Euler systems and special units

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Abstract

Let $F$ be a number field. We investigate the group of Rubin’s special units, $S_F$ defined over $F$. The group of special units is a subgroup of the group of global units containing the group of Sinnott’s cyclotomic units, $C_F$ of $F$. It plays an important role in studying the ideal class group of $F$. Let $(S^n_K)$ be a sequence of decreasing subgroups $S^n_K$ (defined in Section 2) of the group of global units of any real abelian field $K$ which lie between Rubin’s special units and the circular units of $K$. Motivated by a question of whether the group of special units equals the group of cyclotomic units, which is stated by Rubin (Invent. Math. 89 (1987) 511), we propose the following question which relates the group structure of the ideal class group with the group structure of units modulo special units. Are $\frac{Cl_F}{S^n_F}$ and $\bigoplus_{n \geq 0} \frac{S^n_F/S^{n+1}_F}{Zp}$ isomorphic as $\mathbb{Z}[\text{Gal}(F/\mathbb{Q})]$ modules? Let $\mathbb{Z}$ be the set of $p$-adic valued Dirichlet characters of $\text{Gal}(F/\mathbb{Q})$. Let $S^n_F$, $C^n_F$ and $Cl^n_F$ be the $\chi$-eigenspaces of $S_F \otimes \mathbb{Z}_p$, $C_F \otimes \mathbb{Z}_p$ and $Cl_F \otimes \mathbb{Z}_p$ respectively. Using Euler system methods and Thaine’s results we obtain that the $\mathbb{Z}/p\mathbb{Z}$-rank of $\bigoplus_{n \geq 0} (S^n_F/\Zp)^{\chi}$ is less than or equal to the $\mathbb{Z}/p\mathbb{Z}$-rank of $Cl^n_F$ with some inequalities on the cardinalities of both sides. This gives us the following corollary. If $p \nmid \frac{2[F:\mathbb{Q}]}{h_F}$, then for all $\chi \in \mathbb{Z}$, we have $S^n_F = C^n_F \iff Cl^n_F$ is a cyclic group.

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1. Introduction

Let $p$ be an odd prime and $\zeta_p$ be a fixed primitive $p$th root of unity. Let $F := \mathbb{Q}(\zeta_p + \zeta_p^{-1})$ be the maximal real subfield of the cyclotomic field $\mathbb{Q}(\zeta_p)$. Let $Cl_F$ be
the ideal class group of $F$ and let $C_F$ be the group of cyclotomic units of $F$. Rubin defines the group of special units $S_F$ in [6]. The definition of special unit is given in the next section. The group of special units is a subgroup of the group of global units containing the group of cyclotomic units of $F$. The special units give rise to annihilators of certain subquotients of the ideal class group of $F$ (cf. [6]). The motivation for this paper is the following remark given in [6].

Is the group of special units $S_F$ equal to the group of cyclotomic units $C_F$?

The answer to this question is no in general, and this has been known since Kolyvagin. He introduced the idea of Euler Systems which has the special units as its first step. Euler systems and Iwasawa theory have also played an important role in solving a variety of other classical problems in number theory (cf. [2,4,8]).

We want to relate Rubin’s question to a question about the connection between the group structures of the ideal class group and various quotients of the (group of) special units. In order for this we will define a sequence of subgroups $S^n_K$ of the group of global units $E_K$ of any real abelian field $K$ which lie between Rubin’s special units and the circular units $C_K$ of $K$.

$$S^0_K := E_K \supseteq S^1_K \supseteq S^2_K \supseteq \cdots \supseteq S^\infty_K := \bigcap_{n \in \mathbb{N}} S^n_K \supseteq C_K.$$ 

By the class number formula, this sequence must terminate for some non-negative integer $N$,

$$S^\infty_K = \bigcap_{n \in \mathbb{N}} S^n_K = S^N_K.$$

A conjecture of Coleman implies that $S^\infty_K$ is the group of circular units of $K$, $S^\infty_K = C_K$. In fact the conjecture of Coleman contains more information than the above implication. We will not cover Coleman’s conjecture in this paper even though there is a strong connection between special units and Coleman’s conjecture. We refer the reader to [1,9,10] for the details. If we assume that Coleman’s conjecture is true then we can see that the orders of $\text{Cl}_F$ and $\bigoplus_{n \geq 0} \frac{S^n_F}{S^{n+1}_F}$ are equal. This observation together with Rubin’s remark lead us to the following question.

