The critical values of $L$-functions of base change for Hilbert modular forms

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June 6, 2017

Abstract
In this paper we generalize some results, obtained by Shimura, Yoshida and the author, on critical values of $L$-functions of $l$-adic representations attached to Hilbert modular forms twisted by finite order characters, to the critical values of $L$-functions of arbitrary base change to totally real number fields of $l$-adic representations attached to Hilbert modular forms twisted by some general finite-dimensional representations.

2010 MSC: 11F41, 11F80, 11R42, 11R80.
Keywords: $L$-functions, special values, Hilbert modular forms

1 Introduction
For $F$ a totally real number field of degree $n$, let $J_F$ be the set of infinite places of $F$, and let $\Gamma_F := \text{Gal}(\bar{\mathbb{Q}}/F)$. Let $f$ be a normalized Hecke eigenform of $GL(2)/F$ of weight $k = (k(\tau))_{\tau \in J_F}$, where all $k(\tau)$ have the same parity and $k(\tau) \geq 2$. We denote by $\Pi$ the cuspidal automorphic representation of $GL(2)/F$ generated by $f$. In this paper we assume that $\Pi$ is non-CM. We denote by $\rho_{\Pi}$ the $l$-adic representation of $\Gamma_F$ attached to $\Pi$. Define $k_0 = \max\{k(\tau) | \tau \in J_F\}$ and $k^0 = \min\{k(\tau) | \tau \in J_F\}$. Any integer $m \in \mathbb{Z}$ such that $(k_0 - k^0)/2 < m < (k_0 + k^0)/2$ is called a critical value for $f$ or $\Pi$. Let $F'$ be a totally real finite extension of $F$. Consider a finite-dimensional continuous representation

$$\psi : \Gamma_{F'} \to \text{GL}_N(\mathbb{C}).$$

Let $V_\psi$ be the space corresponding to $\psi$. We denote by $d^+_\psi(\psi)$ the dimension of the subspace of $V_\psi$ on which the complex conjugation corresponding to $\tau' \in J_{F'}$ acts by $+1$, and by $d^-_\psi(\psi)$ the dimension of the subspace of $V_\psi$ on which the complex conjugation corresponding to $\tau' \in J_{F'}$ acts by $-1$. Throughout this paper we write $a \sim b$ for $a, b \in \mathbb{C}$ if $b \neq 0$ and $a/b \in \bar{\mathbb{Q}}$.

In this article we prove the following result:
Theorem 1.1. Assume $k(\tau) \geq 3$ for all $\tau \in J_F$ and $k(\tau) \bmod 2$ is independent of $\tau$. Let $F'$ be a totally real finite extension of $F$. Let $\psi$ be a finite-dimensional complex-valued continuous representation of $\Gamma_{F'}$ such that $K := \mathbb{Q}_{\ker \psi}$ is an abelian extension of a totally real number field. Then

$$L(m, \rho_{\Pi}|_{\Gamma_{F'}} \otimes \psi) \sim \pi^m F'[Q] \dim \psi \prod_{\tau' \in J_{F'}} c_{\tau'|F'}^{(-1)^{(m+1)}}(\Pi)^{d_{\tau'}(-1)^m}(\Pi)^{d_{\tau'}^+}$$

for any integer $m$ satisfying

$$(k_0 + 1)/2 \leq m < (k_0 + k^0)/2,$$

where $c_{\tau'|F'}^-$ and $c_{\tau'|F'}^+$ appear in Propositions 2.2 and 2.3 below.

Theorem 1.1 is a generalization of Theorem 4.3 of [S], the main theorem of [Y], Theorem 4 of [Y], Theorems 1.1, 1.2 and 1.3 of [V1] (i.e. Propositions 2.1, 2.2 and 2.3 below; when $\Pi$ is abelian, Theorem 1.1 could be deduced easily from Propositions 2.1, 2.2 and 2.3 below), and of [V4]. It is conjectured that the result obtained in Theorem 1.1 should be true for arbitrary finite-dimensional complex-valued continuous representations $\psi$ of $\Gamma_{F'}$.

We remark that Theorem 1.1 above was proved by Shimura and Yoshida (see Theorem 4.3 of [S], and the Main theorem of [Y]) when $\psi$ has dimension 1, for any integer $m$ satisfying $(k_0 - k^0)/2 < m < (k_0 + k^0)/2$ (i.e. for all the critical values $m$), but because in this paper we use Brauer’s induction theorem, we have to restrict ourselves to the integers $m$ satisfying $(k_0 + 1)/2 \leq m < (k_0 + k^0)/2$, in order to make sure that our $L$-functions that have negative exponent do not vanish at $m$ (see §3 below for details).

