The multi- (or single-) product newsvendor problem is a well-known classical problem in inventory management. The general setting analyzed is as follows. There are multiple (or single) perishable products with random demand in a single-selling season. Given a purchase cost and resale price, the decision maker (aka the newsvendor) chooses the optimal ordering quantity for each product at the beginning of the selling season. If the newsvendor orders too much of any product, all leftovers are sold at salvage value; if the newsvendor orders too little, it incurs lost opportunity of sales. Given uncertain demand, it is obvious that the final profit for the newsvendor is random. Thus prior researchers have focused on maximizing expected profits for the newsvendor in making the ordering decision.

Unless there are resource constraints and/or demand substitution effects, most multi-product newsvendor problems can be decomposed into multiple independent single-product problems. The general solution (for expected profit maximization) is a simple closed-form ratio of the overage and underage "costs" for the newsvendor’s (cumulative) marginal demand distribution. Given such a straightforward result, this approach has been applied in numerous industry settings to address problems of revenue management and/or overbooking.

From a decision-maker’s perspective, maximizing expected profits implies risk neutrality. However, risk neutrality guarantees the best decision only on average. Although the use of this model may be justified by the Law of Large Numbers, one cannot expect that a single realization will be sufficiently close to the expected value. In fact, when actual outcomes deviate greatly from their expected values due to their randomness, it may cause an unacceptably large loss to the newsvendor.
To overcome the drawbacks of risk-neutral models, risk-averse models have been studied in literature. In risk-averse inventory models, inventory managers consider the variability of the outcome in addition to its expected value. For example, under risk aversion, risk-averse inventory managers may prefer more stable outcomes even if the outcomes are worse on average. Since risk-averse models consider inventory models from more conservative standpoint than risk-neutral models, they are a very good fit when inventory managers are more conservative (i.e. energy, environment, sustainability and natural resources industries).

Given different attitudes toward risk, multiple approaches have been taken to improve upon the expected value optimization model. In this literature review, typical risk measures of risk-averse inventory models are categorized into four approaches: utility functions, stochastic dominance, chance constraints, and mean-risk analysis. In the utility function approach, inventory managers optimize the expected value of their utility function instead of the expected outcomes. When the outcomes represent profits, risk-averse decision makers use a concave and nondecreasing utility function (e.g. an exponential utility function). Stochastic dominance is the sequence of the partial orders defined on the space of random variables, and allows pairwise comparison of different random variables. In the sequence of the relations, the second-order stochastic dominance is consistent with risk aversion. Because stochastic dominance is generally very difficult to implement computationally, it has been used mainly as a reference criterion to evaluate the legitimacy of risk-averse inventory models. Chance constraints add constraints to the probabilities that measure the risk. In finance, chance constraints are very popular under the name of Value-at-Risk (VaR).

Mean-risk analysis provides efficient solutions and quantifies the problem using two criteria: the mean (the expected value of the outcome) and the risk (a scalar measuring variability of the outcome). The two important examples of mean-risk analysis are mean-variance and coherent measures of risk. The mean-variance model uses the variance of the return as the risk functional. Due to its popularity and computational tractability, it has been actively used in supply chain management as well as finance. Coherent measures of risk use different risk functionals instead of variance. Thus, coherent measures of risk are extensions of mean-
risk analysis. In addition, it is also known that coherent measures of risk are consistent with the first- and second-order stochastic dominance relations. A formal definition requires that coherent measures of risk satisfy the four axioms (Convexity, Monotonicity, Translation Equivariance and Positive Homogeneity) which guarantee consistency with intuition about rational risk-averse decision making.

These four typical approaches are related and consistent, to some extent, but they are different from each other. There exist advantages and disadvantages of using each risk measure in risk-averse newsvendor models. This dissertation provides a logical reasoning to compare the validity of each risk measure. Consequently, exponential utility functions and coherent measures of risk are selected as two quality risk measures to analyze within a newsvendor model.

The newsvendor problem formulation is as follows. Given products $j = 1, \ldots, n$, let $x = (x_1, x_2, \ldots, x_n)$ be the vector of ordering quantities (decision variables) and let $D = (D_1, \ldots, D_n)$ be the (random) demand vector. We also define model parameters, given $r = (r_1, \ldots, r_n)$ to be the price vector, $c = (c_1, \ldots, c_n)$ to be cost vector, and $s = (s_1, \ldots, s_n)$ to be the vector of salvage values.