**Question.** Are $\text{Cl}_F \cong \bigoplus_{n \geq 0} \frac{S^n_F}{S^{n+1}_F}$ as $\mathbb{Z}[\text{Gal}(F/\mathbb{Q})]$ modules?

The purpose of this note is to study the question given above using standard techniques of Euler systems. We state the main theorem. Let $\hat{\text{Cl}}_F := \text{Cl}_F \otimes \mathbb{Z}_p$. Let $\hat{C}_F$ and $\hat{S}_F$ be the $p$-adic completions of $C_F$ and $S_F$, $\hat{C}_F := C_F \otimes \mathbb{Z}_p$ and $\hat{S}_F := S_F \otimes \mathbb{Z}_p$. Let $\Xi$ be the set of $p$-adic valued Dirichlet characters of $\text{Gal}(F/\mathbb{Q})$. For each $\chi \in \Xi$, we let $C^{\chi}_F, S^{\chi}_F$ be the $\chi$-parts of $\hat{C}_F$ and $\hat{S}_F$ respectively. Let $(C_F)^\chi \cong \bigoplus_{n=1}^k \mathbb{Z}/p^{r_n} \mathbb{Z}$ with $r_1 \geq r_2 \geq r_3 \geq \cdots \geq r_k$ and let $(S^\chi_F)/(S^{\chi+1}_F)^\chi \cong \mathbb{Z}/p^{n+1} \mathbb{Z}$. Let $p$
be an odd prime in the following theorem and corollary. The proof of the following theorem follows from the arguments of Kolyvigin–Rubin–Thaine.

**Theorem A.** For each \( \chi \in \mathcal{Z} \), we have

- \( \text{rk}_{\mathbb{Z}/p\mathbb{Z}} \bigoplus_{n \geq 0} (\mathbb{Z}/p\mathbb{Z})_{\chi}^{n} \leq \text{rk}_{\mathbb{Z}/p\mathbb{Z}} \text{Cl}_{F}^{\chi} \)
- \( \sum_{i=1}^{a} r_i \leq \sum_{i=1}^{a} s_i \) for \( 1 < a < k \), \( \sum_{i=1}^{k} r_i = \sum_{i=1}^{k} s_i \)
- \( r_1 = s_1 \)

From Theorem A, we obtain the following corollary.

**Corollary.** For each \( \chi \in \mathcal{Z} \), we have

\[ S_{F}^{\chi} = C_{F}^{\chi} \iff \text{Cl}_{F}^{\chi} \text{ is a cyclic group.} \]

Kolyvagin proves an equivalent form of Theorem A in Theorem 7 of [3], where Kolyvagin states \( \text{Cl}_{F}^{\chi} \) in terms of weights of certain sets. This result is known to experts, but as far as we know is not written down except possibly in an overly complicated way in Kolyvagin’s paper. We prove Theorem A using results of Thaine and Rubin.

2. Special units

We define special units of level \( n \) following Rubin. Let \( L \) be an abelian extension of a fixed number field \( K \) containing the Hilbert class field of \( K \). (Note that \( K \) is any number field.) Let \( \Sigma_{K} \) be the set of all non-archimedean valuations of \( K \). For a finite subset \( S \) of \( \Sigma_{K} \), we let \( \Sigma_{K}(S) \) be the subset of all non-archimedean valuations of \( K \) not in \( S \) and let \( J_{L/K,n}(S) \) be the subset of fractional ideals \( \alpha = \prod_{i} \mathfrak{p}_{i} \) such that each prime ideal \( \mathfrak{p}_{i} \) dividing \( \alpha \) has absolute degree 1 and splits completely in \( L \) and the number of primes dividing \( \alpha \) is less than \( n + 1 \).

\[ J_{L/K,n}(S) = \left\{ \prod_{i \in I} \mathfrak{q}_{i} \mid \mathfrak{q}_{i} \in \Sigma_{K}(S), \ |I| \leq n, \ \mathfrak{q}_{i} \text{ has absolute degree 1,} \right. \left. \mathfrak{q}_{i} \text{ splits completely in } L. \right\}. \]