2 Known results

Consider $F$ a totally real number field and let $J_F$ be the set of infinite places of $F$. If $\Pi$ is a cuspidal automorphic representation (discrete series at infinity) of weight $k = \{k(\tau)\}_{\tau \in J_F}$ of $\GL(2)/F$, where all $k(\tau)$ have the same parity and all $k(\tau) \geq 2$, then there exists ([T]) a $\lambda$-adic representation

$$\rho_{\Pi} := \rho_{\Pi,\lambda} : \Gamma_F \to \GL_2(O_\lambda) \hookrightarrow \GL_2(\mathbb{Q}_l),$$

which satisfies $L(s, \rho_{\Pi,\lambda}) = L(s - \frac{(k_0 - 1)}{2}, \Pi) = L(s - \frac{(k_0 - 1)}{2}, f)$ (the equality up to finitely many Euler factors) and is unramified outside the primes dividing $n!$ (by fixing a specific isomorphism $i : \mathbb{Q}_l \xrightarrow{\sim} \mathbb{C}$ one can regard $\rho_{\Pi,\lambda}$ as a complex-valued representation). Here $O$ is the coefficients ring of $\Pi$ and $\lambda$ is a prime ideal of $O$ above some prime number $l$, $n$ is the level of $\Pi$ and $f$ is the normalized Hecke eigenform of $\GL(2)/F$ of weight $k$ corresponding to $\Pi$. We denote by $F_\infty^\times$ the archimedean part of the idele group $F_\lambda^\times$ of $F$.

We know (this is Theorem 1.1 of [V1], which is a generalization of Theorem 4.3 of [S]):
 Proposition 2.1. Assume \( k(\tau) \geq 3 \) for all \( \tau \in J_F \) and \( k(\tau) \mod 2 \) is independent of \( \tau \). Let \( F' \) be a totally real finite extension of \( F \). Then for every \( \epsilon = (\epsilon(\tau)) \in (\mathbb{Z}/2\mathbb{Z})^{J_{F'}} \), there exists a constant \( u(\epsilon, \Pi) \in \mathbb{C}^\times / \overline{\mathbb{Q}}^\times \) with the following property. If \( \psi \) is a finite order Hecke character of \( F' \) such that

\[
\psi_\infty(x) = \prod_{\tau \in J_{F'}} \text{sgn}(x_\tau)^{\epsilon(\tau) + m}, \quad x = (x_\tau) \in F'_\infty,
\]

then

\[
L(m, \rho|_{\Gamma_{F'}} \otimes \psi) \sim \pi^m[F':\mathbb{Q}] u(\epsilon, \Pi)
\]

for any integer \( m \) satisfying

\[
(k_0 + 1)/2 \leq m < (k_0 + k^0)/2.
\]

We know (this is Theorem 1.2 of [V1], which is a generalization of the main theorem of [Y]):

 Proposition 2.2. Assume that \( k(\tau) \geq 3 \) for all \( \tau \in J_F \) and \( k(\tau) \mod 2 \) is independent of \( \tau \). Let \( F_0 \) be a totally real finite extension of \( F_0 \). Then for every \( \tau \in J_{F_0} \), there exist constants \( c_\tau^0(\Pi) \in \mathbb{C}^\times \) which are determined uniquely mod \( \overline{\mathbb{Q}}^\times \) such that

\[
u(\epsilon, \Pi) \sim \prod_{\tau \in J_{F'}} c_\tau^{\epsilon(\tau)}(\Pi),
\]

where \( \epsilon = (\epsilon(\tau)) \in (\mathbb{Z}/2\mathbb{Z})^{J_{F'}} \), and \( u(\epsilon, \Pi) \) was defined in Proposition 2.1.

 Here we understand that \( c_\tau^0(\Pi) = c_\tau^+(\Pi), c_\tau^0(\Pi) = c_\tau^-(\Pi) \) by identifying \( \mathbb{Z}/2\mathbb{Z} \) with \( \{0, 1\} \).

