of paying the cost of a public good, which arises quite generally with indivisible events. If the instruction does not have this nonconservative property, but the consumption remains tied together and must be equal, then the final control cannot vary between the two learners, as in the preceding example, but must be divided 0.5 and 0.5. Some indivisible events have this property, while others do not. As an example of indivisible events that do not have fixed ratios of final control, joint purchase of a yacht by two persons may be through unequal shares, leading to unequal rights of usage by the two persons. In this case of variable rights of usage, the mathematical analysis remains the same as the simple divisible internal event analysis, so long as all the persons who, in the analysis, end up with some final control are able to divide usage according to their degree of control. This requires some organization (which the model assumes), but does not involve the problem of paying the cost of a public good, since each actor gives up resources in proportion to his rights of usage.

For the other variations discussed above, however, the present mathematical system is not sufficient, and it will be necessary in future work to introduce appropriate modifications to allow these variations to be mirrored.

There are other kinds of situations in informal groups in which events may b[e] thought of as indivisible, but in which the question of final control does not arise allowing them to be studied as if they were divisible events. When all actors favo[r] the same outcomes for all events, then the question of what a fraction of contro[l] represents behaviorally need not be resolved in order to calculate the power of eac[h] actor and the value of each event. This can be illustrated by use of an example.

Suppose in a group each member is asked his interest in each other member's participation in the group. Although participation vs. nonparticipation is an indivisible event, interests could be elicited from group members as if it were divisible: 'Suppose you knew that altogether there would be 100 hours of the presence of all other members of the group in the next 100 hours of its activities. If you had your preference, what would be the distribution of amounts of time present among the other members, summing to 100 hours altogether?'

Such a question posed to all members would give data that could be directly interpreted as an interest matrix. Since initially each person has control of his own presence, the control matrix C would have 1's in the main diagonal and zeroes elsewhere. If actor 1 were quite popular with some of the group members, and 2 with others, interest and control matrices for a group of 5 members might look like this, where event i is the presence of actor i in the group.

X		Events 1	2	3	4	5
	1	0	.25	.25	.25	.25
actors	2	.4	0	.2	.1	.3
	3	.5	0	0	.2	.3
	4	.3	.6	.1	0	0
	5	.3	.7	0	0	0

C		Actors 1	2	3	4	5
	1	1	0	0	0	0
events	2	0	1	0	0	0
	3	0	0	1	0	0
	4	0	0	0	1	0
	5	0	0	0	0	1

Calculation of the power of each actor in this group would show power as follows:

$$R = .273, .276, .136, .123, .192$$

Values of events 1–5 have the same distribution as power of actors. The power of members 1 and 2 is about equal, the power of 3 and 4 is about equal, and the power of 5 is intermediate between these two levels. This means that 1 and 2 could get their ways more often than 3 or 4, using the threat of nonparticipation. It means that if there were a formal constitution to the group and votes were allocated among the members, then the distribution of votes that would preserve the interests of each in the others' participation is a distribution which give actor 1 a power of .273 of the whole, and so on for the others. (This is not the same as .273 of the total votes, because the discontinuous character of most voting rules makes power nonlinearly related to number of votes. However, if the voting rule were probabilistic, with a chance mechanism giving a positive outcome with probability equal to the proportion of votes cast in favor, then power to control the outcome is equal to the proportion of votes held.)

This example suggests a rational basis for the allocation of power to group members in establishing a constitution. The power that each member has, in a constitution created in this way, is merely the embedded interests of all other members in his participation in the group. Thus his withdrawl from the group would reduce the others' interests in the group by that amount. Giving him the power indicated is giving power toward the direct satisfaction of his interests in future activities, but also toward the indirect satisfaction of others' interests.

DIVISIBLE EVENTS WITH EXTERNALITIES

There are some divisible events which nevertheless retain, even after having been divided, interest of more than one actor in events controlled by that actor. Private goods that exhibit externalities exemplify this kind of event. Use of water from a stream by an upstream actor, a town, a firm, or a family, changes the quality of the water for the downstream user. The situation can be mirrored by a very simple application of the theory, which is like that for divisible internal events except that now each actor's use or consumption of a good must be designated as a distinct event. This means that divisible events, such as private goods, are redefined in such a way that they become indivisible events, by being broken down into each actor's consumption or use. In this example, there are two actors and three events:

Actor 1: The upstream water user
Actor 2: The downstream water user
Event 1: Use of water by the upstream user
Event 2: Use of water by the downstream user
Event 3: A generalized resource (money) divided in some ratio between them

The upstream user has control over his water use and some fraction of the total noney; the downstream user has control over his water use and some fraction of he total money. The upstream user has interest only in his water use and in money; he downstream user has interests opposed to the upstream user's water use, and nterests in his own use and in money. Interest and control matrices might be:

X		actor 1's use	actor 2's use	money
pstream user	1	.5	0	.5
ownstream user	2	(−).2	.3	.5

C	Upstream user	Downstream user
actor 1's use	1	0
actor 2's use	0	1
money	.3	.7

The negative sign in parentheses indicates that outcome 2 of event 1 (non-use of ter by actor 1) is the outcome desired by actor 2. (Formally, this could be introced by a third matrix S with $+1$, -1, and 0, showing the sign of the directed erest of each actor in each event. In this case, the S matrix has a -1 only in s_{21}.) The question in cases like this is what will happen: will the upstream user continue pollute, or will the downstream user be able to induce him to stop his use? Note t in this application of the model, there is no mechanism other than use of the eralized resource, money, by which the downstream user can induce the upstream

user to stop. If a political process had been included in the model, then depending on actor 2's control over other political events in which actor number one was interested, he might be able to gain passage of a bill to prevent actor 1 from polluting. The general means by which this would be done would be political exchange, mirrored by this theory in much the same way as here, except with an expanded set of events and actors.

The application of the model in this case is somewhat different than for events with no externalities. Since we want to see whether the action will be carried out, we consider two sets of event outcomes, two possible 'regimes': $+++$ and $-++$. It is only these outcomes which are desired, the first set more desired by actor 1 and the second set more desired by actor 2. The model is applied twice, first by excluding the interests of actor 2 opposed to 1's water use, and then by excluding the interests of actor 1 favoring his water use. Then it is possible to see whether actor 1's power (under the first regime) that he is willing to devote to water use (as measured by x_{11} times his power) is greater than actor 2's power (under the second regime) that he is willing to devote to opposing 1's water use (measured by x_{21} times his power). Calculations are as follows:

Regime A: Outcome $+++$

The interest matrix is revised so that the second row is 0, .375, .625.

The equation $R_a = R_a X_a C$ gives:

$$r_{1a} = .65r_{1a} + .1875r_{2a}$$
$$r_{2a} = .35r_{1a} + .8125r_{2a}$$

Solving gives $r_{1a} = .35$, $r_{2a} = .65$

The ability of actor 1 to implement this outcome is $r_{1a}x_{11}$, or

$$.35 \times .5 = .175.$$

Regime B: Outcome $-++$

The interest matrix is revised so that the first row is 0 0 1; the second row is as in X.

The equation $R_b = R_b X_b C$ gives:

$$r_{1b} = .3r_{1b} + .35r_{2b}$$
$$r_{2b} = .7r_{1b} + .65r_{2b}$$

Solving gives $r_{1b} = .333$, $r_{2b} = .667$.

The ability of actor 2 to implement this outcome is $r_{2b}x_{21}$, or

$$.667 \times .2 = .133.$$

Comparison of $r_{1a}x_{11}$ with $r_{2b}x_{21}$ shows that the former is larger, so that outcome $+++$ can be successfully implemented by actor 1, in opposition to outcome $-++$. Thus the pollution will continue, with actor 2's resources not quite great enough to overcome the greater interest that actor 1 has in his use of the water than actor 2 has in its being stopped. Thus the correct interest matrix to use for assessing final control is the one in which the downstream user's interests opposed to the upstream user are deleted. He cannot gain control of that event, and thus must allocate his resources elsewhere. The matrix is

.5	0	.5
0	.375	.625

If actor 2's proportion of the generalized resource were even greater than it is here, he could have enough power to successfully oppose actor 1, paying him to stop polluting the water—and paying a price high enough that actor 1 would find it to his interest to stop. In that case, the appropriate interest matrix to use would be

0	0	1
.2	.3	.5,

and since the upstream user began with control over the event, the downstream user's interests would be realized only through purchasing that control by use of event 3, his generalized resource. In that case, actor 1 would end up with .5 of the money, rather than .3, in return for his loss of control of his water usage.

In informal groups, there are many types of divisible events with externalities. Some of these may be treated by the above kind of analysis; others require some modification of this analysis. One such modification involves conflict processes, as discussed below.