For each prime ideal \( \mathfrak{q} \) and fractional ideal \( \mathfrak{a} \) in \( J_{L/K,n}(S) \), we let

\[ K(\mathfrak{q}) = \text{the ray class field of } K \text{ modulo } \mathfrak{q}, \]

\[ K(\mathfrak{a}) = \prod_{k=1}^{r} K(\mathfrak{q}_{k}) \text{ when } \mathfrak{a} = \prod_{k=1}^{r} \mathfrak{q}_{k}, \]

\[ L(\mathfrak{a}) = \text{the compositum of } L \text{ and } K(\mathfrak{a}), \]

\[ w(\mathfrak{q}) = \text{the order of the image of } \mathcal{O}_{K}^{\chi} \text{ in } (\mathcal{O}_{K}/\mathfrak{q})^{\chi}. \]
Let $\text{Map}_{L/K,n}(S)$ be the set of maps $u$ from $J_{L/K,n}(S)$ to a fixed algebraic closure $L^{\text{al}}$ of $L$ such that for each prime $q$ in $J_{L/K,n}(S)$ and $a \in J_{L/K,n-1}(S)$,

$$N_{L/(aq)/L(a)} u(aq) = u(a)^{F_{q^{-1}}},$$

and

$$u(aq) \equiv u(a)^{w(q)} \mod \text{(primes over } q).$$

Following Rubin’s definition (cf. [6]) of special numbers, we define the special numbers of level $n$, $\tilde{S}_n^L$ as

$$\{u(a) \mid u \in \text{Map}_{L/K,n}(S), a \in J_{L/K,n}(S)\}$$

and define the special units of level $n$, $S_n^L$ as $\tilde{S}_n^L \cap E_L$. Then Rubin’s special units are the special units of level 1. We write $S_n^L := S_n^L/Q$.

In what follows we fix $L = \mathbb{Q}((\zeta_{p^n} + \zeta_{p^n}^{-1}))$, the maximal real subfield of the $p^n r$-th cyclotomic field $\mathbb{Q}(\zeta_{p^n r})$. We denote by $E_L$ the group of global units of $L$. We have the following sequence of subsets of global units with inclusions,

$$S_0^L := E_L \supseteq S_1^L \supseteq S_2^L \supseteq \cdots \supseteq S_\infty^L := \bigcap_{n \in \mathbb{N}} S_n^L \supseteq C_L.$$

The main result of [10] asserts that $S_\infty^L$ is essentially equal to the cyclotomic units $C_L$.

**Theorem 2.1.** Let $p$ be an odd prime. If $p \nmid \phi(r)$ then $(S_\infty^L/C_L) \otimes \mathbb{Z}_p = 1$.

**Proof.** See Theorem A of [10]. \hfill \Box

Recall the sequence $S_0^L \supseteq S_1^L \supseteq S_2^L \supseteq S_3^L \supseteq \cdots \supseteq S_\infty^L$ stops for some number $N$ by the class number formula,

$$S_\infty^L = \bigcap_{n \in \mathbb{N}} S_n^L = S_N^L.$$ 

We denote the smallest such integer by $N(L)$, so $S_{L}^{N(L)} = S_{L}^{N(L)+1}$ for all $i \geq 0$. We want to determine the invariant $N(L)$ for various fields $L$. In particular, it follows from Theorem 2.1 that the question in [6] is essentially whether $N(\mathbb{Q}(\zeta_{p} + \zeta_{p}^{-1})) = 1$.

For each $\mathbb{Z}[\text{Gal}(L/\mathbb{Q})]$-module $M$, we let $\hat{M} := M \otimes \mathbb{Z}_p$.