 We know (this is Theorem 1.3 of [V1], which is a generalization of Theorem 4 of [Y]):

 Proposition 2.3. Assume that \( k(\tau) \geq 3 \) for all \( \tau \in J_F \) and \( k(\tau) \mod 2 \) is independent of \( \tau \). Let \( F' \) be a totally real finite extension of \( F \). Then we have

\[
c_\tau^+(\Pi) = c_\tau^0(\Pi) \quad \text{for every} \quad \tau \in J_{F'}.
\]

3 The proof of Theorem 1.1

We fix a non-CM cuspidal automorphic representation \( \Pi \) of \( \text{GL}(2)/F \) as in Theorem 1.1, and let \( F'/F \) be a totally real finite extension. Let \( \psi \) be a finite-dimensional representation of \( \Gamma_{F'} \) as in Theorem 1.1, such that \( K := \overline{\mathbb{Q}}^{\ker \psi} \) is an abelian extension of a totally real number field. We denote by \( F'' \) the maximal totally real subfield of \( K := \overline{\mathbb{Q}}^{\ker \psi} \). Obviously \( F''/F' \) is Galois and \( K \) is an abelian extension of \( F'' \).

From the beginning of §15 of [CR] we know that there exist some subfields \( F_i \subseteq F'' \) such that \( \text{Gal}(F''/F_i) \) are cyclic, and some integers \( n_i \), such that the trivial representation

\[
1_{F'} : \text{Gal}(F''/F') \to \mathbb{C}^\times
\]
can be written as

$[F'' : F']_{F_i} = \sum_{i=1}^{n_i} n_i \text{Ind}_{\text{Gal}(F''/F') \rightarrow \text{Gal}(F'/F_i)}^{\text{Gal}(F''/F')} 1_{F_i},$

where $1_{F_i} : \text{Gal}(F''/F_i) \rightarrow \mathbb{C}^\times$ is the trivial representation. In particular we have $[F'' : F'] = \sum_{i=1}^{n} n_i [F_i : F']$. Then

$L(s, \rho|\gamma_p \otimes \psi)|_{F'' : F'} = \prod_{i=1}^{n} L(s, \rho|\gamma_p \otimes \psi \otimes \text{Ind}_{\text{Gal}(F''/F') \rightarrow \text{Gal}(F'/F_i)}^{\text{Gal}(F''/F')} 1_{F_i})^{n_i}$

$= \prod_{i=1}^{n} L(s, \text{Ind}_{\text{Gal}(F''/F') \rightarrow \text{Gal}(F'/F_i)}^{\text{Gal}(F''/F')} (\rho|\gamma_p \otimes \psi|\gamma_p))^{n_i} = \prod_{i=1}^{n} L(s, \rho|\gamma_p \otimes \psi|\gamma_p)^{n_i}$

$= \prod_{i=1}^{n} L(s, \rho|\gamma_p \otimes \psi|\gamma_p)^{n_i}.$

We write

$\psi|\gamma_p = \bigoplus_{j=1}^{u_i} \psi_{ij},$

where $\psi_{ij}$ are irreducible representations of $\Gamma_{F_i}$. Since $\text{Gal}(F''/F_i)$ is cyclic, $\psi_{ij}|\gamma_p$ is abelian and $\psi_{ij}$ is irreducible, we get that $\psi_{ij} \simeq \text{Ind}_{\Gamma_{F_{ij}} \rightarrow \mathbb{C}^\times} \phi_{ij}$ for some continuous character

$\phi_{ij} : \Gamma_{F_{ij}} \rightarrow \mathbb{C}^\times,$

where $F_{ij}$ is a subfield of $F''$ which contains $F_i$. This is true, because if $\sigma$ be a generator of $\text{Gal}(F''/F_i)$, then since $F''/F_i$ is Galois, $\sigma$ permutes the irreducible components of $\psi_{ij}|\gamma_p$. The representation $\psi_{ij}|\gamma_p$ is abelian, and thus a direct sum of characters. Let $\phi$ be one of these characters. We denote by $F_{ij}$ the subfield of $F''$ which contains $F_i$ having the property that $\text{Gal}(F''/F_{ij})$ is the stabiliser of $\phi$ under the action of $\text{Gal}(F''/F_i) = \langle \sigma \rangle$. The character $\phi$ extends to a character $\phi_{ij}$ of $\Gamma_{F_{ij}}$. Then, because $\psi_{ij}$ is irreducible, $\sigma \in \text{Gal}(F_{ij}/F_i)$ permutes simply-transitively all the components of the representation $\psi_{ij}|\gamma_p$, so that it is a sum of conjugates of $\phi_{ij}$ and hence abelian, and we have that $[F_{ij} : F_i] = \dim \psi_{ij}$. Let $V_{\phi_{ij}}$ be the space corresponding to $\psi_{ij}$, and $V_{\psi_{ij}}$ be the space corresponding to $\phi_{ij}$. Since $\text{Hom}^{\Gamma_{F_{ij}}} (V_{\psi_{ij}}, V_{\phi_{ij}})$ is nontrivial, by Frobenius reciprocity we get that $\text{Hom}^{\Gamma_{F_{ij}}} (V_{\psi_{ij}}, \text{Ind}_{\Gamma_{F_{ij}} \rightarrow \mathbb{C}^\times} \phi_{ij})$ is also non-trivial. But $\dim \text{Ind}_{\Gamma_{F_{ij}}} \phi_{ij} = \dim \psi_{ij}$, and thus we obtain $\psi_{ij} \simeq \text{Ind}_{\Gamma_{F_{ij}} \rightarrow \mathbb{C}^\times} \phi_{ij}$. Therefore we have

