# ON THE UNIVERSAL NORM GROUPS OF THE GLOBAL UNITS AND THE $p$-UNITS

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Abstract. Following Kuz'min, we investigate the Tate module of a number field $k$ in terms of the universal norm property of the $p$-units. More precisely we relate the condition of finiteness of the Galois invariant of the Tate module which is known to be equivalent to the generalized Gross conjecture with the universal norm subgroups of the global units and the $p$-units over the cyclotomic $\mathbb{Z}_p$-extension of $k$.

## 1. Introduction

Let $k_\infty^{\text{cyc}} = \bigcup_n k_n$ be the cyclotomic $Z_p$-extension of a number field $k$ with the Galois group $\Gamma := G(k_\infty^{\text{cyc}}/k)$. Let $T_p(k)^\Gamma$ be the Galois invariant of the Tate module $T_p(k)$ of $k$ over $k_\infty^{\text{cyc}}/k$. The structure of $T_p(k)^\Gamma$ is related via infinite class field theory with deep arithmetic properties of the $p$-units. One of the related study was done by Kuz'min in [14] in terms of the universal norm group of the $p$-units for the norm map $N_n := N_{k_n/k}$ of $K_n/k$.

This finiteness condition of $T_p(k)^\Gamma$ is known to be equivalent to the generalized Gross conjecture (cf. [13]). When the base field is a CM-field, an equivalent form of the conjecture was made by Gross (cf. [8]) and studied by Coates and Lichtenbaum (cf. [3]) in relation with special values at zero of $p$-adic $L$-functions. The conjecture was proved for abelian fields by Greenberg (cf. [6]). The study of $T_p(k)^\Gamma$, for any number field, is due to Kuzmin (cf. [14]). The Kuzmin isomorphism in Proposition 2.1 is a special case of a general spectral sequence (see p.557 of [12] and the references therein).

The purpose of this paper is to extend Kuz'min's point of view and to relate the generalized Gross conjecture not only using the $p$-units but also using the global units. The proof will provide an information on the quotient of the subgroup

$$E_k \cap k^{\text{univ}} = E_k \cap \bigcap_n N_n k_n^\times$$

of the global units $E_k$ which are the universal norm elements for the norm map $N_n := N_{k_n/k}$ of $k_n/k$ and the subgroup of the universal norm elements

$$E_k^{\text{univ}} = \bigcap_n N_n E_n$$

which are coming from the global units $E_n$ of $k_n$ over the $\mathbb{Z}_p$-extension.

Under the assumption of the generalized Gross conjecture, we can show that the quotient group is finite as shown in Theorem 1.1 Moreover, after tensoring with $\mathbb{Z}_p$ to the global $p$-units of each intermediate field $k_n$, it can be shown that the finiteness condition of the quotient of certain universal norm groups is equivalent to the generalized Gross conjecture.

For the unique subfield $k_n$ of $k_\infty^{\mathrm{cyc}}$ of degree $p^n$ over $k$, let $N_{m,n} = N_{k_m/k_n} (m \geq n)$ denote the norm map from $k_m$ to $k_n$ and let $N_m = N_{m,0}$ denote the norm map from $k_m$ to the ground field $k_0 = k$.

For a subgroup $\mathcal{F}_n$ of $k_n^\times$, we define the norm compatible subgroup $\mathcal{F}_n^{\mathrm{comp}}$ and the universal norm subgroup $\mathcal{F}_n^{\mathrm{univ}}$ as follows

$$\mathcal{F}_n^{\mathrm{univ}} := \bigcap_{m \geq n} N_{m,n} \mathcal{F}_m, \quad \mathcal{F}_n^{\mathrm{comp}} := \pi(\varprojlim_{m \geq n} \mathcal{F}_m)$$

where the inverse limits are taken with respect to the norm maps and $\pi = \pi_n$ denotes the natural projection from $\varprojlim_{m \geq n} \mathcal{F}_m$ to $\mathcal{F}_n$ defined as

$$\pi((b_m)_{m \geq n}) = b_n.$$

By taking ramifications over a $\mathbb{Z}_p$-extension into account (see §2.2 of [1]), we notice that

$$\mathcal{F}_n^{\mathrm{univ}} = \mathcal{F}_n^{\mathrm{univ}} \cap U_n', \quad \mathcal{F}_n^{\mathrm{comp}} = (\mathcal{F}_n \cap U_n')^{\mathrm{comp}}$$

where $U_n'$ denotes the $p$-units of $k_n$ and

$$(\mathcal{F}_n \cap U_n')^{\mathrm{comp}} = \pi(\varprojlim_{m \geq n} (\mathcal{F}_m \cap U_n')).$$

For $\{\mathcal{F}_n \otimes \mathbb{Z}_p\}_{m \geq n}$, let $\pi$ also denote the natural projection

$$\pi : \varprojlim_{m \geq n} (\mathcal{F}_m \otimes \mathbb{Z}_p) \to \mathcal{F}_n \otimes \mathbb{Z}_p.$$

The norm compatible subgroup $(\mathcal{F}_n \otimes \mathbb{Z}_p)^{\mathrm{comp}}$ and the universal norm subgroup $(\mathcal{F}_n \otimes \mathbb{Z}_p)^{\mathrm{univ}}$ of $\mathcal{F}_n \otimes \mathbb{Z}_p$ are defined as

$$(\mathcal{F}_n \otimes \mathbb{Z}_p)^{\mathrm{univ}} := \bigcap_{m \geq n} N_{m,n}(\mathcal{F}_m \otimes \mathbb{Z}_p), \quad (\mathcal{F}_n \otimes \mathbb{Z}_p)^{\mathrm{comp}} := \pi(\varprojlim_{m \geq n} (\mathcal{F}_m \otimes \mathbb{Z}_p)).$$

For the procyclic group $\Gamma = G(k_\infty/k)$ and for each $m \geq n \geq 0$, let $\Gamma_n = G(k_\infty/k_n)$ be the unique subgroup of $\Gamma$ with index $p^n$ and let

$$G_{m,n} := \Gamma_n/\Gamma_m \cong G(k_m/k_n), \quad G_n := \Gamma/\Gamma_n \cong G(k_n/k).$$

