The Forward Method as a Solution Refinement in Rational Expectations Models *

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Abstract

This paper generalizes the standard forward method of recursive substitution to a general class of linear Rational Expectations models with potentially multiple fundamental solutions. It is shown that the existence and uniqueness of the well-known forward solution are preserved in a general context. We also propose a key property embedded in the forward solution –the no-bubble condition– as an economically sensible solution refinement in the class of fundamental solutions. In the literature, the no-bubble condition has been assumed to rule out non-fundamental bubble solutions. We show that the forward solution is the only Rational Expectations equilibrium satisfying the no-bubble condition and consequently, it is the most relevant fundamental solution within the class of fundamental equilibria. Several economic examples are provided where the fundamental solutions obtained by other solution methods and refined by other solution selection criteria violate the no-bubble condition.

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1 Introduction

It is well-known that Rational Expectations (RE) models can have multiple solutions. There can be an infinity of non-fundamental (also known as bubble or sunspot) solutions and a multiple, but finite number of fundamental solutions that depend only on the state variables. While the literature has developed a number of mathematical solution methods and economic solution refinement schemes to identify the economically sensible solutions, two important issues remain unresolved. First, when both fundamental and non-fundamental solutions coexist, which class of RE solutions is more relevant to a given model? Second, out of the potentially multiple fundamental solutions, which is the most economically sensible one and in what sense it is so? This paper is an attempt to give an answer to the second question. To do so, we first generalize the traditional forward method of recursive substitution to a broad class of multivariate linear RE models. Then we show that the nature of the forward method provides an economically meaningful solution refinement for linear RE models among the fundamental solutions.

In textbook-style univariate RE models without predetermined variables, such as the Cagan (1956) or Samuelson (1958) models, where the state variables are simply exogenous processes, it is standard to solve these models forward recursively and obtain the forward (also known as forward-looking) solution. The forward method is straightforward to implement and the concept of the forward solution is well-understood in the literature.

However, the forward method has not been formally developed or analyzed for modern macroeconomic models such as linear dynamic stochastic general equilibrium models with lagged predetermined variables. Instead, alternative solution techniques have been developed based on mathematical devices such as eigenvalue-eigenvector decompositions (see, for instance, Blanchard and Kahn (1980), Uhlig (1997), King and Watson (1998), McCallum (1999), Klein (2000) and Sims (2002), among others).

While these solution methods can easily and completely characterize the set of fundamental solutions, the problem of multiplicity of stationary fundamental solutions can naturally arise in RE models. As a result, researchers have proposed several solution selection criteria to identify the economically meaningful solution(s). Among the best known criteria are the minimum state variable (MSV) criterion, designed by McCallum (1983) and the E-Stability criterion, devised by Evans and Honkapohja (2001). Another popular criterion is to select the solution associated with the smallest roots (generalized eigenvalues), the MOD solution of McCallum (2004). However, no consensus has been reached on the "right" Rational Expectations Equilibrium (REE). McCallum (2004) provides an example where the MSV solution does not coincide with the MOD solution to the model.[1] As Evans and Honkapohja (2001) show, several RE solutions can pass the E-Stability criterion under adaptive learning. Even under determinacy with a unique stable fundamental REE, the economic relevance of the determinate (MOD) solution should not be taken for granted, as McCallum (2008) shows an example where the MOD solution is not E-stable (see also Cho and McCallum (2009)).

In order to resolve these problems, we apply the standard forward method to a general class of linear RE models with predetermined variables. The forward method consists simply of deriving the model-implied forward representation, where the current endogenous variables are related to 1) the expected future endogenous variables and 2) the state variables (both predetermined and exogenous variables), whenever the stochastic processes of the exogenous variables are known. The forward solution is defined as the

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[1] The solution obtained via the MSV criterion is often called the MSV solution in the literature. However, the fundamental solutions are also called MSV solutions, because they depend exclusively on the minimal set of state variables. To avoid confusion throughout the paper, we restrict the term MSV solution to that obtained via the solution method proposed by McCallum, whereas fundamental solutions will denote the solutions that depend on the minimal set of state variables.

relation between the current endogenous variables and the state variables if the relation becomes stable as the forward recursion goes to infinity, which is what we call the Forward Convergence Condition (FCC hereafter).[2] By definition, the forward solution is a fundamental REE. Its existence is fully governed by the model property, FCC and if it exists, it is unique.

Behind the concept of the forward solution lies the *assumption* that the expected future endogenous variables do not affect the current endogenous variables, as the forward recursion goes to infinity. This condition is typically referred to as the No-Bubble Condition, transversality condition, or, in some contexts, no-Ponzi scheme or zero terminal condition. It should be stressed that the no-bubble condition (NBC hereafter) may or may not hold depending on the particular solution with which expectations are formed in the forward representation. In this paper, we prove that the forward solution indeed satisfies the NBC. Intuitively, the recursive structure present in a model implies that the forward-looking agents deduce the relation between the endogenous and the state variables recursively in a forward-looking manner. If such a relation is stable in the limit, it is natural that the expectations of the endogenous variables very far in the future do not affect the current endogenous variable. Hence, the forward solution is by itself a very natural fundamental REE.