**Lemma 2.2.** Under the assumption of Theorem 2.1, we have

$$|\hat{C}_L| = \bigoplus_{n \geq 0} \frac{|\hat{S}_n^L|}{|\hat{S}_{n+1}^L|}.$$
Proof. It follows from Theorem 2.1 and the class number formula that
\[ |\widehat{\text{Cl}}_L| = [\widehat{E}_L : \widehat{C}_L] = \prod_{n=0}^{\infty} \left[ S^n_L : S^{n+1}_L \right] = \bigoplus_{n \geq 0} \left[ S^n_L : S^{n+1}_L \right]. \]

Now we are ready to generalize Theorem A in the following form. Let \( Z' \) denote either \( Z \) or the subring of \( \mathbb{Q} \) generated over \( Z \) by \( 2^{-1} \), according as \( pr \) is divisible at most by two distinct primes or not.

Question. Are \( \widetilde{\text{Cl}}_L \cong \bigoplus_{n \geq 0} \frac{S^n_L}{S^{n+1}_L} \) as \( \mathbb{Z}'[\text{Gal}(L/\mathbb{Q})] \) modules (cf. [11])?

Remark. The question is equivalent to the assertion that for all odd prime \( p \), \( \widetilde{\text{Cl}}_L \) and \( \bigoplus_{n \geq 0} \frac{S^n_L}{S^{n+1}_L} \) are equal as \( \mathbb{Z}_p[\text{Gal}(L/\mathbb{Q})] \)-modules.

One could generalize Theorem A to \( L \), and see Theorem 3.2 for the precise statement. However, the computation of the least exponent of \( \text{Cl}_L \) would require more calculations. One could follow the method of Thaine in [12].

In the next section, we will briefly recall the definition of Euler systems of Kolyvagin and Rubin. We give a relation of special units and Euler systems. One can define an Euler system as a Galois equivariant map from \( \mathcal{J}_L(S) := \bigcup_{n=1}^{\infty} \mathcal{J}_L/\mathbb{Q},n(S) \) to an algebraic closure of \( L \) satisfying the axioms of Section 3. Let \( \mathcal{E}_L := \mathcal{E}_L(\phi) \) be the set of all Euler systems of \( L \) as defined in Section 3. Each Euler system \( \zeta \in \mathcal{E}_L \) produces an element in the intersection of the special numbers of level \( n \),
\[ \mathcal{E}_L \rightarrow \bigcap_{n=1}^{\infty} \tilde{S}^n_L \rightarrow 0. \]

3. Proof of Theorem A

We begin with the definition of Euler systems of Kolyvagin and Rubin (cf. [7]). Let \( K \) be a totally real abelian number field of conductor \( t \). Let \( \mathcal{J}(t) \) be the set of positive square free integers divisible only by primes \( \ell \equiv 1 \pmod{t} \). Given any finite subset \( S \) of \( \mathbb{N} \), we denote \( \mathcal{J}_K(S) := \mathcal{J}(t) \setminus S \). An Euler system for \( K \) is a function \( \zeta : \mathcal{J}_K(S) \rightarrow \mathbb{Q} \) with the following properties:

- \( \zeta(v) \in K(\mu_\infty)^X \) for all \( v \in \mathcal{J}_K(S) \).
- If \( v \neq 1 \), \( \zeta(\ell v) \) is a global unit.
We denote the set of all Euler systems defined over \( J_K(S) \) by \( \mathcal{E}_K(S) \).

Let \( p \) be an odd prime and \( F := \mathbb{Q}(\zeta_p + \zeta_p^{-1}) \). We write \( \Delta := \text{Gal}(F/\mathbb{Q}) \). Let \( p \mid |\Delta| \). Let \( \mathfrak{M} \) be a \( \mathbb{Z}_p[A] \)-module. Let \( \chi, \psi \) be \( p \)-adic characters of \( \Delta \). We say \( \chi \) is conjugate to \( \psi \) over \( \mathbb{Q}_p \) if there is a \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \) such that

\[
\chi = \sigma \psi.
\]