Let $S_n = \{v|p\}$ be the set of primes of $k_n$ dividing $p$ and let

$$A_n := Cl(k_n) \otimes \mathbb{Z}_p$$

be the $p$-primary part of the ideal class group $Cl(k_n)$ of $k_n$. Let

$$A_n' := Cl_S(k_n) \otimes \mathbb{Z}_p$$

be the $p$-primary part of the $S_n$-ideal class group

$$Cl_S(k_n) := Cl(k_n)/\langle S_n \rangle$$

of $k_n$ where $\langle S_n \rangle = \langle cl(v) \mid v \in S_n \rangle$ is the subgroup of the ideal class group which is generated by the ideal class $cl(v) \in Cl(k_n)$ containing $v$ for each $v \in S_n$. We define the Tate module $T_p(k)$ of $k$ following Kuz'min as the inverse limit of $A_n'$ with respect to the norm maps,

$$T_p(k) := \varprojlim_n A_n'.$$

As mentioned already, it is well known that the generalized Gross conjecture is equivalent to

$$T_p(k)^\Gamma < \infty.$$

Using infinite class field theory we know the following isomorphism of Kuz'min(see Proposition 1.1 of [18] or Proposition 7.5 of [14])

$$T_p(k)^\Gamma \cong \frac{(k^\times \otimes \mathbb{Z}_p)^{\text{univ}}}{(U'_k \otimes \mathbb{Z}_p)^{\text{univ}}}.$$

where $U'_k = U'_0$ the $p$-units of $k$. For pairs of positive integers $a_n, b_n$, we will write

$$a_n \sim b_n$$

if there is a positive real number $c$ such that for all $n$,

$$c^{-1} < a_n b_n^{-1} < c.$$

**Theorem 1.1.** *The generalized Gross conjecture is equivalent to*

$$(U'_k \cap N_n k_n^\times : N_n(U'_n)) \sim 1.$$

*If $k$ satisfies the generalized Gross conjecture, then we have*
(i) $\text{rank}_{\mathbb{Z}_p}\ ((k^\times \otimes \mathbb{Z}_p)^{\text{univ}} \cap (E_k \otimes \mathbb{Z}_p)) = \text{rank}_{\mathbb{Z}_p}\ (E_k \otimes \mathbb{Z}_p)^{\text{univ}}$
(ii) $(k^{\text{univ}} \cap E_k : E_k^{\text{univ}}) < \infty$
(iii) $(k^{\text{univ}} : {U'_k}^{\text{univ}}) < \infty$.

For the structure of the quotient groups above, we show that they are isomorphic for all sufficiently large layers.

**Proposition 1.2.** *For all sufficiently large $m \geq n$, the natural map $\lambda_{n,m}$ induced by the embedding induces an isomorphism*

$$\frac{(k_n^\times \otimes \mathbb{Z}_p)^{\text{univ}}}{(U'_n \otimes \mathbb{Z}_p)^{\text{univ}}} \cong \frac{(k_m^\times \otimes \mathbb{Z}_p)^{\text{univ}}}{(U'_m \otimes \mathbb{Z}_p)^{\text{univ}}}$$

$$\frac{k_n^{\text{univ}}}{{U'_n}^{\text{univ}}} \cong \frac{k_m^{\text{univ}}}{{U'_m}^{\text{univ}}}$$

*and there is an exact sequence*

$$1 \longrightarrow \frac{k_n^{\text{univ}}}{{U'_n}^{\text{univ}}} \longrightarrow \frac{(k_n^\times \otimes \mathbb{Z}_p)^{\text{univ}}}{(U'_n \otimes \mathbb{Z}_p)^{\text{univ}}}.$$

*If the generalized Gross conjecture holds for $\{k_n\}_n$, then all the groups above are finite.*

## 2. The Tate module and the generalized Gross conjecture

Let $k_\infty^{\text{cyc}} = \bigcup_n k_n$ be the cyclotomic $Z_p$-extension of $k$. We define the Tate module $T_p(k)$ of $k$,

$$T_p(k) = \varprojlim_n G(L_n/k_n)$$

as the inverse limit of $G(L_n/k_n)$ where $L_n$ is the maximal abelian $p$-extension of $k_n$ which is unramified and all primes dividing $p$ split completely over $k_n$. By Class field theory, this is equal to the one defined in the introduction.

For the classical version of the generalized Gross conjecture and its related versions, we refer to [4], [6], [8] and [11]. We briefly introduce following Iwasawa (see §4 of [10]) and Kolster (see §1 of [13]) equivalent forms of the generalized Gross conjecture. Let $S_\infty = \{v|p\}$ be the set of primes of $k_\infty^{\text{cyc}}$ dividing $p$ which is finite.

It is well known that the generalized Gross conjecture for $k$ is equivalent to one of the following equivalent statements;

i) ${A'}_\infty^\Gamma < \infty$,
where $A'_\infty = \varinjlim A'_n$ is the $p$-part of the $S_\infty$-ideal class group of $k_\infty^{\rm cyc}$,
ii) $H^1(\Gamma, A'_\infty) = 0$,
iii) $T_p(k)_\Gamma < \infty$,
iv) $T_p(k)^\Gamma < \infty$.
The equivalence of i), ii) and iii) is explained in §4 of [10] and the equivalence of ii) and iv) is explained in Theorem 1.14 of [13].

2.1. **The structure of the Tate module.** For a finite cyclic extension $K/k$ and for a finite set $S$ of primes of $k$, let $U_k(S)$ be the global $S$-units of $k$. Let $S'$ be the set of primes of $K$ lying over each prime $v \in S$,

$$S' = \{w|v \ ; \ v \in S\}.$$

We also let $U_K(S) := U_K(S')$ denote the global $S'$-units of $K$. Let

$$J_{K,S} := \prod_{w \notin S'} U_v \times \prod_{w \in S'} k_v^\times$$

be the $S$-idele group where we identify $U_K(S)$ with a subgroup of $J_{K,S}$ via the the diagonal imbedding $\phi_{K,S} : U_K(S) \longrightarrow J_{K,S}$.

For $K = k_n$ and $S = \{v|p\}$, we write

$$U'_n := U_{k_n}(S), \quad \phi_n := \phi_{k_n,S}.$$

We have the following exact sequence

$$1 \longrightarrow \ker(\phi_n) \longrightarrow U_k(p) \xrightarrow{\phi_n} \widehat{H}^0(G_n, J_{k_n,S})$$

where $G_n := G(k_n/k)$ denotes the Galois group of $k_n/k$.