$L(s, \rho|\gamma_p \otimes \psi)|_{F'' : F'} = \prod_{i=1}^{n} L(s, \rho|\gamma_p \otimes \psi|\gamma_p)^{n_i}$

$= \prod_{i=1}^{n} \prod_{j=1}^{u_i} L(s, \rho|\gamma_p \otimes \phi_{ij})^{n_i}.$

4
Using the potential modularity of the representation \( \rho_{n|F_{ij}} \) (see Theorem A of [BGGT], Theorem 2.1 of [V2] or Theorem 1.1 of [V3]), one can prove easily the meromorphic continuation to the entire complex plane of the functions \( L(s, \rho_{n|F_{ij}} \otimes \psi) \) (for details see for example the proof of Theorem 1.1 of [V1]), and hence one gets the meromorphic continuation to the entire complex plane of the function \( L(s, \rho_{n|F_{ij}} \otimes \psi)[F^\prime:F] \). Moreover, from the proof of Theorem 1.1 of [V1] we know that the function \( L(s, \rho_{n|F_{ij}} \otimes \psi) \) has no poles or zeros at \( s = m \) for each integer \( m \) satisfying \((k_0 + 1)/2 < (k_0 + k_0')/2\). Thus for each integer \( m \) satisfying \((k_0 + 1)/2 < (k_0 + k_0')/2\), we get the identity

\[
L(m, \rho_{n|F_{ij}} \otimes \psi)[F^\prime:F] = \prod_{i=1}^{u} \prod_{j=1}^{u_i} L(m, \rho_{n|F_{ij}} \otimes \phi_{ij})^{n_i}.
\]

We have:

\[
[F^\prime : F^\prime] \psi = \sum_{i=1}^{u} n_i \text{Ind}_{\text{Gal}(K/F)}^{\text{Gal}(K/F')} \psi|_{F_{ij}} = \sum_{i=1}^{u} n_i \text{Ind}_{\text{Gal}(K/F)}^{\text{Gal}(K/F')} (\sum_{j=1}^{u_i} \psi_{ij})
\]

\[
= \sum_{i=1}^{u} n_i \text{Ind}_{\text{Gal}(K/F)}^{\text{Gal}(K/F')} (\sum_{j=1}^{u_i} \text{Ind}_{F_{ij}}^{F_{ij}} \phi_{ij}) = \sum_{i=1}^{u} \sum_{j=1}^{u_i} n_i \text{Ind}_{\text{Gal}(K/F)}^{\text{Gal}(K/F')} \phi_{ij}.
\]

Thus

\[
[F^\prime : F^\prime] d_{\tau_{ij}}^-(\psi) = \sum_{i=1}^{u} \sum_{j=1}^{u_i} n_i \sum_{\{\tau_{ij} \in J_{F_{ij}}| \tau_{ij} = \tau'\}} d_{\tau_{ij}}^-(\phi_{ij})
\]

and

\[
[F^\prime : F^\prime] d_{\tau_{ij}}^+(\psi) = \sum_{i=1}^{u} \sum_{j=1}^{u_i} n_i \sum_{\{\tau_{ij} \in J_{F_{ij}}| \tau_{ij} = \tau'\}} d_{\tau_{ij}}^+(\phi_{ij}),
\]

for any \( \tau' \in J_{F_{ij}} \). Also we have that \([F^\prime : F^\prime] \dim \psi = \sum_{i=1}^{u} \sum_{j=1}^{u_i} n_i [F_{ij} : F^\prime]\), and thus \([F^\prime : \mathbb{Q}] \dim \psi = \sum_{i=1}^{u} \sum_{j=1}^{u_i} n_i [F_{ij} : \mathbb{Q}]\).