Let \( \Xi \) be the set of \( p \)-adic characters of \( \Delta \) modulo \( \mathbb{Q}_p \)-conjugacy. Then each character in \( \Xi \) corresponds to an irreducible \( \mathbb{Z}_p \)-representation of \( \Delta \). Let \( \mathbb{Z}_p(\chi) \) be the ring of integers of \( \mathbb{Q}_p(\chi) := \mathbb{Q}_p(\chi(\delta)) | \delta \in \Delta \). We let

\[
\mathfrak{M}^\chi := \mathfrak{M} \otimes_{\mathbb{Z}_p[A]} \mathbb{Z}_p(\chi)
\]

be the \( \chi \)-part of \( \mathfrak{M} \) where \( G \) acts on \( \mathbb{Z}_p(\chi) \) via \( \chi \). Let \( e_\chi \) be the idempotent of \( \mathbb{Z}_p[A] \) corresponding to \( \chi \)

\[
e_\chi = \frac{1}{|\Delta|} \sum_{\delta \in \Delta} \text{Tr}_{\mathbb{Q}_p(\chi)/\mathbb{Q}_p} \chi(\delta^{-1}) \delta,
\]

where \( \text{Tr}_{\mathbb{Q}_p(\chi)/\mathbb{Q}_p} \) is the trace map from \( \mathbb{Q}_p(\chi) \) to \( \mathbb{Q}_p \). The group ring \( \mathbb{Z}_p[A] \) decomposes as a product of discrete valuation rings \( \mathbb{Z}_p[A] \cong \prod_{\chi \in \Xi} e_\chi \mathbb{Z}_p[A] \cong \prod_{\chi \in \Xi} \mathbb{Z}_p(\chi) \) and \( \mathfrak{M} = \prod_{\chi \in \Xi} \mathfrak{M}^\chi \) with \( \mathfrak{M}^\chi = \mathfrak{M} \otimes_{\mathbb{Z}_p[A]} \mathbb{Z}_p(\chi) \cong \mathfrak{M} \otimes_{\mathbb{Z}_p[A]} e_\chi \mathbb{Z}_p[A] \cong e_\chi \mathfrak{M} \).

For a finite \( \mathbb{Z}[A] \)-module \( A \) we recall \( \hat{A} := A \otimes_{\mathbb{Z}} \mathbb{Z}_p \). Let \( \text{rk}_p(A^\chi) \) denote the \( \mathbb{Z}/p\mathbb{Z} \)-rank of the \( \chi \)-part \( A^\chi \) of \( \hat{A} \). We let \( \text{Rk}_p(A) \) denote the maximum rank of \( \text{rk}_p(A^\chi) \) as \( \chi \) runs through the characters in \( \Xi \), \( \text{Rk}_p(A) = \max_{\chi \in \Xi} \text{rk}_p(A^\chi) \) and let \( \text{Rk}(A) \) be the maximum rank of \( \text{rk}_p(A) \) as \( p \) ranges over the primes dividing the order \( |A| \) of \( A \), \( \text{Rk}(A) = \max_{p||A|} \text{rk}_p(A) \). As a first step we state one of the fundamental theorems on Euler systems.

**Theorem 3.1.** Let \( \text{Cl}_F \) be the ideal class group of \( F \). If \( \text{Rk}(\text{Cl}_F) = m \) then we have \( S'_F \otimes \mathbb{Z}_p = C_F \otimes \mathbb{Z}_p \) for all odd prime \( p \) and for all \( r \geq m \). Hence if the class number of \( F \) is prime to \( 2[F : \mathbb{Q}] \) then

\[
S'_F = C_F \quad \text{for all } r \geq m.
\]

As a special case, if we assume that \( \text{Rk}(\text{Cl}_F) \leq 1 \), for instance, \( \text{Cl}_F \) is a cyclic \( \mathbb{Z}[A] \) module then \( \hat{S}_F = \hat{C}_F \) for all odd prime \( p \). In particular, if the class number of \( F \) is prime to \( 2[F : \mathbb{Q}] \) then we have

\[
S_F = C_F.
\]

**Proof.** This theorem can be proved using Rubin’s argument of Euler systems. We give here a sketch of the proof. If \( \text{Cl}_F \) satisfies the assumption then one can choose a
special unit in $S_{F}^{m}$ such that the cardinality $|Cl_{F}\mathcal{F}|$ divides the cardinality $(E_{F}/S_{F}^{m})^{\mathcal{Z}}$ for all irreducible $\mathbb{Z}_{p}$-characters of $\mathcal{A}$ from Euler system arguments. Then it follows from the class number formula and the inclusion $C_{F} \subset S_{F}^{m}$ that $S_{F}^{r} = C_{F}$, for all $r \geq m$. For the details we refer to Chapter 2 of Rubin’s book [7]. □

Next we extend Theorem 3.1 to high special units. Given $\mathcal{Z}$-part $(S_{F}^{n})^{\mathcal{Z}}/(S_{F}^{n+1})^{\mathcal{Z}}$ be isomorphic to $\mathbb{Z}/p^{n+1}\mathbb{Z}$. The following theorem follows from the arguments of Kolyvigin–Rubin–Thaine.