Then Hasse's theorem for $k^\times$ implies that

$$N_n U'_n \subset \ker(\phi_n) = U'_k \cap N_n k_n^\times.$$

By tensoring with $\mathbb{Z}_p$, the exact sequence induces

$$1 \longrightarrow \ker(\overline{\phi}_n) = \ker(\phi_n) \otimes \mathbb{Z}_p \longrightarrow U'_k \otimes \mathbb{Z}_p \xrightarrow{\overline{\phi}_n} \widehat{H}^0(G_n, J_{k_n,S})$$

together with

$$\begin{aligned} N_n(U'_n \otimes \mathbb{Z}_p) \subset \ker(\overline{\phi}_n) &= (U'_k \cap N_n k_n^\times) \otimes \mathbb{Z}_p \\ &= (U'_k \otimes \mathbb{Z}_p) \cap ((N_n k_n^\times) \otimes \mathbb{Z}_p) \\ &= (U'_k \otimes \mathbb{Z}_p) \cap (N_n(k_n^\times \otimes \mathbb{Z}_p)) \end{aligned}$$

since $\mathbb{Z}_p$ is flat (cf. Proposition 6 of Ch I of [2]). Then

$$\begin{aligned} \ker(\overline{\phi}_\infty) := \bigcap_{n \geq 0} \ker(\overline{\phi}_n) &= \bigcap_{n \geq 0} ((U'_k \cap N_n k_n^\times) \otimes \mathbb{Z}_p) \\ &= (k^\times \otimes \mathbb{Z}_p)^{\rm univ} \cap (U'_k \otimes \mathbb{Z}_p) \\ &= (k^\times \otimes \mathbb{Z}_p)^{\rm univ}. \end{aligned}$$

Using infinite class field theory we know the following result of Kuz'min(see Proposition 1.1 of [18] or Proposition 7.5 of [14]).

**Proposition 2.1.** *Let $k$ be a number field. Then there is an isomorphism*

$$T_p(k)^\Gamma \cong \frac{(k^\times \otimes \mathbb{Z}_p)^{\rm univ}}{(U'_k \otimes \mathbb{Z}_p)^{\rm univ}}.$$

For a finitely generated $\mathbb{Z}$-module $M$, let $M/\mathrm{tor}(M)$ be the quotient of $M$ by its torsion $\mathrm{tor}(M)$ and let

$$\mathrm{rank}_{\mathbb{Z}}\, M := \mathrm{rank}_{\mathbb{Z}}\, (M/\mathrm{tor}(M))$$

be the $\mathbb{Z}$-rank of $M/\mathrm{tor}(M)$. Similarly we define the $\mathbb{Z}_p$-rank $\mathrm{rank}_{\mathbb{Z}_p} M$ for a finitely generated $\mathbb{Z}_p$-module $M$.

For abelian groups $B_n, C_n$, we will write

$$B_n \sim C_n$$

if there is a homomorphism $f_n : B_n \to C_n$ for each $n$ such that the kernel and cokernel of $f_n$ is bounded independently of $n$. Finally let $\sharp(M)$ denote the cardinality of a finite set $M$.

We recall Theorem 1.1 of the introduction and remark that it would be interesting to classify the cases or conditions under which one of the following conditions (i), (ii) or (iii) is equivalent to the generalized Gross conjecture.

**Theorem 2.2.** *The generalized Gross conjecture is equivalent to*

$$(U'_k \cap N_n k_n^{\times} : N_n(U'_n)) \sim 1.$$

*If $k$ satisfies the generalized Gross conjecture, then we have*
(i) $\mathrm{rank}_{\mathbb{Z}_p}\, ((k^{\times} \otimes \mathbb{Z}_p)^{\mathrm{univ}} \cap (E_k \otimes \mathbb{Z}_p)) = \mathrm{rank}_{\mathbb{Z}_p}\, (E_k \otimes \mathbb{Z}_p)^{\mathrm{univ}}$
(ii) $(k^{\mathrm{univ}} \cap E_k : E_k^{\mathrm{univ}}) < \infty$
(iii) $(k^{\mathrm{univ}} : {U'_k}^{\mathrm{univ}}) < \infty$.

*Proof.* Let $L/k$ be a finite Galois extension with Galois group $G := G(L/k)$ and let $N_{L/k}$ the norm map of $L/k$. For a finite set $S$ of finite primes of $k$, let $I_k(S), P_k(S)$ and $U_k(S)$be the the group of $S$-ideals of $k$, the subgroup of principal $S$-ideals in $I_k(S)$ and the $S$-units of $k$ respectively.

Similarly we define $I_L(S), P_L(S)$ and $U_L(S)$ as the the group of $S'$-ideals of $L$, the subgroup of principal $S'$-ideals in $I_L(S)$ and the $S'$-units of $L$ respectively where $S'$ denotes the prime ideals of $L$ dividing the prime ideals of $S$. It follows from $P_L(S) = L^{\times}/U_k(S)$ and the $S'$-ideal class group $Cl_L(S) = I_L(S)/P_L(S)$ that there exist exact sequences of $G$-modules

$$1 \longrightarrow P_L(S)^G \longrightarrow I_L(S)^G \longrightarrow Cl_L(S)^G \longrightarrow H^1(G, P_L(S)) \longrightarrow 1 \tag{1}$$

$$1 \longrightarrow U_k(S) \longrightarrow k^{\times} \longrightarrow P_L(S)^G \longrightarrow H^1(G, U_L(S)) \longrightarrow 1 \tag{2}$$

$$1 \longrightarrow H^1(G, P_L(S)) \longrightarrow H^2(G, U_L(S)) \longrightarrow H^2(G, L^{\times}). \tag{3}$$

From the equation (1) and $Cl_k(S) = I_k(S)/P_k(S)$, there exist the commutative diagram

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & P_k(S) & \longrightarrow & I_k(S) & \longrightarrow & Cl_k(S) & \longrightarrow & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \iota_{k,L}} & & & & \\ 1 & \longrightarrow & P_L(S)^G & \longrightarrow & I_L(S)^G & \longrightarrow & Cl_L(S)^G & \longrightarrow & H^1(G, P_L(S)) & \longrightarrow & 1. \end{array}$$

Applying the snake lemma to the above diagram, we obtain

$$\begin{aligned} 1 \longrightarrow \ker(\iota_{k,L}) \longrightarrow \quad & H^1(G, U_L(S)) \longrightarrow I_L(S)^G/I_k(S) \\ \longrightarrow \quad & \mathrm{coker}(\iota_{k,L}) \longrightarrow H^1(G, P_L(S)) \longrightarrow 1 \end{aligned} \tag{4}$$

where we used

$$P_L(S)^G/P_k(S) \cong H^1(G, U_L(S))$$

from the equation (2).