CONFLICT

All the analysis to this point has assumed that one or the other interest in the event with externalities will be pursued, and the other interest will be withdrawn by the actors who hold it, in return for compensation if they initially hold the rights to the action. Such withdrawal in the presence of an opposition that can mobilize more powerful resources is a rational action, while expenditure of resources that are either less than those of the opponent, or that are excessive, constituting a diversion of resources from other events that would bring greater gain, is not rational. That is if for actor 2, $r_{2b}x_{21}$ is less than $r_{1a}x_{11}$, it is not rational to expend resources $r_{2b}x_{21}$ on the event, because his opponent will spend more, and win. Neither is it rational to spend more than $r_{2b}x_{21}$, say enough to exceed $r_{1a}x_{11}$, because the extra resources spent in that way will bring less satisfaction, even if they are sufficient to gain control of the event, than if they are employed to gain control of other events in proportion to his interests.

However, it may well be that $r_{1a}x_{11}$ and $r_{2b}x_{21}$ are close enough that both sides estimate that they will be able to gain control of event 1. If control is gained merely through a market process in which the losing side can recover his offered resources, then the system will operate as described earlier, and those resources will be deployed in alternative ways. But if employment of these resources constitutes a struggle for control over the event, as is often the case in noneconomic transactions in society, the resources, or some large part of them, are lost and constitute a waste. We can, in effect, specify three levels of social functining in a system where $v_{1a} > v_{1b}$. The first level is for the action to take place without employment of any opposing resources by the aggrieved parties—who would be compensated if they initially held rights to the action, but in all cases would use the resources $r_{2b}x_{21}$ in other ways. The second level is for the action *not* to take place, and for actor 1 to employ the resources that he would have used for event 1, that is resources $r_{1a}x_{11}$, in other ways. In this case, the magnitude of the loss is a function of $v_{1a}-v_{1b}$ (or $r_{1a}x_{11}-r_{2b}x_{21}$), for it is this extra amount of resources that is being redirected to events that produce lesser utility.

The third level of social functioning occurs when *both* sides employ their resources for control of the event, in a struggle for control in which the resources of the losing side are used up in the struggle. In this case, the loss is not merely the deployment of resources on events that bring lesser utility, as in the second level of functioning; it is a total loss of the resources.

The resources lost in this case cannot be calculated under regime *a* or regime *b*, because both of those regimes assume a redeployment of resources of the losing side, to give a set of values v_{ai} or v_{bi} for events *i* which sum to 1.0, the total set of resources in the system. What is necessary is to assume a regime *c*, in which the event with externalities is represented by *two* events, one in which a positive outcome is carrying out the offending action, and the other in which a positive outcome is not carrying out the offending action. Each of these events is fully controlled by each of two new hypothetical actors, actor $n+1$ for event *i*, and actor $n+2$ for event *k*. These actors have exactly the same interest distribution as the real actor that initially controls action *i*; but one of these actors, whose event is of lower value, will withdraw his resources from the system, representing a resource loss. The other, whose event is of higher value, will combine his resources with the real actor who initially controls right to event *i*, to give his final power.

The winning side is actor 1 if $v_{ic} > v_{kc}$, and actor 2 if $v_{ic} < v_{kc}$. This will ordinarily, but not necessarily, be the same side that will win when there is a non-destructive use of resources to gain control of the event, as in regime *a* or regime *b*. The lack of complete correspondence arises because under this different resource deployment, other events will have different values, and thus those who control them different resources.

The functioning of such a system in which there is a struggle for control with destruction of the resources of the losing side can be illustrated by use of earlier example under which regimes *a* and *b* were compared.

EXAMPLE WITH CONFLICT: REGIME *C*

In this example, the interest and control matrices are like those in the preceding example, except that event 1 (actor 1's use of water) becomes two events, 1 and 4. In addition, there are now two hypothetical actors, 3 and 4, whose interest distribution is identical to that of actor 1, and who control respectively events 1 and 4. The revised interest and control matrices are:

X

		E_1	E_2	E_3	E_4
actors	A_1	.5	0	.5	0
	A_2	0	.3	.5	.2
	A_3	.5	0	.5	0
	A_4	.5	0	.5	0