Setting $\bar{c}_j = c_j - s_j$ and $\bar{r}_j = r_j - s_j$, the total profit function is written as follows:

$$\Pi(x, D) = \sum_{j=1}^{n} \Pi_j(x_j, D_j),$$  \hspace{1cm} (1)

where

$$\Pi_j(x_j, D_j) = -\bar{c}_j x_j + \bar{r}_j \min\{x_j, D_j\}$$  \hspace{1cm} (2)

$$= (r_j - c_j)x_j - (r_j - s_j)(x_j - D_j)_+, \quad j = 1, \ldots, n. \hspace{1cm} (3)$$

Assuming that the demand vector $D$ is random with a continuous joint probability distribution, for every $x \geq 0$ the profit $\Pi(x, D)$ is a real random variable.

Then the original newsvendor model under risk neutrality is:

$$\max_{x \geq 0} \mathbb{E}[\Pi(x, D)].$$  \hspace{1cm} (4)
The multi-product risk-averse newsvendor optimization model with coherent measures of risk is:

$$\min_{x \geq 0} \rho[\Pi(x, D)],$$  \hspace{1cm} (5)

where $\rho[\cdot]$ is a law-invariant coherent measure of risk, and $\Pi(x, D)$ represents the profit of the newsvendor, as defined in (1).

The exponential utility function of a profit $z \in \mathbb{R}$ is defined as follows:

$$u(z) = -e^{-\lambda z}. \hspace{1cm} (6)$$

It is nondecreasing and concave. Here, $\lambda$ is a positive degree of risk aversion. The expected utility of a random profit $Z$ is defined as follows:

$$U(Z) = \mathbb{E}[-e^{-\lambda Z}]. \hspace{1cm} (7)$$

Setting $Z = \Pi(x, D)$, I obtain the expected utility in the newsvendor problem,

$$U(\Pi(x, D)) = \mathbb{E}[-e^{-\lambda \Pi(x, D)}]. \hspace{1cm} (8)$$

Thus, the problem to optimize the expected utility can be represented equivalently as follows:

$$\min_{x \geq 0} \mathbb{E}[e^{-\lambda \Pi(x, D)}]. \hspace{1cm} (9)$$

With the exponential utility function in (9), the multi-product model is not decomposed equivalently into multiple single-product models unless product demands are independent. On the other hand, the multi-product models can never be decomposed in any cases with the coherent measure of risk in (5). Therefore, in general, one needs to consider the multi-product models as product-mix portfolios under risk aversion. This is one of the fundamental structural differences between risk-neutral and risk-averse newsvendor models. This dissertation analyzed three different newsvendor models with coherent measures of risk and exponential utility function: the single-product model (Case 1), a multi-product model with independent demand (Case 2), and a multi-product model with dependent demand (Case 3). Then similarities and differences between the newsvendor models and financial portfolio optimization models are discussed. The key results are as follows.
For single-product models, the impact of degree of risk aversion was examined with coherent measures of risk and exponential utility function. In both cases, it was proven that higher degree of risk-aversion results in lower optimal ordering quantity. In newsvendor problem formulation, higher ordering quantity implies higher variability of profit, so the risk-averse newsvendor tends to decrease optimal ordering quantity to avoid higher risk. A closed-form optimal solution was obtained with coherent measures of risk and closed-form approximation with exponential utility functions.

For multi-product models with independent demands, the convexity of the objective functions was first established with both risk measures. For the impact of degree of risk aversion, higher degree of risk aversion leads to lower optimal solution with exponential utility functions. However, with coherent measures of risk, this result is consistent with the case of exponential utility functions, provided that the products are identical. Next, a closed-form approximation was obtained taking each risk measure into account (for a large but finite number of products). This approximation is no more difficult to compute than the risk-neutral solutions and shows a rapid convergence rate as the number of products increases. Finally, the asymptotic behavior of the solution was examined and results revealed that as the number of products approaches infinity, risk-neutral solutions become asymptotically optimal, implying that in the limit risk aversion has no impact.

For multi-product models with dependent demands it was found that for both risk measures, the optimal ordering quantity with negatively (or positively) correlated demand is higher (or lower) than the optimal solution with independent demand. This proposition implies portfolio effects, which was proven for an arbitrary number of products with an exponential utility function. However, due to the significant analytical challenge, the proposition was only proven for a two-identical-product system with coherent measures of risk. Demand correlation can also significantly affect the impact of degree of risk aversion on optimal order quantities. When the demand is independent or positively correlated, higher degree of risk aversion leads to lower optimal ordering quantity. However, for a strongly negative correlation, risk aversion may increase the optimal ordering quantity.
Acknowledgement: I am very grateful to Professor Nallan C. Suresh (Coordinator of 2010 Elwood S. Buffa DSI Doctoral Dissertation Competition) and the 45 reviewers for their efforts in the evaluation process and my dissertation advisor, Professor Andrzej Ruszczyński, for his kind help and support during my dissertation work.

Sungyong Choi is a visiting assistant professor of management science and information systems at Rutgers University. His research addresses supply chain management problems under risk. His research has been published in Operations Research and Operations Research Letters.