By counting cardinalities of each terms of the equation (4), we have a version of the genus theory for the $S$-ideal class group (cf. Corollary 4.2.2 of Ch. IV of of [5]))

$$\sharp(Cl_L(S)^G) = \frac{\sharp(Cl_k)\sharp(I_L(S)^G/I_k(S))\sharp(H^1(G, P_L(S)))}{\sharp(H^1(G, U_L(S)))}.$$

It is clear from the definition that

$$\sharp(I_L(S)^G/I_k(S)) = \prod_{v\notin S} e_v$$

where the product is taken over all the primes $v$ not in $S$ such that the ramification index $e_v := e_v(L/k)$ is nontrivial over $L/k$ (cf. Ch. 6 of [16]).

Moreover if $G = G(L/k)$ is cyclic, then from the equation (3), it follows that

$$\begin{aligned} H^1(G, P_L(S)) \cong H^{-1}(G, P_L(S)) &\cong \text{kernel} : \widehat{H}^0(G, U_L(S)) \to \widehat{H}^0(G, L^\times) \\ &\cong \frac{U_k(S) \cap N_{L/k}L^\times}{N_{L/k}U_L(S)} \end{aligned}$$

where the isomorphism is the connecting homomorphism defined by $(\alpha) = \alpha\mathcal{O}_L(S) \mapsto N_{L/k}(\alpha)$ for the ring $\mathcal{O}_L(S)$ of $S$-integers of $L$.

Putting all these together, we are led to

$$(5)\quad \sharp(Cl_L(S)^G) = \sharp(Cl_k(S)) \cdot \prod_{v\notin S} e_v \cdot \frac{(U_k(S) \cap N_{L/k}L^\times : N_{L/k}U_L(S))}{\sharp(H^1(G, U_L(S)))}.$$

We now see the case $L = k_n$ and $G(L/k) = G(k_n/k) = G_n$ for $k_\infty^{\text{cyc}} = \bigcup k_n$. Let $I_n(S)$, $P_n(S)$ and $U_n(S)$ denote the the group of $S$-ideals of $k_n$, the subgroup of principal $S$-ideals in $I_n(S)$ and the $S$-units of $k_n$ respectively.

Now let $S$ be the set of primes dividing $p$. Since all primes prime to $p$ are unramified over $k_\infty^{\text{cyc}}/k$ (see Ch. 13 of [20] and §4 of Ch. 5 of [15]), it follows that

$$\prod_{v\notin S} e_v = 1.$$

For $U'_\infty := \varinjlim U'_n$, the inflation map

$$1 \longrightarrow H^1(G_n, U'_n)) \longrightarrow H^1(\Gamma, U'_\infty))$$

and

$$H^1(\Gamma, U'_\infty)) < \infty$$

which is Proposition 3 of of [10], show that

$$H^1(G, U'_n)) \sim 1.$$

Since $U'_k \cap N_n k_n^\times / N_n(U'_n)$ is a $p$-group, for the $p$-primary part $A'_n$ of $Cl_n(p)$, we obtain the following lemma (cf. [19]).

**Lemma 2.3.** *We have the following asymptotic formula*

$$\begin{aligned} \sharp({A'_n}^{G_n}) &\sim \sharp(H^1(G_n, P_n(p))) \\ &\sim \sharp(\frac{U'_k \cap N_n k_n^\times}{N_n(U'_n)}). \end{aligned}$$

By Proposition 1.4 and Lemma 1.5 of [13], the generalized Gross conjecture is equivalent to

$$A_n'^{G_n} \sim 1. \tag{6}$$

Hence Lemma 2.3 completes the proof of the first statement of the theorem.

For the proof of the second statement, let

$$[S_n] := \{\prod_{\mathfrak{p}|p} \mathfrak{p}^{n_\mathfrak{p}} \mid n_\mathfrak{p} \in \mathbb{Z}\} \subset I_n$$

denote the subgroup of the fractional ideal group $I_n$ which is generated by the primes of $k_n$ dividing $p$ and hence for the principal fractional ideal group $P_n$ of $I_n$,

$$\langle S_n\rangle = [S_n]P_n/P_n, \; Cl_n(p) = I_n/[S_n]P_n.$$

From the exact sequence

$$1 \longrightarrow ([S_n]P_n \otimes \mathbb{Z}_p)^{G_n} \longrightarrow (I_n \otimes \mathbb{Z}_p)^{G_n} \longrightarrow A_n'^{G_n}$$

the generalized Gross conjecture implies that

$$\frac{([S_n]P_n \otimes \mathbb{Z}_p)^{G_n}}{(P_n \otimes \mathbb{Z}_p)^{G_n}} \sim \frac{(I_n \otimes \mathbb{Z}_p)^{G_n}}{(P_n \otimes \mathbb{Z}_p)^{G_n}}.$$

We claim that

$$\frac{([S_n]P_n \otimes \mathbb{Z}_p)^{G_n}}{(P_n \otimes \mathbb{Z}_p)^{G_n}} \sim (\langle S_n\rangle \otimes \mathbb{Z}_p)^{G_n} = (\frac{[S_n]P_n \otimes \mathbb{Z}_p}{P_n \otimes \mathbb{Z}_p})^{G_n}.$$

The claim shows that the generalized Gross conjecture implies

$$\frac{(I_n \otimes \mathbb{Z}_p)^{G_n}}{(P_n \otimes \mathbb{Z}_p)^{G_n}} \sim (\langle S_n\rangle \otimes \mathbb{Z}_p)^{G_n}. \tag{7}$$

We prove the claim. Fix a positive integer $n_0$ such that all primes dividing $p$ are totally ramified $k_\infty/k_{n_0}$. Then all the primes of $k_n$ dividing $p$ are fixed by the Galois action of $G(k_n/k_{n_0})$ for all $n \geq n_0$. Hence the following surjection is an isomorphism

$$\frac{([S_n]P_n \otimes \mathbb{Z}_p)^{G_{n,n_0}}}{(P_n \otimes \mathbb{Z}_p)^{G_{n,n_0}}} \xrightarrow{\cong} (\frac{[S_n]P_n \otimes \mathbb{Z}_p}{P_n \otimes \mathbb{Z}_p})^{G_{n,n_0}} = \frac{[S_n]P_n \otimes \mathbb{Z}_p}{P_n \otimes \mathbb{Z}_p}.$$

since $S_n$ is fixed by $G_{n,n_0}$. By taking the invariant of $G_{n_0}$ in the above identity, we have

$$(\frac{([S_n]P_n \otimes \mathbb{Z}_p)^{G_{n,n_0}}}{(P_n \otimes \mathbb{Z}_p)^{G_{n,n_0}}})^{G_{n_0}} \xrightarrow{\cong} (\frac{[S_n]P_n \otimes \mathbb{Z}_p}{P_n \otimes \mathbb{Z}_p})^{G_n}. \tag{8}$$