C

		A_1	A_2	A_3	A_4
events	E_1	0	0	1	0
	E_2	0	1	0	0
	E_3	.3	.7	0	0
	E_4	0	0	0	1

Calculation of resources of the five actors, using equation (5) gives

$$r_{1c} = .15$$
$$r_{2c} = .5$$
$$r_{3c} = .25$$
$$r_{4c} = .1$$

Resources devoted to event 1 by actor 1 (which includes resources of hypothetical actors 3 and 4) and to event 4 by actor 2 are:

$$v_{1c} = (r_{1c}+r_{3c}+r_{4c})x_{11} = .5\times.5 = .25$$
$$v_{4c} = r_{2c}x_{21} = .5\times.2 = .10$$

Since $v_{1c} > v_{4c}$, event 1 has a positive outcome (the upstream user uses the water), and event 4, which is logically incompatible with it (non-use of the water) has a negative outcome. This means that resources of value .10 are lost through conflict between actors 1 and 2. Hypothetical actor 4 withdraws his resources from the system, and the power of actor 1 is augmented by that of actor 3, whose resources remain in. Thus the power of actor 1 is $r_{1c}+r_{3c}$, or .40. The total power in the system is now .40, .50, summing to .90 rather than 1.00; and the total value in the system is identical. The greater power of actor 1 derives from his control of events 1 and 4 (through hypothetical actors 3 and 4), which have, in this case, more interest concentrated on them than in the case where one side invested no resources in the event.

It is not in fact necessary to explicitly introduce the two new actors into the system, for they behave just like the actor who initially controls the event. They serve merely as a conceptually clarifying device, showing how resources are subtracted from the system, being withdrawn both from the value of events and from the power held by the actor who controls the event in question.

The introduction of conflict over divisible events with externalities is the opening wedge of a much broader investigation of collective decisions and conflict, involving many actors on both sides and many issues. This section has given an indication of how some aspects of such conflicts may be treated within the present framework of ideas. But this broader examination of conflict must be deferred to a subsequent paper. Here the principal point is that when there are events with external effects opposite to the effects of the event for the actor himself, two kinds of social processes might occur: market valuation of the two outcomes of the event, with only one side devoting resources to gaining or keeping control of the event; or conflict, in which both sides devote resources to the event, and those of the losing side are wasted resources. In the latter case, the level of social efficiency of the system is below that in which the same side won, but with a redeployment of the potentially opposing resources by the losing side.

EXTENSIONS

Extension to collective decisions, with or without explicit decision rules, is possibl and some work has been carried out on those extensions, without, however, solvi certain of the central problems. An initial statement is found in Coleman (1964, 1966 and further work is reported in Coleman (1973a). Other extensions, in particular imperfect exchange processes, have not been carried out. Work in both these directio is important to further development of the theory.

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APPENDIX

1. INTEREST IN DIVISIBLE EVENTS

In a system of action involving divisible events, it was assumed that the fraction of an actor's total resources allocated to gaining control of event i was independent of the cost of control of event i and independent of the actor's total resources. This can be derived from the Weber-Fechner law in conjunction with the behavioral postulate that the actor will act so as to maximize his satisfaction given his resources.

The Weber-Fechner law states that the increment of subjective state experienced by a given increment of objective stimulus is inversely proportional to the existing level of the objective stimulus. If the subjective state is taken as actor j's satisfaction with event i, denoted by s_{ij}, and the existing level of the objective stimulus is taken as the amount of i over which he has control, c_{ij}, then the Weber-Fechner law states that

$$\frac{\Delta s_{ij}}{\Delta c_{ij}} = k_{ji}\frac{1}{c_{ij}} \qquad \text{(A.1)}$$

where k_{ij} is a nonnegative constant for actor j associated with event i showing the amount of satisfaction derived from a 100% increase in the amount of event i controlled. The amount of resources required to gain an increment of control Δc_{ij} over event i is that increment times the value of i in the system, v_i, or $v_i \Delta c_{ij}$. Thus the increment of satisfaction per increment of resources expended is $\Delta s_{ij}/(v_i \Delta c_{ij})$, or

$$\frac{\Delta s_{ij}}{v_i \Delta c_{ij}} = \frac{k_{ji}}{v_i c_{ij}} \qquad \text{(A.2)}$$

The behavioral postulate of maximization of satisfaction given resources implies that the individual will gain control of each event to the point that the marginal satisfaction from expenditure of a given amount of resources is equal for all, or

$$\frac{\Delta s_{ij}}{v_i \Delta c_{ij}} = \frac{\Delta s_{hj}}{v_h \Delta c_{hj}} \quad \text{for all events } i, h. \qquad \text{(A.3)}$$

Thus for all i and h, he will gain control of events i and h to the point that the final or equilibrium control, c^*_{ij}, which he has over each event is such that the following equation holds for all events i and h:

$$\frac{k_{ji}}{v_i c^*_{ij}} = \frac{k_{jh}}{v_h c^*_{hj}} \qquad \text{(A.4)}$$

Expressed in terms of c^*_{ij}, this becomes

$$c^*_{ij} = \frac{k_{ji}}{v_i}\frac{v_h c^*_{hj}}{k_{jh}}, \qquad \text{(A.5)}$$

where h is any other event in the system.