**Theorem 3.2.** Let $p$ be an odd prime and $\chi \in \mathcal{E}$. Then we have

- $\text{rk}_{\mathbb{Z}/p\mathbb{Z}} \bigoplus_{n \geq 0} (S_{F}^{n})^{\mathcal{Z}}/(S_{F}^{n+1})^{\mathcal{Z}} \leq \text{rk}_{\mathbb{Z}/p\mathbb{Z}} Cl_{F}^{\mathcal{Z}}$;
- $\sum_{i=1}^{a} r_{i} \leq \sum_{i=1}^{a} s_{i}$ for $1 < a < k$, $\sum_{i=1}^{k} r_{i} = \sum_{i=1}^{k} s_{i}$;
- $r_{1} = s_{1}$.

**Proof.** We know from the arguments of Euler systems (cf. Section 2 of [7]) that any subgroup of $Cl_{F}^{\mathcal{Z}}$ generated by $k$-elements has order dividing $[(S_{F}^{0})^{\mathcal{Z}} : (S_{F}^{k})^{\mathcal{Z}}]$. It follows from this fact that

$$p^{r_{1}} | p^{s_{1}}, ~ p^{r_{1}+r_{2}} | p^{s_{1}+s_{2}}, \ldots, ~ p^{r_{1}+r_{2}+r_{3}+\cdots+r_{k}} | p^{s_{1}+s_{2}+s_{3}+\cdots+s_{k}}.$$ 

From the class number formula (Gras’ conjecture) the last divisibility becomes equality and hence

$$r_{1} \leq s_{1},$$

$$r_{1} + r_{2} \leq s_{1} + s_{2},$$

$$\vdots$$

$$r_{1} + r_{2} + r_{3} + \cdots + r_{k-1} \leq s_{1} + s_{2} + s_{3} + \cdots + s_{k-1},$$

$$r_{1} + r_{2} + r_{3} + \cdots + r_{k} = s_{1} + s_{2} + s_{3} + \cdots + s_{k}$$

and $s_{n} = 0$ for all $n > k$. This shows that $\mathbb{Z}/p\mathbb{Z}$-rank of $\bigoplus_{n \geq 0} S_{F}^{n}/S_{F}^{n+1}$ is less than or equal to $\mathbb{Z}/p\mathbb{Z}$-rank of $Cl_{F}$ and $|(E_{F}/S_{F}^{1})^{\mathcal{Z}}|$ annihilates $Cl_{F}^{\mathcal{Z}}$. 
Remark. Note that the left-hand side of the first inequality in the above theorem is the quantity $N(F)$ defined on page 4,

$$\text{rk}_{\mathbb{Z}/p\mathbb{Z}} \bigoplus_{n \geq 0} \left( \frac{(S_F^n)^{\mathcal{I}}}{(S_F^{n+1})^{\mathcal{I}}} \right) = N(F).$$

Next we show that $|E_F/S_{\mathcal{I}}^\mathcal{I}|$ is the least exponent of $\text{Cl}_{F}^\mathcal{I}$. Let $\mathfrak{B} \in \text{Cl}_{F}^\mathcal{I}$ and let $q \in \mathfrak{B}$ be a prime ideal above a rational prime $q$, $q \equiv 1 \mod p$. We recall the definition of the special unit for $q$ given by Thaine in [12]. A special unit $S_{F,q}$ for $q$ is any element $z \in E_F$ with the following property. There exists an element $\hat{z} \in F(\mu_q)$, $N_{F(\mu_q)/F}(\hat{z}) = 1$ and $\hat{z} \equiv z \mod (\zeta_q - 1)$. Note that the definition of special unit $S_{F,q}$ is not exactly equal to that of Thaine in [12] and that the difference of these definitions has no effect in our arguments. The following theorem is Theorem 1 of Thaine in [12].