By taking $G_{n_0}$-cohomology from the isomorphism above, it follows from the exact sequence

$$1 \longrightarrow (P_n \otimes \mathbb{Z}_p)^{G_n} \longrightarrow ([S_n]P_n \otimes \mathbb{Z}_p)^{G_n} \longrightarrow (\frac{([S_n]P_n \otimes \mathbb{Z}_p)^{G_{n,n_0}}}{(P_n \otimes \mathbb{Z}_p)^{G_{n,n_0}}})^{G_{n_0}} \longrightarrow$$

$$\xrightarrow{d_n} H^1(G_{n_0}, (P_n \otimes \mathbb{Z}_p)^{G_{n,n_0}}).$$

Note that the cokernel of the connecting map $d_n$ is finite independently of $n$ since it is a $\sharp(G_{n_0})$-torsion group as a subgroup of $H^1(G_{n_0}, (P_n \otimes \mathbb{Z}_p)^{G_{n,n_0}})$ and of finite rank bounded by $\sharp(S_n)$ which is also finite independently of $n$ over $k_\infty^{\rm cyc}/k$. It follows that

$$\frac{([S_n]P_n\otimes\mathbb{Z}_p)^{G_n}}{(P_n\otimes\mathbb{Z}_p)^{G_n}}\sim(\frac{([S_n]P_n\otimes\mathbb{Z}_p)^{G_{n,n_0}}}{(P_n\otimes\mathbb{Z}_p)^{G_{n,n_0}}})^{G_{n_0}}\sim(\frac{[S_n]P_n\otimes\mathbb{Z}_p}{P_n\otimes\mathbb{Z}_p})^{G_n}$$

as claimed.

For an abelian group $G$ and $G$-module $A$, by tensoring with the flat $\mathbb{Z}$-module $\mathbb{Z}_p$, the exact sequence

$$1\longrightarrow H^{-1}(G,A)\longrightarrow A/(\sigma-1)A\longrightarrow N_G(A)\longrightarrow 1$$

leads to

$$H^{-1}(G,A)\otimes\mathbb{Z}_p\cong H^{-1}(G,A\otimes\mathbb{Z}_p).$$

If $G$ is a $p$-group, then we have

$$H^{-1}(G,A)\cong H^{-1}(G,A)\otimes\mathbb{Z}_p\cong H^{-1}(G,A\otimes\mathbb{Z}_p).$$

It follows from

$$1\longrightarrow(\langle S_n\rangle\otimes\mathbb{Z}_p)^{G_n}\longrightarrow A_n^{G_n}\longrightarrow {A'_n}^{G_n}$$

that the generalized Gross conjecture implies that

$$(\langle S_n\rangle\otimes\mathbb{Z}_p)^{G_n}\sim A_n^{G_n} \tag{9}$$

since the generalized Gross conjecture is equivalent via the equation (6) to

$$\sharp({A'_n}^{G_n})\sim 1.$$

From the exact sequence

$$1\longrightarrow(P_n\otimes\mathbb{Z}_p)^{G_n}\longrightarrow(I_n\otimes\mathbb{Z}_p)^{G_n}\longrightarrow A_n^{G_n}\longrightarrow H^1(G_n,P_n)\longrightarrow 1$$

and the equations (7) and (9), it follows that

$$1\sim H^1(G_n,P_n)$$

and from the equation (3) with $S=\phi$, that

$$1\sim H^1(G_n,P_n)\cong\frac{E_k\cap N_n(k_n^\times)}{N_n(E_n)}. \tag{10}$$

Since the above quotient is a $p$-group, by tensoring $\mathbb{Z}_p$, the generalized Gross conjecture implies

$$\frac{(E_k\cap N_nk_n^\times)\otimes\mathbb{Z}_p}{N_n(E_n\otimes\mathbb{Z}_p)}\sim(\frac{E_k\cap N_nk_n^\times}{N_nE_n})\otimes\mathbb{Z}_p\cong\frac{E_k\cap N_nk_n^\times}{N_nE_n}\sim 1.$$

By taking inverse limits with respect to the inclusions in the exact sequence

$$1\longrightarrow(E_k\cap N_nk_n^\times)\otimes\mathbb{Z}_p\longrightarrow N_n(E_n\otimes\mathbb{Z}_p)\longrightarrow(\frac{E_k\cap N_nk_n^\times}{N_nE_n})\otimes\mathbb{Z}_p\longrightarrow 1$$

we have

$$\mathrm{rank}_{\mathbb{Z}_p}\ ((k^\times\otimes\mathbb{Z}_p)^{\rm univ}\cap(E_k\otimes\mathbb{Z}_p))=\mathrm{rank}_{\mathbb{Z}_p}\ (E_k\otimes\mathbb{Z}_p)^{\rm univ} \tag{11}$$

which shows (i).

From the equation (10) and the exact sequence

$$1\longrightarrow\varprojlim(E_k\cap N_nk_n^\times)\longrightarrow\varprojlim N_n(E_n)\longrightarrow\varprojlim(\frac{E_k\cap N_nk_n^\times}{N_nE_n}),$$

it follows that the generalized Gross conjecture implies (ii) of the theorem

$$\mathrm{rank}_{\mathbb{Z}}\ (k^{\mathrm{univ}} \cap E_k) = \mathrm{rank}_{\mathbb{Z}}\ E_k^{\mathrm{univ}}.$$

Similarly since $k^{\mathrm{univ}} \subset U_k'$, Lemma 2.3, the equation (6) and the exact sequence

$$1 \longrightarrow N_n(U_n') \longrightarrow (U_k' \cap N_n k_n^\times) \longrightarrow \frac{U_k' \cap N_n k_n^\times}{N_n(U_n')} \longrightarrow 1$$

lead to (iii) of the theorem

$$\mathrm{rank}_{\mathbb{Z}}\ k^{\mathrm{univ}} = \mathrm{rank}_{\mathbb{Z}}\ {U_k'}^{\mathrm{univ}}.$$

This completes the proof of Theorem 2.2. □

Using the ambiguous class number for the cyclic extension $L/k$ with $S = \phi$, the empty set so that $U_k(S) = E_k, U_L(S) = E_L$. The equation (5) reads off

$$\begin{aligned}\sharp(Cl_L^{G(L/k)}) &= h_k \frac{(E_k \cap N_{L/k} L^\times : N_{L/k} E_L)}{\sharp H^1(G_n, E_L)} \prod_v e_v \\ &\sim \frac{(E_k \cap N_{L/k} L^\times : N_{L/k} E_L)}{\sharp H^1(G_n, E_L)} \prod_v e_v.\end{aligned}$$

where $h_k$ the class number of $k$ and $e_v$ the ramification index over $L/k$.