Now from Propositions 2.1, 2.2 and 2.3 one gets easily that

\[
L(m, \rho_{n|F_{ij}} \otimes \phi_{ij}) \sim \pi^{m[F_{ij} : \mathbb{Q}]} \prod_{\tau_{ij} \in J_{F_{ij}}} c_{\tau_{ij}|F^\prime}^{(-1)(m+1)} (\Pi) d_{\tau_{ij}}^-(\phi_{ij}) c_{\tau_{ij}|F^\prime}^{(-1)m} (\Pi) d_{\tau_{ij}}^+(\phi_{ij}),
\]

and hence

\[
L(m, \rho_{n|F_{ij}} \otimes \psi)[F^\prime:F] = \prod_{i=1}^{u} \prod_{j=1}^{u_i} L(m, \rho_{n|F_{ij}} \otimes \phi_{ij})^{n_i}
\]

\[
\sim \prod_{i=1}^{u} \prod_{j=1}^{u_i} \pi^{m[F_{ij} : \mathbb{Q}]} \prod_{\tau_{ij} \in J_{F_{ij}}} c_{\tau_{ij}|F^\prime}^{(-1)(m+1)} (\Pi)^{n_i} d_{\tau_{ij}}^-(\phi_{ij})^c_{\tau_{ij}|F^\prime}^{(-1)m} (\Pi)^{n_i} d_{\tau_{ij}}^+(\phi_{ij})
\]
\[ = \pi^m[F', \mathbb{Q}] \dim \psi \prod_{\tau' \in \mathcal{J}_{\mathbb{Q}}} (\Pi)^{\sum_{i=1}^{n_i} \sum_{\tau_{ij} \in \mathcal{J}_{F|_{\mathcal{J}_{\mathbb{Q}}}}} (\Pi)^d_{\tau_{ij}} (\phi_{ij})} \]
\[ \times \prod_{\tau' \in \mathcal{J}_F} (\Pi)^{\sum_{i=1}^{n_i} \sum_{\tau_{ij} \in \mathcal{J}_{F|_{\mathcal{J}_F}}} (\Pi)^{\sum_{\tau_{ij} \in \mathcal{J}_{F|_{\mathcal{J}_F}}} (\Pi)} \]
\[ = \pi^m[F', \mathbb{Q}] \dim \psi \prod_{\tau' \in \mathcal{J}_F} (\Pi)^{\sum_{i=1}^{n_i} \sum_{\tau_{ij} \in \mathcal{J}_{F|_{\mathcal{J}_F}}} (\Pi)^{\sum_{\tau_{ij} \in \mathcal{J}_{F|_{\mathcal{J}_F}}} (\Pi)} \]
\[ = \pi^m[F', \mathbb{Q}] \dim \psi \prod_{\tau' \in \mathcal{J}_F} (\Pi)^{\sum_{i=1}^{n_i} \sum_{\tau_{ij} \in \mathcal{J}_{F|_{\mathcal{J}_F}}} (\Pi)^{\sum_{\tau_{ij} \in \mathcal{J}_{F|_{\mathcal{J}_F}}} (\Pi)}} \]
\[ = \pi^m[F', \mathbb{Q}] \dim \psi \prod_{\tau' \in \mathcal{J}_F} (\Pi)^{\sum_{i=1}^{n_i} \sum_{\tau_{ij} \in \mathcal{J}_{F|_{\mathcal{J}_F}}} (\Pi)^{\sum_{\tau_{ij} \in \mathcal{J}_{F|_{\mathcal{J}_F}}} (\Pi)}} \]
and thus
\[ L(m, \rho|_{\mathcal{J}_F} \otimes \psi) \sim \pi^m[F', \mathbb{Q}] \dim \psi \prod_{\tau' \in \mathcal{J}_F} (\Pi)^{\sum_{i=1}^{n_i} \sum_{\tau_{ij} \in \mathcal{J}_{F|_{\mathcal{J}_F}}} (\Pi)^{\sum_{\tau_{ij} \in \mathcal{J}_{F|_{\mathcal{J}_F}}} (\Pi)}} \]
which proves Theorem 1.1. ■

References