There may well exist other fundamental solutions to a model regardless of whether the model satisfies the FCC or not. However, we show that all fundamental REEs except the forward solution violate the NBC, despite the fact that they are fundamental solutions that depend only on the state variables. The term involving the expectations of the future endogenous variables in the forward representation is often called a "bubble term" (see,

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[2] This condition is known as boundedness of the state variables. See, for instance, Blanchard (1979), McCallum (1983) and Woodford (2003)). Flood and Garber (1980) explore the importance of this convergence property in their monetary model and refer to it as process consistency.

for instance, Ljungqvist and Sargent (2004)). They interpret the term as an asset bubble in their stock pricing model. In general, the expectational term may also be called a "bubble" in the sense that it is anything unrelated to the fundamentals in a given model. By examining the behavior of this "bubble term" when expectations are formed with the other fundamental solutions, we show that the forward solution is indeed the most natural fundamental REE.

An important advantage of the forward method is that it directly yields the forward solution whenever it exists by examining the FCC only, and we do not need to obtain the alternative REEs to examine the NBC. Our results also show that the solutions obtained by the existing solution techniques and refined by existing solution selection criteria must violate the NBC if they are not the forward solution. Indeed, using several economic examples, we show that the REEs chosen by the MSV, MOD or E-stability criteria may fail to satisfy the NBC.

A closely related work is Binder and Pesaran (1997). They impose certain terminal conditions on the expectations of the future endogenous variables and solve RE models backward recursively. Driskill (2006) also proposes a similar method based on backward induction and illustrates his technique using several popular examples in a univariate context. Our method is different from theirs in that we solve a general linear RE model forward without imposing a specific terminal condition. We also explore the economic implications of the NBC.

In addition to the set of fundamental solutions, there may exist an infinite number of non-fundamental REEs, as analyzed by Farmer and Guo (1994) and Lubik and Schorfheide (2004). These solutions must violate the NBC, as they are defined to do so. Therefore, we do not claim that the NBC is a solution refinement for non-fundamental REEs. As mentioned above, when both classes of solutions coexist, it is an open question which class is more relevant to a given model. In this paper, we do not take a stand on this issue, as the scope of this paper is confined to the class of fundamental REEs.

The paper proceeds as follows. Section 2 reviews and clarifies the key properties of the standard forward method in a univariate RE model without predetermined variables. Section 3 illustrates how our method can be extended to the univariate models with predetermined variables. We also provide a graphical analysis of the forward method. In section 4, we generalize the forward method to a class of linear multivariate RE models. Section 5 provides several useful applications of the forward method. Section 6 concludes.

2 Univariate Models Without Predetermined Variables

We start with a simple univariate linear Rational Expectations (RE) model in the absence of predetermined variables. Even though this is a well-understood model, it is instructive to do so for two reasons: First, it provides very clear economic intuition for the subsequent discussion of the forward method in more general models. Second, it allows us to introduce the two key concepts needed to characterize the forward method.

A univariate RE model without predetermined variables can be expressed as:

$$x_t = aE_t x_{t+1} + z_t, \tag{1}$$

where x_t is an endogenous variable and E_t is the mathematical expectation operator conditional on information available at time t. z_t is an exogenous forcing variable, the only state variable in this model. A typical example of this kind is a stock price model

$p_t = \beta E_t(p_{t+1} + d_{t+1})$ where p_t is the stock price, d_t is a known exogenous dividend process and β is a time discount factor. The state variable z_t can be flexibly defined as $z_t = \beta E_t d_{t+1}$ in this case. We leave a as unrestricted in order to encompass various popular models.[3] Here we assume that z_t follows a stationary AR(1) process:

$$z_t = \rho z_{t-1} + \epsilon_t, \tag{2}$$

where $|\rho| < 1$ and ϵ_t is a white noise process. The class of REEs discussed in this paper is the set of fundamental solutions where the endogenous variables exclusively depend on the minimal set of the state variables, which in this model is simply z_t:

$$x_t = \gamma z_t, \tag{3}$$

where $\gamma = 1/(1 - a\rho)$. The fundamental solution does not exist when $a\rho = 1$. It is important to note that the number of fundamental solutions is finite and here it is at most one, if it exists. In addition to the fundamental solutions, there may exist a class of non-fundamental, bubble solutions, such as:

$$x_t = \gamma z_t + w_t, \tag{4}$$

where w_t is often called an arbitrary "bubble term" such that $w_t = aE_t w_{t+1}$ (see, for instance, Ljungqvist and Sargent (2004)).[4]

We now briefly review the standard forward method of recursive substitution and the

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[3] For instance, in the Cagan (1956) model, x_t is the (log) price level, z_t is the (log) nominal money stock and a is typically smaller than 1. But a can be greater than 1 in the overlapping generations model of Samuelson (1958). In the intertemporal IS equation, x_t and z_t are the log consumption and the expected (exogenous) real interest rate, respectively, and $a = 1$.