We now apply this for $L = k_n$. Since for all sufficiently large $n$, all the primes lying over $p$ are totally ramified over $k_\infty/k_n$, we may assume that all the primes lying over $p$ are totally ramified over $k_\infty/k$ by replacing $k$ by $k_n$. Moreover if we assume the generalized Gross conjecture, then by (10), the formula above leads to

$$\begin{aligned}\sharp(A_n^{G_n}) &\sim \frac{(E_k \cap N_n k_n^\times : N_n E_n)}{\sharp H^1(G_n, E_n)} \prod_{v \nmid \infty} p^n \\ &\sim \frac{\prod_{v|p} p^n}{\sharp H^1(G_n, E_n)}.\end{aligned}$$

We need a formula of Iwasawa (see §5 of [10]) for the bound of the first cohomology of $\sharp H^1(G_n, E_n)$. Let $r$ be the number of primes of $k_n$ dividing $p$. Then there exists a number $r' \leq r$ such that

$$\sharp H^1(G_n, E_n) \sim p^{r'n}.$$

Since all primes prime to $p$ are unramified over the $\mathbb{Z}_p$-extension, we have

$$\begin{aligned}\sharp(A_n^{G_n}) &\sim \frac{\prod_{v|p} p^n}{\sharp H^1(G_n, E_n)} \\ &\sim p^{(r-r')n}.\end{aligned}$$

It follows from the fact that the generalized Gross conjecture is equivalent to ${A_n'}^{G_n} \sim 1$ and from the exact sequence

$$1 \longrightarrow (\langle S_n \rangle \otimes \mathbb{Z}_p)^{G_n} \longrightarrow A_n^{G_n} \longrightarrow {A_n'}^{G_n}$$

induced from $1 \longrightarrow \langle S_n \rangle \otimes \mathbb{Z}_p \longrightarrow A_n \longrightarrow A_n' \longrightarrow 1$ that

$$\sharp(A_n^{G_n}) \sim \sharp(\langle S_n \rangle \otimes \mathbb{Z}_p)^{G_n} \sim p^{(r-r')n}.$$

We have the following remark.

**Remark.** Let $r$ be the number of primes of $k_n$ dividing $p$. If the generalized Gross conjecture is true, then there exists $r' \leq r$ such that

$$\sharp H^1(G_n, E_n) \sim p^{nr'},\ \sharp(A_n^{G_n}) \sim \sharp(\langle S_n \rangle \otimes \mathbb{Z}_p)^{G_n} \sim p^{(r-r')n}.$$

Moreover if we choose $n_0$ as a number such that $k_\infty/k_{n_0}$ is totally ramified at all primes dividing $p$, then for all $n \geq n_0$,

$$\sharp(A_n^{G_{n,n_0}}) \sim \sharp(\langle S_n \rangle \otimes \mathbb{Z}_p) \sim p^{(r-r')n}.$$

Note that this shows the asymptotic formula for the Galois invariant $\sharp(A_n^{G_n})$ of the ideal class group of $k_n$, whereas the generalized Gross conjecture predicts that of the Galois invariant $\sharp({A'_n}^{G_n})$ of the $p$-ideal class group of $k_n$ over $k_\infty^{\mathrm{cyc}}/k$.

2.2. **The structure of the quotient of the universal norm groups.** In this subsection we show that the quotient groups of universal norm groups are isomorphic for all large layers and finite under the assumption of the generalized Gross conjecture. Note that the proof uses essentially the same argument of Kuz'min (see §8 of [14]).

**Proposition 2.4.** *For all sufficiently large $m \geq n$, the natural map $\lambda_{n,m}$ induced by the embedding induces an isomorphism*

$$\frac{(k_n^\times \otimes \mathbb{Z}_p)^{\mathrm{univ}}}{(U'_n \otimes \mathbb{Z}_p)^{\mathrm{univ}}} \cong \frac{(k_m^\times \otimes \mathbb{Z}_p)^{\mathrm{univ}}}{(U'_m \otimes \mathbb{Z}_p)^{\mathrm{univ}}}$$

$$\frac{k_n^{\mathrm{univ}}}{{U'_n}^{\mathrm{univ}}} \cong \frac{k_m^{\mathrm{univ}}}{{U'_m}^{\mathrm{univ}}}$$

*and there is an exact sequence*

$$1 \longrightarrow \frac{k_n^{\mathrm{univ}}}{{U'_n}^{\mathrm{univ}}} \longrightarrow \frac{(k_n^\times \otimes \mathbb{Z}_p)^{\mathrm{univ}}}{(U'_n \otimes \mathbb{Z}_p)^{\mathrm{univ}}}.$$

*If the generalized Gross conjecture holds for $\{k_n\}_n$, then all the groups above are finite.*

*Proof.* Let

$$\mathcal{U}'_\infty = \varprojlim F(U'_n \otimes \mathbb{Z}_p)$$

denote the inverse limit of $F(U'_n \otimes \mathbb{Z}_p)$ with respect to the norm maps.

The rank of $\mathcal{U}'_\infty$ as a $\Gamma$-module is computed in the following theorem which is Theorem 7.2 of [14].

**Theorem 2.5.** *Let $k$ be a number field and let $r_1$ and $r_2$ be the number of real and complex places of $k$. Then $\mathcal{U}'_\infty$ is a free $\Gamma$-module of rank $r_1 + r_2$.*

*Proof.* See the proof of Theorem 7.2 of [14]. Since the weak Leopoldt conjecture holds for the cyclotomic $\mathbb{Z}_p$-extension of a number field, we can find an another proof from Corollary 10.3.24 and Theorem 11.3.11 of [17]. □

In Theorem 7.3 of [14], Kuz'min proved an isomorphism

$$\pi : (\mathcal{U}'_\infty)_\Gamma \xrightarrow{\cong} F(U'_k \otimes \mathbb{Z}_p)^{\mathrm{univ}}$$

which is also proved by Greither and others in different ways (see [7] and [18]).

**Theorem 2.6.** $(\mathcal{U}'_\infty)_\Gamma \cong F(U'_k \otimes \mathbb{Z}_p)^{\mathrm{univ}}$.

From Theorems 2.5 and 2.6, it leads to the following result of Kuz'min (cf. [14]).