Since the quantity $v_h c^*_{hj}/k_{jh}$ is independent of event h, it may be replaced by a sing constant for actor j, K_j, so that equation (A.5) becomes

$$c^*_{ij} = \frac{k_{ji}}{v_i} K_j. \qquad \text{(A.}$$

Equation (A.6) gives the proportion of control over event i that actor j will contr after exchange. It is necessary now to show that this fraction of control, c^*_{ij}, is suc that the fraction of his total resources allocated to i is independent of the unit co of i, v_i, and of his total resources, r_j. Multiplying equation (A.6) through by v_i give the amount of resources, $v_i c^*_{ij}$, devoted to event i, and summing over all events give his total resources:

$$\sum_{i=1}^{m} v_i c^*_{ij} = K_j \sum_{i=1}^{m} k_{ji}. \qquad \text{(A.7}$$

The quantity on the left of the equation is equal to his total resources r_j, so that i we impose a scale on k_{ji} such that $\sum_{i=1}^{m} k_{ji} = 1$ (which equation (A.1) shows can b done without loss of generality, since the scale on s_{ij} is arbitrary), then $K_j = r_j$ This means also that k_{ji} represents a fraction of resources devoted to event i, as is evident by multiplying equation (A.6) through by v_i and substituting r_j for K_j:

$$v_i c^*_{ij} = k_{ji} r_j. \qquad \text{(A.8)}$$

As equation (A.8) shows, since k_{ji} is a constant, independent of v_i or r_j, this means that the fraction of resources devoted to event i is independent of the cost or price, v_i, and of the total amount of resources held, r_j. Thus k_{ji}, when scaled so that $\sum_i k_{ji} = 1$, has all the properties attributed in the text to x_{ji}, j's interest in event i. Thus $x_{ji} = k_{ji}$, and its fundamental definition can be regarded as given by equation (A.1) together with the definition of scale, $\sum_i x_{ji} = 1$.

2. INTEREST IN INDIVISIBLE EVENTS

Interest in indivisible events cannot be assessed in the same way as divisible events. The indivisibility makes impossible a quantitative division of consequences through a division of control, and thus impossible a quantitative measure of the subjective impact of those consequences, such as shown in equation (A.1). Thus it is necessary to attach discrete levels of satisfaction or utility to their discrete outcomes. If their outcomes are described as a positive outcome and a negative outcome, then we may think of two utility levels, or for the event a utility difference between positive and negative outcomes.

Although I have described 'utility differences' as if a number can be associated with such a difference, the possibility of doing so depends on the existence of a measurement procedure to do so, which allows assignment of a unique number, or

B

a number unique within a certain set of transformations, to the utility difference. The measurement procedure, to be valid for the theory at hand, must contain operations that are intrinsic to the theory itself, not imported from outside. There are two questions in such measurement: first, is it possible within the framework of the theory (or equivalently, does it have operational meaning within the theory) to specify a set of operations that will give a particular level of measurement? And, second, does the empirical use of these operations result in measurement which in fact has the properties of numbers specified in the level of measurement? Only the first is at question here; the second depends upon empirical examination.

Three levels of measurement, corresponding to three conditions that can exist in the theory, will be examined. These imply, for their verification, increasingly strong assumptions about behavior. The quantity being measured is the *directed interest* of actor j in event i, which will be denoted y_{ji}. This is intended to be actor j's interest in seeing outcome 1 occur rather than outcome 2. It may be thought of as deriving from the difference in utilities between positive and negative outcomes, but need not be.

We assume that actors act so as to maximize, subject to their initial resources, their realized interest, where interest is realized through attaining the outcome in which the actor has a positive interest (or equivalently, the outcome for which his utility is greater).