[4] In their simple stock pricing model, Ljungqvist and Sargent (2004) use the alternative expression ξ_t/β^t such that $w_t = \xi_t/\beta^t$, where ξ_t is a martingale, i.e., $\xi_t = E_t\xi_{t+1}$.

resulting forward solution, and outline the key features of the forward method. Solving the model forward using the law of iterative expectations is equivalent to deriving a forward representation of the model as follows:

$$\begin{aligned} x_t &= a^k E_t x_{t+k} + \gamma_k z_t, && (5) \\ \gamma_k &= \sum_{i=1}^{k} (a\rho)^{i-1}, && (6) \end{aligned}$$

for $k = 1, 2, 3, ...$ Note that any RE solution, either fundamental or non-fundamental, must satisfy the forward representation (5) for all k, because it is implied by the model. The forward solution in the literature is defined as follows: When the coefficient of the state variable, γ_k, converges as k goes to infinity, its limiting value is given by:

$$\gamma^* = \lim_{k \to \infty} \gamma_k = 1/(1 - a\rho). \tag{7}$$

Then, the forward solution is defined as the model-implied relation between the endogenous variable and the state variable in the forward representation in the limit:

$$x_t = \gamma^* z_t, \tag{8}$$

by *assuming* that the expectational term dies out in the forward representation as follows:

$$\lim_{k \to \infty} a^k E_t x_{t+k} = 0. \tag{9}$$

The two key concepts embedded in the forward method are 1) the convergence property of the state variable, which is completely characterized by the sequence γ_k, and 2) the limiting behavior of the expectational term $\lim_{k \to \infty} a^k E_t x_{t+k}$ in the forward representation of the model. In order to explore their economic implications, we formally define these two concepts in a general context:[5]

Definition 1 *A linear Rational Expectations model is said to satisfy the Forward Convergence Condition (FCC) if the coefficients of the state variables in the forward representation of the model converge as the forward recursion goes to infinity.*

Definition 2 *The forward solution to a linear Rational Expectations model is defined as the limiting relation between the current endogenous variables and the state variables in the forward representation as the forward recursion tends to infinity, as long as the model satisfies the FCC.*

By definition, the forward solution exists if and only if the model satisfies the FCC. For the model (1), the sequence γ_k is uniquely defined by the model. Therefore, the forward solution is unique if it exists. This uniqueness and existence of the forward solution are the key elements of the forward method and we will show that these properties are preserved in the class of general models.

In the aforementioned stock price example, forward convergence means that the stock can be priced by the present value of the dividend process. Another important example is taken from Flood and Garber (1980) where p_t is the log price level, z_t is a money supply process and a is a function of some parameters. They introduce "process consistency", an essential characteristic of anything which pretends to serve as money. They argue that any "process inconsistent" money supply will be rejected by the public as it does not provide a finite solution for the price level in a Cagan-type hyperinflationary money market. Our FCC generalizes their process consistency. Therefore, the forward

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[5] Throughout this paper, we assume that the dynamics of the exogenous variables are known so that the convergence property of the state variables in the forward representation can be characterized by the coefficients of the state variables.

convergence is a minimal requirement that any reasonable RE model should possess, and thus the forward solution is a very natural RE equilibrium.

For our forward method to be a complete solution refinement scheme to a general class of RE models, it must provide an economic rationale against alternative solutions. This is completely provided by the following no-bubble condition:

Definition 3 *A Rational Expectations Equilibrium (solution) to a given model is said to satisfy the No-Bubble Condition (NBC) if the values of the term involving the expectation of the future endogenous variables in the forward representation converge to zeros as the forward recursion tends to infinity when expectations are formed with the solution.*

For the model (1), (9) is the NBC.[6] The NBC has a natural economic interpretation. From equation (5), it is clear that the NBC can only hold for the models that satisfy the FCC, since the expectational term will not converge if $\gamma_k z_t$ does not converge. If the model (1) satisfies the FCC, then we have the following:

$$x_t = \lim_{k\to\infty} a^k E_t x_{t+k} + \gamma^* z_t. \tag{10}$$

Since the forward solution must solve (10), when the expectational term is evaluated with this bubble-free, fundamental solution, $\lim_{k\to\infty} a^k E_t x_{t+k} = 0$. We can also directly prove this: the FCC holds if and only if $|a\rho| < 1$. Thus, $\lim_{k\to\infty} a^k E_t x_{t+k} = \left[\lim_{k\to\infty} a^k \gamma^* \rho^k\right] z_t = 0$ as $\lim_{k\to\infty}(a\rho)^k = 0$, leading to the NBC.[7]

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[6] There seems to be no single terminology to denote this condition in the literature. No-bubble condition is common in asset pricing equations. In the context of fiscal policy, it is called the No-Ponzi game or the intertemporal budget condition (see Walsh (2003)), or also zero terminal condition (Devereux and Mansoorian (1992)). In alternative macroeconomic models it is also called the transversality condition (Romer (1996)) or a zero boundary condition (Driskill (2006)) to pin down a solution.

[7] Blanchard (1979) shows that even when $|a| > 1$, if z_t is expected to return to its mean fast enough (here ρ is small enough such that $|a\rho| < 1$), the forward solution exists.

Recall that the non-fundamental solution (4) with $\gamma = \gamma^*$ must also solve (10). This implies that the expectational term survives for the non-fundamental, bubble component:

$$\lim_{k\to\infty} a^k E_t x_{t+k} = \lim_{k\to\infty} a^k E_t w_{t+k} = w_t. \tag{11}$$

This clearly shows what the expectational term means in the forward representation. For the asset pricing model above, w_t is called "asset bubble"(see chapter 13 of Ljungqvist and Sargent (2004) for instance). In a general context, this expectational term in the limit may also be called a "bubble term" where "bubble" is anything unrelated to the fundamentals or state variables in a given model.