**Corollary 2.7.** *As $G(k_n/k)$-modules, there exists an isomorphism*

$$\phi_n : (U'_n \otimes \mathbb{Z}_p)^{\mathrm{univ}} \xrightarrow{\sim} \mathbb{Z}_p[G(k_n/k)]^{r_1+r_2} \oplus \operatorname{tor}_p(U'_n)$$

*where* $\operatorname{tor}_p(U'_n)$ *denotes the $p$-power roots of unity in* $U'_n$.

For a $\mathbb{Z}_p$-module $M$, we write

$$F(M) := M/\operatorname{tor}(M)$$

for the quotient of $M$ by its $\mathbb{Z}_p$-torsion $\operatorname{tor}(M)$. For all $m \geq n$, there exists a natural map

$$\lambda_{n,m} : F(U'_n \otimes \mathbb{Z}_p)/F(U'_n \otimes \mathbb{Z}_p)^{\mathrm{univ}} \to F(U'_m \otimes \mathbb{Z}_p)/F(U'_m \otimes \mathbb{Z}_p)^{\mathrm{univ}}$$

which is induced from the natural embedding. Let $s = \sum_{g \in G(k_m/k_n)} g$ and let $G(k_m/k) = \bigcup_{i \in I} \sigma_i G(k_m/k_n)$. Since the isomorphism of Corollary 2.7 is Galois equivariant, we have the following commutative diagram

$$\begin{array}{ccc} F(U'_n \otimes \mathbb{Z}_p)^{\mathrm{univ}} & \xrightarrow{\phi_n} & \mathbb{Z}_p[G(k_n/k)]^{r_1+r_2} \cong \mathbb{Z}_p[\{\sigma_i s : i \in I\}]^{r_1+r_2} \\ \downarrow & & \downarrow \\ F(U'_m \otimes \mathbb{Z}_p)^{\mathrm{univ}} & \xrightarrow{\phi_m} & \mathbb{Z}_p[G(k_m/k)]^{r_1+r_2}. \end{array}$$

where the vertical maps are the inclusion map. We obtain the following corollary.

**Corollary 2.8.** $H^0(G(k_m/k_n), (U'_m \otimes \mathbb{Z}_p)^{\mathrm{univ}}) = (U'_n \otimes \mathbb{Z}_p)^{\mathrm{univ}}$.

It follows from Corollary 2.8 that $\lambda_{n,m}$ is injective. Note that if we choose $n_0$ such that $k_\infty/k_{n_0}$ is totally ramified, then for all $n \geq n_0$,

$$\operatorname{rank}(F(U'_n \otimes \mathbb{Z}_p)/F(U'_n \otimes \mathbb{Z}_p)^{\mathrm{univ}}) = r - 1$$

where $r = r_{n_0}$ is the number of primes of $k_n$ dividing $p$.

In the following, we define $(k_n^\times \otimes \mathbb{Z}_p)^{\mathrm{univ}}$ as the same way as $(k^\times \otimes \mathbb{Z}_p)^{\mathrm{univ}}$ by replacing the base field $k$ with $k_n$. Write

$$\Omega_n := F(k_n^\times \otimes \mathbb{Z}_p)^{\mathrm{univ}}/F(U'_n \otimes \mathbb{Z}_p)^{\mathrm{univ}}.$$

Since $\lambda_{n,m}$ is injective, the induced map

$$\lambda_{n,m} : \Omega_n \to \Omega_m$$

is also injective and for all $n \geq 0$,

$$\operatorname{rank}(\Omega_n) \leq r - 1.$$

Hence the torsion subgroups $\operatorname{tor}(\Omega_n)$ are isomorphic for all sufficiently large $n$ and from Proposition 2.1, the torsion subgroups $\operatorname{tor}(T_p(k_n)^{\Gamma_n}) = \operatorname{tor}(T_p(k_n))^{\Gamma_n}$ are also isomorphic for all sufficiently large $n$.

Since $F(U'_n \otimes \mathbb{Z}_p)^{\mathrm{univ}}$ is a cohomologically trival $\mathbb{Z}_p[G(k_n/k)]$-module by Corollary 2.7, it follows that the image $\operatorname{im}(\lambda_{n,m})$ of $\lambda_{n,m}$ can be identified with

$$\operatorname{im}(\lambda_{n,m}) \cong H^0(G(k_m/k_n), \Omega_m).$$

We choose a constant $p^t$ such that $\Omega_n^{p^t}$ is torsion free for all $n \geq 0$. Now $\Gamma_n$ acts trivially on

$$\Omega_m^{p^t} \cap \operatorname{im}(\lambda_{n,m})$$

and hence $\Gamma_n$ acts trivially on $\Omega_m^{p^t}$ since $\Omega_m^{p^t}$ is a torsion free module and

$$(\Omega_m^{p^t} : \Omega_m^{p^t} \cap \operatorname{im}(\lambda_{n,m})) < \infty.$$

Since

$$\mathrm{cok}(\lambda_{n,m}) = (\Omega_m : \Omega_m^{\Gamma_n}) \le (\Omega_m : \Omega_m^{p^t}) \le p^t(r-1)$$

the cokernel $\mathrm{cok}(\lambda_{n,m})$ of $\lambda_{n,m}$ is bounded by a constant for all sufficiently large $m \ge n$. We hence conclude that for all sufficiently large $m \ge n$, the cokernel of $\lambda_{n,m}$ is trivial and hence $\lambda_{n,m}$ is isomorphic. The following corollary is essentially due to Kuz'min (cf. [14]).

**Corollary 2.9.** *For all sufficiently large $m \ge n$, $\lambda_{n,m}$ induces an isomorphism*

$$\lambda_{n,m} : \frac{(k_n^\times \otimes \mathbb{Z}_p)^{\mathrm{univ}}}{(U_n' \otimes \mathbb{Z}_p)^{\mathrm{univ}}} \cong \frac{(k_m^\times \otimes \mathbb{Z}_p)^{\mathrm{univ}}}{(U_m' \otimes \mathbb{Z}_p)^{\mathrm{univ}}}.$$

Since $k_n^{\mathrm{univ}} \otimes \mathbb{Z}_p \subset (k_n^\times \otimes \mathbb{Z}_p)^{\mathrm{univ}}$ and ${U_n'}^{\mathrm{univ}} \otimes \mathbb{Z}_p \subset (U_n' \otimes \mathbb{Z}_p)^{\mathrm{univ}}$, it follows that

$$\mathrm{rank}_{\mathbb{Z}} \frac{U_n'}{{U_n'}^{\mathrm{univ}}} = \mathrm{rank}_{\mathbb{Z}_p} (\frac{U_n'}{{U_n'}^{\mathrm{univ}}} \otimes \mathbb{Z}_p) \ge \mathrm{rank}_{\mathbb{Z}_p} \frac{U_n' \otimes \mathbb{Z}_p}{(U_n' \otimes \mathbb{Z}_p)^{\mathrm{univ}}}$$

$$\mathrm{rank}_{\mathbb{Z}} \frac{U_n'}{k_n^{\mathrm{univ}}} \ge \mathrm{rank}_{\mathbb{Z}_p} \frac{U_n' \otimes \mathbb{Z}_p}{(k_n^\times \otimes \mathbb{Z}_p)^{\mathrm{univ}}}.$$