Level 1: Sign of y_{ji}

Assume a set of actors indexed $j = 1, \ldots, n$, and a set of events indexed $i = 1, \ldots, m$. Actor's behavior is constrained to that of casting a vote, when he does not know the votes of other actors. A vote for a given outcome has a normal meaning, i.e., an outcome will be achieved if the proportion of votes favoring it exceeds a certain minimum, or the probability of that outcome is increased as the proportion of votes for it increases.

If the events are separable, *i.e.*, independent in their consequences, then the actor may consider each event separately.

The first level of measurement is achieved by giving y_{ji} a number with a sign depending on whether he voted for a positive outcome, a negative one, or did not vote. (Rational behavior dictates that he vote for that outcome which he prefers [i.e., in which he has positive interest or for which he has higher utility], because given lack of knowledge of others' votes, his subjective probability of achieving an outcome is increased by voting for it. The result of this level of measurement is assignment of positive or negative numbers or zero to y_{ji} (which may be $+1$, -1, 0, without loss of information, since any positive or negative numbers are equally valid).

Validation of this level of measurement occurs if there is consistency in his voting, independent of the order of events or other variations. The principal source of non-validation would probably be non-separability of events.

Level 2: Order relation among the absolute values of y_{ji}

The assumptions of level 1 measurement are continued here, but one constraint on behavior is removed. Actors are free to give up a vote on any event in exchange for a single vote on another event. The second level of measurement is achieved by

assigning numbers to $|y_{ji}|$ such that $|y_{ji}| > |y_{jk}|$ if and only if the actor is willing to give up a vote on event k in return for a vote on event i, and $|y_{ji}| = |y_{jk}|$ if he is not willing to make a trade in either direction. His absence of knowledge about others' vote intentions makes his estimate of the probability that an additional vote will change the outcome the same for all events. Thus his implicit comparison in the trade is a loss of expected utility due to giving up a vote on k, $\Delta p_k|y_{jk}|$, where Δp_k is the subjective probability that this vote will change the outcome of k in the non-desired direction, versus a gain in expected utility due to gaining a vote on i, $\Delta p_i|y_{ji}|$, where Δp_i is the probability that this vote will change the outcome of i in the desired direction. Since nothing is known about others' votes, $\Delta p_k = \Delta p_i$, and the expected utility of the trade is positive if and only if $|y_{ji}| > |y_{jk}|$.

Validation of this level of measurement occurs if there is consistency in his exchanges, so that there are some numbers that can be assigned such that the relation '$>$' has the properties of an order relation, including transitivity.†

Level 3: A metric on y_{ji}

The assumptions of level 2 measurement are continued here, but another constraint on behavior is removed. Actors are free to give up any number of votes on one or more events in exchange for any number of votes on one or more other events. This provides a 'combination' operation analogous to that in classical physical measurement of combining weights, assigning numbers to $|y_{ji}|$, $|y_{jk}|$, and $|y_{jh}|$ such that the properties of addition are preserved. For example, he will give up one vote on h for one on k and one on i if and only if $|y_{ji}|+|y_{jk}| > |y_{jh}|$. It is assumed that his control of any event is so small that even with the proposed exchanges, the probability of an outcome change due to gain or loss of a vote is the same for all events.

Validation of this level of measurement occurs if the actor's behavior in agreeing to combined exchanges is like that for simple one-vote exchanges, but with $\Delta p_i|y_{ji}| + \Delta p_k|y_{jk}|$ replacing $\Delta p_i|y_{ji}|$ in expected utility calculations when votes on i and k are offered together. This is in effect a validation that the combination operation in behavior (votes on two or more events) has the same properties as addition.

The metric resulting from the above operation will be unique up to a positive scale constant. That is, there is nothing in observed behavior to distinguish between measures that have different (positive) scale constants. Thus an arbitrary positive scale constant can be applied, such that, over all events in a given system, $\sum_i |y_{ji}| = 1.0$. With this scaling, $|y_{ji}|$ can be thought of as relative utility differences between positive and negative outcomes. They have the properties of interest in indivisible events as described in the text and denoted by x_{ji}. Thus y_{ji} is a directed interest, with a sign depending upon the more desired outcome (and deriving from level of measurement), and an absolute value equal to x_{ji}, which is nonnegative, and is the size of his relative interest in event i.

† The niceties of measurement, involving for example the possibilities of intransitivity due to the cumulation of small differences, will not be treated here. These are the same as in any calibration with an insensitive instrument in physical sciences.