Do there exist other fundamental solutions than the forward solution? In this context without predetermined variables, this can only happen when the FCC does not hold so that the forward solution does not exist. We now show that any fundamental REE that differs from the forward solution must violate the NBC despite the fact that it is a fundamental solution which depends only on fundamentals. For instance, consider the model (1) with $|a\rho| \geq 1$. In this instance, a fundamental solution exists and can be obtained by other methods, for instance, the method of undetermined coefficients as $x_t = \gamma z_t$, where $\gamma = 1/(1 - a\rho)$. Thus, except for the case $a\rho = 1$, such a solution exists. Could this solution be regarded as a plausible equilibrium path? Suppose that this is the monetary model of Flood and Garber (1980). If z_t is an AR(1) money supply process and it rises, we should expect the price level to be higher. But since $\gamma < 0$ when $a\rho > 1$, this solution implies a fall in the price level, which is clearly counterintuitive.

There are many other examples such that this kind of solution is not considered as an economically relevant REE in the literature, as shown by Blanchard (1979), McCallum (1983), or Woodford (2003). In terms of our terminology, these solutions violate the NBC. Specifically, if expectations are formed with this solution, $a^k E_t x_{t+k} = (a\rho)^k \gamma z_t$ explodes when $a\rho > 1$ or oscillates when $a\rho = -1$. Clearly, such a non-convergent expectational term is not easy to interpret economically, given that the solution is a fundamental REE.

We now turn to univariate models with predetermined variables where multiple fundamental REEs can exist and then explore the economic implications of the NBC.

3 Univariate Models with Predetermined Variables

To our knowledge, the forward representation of a model with predetermined variables has not been formally developed. In this section we develop the forward method in a univariate framework with predetermined variables. We then provide a graphical analysis of our method and illustrate how the forward method and the existing methods differ. While the generalization to the multivariate framework is provided in section 4, the univariate models with predetermined variables will suffice to show the essential features of our methodology.

Consider a simple univariate model with a predetermined variable:

$$\begin{aligned} x_t &= aE_t x_{t+1} + bx_{t-1} + z_t, & (12) \\ z_t &= \rho z_{t-1} + \epsilon_t. & (13) \end{aligned}$$

The only difference with the model (1) is that the state variables are the predetermined variable, x_{t-1}, as well as z_t. The class of fundamental solutions can be easily characterized, for instance, by using the method of undetermined coefficients, and it is given by:

$$x_t = \omega x_{t-1} + \gamma z_t, \tag{14}$$

where (ω, γ) must belong to the following set of solution candidates:

$$\mathcal{S}=\{(\omega,\gamma)|\ \omega=(1-a\omega)^{-1}b,\quad \gamma=(1-a\omega)^{-1}(1+a\gamma\rho),\ \ (\omega,\gamma)\in\mathcal{R}\times\mathcal{R}\}, \tag{15}$$

provided that $1 - a\omega \neq 0$. $\mathcal{S}$ is the exhaustive and finite (here at most two) set of (ω, γ) consistent with (14). The class of non-fundamental, bubble solutions, is of the form:

$$x_t = \omega x_{t-1} + \gamma z_t + w_t, \tag{16}$$

where w_t is an arbitrary process such that $w_t = (1 - a\omega)^{-1} a E_t w_{t+1}$.

3.1 The Forward Method

As in the previous model, the forward method can be characterized by three ingredients: the forward representation of the model, the forward convergence condition together with the forward solution, and the no-bubble condition.

Forward Representation: Rewrite the model (12) with $m_1 = a$, $\omega_1 = b$, and $\gamma_1 = 1$, such that $x_t = m_1 E_t x_{t+1} + \omega_1 x_{t-1} + \gamma_1 z_t$. Shifting this equation forward one period and taking conditional expectations yields $E_t x_{t+1} = m_1 E_t x_{t+2} + w_1 x_t + \rho\gamma_1$, which depends on x_t. Replacing $E_t x_{t+1}$ and rearranging the model (12), we can derive $x_t = m_2 E_t x_{t+2} + \omega_2 x_{t-1} + \gamma_2 z_t$ where $m_2 = (1 - a\omega_1)^{-1} a m_1$, $\omega_2 = (1 - a\omega_1)^{-1} b$ and $\gamma_2 = (1 - a\omega_1)^{-1}(1 + a\gamma_1\rho)$. In this way, we can construct the *unique* set of sequences, $\{m_k, \omega_k, \gamma_k\}$ recursively as functions of the structural parameters, a, b and ρ, such that:

$$x_t = m_k E_t x_{t+k} + \omega_k x_{t-1} + \gamma_k z_t, \tag{17}$$

where $m_1 = a$, $\omega_1 = b$, and $\gamma_1 = 1$, and for all $k = 2, 3, 4, ...$,

$$\begin{aligned} m_k &= (1 - a\omega_{k-1})^{-1} a m_{k-1}, && (18) \\ \omega_k &= (1 - a\omega_{k-1})^{-1} b, && (19) \\ \gamma_k &= (1 - a\omega_{k-1})^{-1}(1 + a\rho\gamma_{k-1}). && (20) \end{aligned}$$

Notice the similarity of the sequence (ω_k, γ_k) in (19) and (20) with the conditions in the set of solutions, $\mathcal{S}$. When (ω_k, γ_k) converge, the limiting values will be a member of $\mathcal{S}$ in (15). The only necessary condition for the existence of the forward representation is:

$$1 - a\omega_k \neq 0, \tag{21}$$

for all $k = 1, 2, 3, ...$ For ease of exposition, we call this condition the regularity condition. One can show that when $ab \leq 1/4$, (ω, γ) is real-valued and (21) always holds.