Write $\mathcal{N}_n(m) := N_{m,n}\, U_m'$. Then the inverse limit of $\mathcal{N}_n(m)$ with respect to the inclusion maps is equal to

$$\varprojlim_{m \ge n} \mathcal{N}_n(m) = {U_n'}^{\mathrm{univ}}.$$

Similarly since $\mathcal{N}_n(m) \otimes \mathbb{Z}_p = (N_{m,n}\, U_m') \otimes \mathbb{Z}_p = N_{m,n}\, (U_m' \otimes \mathbb{Z}_p)$, the inverse limit of $\mathcal{N}_n(m) \otimes \mathbb{Z}_p$ with respect to the inclusion maps is equal to

$$\varprojlim_{m \ge n} (\mathcal{N}_n(m) \otimes \mathbb{Z}_p) = (U_n' \otimes \mathbb{Z}_p)^{\mathrm{univ}}.$$

By taking inverse limits with respect to the inclusions, the exact sequence

$$1 \longrightarrow \mathcal{N}_n(m) \otimes \mathbb{Z}_p \longrightarrow U_n' \otimes \mathbb{Z}_p \longrightarrow (\frac{U_n'}{\mathcal{N}_n(m)}) \otimes \mathbb{Z}_p \longrightarrow 1$$

leads to

$$1 \longrightarrow (U_n' \otimes \mathbb{Z}_p)^{\mathrm{univ}} \longrightarrow U_n' \otimes \mathbb{Z}_p \longrightarrow \varprojlim_{m \ge n} (\frac{U_n'}{\mathcal{N}_n(m)} \otimes \mathbb{Z}_p) \longrightarrow 1$$

where the inverse limit is exact since $U_n' \otimes \mathbb{Z}_p$ is compact.

Since $U_n'/\mathcal{N}_n(m)$ is a $p$-group, the above identity will lead to

$$\varprojlim_{m \ge n} (\frac{U_n'}{\mathcal{N}_n(m)}) \cong \varprojlim_{m \ge n} (\frac{U_n'}{\mathcal{N}_n(m)} \otimes \mathbb{Z}_p).$$

By taking inverse limits to

$$1 \longrightarrow \mathcal{N}_n(m) \longrightarrow U_n' \longrightarrow \frac{U_n'}{\mathcal{N}_n(m)} \longrightarrow 1$$

we have the left exact sequence

$$1 \longrightarrow {U_n'}^{\mathrm{univ}} \longrightarrow U_n' \longrightarrow \varprojlim_{m \ge n} (\frac{U_n'}{\mathcal{N}_n(m)}).$$

Hence it follows from the equation (12) that the following inclusion

$$1 \longrightarrow \frac{U'_n}{{U'_n}^{\mathrm{univ}}} \longrightarrow \frac{U'_n \otimes \mathbb{Z}_p}{(U'_n \otimes \mathbb{Z}_p)^{\mathrm{univ}}}$$

leads to

$$1 \longrightarrow \frac{U'_n}{{U'_n}^{\mathrm{univ}}} \longrightarrow \frac{U'_m}{{U'_m}^{\mathrm{univ}}}.$$

We proved the following lemma.

**Lemma 2.10.** $H^0(G_{m,n}, {U'_m}^{\mathrm{univ}}) = {U'_n}^{\mathrm{univ}}$.

For $N_{m,n}k_m^{\circ} := N_{m,n}k_m^{\times} \cap U'_n$, we have

$$\varprojlim_{m \geq n} N_{m,n}k_m^{\circ} = \varprojlim_{m \geq n} N_{m,n}k_m^{\times} = k_n^{\mathrm{univ}}$$

$$\varprojlim_{m \geq n} N_{m,n}(k_m^{\circ} \otimes \mathbb{Z}_p) = (k_n^{\times} \otimes \mathbb{Z}_p)^{\mathrm{univ}}.$$

By taking the inverse limits to the following exact sequence

$$1 \longrightarrow N_{m,n}U'_m \longrightarrow N_{m,n}k_m^{\circ} \longrightarrow (\frac{N_{m,n}k_m^{\circ}}{N_{m,n}U'_m}) \otimes \mathbb{Z}_p \longrightarrow 1$$

we have

$$1 \longrightarrow {U'_n}^{\mathrm{univ}} \longrightarrow k_n^{\mathrm{univ}} \longrightarrow \varprojlim_{m \geq n} (\frac{N_{m,n}k_m^{\circ}}{N_{m,n}U'_m} \otimes \mathbb{Z}_p).$$

The last term is isomorphic to

$$\varprojlim_{m \geq n} (\frac{N_{m,n}k_m^{\circ}}{N_{m,n}U'_m} \otimes \mathbb{Z}_p) \cong \frac{(k_n^{\times} \otimes \mathbb{Z}_p)^{\mathrm{univ}}}{(U'_n \otimes \mathbb{Z}_p)^{\mathrm{univ}}}$$

by taking the inverse limits to the following exact sequence

$$1 \longrightarrow N_{m,n}U'_m \otimes \mathbb{Z}_p \longrightarrow N_{m,n}k_m^{\circ} \otimes \mathbb{Z}_p \longrightarrow (\frac{N_{m,n}k_m^{\circ}}{N_{m,n}U'_m}) \otimes \mathbb{Z}_p \longrightarrow 1.$$

Hence it follows that there is an exact sequence

$$1 \longrightarrow \frac{k_n^{\mathrm{univ}}}{{U'_n}^{\mathrm{univ}}} \longrightarrow \frac{(k_n^{\times} \otimes \mathbb{Z}_p)^{\mathrm{univ}}}{(U'_n \otimes \mathbb{Z}_p)^{\mathrm{univ}}}$$

and from Lemma 2.10 that the inclusion induces an injective map

$$1 \longrightarrow \frac{k_n^{\mathrm{univ}}}{{U'_n}^{\mathrm{univ}}} \longrightarrow \frac{k_m^{\mathrm{univ}}}{{U'_m}^{\mathrm{univ}}}.$$

This completes the proof of Proposition 2.4. □

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