**Theorem 3.3** (Thaine [12]). Let $p$ be an odd prime. Let $\rho$ be a generator of $(E_F/C_F)^\mathcal{I}$. If $k$ is the smallest integer such that $\rho^{p^k}$ is a special unit for $q$, then $p^k$ is the order of $\mathfrak{B}$.

**Proof.** See the proof of Theorem 1 of Thaine in [12]. □

It is well known that $E_F^\mathcal{I}/C_F^\mathcal{I}$ is a cyclic group which can be proved by direct computations (cf. [13]). Let $\rho$ be a generator of $(E_F/C_F)^\mathcal{I}$ as in Theorem 3.3. If $p^a$ is the smallest integer such that $\rho^{p^a} \in S_{F,q}$ for all $q \in \mathfrak{B}$ and all $\mathfrak{B} \in \text{Cl}_{F}^\mathcal{I}$ then $p^a$ is the least exponent of $\text{Cl}_{F}^\mathcal{I}$. Now it follows from

$$\bigcap_{q \in \mathfrak{B}} S_{F,q}^\mathcal{I} = S_F^\mathcal{I}$$

that $|E_F^\mathcal{I}/S_F^\mathcal{I}|$ is the least exponent of $\text{Cl}_{F}^\mathcal{I}$. This completes the proof of Theorem 3.2. □

**Corollary 3.4.** Let $\chi \in \Xi$. Then

$$S_F^\chi = C_F^\chi \iff \text{Cl}_{F}^\chi \text{ is a cyclic group.}$$

**Proof.** Suppose that $S_F^\chi = C_F^\chi$. Then $|E_F^\chi/S_F^\chi| = |E_F^\chi/C_F^\chi|$ is the least exponent of $\text{Cl}_{F}^\chi$. It follows from Theorem 3.2 that $\text{Cl}_{F}^\chi$ is cyclic. The other direction follows from Theorem 3.2. □

**Corollary 3.5.** Suppose that $(h_F, 2[F : \mathbb{Q}]) = 1$. Then

$$S_F = C_F \iff \text{Cl}_{F} \text{ is a cyclic } \mathbb{Z}[\Delta] \text{ module.}$$
Proof. From Theorem 3.2 we obtain the following implications.

\[ S_F = C_F \iff \hat{S}_F = \hat{C}_F \text{ for all } p \]
\[ \iff S_F^\chi = C_F^\chi \text{ for all } \chi \text{ and } p \]
\[ \iff \text{Cl}_F^\chi \text{ is a cyclic group for all } \chi \text{ and } p \]
\[ \iff \text{Cl}_F \otimes \mathbb{Z}_p \text{ is a cyclic } \mathbb{Z}_p[A] \text{ module for all } p \]
\[ \iff \text{Cl}_F \text{ is a cyclic } \mathbb{Z}[A] \text{ module.} \]

Corollary 3.6. Suppose that \((h_F, 2[F : \mathbb{Q}]) = 1\). Then we obtain

\[ \bigoplus_{n \geq 0} \frac{S_F^n}{S_F^{n+1}} \text{ is a cyclic } \mathbb{Z}[A] \text{ module} \iff \text{Cl}_F \text{ is a cyclic } \mathbb{Z}[A] \text{ module.} \]

In this case they have the same Galois module structure.

Proof. For all primes \(p\) and \(p\)-adic characters \(\chi\) of \(A\), it follows from Theorem 3.2 and the assumption that

\[ \bigoplus_{n \geq 0} \frac{(S_F^n)^\chi}{(S_F^{n+1})^\chi} \text{ is a cyclic group} \iff \bigoplus_{n \geq 0} \frac{(S_F^n)^\chi}{(S_F^{n+1})^\chi} \cong \frac{E_F^\chi}{C_F^\chi} \]
\[ \iff S_F^\chi = C_F^\chi \]
\[ \iff \text{Cl}_F^\chi \text{ is a cyclic group.} \]

In the first \(\iff\), we used the fact that \(E_F^\chi/C_F^\chi\) is a cyclic group. Since these \(\chi\)-eigenspaces have the same cardinality, this completes the proof. \(\square\)

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