Forward Convergence Condition: The model (12) satisfies the forward convergence condition (FCC) if the sequence (ω_k, γ_k) converges to (ω^*, γ^*) since these are the coefficients of the state variables x_{t-1} and z_t. Under the FCC, the model implies:

$$x_t = \lim_{k\to\infty} m_k E_t x_{t+k} + \omega^* x_{t-1} + \gamma^* z_t. \tag{22}$$

The forward solution is defined as the model-implied forward representation in the limit where the endogenous variable is a function of the state variables only:

$$x_t = \omega^* x_{t-1} + \gamma^* z_t. \tag{23}$$

A crucial part of the FCC is that $(\omega^*, \gamma^*) \in \mathcal{S}$, thus the forward solution is a fundamental

REE. Just as in the model without predetermined variables, the forward solution exists if and only if the model satisfies the FCC and it is unique because the sequence (ω_k, γ_k) is uniquely defined by the model.

No-Bubble Condition: The term involving the expectations of the future endogenous variable in the forward representation is $m_k E_t x_{t+k}$. Therefore, the no-bubble condition (NBC) is given by:

$$\lim_{k \to \infty} m_k E_t x_{t+k} = 0. \tag{24}$$

Even for the models with predetermined variables where multiple fundamental solutions can exist, the forward solution is still the only REE that satisfies the NBC. From equation (22), it is clear that the forward solution satisfies the NBC. If other distinctive fundamental solutions exist, the expectational term evaluated with any such one cannot be zeros, violating the NBC.

Our forward method is completely derived using the model parameters and the assumption of Rational Expectations only, without reference to the information about set of solutions (14). The sequence $(m_k, \omega_k, \gamma_k)$ is uniquely determined by the model parameters and independent of any particular solution. Just examining convergence properties of (ω_k, γ_k), we have the following important result. *The forward solution exists to (12) if and only if it satisfies the FCC and it is the unique REE that satisfies the NBC.* The concept of the forward solution is precisely the same as the one in the model without predetermined variables. The way in which the forward solution is derived fully reflects the recursive way the forward looking, rational agents deduce the relation between the endogenous and state variables. Hence, the forward solution is clearly a very relevant fundamental REE to a model with lagged variables. In what follows, we further discuss what the failure of the NBC means and show that the forward solution is the most sensible fundamental REE. We also compare our methodology with the existing popular techniques. However, it should be stressed that solving for all the fundamental REEs and examining the NBC is not required for implementing the forward method.

3.2 NBC as a Solution Refinement

RE models can be categorized into two groups: one where the FCC holds and the other in which it does not. In the latter case, it is straightforward to reject both fundamental solutions to the model, just as we did for the model without lagged variables such as the stock pricing model of Ljungqvist and Sargent (2004), or the monetary model of Flood and Garber (1980). Hence, we focus on the case where the FCC holds.

When (12) is forward-convergent, $\lim_{k\to\infty} m_k E_t x_{t+k}$ becomes a function of the state variables. To see this, suppose that the model has two real-valued solutions and for simplicity, assume away z_t. Then, the solution can be written as $x_t = \omega(i)x_{t-1}$, $i = 1, 2$, where $\omega(i)$ is a root of $a\omega^2 - \omega + b = 0$. Suppose, without loss of generality, that $|\omega(1)| < |\omega(2)|$. Then, the forward solution corresponds to the one with the smallest root.[8] To see this, compute the expectational term with both solutions. Then, we have $|\lim_{k\to\infty} m_k\omega(1)^k x_t| < |\lim_{k\to\infty} m_k\omega(2)^k x_t|$. This implies that $\lim_{k\to\infty} m_k\omega(2)^k x_t \neq 0$ and thus the forward solution ω^* must be $\omega(1)$. Note also that $l_x(2) = \lim_{k\to\infty} m_k\omega(2)^k (\neq 0)$ is a k-invariant constant. Therefore, with the other solution, $\lim_{k\to\infty} m_k E_t x_{t+k} = l_x(2)x_t$. This implies that even if the expectation is formed with a purely fundamental REE, such an expectational term survives and affects the current endogenous variable x_t. It is not very plausible to have a time-dependent terminal condition for the vast majority of infinite horizon models. This very much resembles the effect of the bubble. In his recent study, McCallum (2009) argues that different fundamental solutions represent different causal

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[8] It should be noted that the forward solution does not necessarily correspond to the MOD solution associated with the smallest eigenvalues in multivariate models as we will show in the following sections.

relations between x_t and x_{t-1}. He shows that the fundamental solution with a larger root is associated with a non-fundamental bubble solution, although it has a functional form of fundamentals. To sum up, the forward solution is the only fundamental solution that is truly bubble-free and consequently, it is the most economically sensible REE in the class of fundamental solutions.

Now we compare our methodology with the standard solution techniques and refinement schemes. The forward method utilizes the recursive structure of the underlying model and examines a model-implied relation whereas alternative methods first characterize the set of solutions mathematically. Consequently, and in contrast to the forward method, these other methods necessarily face the problem of multiple stationary solutions, and a particular solution refinement scheme or selection criterion is needed, such as the E-stability criterion of Evans and Honkapohja (2001) or the MSV criterion of McCallum (1983), in order to choose one solution.[9] These criteria may still yield a solution that violates the NBC to a model, in particular even when it fails to satisfy the FCC. We will provide some numerical examples later in this paper.

A very similar solution technique and solution selection criterion has been proposed by Driskill (2006). Essentially, he imposes a restriction that the expectational effect disappears at some finite horizon of a univariate infinite horizon model and solves the model backward to see if the relation between the endogenous variable and the state variable becomes stable. In some sense, he seeks for a solution subject to the NBC. One critical difference is that his methodology is silent about the economic sensibility of the other solution. Additionally, our methodology is generalized to the multivariate case as well.

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[9] Notice that we do not require *a priori* information about the stationarity of the solution. If the forward solution exists, we conclude that it is stationary if $|\omega^*| < 1$.

3.3 Graphical Representation of the Forward Method

The actual working of the forward method can be crisply understood as follows. Consider the univariate model of the form (12) without z_t. Then we have the two solutions $x_t = \omega(i)x_{t-1}$ for $i = 1, 2$. The forward representation and the sequence of the state variable coefficient are, respectively, given by:

$$\begin{aligned} x_t &= m_k E_t x_{t+k} + \omega_k x_{t-1}, && (25) \\ \omega_k &= (1 - a\omega_{k-1})^{-1} b, && (26) \end{aligned}$$

for $k \geq 2$, with $m_1 = a$ and $\omega_1 = b$. Let $v_k = (1 - a\omega_k)$ in equation (26). Then $v_1 = 1 - \theta$ where $\theta = ab$ and v_k is given by:

$$v_{k+1} = 1 - \theta / v_k, \tag{27}$$

for $k = 1, 2, 3, ...$ Let $v(1) = \frac{1+\sqrt{1-4\theta}}{2}$ and $v(2) = \frac{1-\sqrt{1-4\theta}}{2}$ be the solutions of $v = 1 - \theta/v$. Then it is easy to see that $\omega(j) = v(j)^{-1}b$ for $j = 1, 2$.

Figure 1 depicts how the forward solution is defined and obtained for the forward-convergent model. For the range of $0 < \theta \leq 1/4$, panel A of figure 1 shows that v_k is well defined for all $k = 1, 2, ...$ and starting from v_1, it converges monotonically to $v(1)$. Note that since $v(1) > v(2) > 0$, $|\omega(1)| < |\omega(2)|$. This implies that ω_k converges to ω^*, which coincides with $\omega(1) = v(1)^{-1}b$, the smallest root of ω.[10] Panel B shows that for the range of $\theta < 0$, v_k is also well defined and starting from v_1, it converges to $v(1)$ with

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[10] The case of $\theta = 0$ becomes trivial. This can happen when $a = 0$ or $b = 0$. If $a = 0$, the model becomes purely backward-looking, whereas if $b = 0$, it is purely forward-looking. In both cases, $v_k = 1$ and $\omega_k = b$ for all k.

oscillation.[11,12] In either case, v_k converges to $v(1)$. We emphasize that the initial value of v_1 is given by the model parameters as $1-\theta$, not as an arbitrary value. This is the reason why the forward solution is always unique if it exists and it is a model-implied relation between the endogenous and state variables. In contrast, standard solution methods essentially characterize the two roots of v and thus ω.

Second, suppose that the model does not satisfy the FCC. In this simple model, the FCC is violated only if $\theta > 1/4$, the case where real-valued solutions do not exist. However, there may exist real-valued solutions in general even when the FCC does not hold. Figure 2 illustrates the path of v_k when $\theta > 1/4$. Panel A shows that as long as the regularity condition is not violated, v_k explodes with oscillation, implying that ω_k does not converge. Panel B illustrates the case where the regularity condition is violated at $k = K > 2$. $v_k \neq 0$ for $k = 1, 2, ..., K-2$, and $v_{K-1} = 0$. Since $\omega_k = v_{k-1}^{-1}b$, from equation (26), ω_1 through ω_{K-1} are well defined. But ω_K and $v_K = 1 - a\omega_K$ cannot be defined.[13] We may interpret that v_K jumps to infinity (or minus infinity) and correspondingly $\omega_K = -\infty$ (∞). Then, from (27), $v_{K+1} = 1$ and $\omega_{K+1} = 0$. Finally, $v_{K+2} = v_1$ and

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[11] When $\theta \leq 1/4$, the regularity condition is always satisfied. Here the regularity condition amounts to $v_k = (1 - a\omega_k) \neq 0$. We now prove that $v_k \geq 1/2 > 0$ for all $k = 1, 2, 3, ...$ When $0 \leq \theta \leq 1/4$, suppose that $v_K \geq 1/2$ for some $K \geq 1$. Then $-\theta/v_K \geq -2\theta \geq -1/2$. Therefore, $v_{K+1} = 1 - \theta/v_K \geq 1/2$. Since $v_1 = 1 - \theta \geq 1/2$, $v_k = 1 - a\omega_k \geq 1/2$ for all $k \geq 1$. When $\theta < 0$, suppose that $v_K > 1$ for some $K \geq 1$. Then $v_{K+1} = 1 - \theta/v_K > 1$. Since $v_1 = 1 - \theta > 1$, $v_k = 1 - a\omega_k > 1$ for all $k \geq 1$.

[12] The forward method can also be applied to a model with repeated eigenvalues, i.e., when $\theta = 1/4$. In this instance, equation (27) and $v_{k+1} = v_k$ are tangent at $v = 1/2$ and v_k converges to $1/2$ from $v_1 = 3/4$. However, since the slope at $v = 1/2$ is 1, we conjecture that the speed of convergence would be much lower. Indeed, the convergence speed of the forward solution is faster the more distant the two roots of v (or ω in $\mathcal{S}$) are, because the slope of v_k in panel A is flatter at $v(1)$. When $a = 0.75$ and $b = 1/3$, $\omega(1) = \omega(2) = 2/3$. In this case, more than 1000 recursions are needed to attain a precision to the third decimal point. However, when $a = 0.749$, $\omega(1) = 0.643$, $\omega(2) = 0.692$, it only takes 57 recursions to reach the solution with the same precision.

[13] In this case, violation of the regularity condition implies that the forward representation of the model becomes:

$$0 = (1 - a\omega_{K-1})x_t = am_{K-1}E_t x_{t+K} + bx_{t-1}.$$

Therefore, economic agents cannot relate the current variable to the K-th and higher order forward-looking terms recursively, even if they have been able to do so up to the $(K-1)$-th order. An implication of this point is that the regularity condition is a property that a well defined RE model needs to satisfy.

$\omega_{K+2} = \omega_1 = b$. That is, when the regularity condition is violated at $k = K$, the patterns of $\{\omega_k\}_{k=1}^{K+1}$ are periodically repeated. This implies that when the regularity condition is violated, ω_k does not converge and the forward solution does not exist.

4 Forward Method in General Linear Rational Expectations Models

In this section we present a formal derivation of the forward method in the class of linear multivariate RE models with predetermined variables. The forward method and the forward solution are analogous to those in univariate models.

4.1 The Model and the Class of Fundamental Solutions

Consider the following standard model:

$$B_1 x_t = A_1 E_t x_{t+1} + B_2 x_{t-1} + C_1 z_t, \tag{28}$$

where x_t is an $n \times 1$ vector of endogenous variables, B_1, A_1 and B_2 are $n \times n$ coefficient matrices of structural parameters. We assume that B_1 is a non-singular matrix but A_1 and B_2 can be singular. z_t is an $m \times 1$ vector of exogenous variables whose data generating process is known. C_1 is an $n \times m$ coefficient matrix of z_t. The information set available at time t includes all the current and past endogenous and exogenous variables. Pre-multiplying both sides by B_1^{-1} and assuming that z_t follows a VAR(1) law of motion, the model can be represented as:

$$x_t = AE_t x_{t+1} + Bx_{t-1} + Cz_t, \tag{29}$$

$$z_t = Rz_{t-1} + \epsilon_t, \quad E_t \epsilon_{t+1} = 0_{m\times 1}, \tag{30}$$

where $A = B_1^{-1}A_1$, $B = B_1^{-1}B_2$, $C = B_1^{-1}C_1$ and R is an $m \times m$ coefficient matrix. The eigenvalues of R are assumed to be inside the unit circle. As Binder and Pesaran (1997) show, this model is quite general in the sense that it nests models with an arbitrary number of leads in the forward-looking variables, an arbitrary number of lags in the predetermined variables, and an arbitrary time at which expectations are formed.

Since the state variables are the vector of predetermined variables, x_{t-1}, and the exogenous variables, z_t, the class of fundamental REEs has the following form:

$$x_t = \Omega x_{t-1} + \Gamma z_t, \tag{31}$$

where Ω and Γ are $n \times n$ and $n \times m$ matrices, respectively. The complete set of real-valued solutions for Ω and Γ is given by:

$$\mathcal{S}=\{(\Omega,\Gamma)|\ (\Omega,\Gamma) \in \mathcal{R}^{n\times n} \times \mathcal{R}^{n\times m}\}, \tag{32}$$

where Ω and Γ solve the following equations:

$$\Omega = (I_n - A\Omega)^{-1}B, \tag{33}$$

$$\Gamma = (I_n - A\Omega)^{-1}(C + A\Gamma R), \tag{34}$$

provided that $|I_n - A\Omega| \neq 0$, where I_n denotes an identity matrix of order n.[14] $\mathcal{S}$ is

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[14] A class of non-fundamental solutions can also be described as $x_t = \Omega x_{t-1} + \Gamma z_t + w_t$, where w_t is

the exhaustive set of (Ω, Γ) consistent with (31). We can rewrite equation (33) as the following matrix quadratic form:

$$A\Omega^2 - \Omega + B = 0_{n\times n}. \tag{35}$$

It is now standard to solve the model through the methods developed by Blanchard and Kahn (1980), Uhlig (1997), King and Watson (1998), McCallum(1983,1999), Klein (2000) and Sims (2002). Those solution techniques are by and large based on eigenvalue-eigenvector decomposition theory and they can completely characterize the solutions in $\mathcal{S}$. For example, the QZ method based on the generalized Schur decomposition can easily characterize the set of solution candidates for Ω, and thus $\mathcal{S}$, with the generalized eigenvalues implied by the matrices of structural parameters A and B.[15] By inspecting the generalized eigenvalues, one can easily detect whether there is a unique or a multiple number of real-valued stationary fundamental solutions (see, for instance, Theorem 3 of Uhlig (1997)). It can be shown that there are at most ${}_{2n}C_n$ elements of $\mathcal{S}$.

4.2 The Forward Method

As in the previous section, we first derive the forward representation of the model. Then we define the forward solution using the FCC and NBC concepts and show that the forward solution is the only REE satisfying the NBC.

Forward Representation: We first show that the model can be solved forward under a regularity condition. The forward representation of the model can be derived as follows.

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an arbitrary process satisfying $w_t = (I - A\Omega)^{-1} A E_t w_{t+1}$.

[15] Following Klein (2000), we define the generalized eigenvalues as the elements of the set $\lambda(\mathcal{N}, \mathcal{M}) = \{v \in \mathcal{C} : |\mathcal{N} - v\mathcal{M}| = 0\}$, where $\mathcal{M} = \begin{bmatrix} A & 0_{n\times n} \\ 0_{n\times n} & I_n \end{bmatrix}$ and $\mathcal{N} = \begin{bmatrix} I_n & -B \\ I_n & 0_{n\times n} \end{bmatrix}$.

Claim: *Consider the model in* (29) *and* (30). *Suppose that* A, B, C *and* R *are real-valued. Then, there exists a unique sequence of real-valued matrices* $\{M_k,\ \Omega_k,\ \Gamma_k,\ k = 1, 2, 3, ...\}$ *such that:*

$$x_t = M_k E_t x_{t+k} + \Omega_k x_{t-1} + \Gamma_k z_t, \tag{36}$$

where $M_1 = A$, $\Omega_1 = B$, $\Gamma_1 = C$, *and for* $k = 2, 3, ...,$

$$\begin{aligned} M_k &= (I_n - A\Omega_{k-1})^{-1} A M_{k-1}, & (37) \\ \Omega_k &= (I_n - A\Omega_{k-1})^{-1} B, & (38) \\ \Gamma_k &= (I_n - A\Omega_{k-1})^{-1} (C + A\Gamma_{k-1} R), & (39) \end{aligned}$$

if the following regularity condition is satisfied for all $k = 1, 2, 3, ...$:

$$|I_n - A\Omega_k| \neq 0. \tag{40}$$

Proof. *See Appendix A.* ■

Notice the similarity between (38), (39) and (33), (34), respectively. If these sequences converge, their limits must be a member of $\mathcal{S}$. The regularity condition is a necessary condition under which a given RE model can be solved forward recursively for all leads. If the regularity condition is violated, then the model does not satisfy the FCC.[16]

FCC, NBC and the Forward Solution: The definitions of the forward convergence condition (FCC), the forward solution and the no-bubble condition (NBC) are all pre-

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[16] King and Watson (1998) provide a necessary condition for the existence of solutions which is equivalent to $|I_n - A\Omega| \neq 0_{n\times n}$ in our model. Note that this is the limiting case of our regularity condition, i.e. $|I_n - A\Omega^*| \neq 0_{n\times n}$. Our condition is stronger than theirs in that it requires that $|I_n - A\Omega_k| \neq 0_{n\times n}$ for all k. However, the regularity condition does not need to hold for other $\Omega \neq \Omega^*$ in $\mathcal{S}$. This condition is similar to that of Binder and Pesaran (1997) up to a finite k. They informally stated that the regularity condition holds for all the economic examples they examine.

served in this general framework. The model, (29) and (30), satisfies the FCC if the sequence (Ω_k, Γ_k) of the state variables defined in (38) and (39) converges in the forward representation of the model. Under the FCC, the model implies:

$$x_t = \lim_{k\to\infty} M_k E_t x_{t+k} + \Omega^* x_{t-1} + \Gamma^* z_t, \tag{41}$$

where $\Omega^* = \lim_{k\to\infty} \Omega_k$ and $\Gamma^* = \lim_{k\to\infty} \Gamma_k$. The forward solution is the model-implied forward representation of the model in the limit without the expectational term in (41):

$$x_t = \Omega^* x_{t-1} + \Gamma^* z_t. \tag{42}$$

Note that $(\Omega^*, \Gamma^*) \in \mathcal{S}$ because equations (38) and (39) fulfill the conditions in $\mathcal{S}$. Hence, the forward solution is one fundamental REE to the model. Next, the NBC is given by:

$$\lim_{k\to\infty} M_k E_t x_{t+k} = 0_{n\times 1}. \tag{43}$$

All the properties of the FCC and the NBC in general linear RE models are isomorphic to those in univariate models. The only difference is that the multivariate model can have more than two but up to ${}_{2n}C_n$ fundamental solutions. The following proposition formally states our finding.

Proposition 1: *Consider the linear Rational Expectations model, (29) and (30). Then, the forward solution (42) is the unique real-valued fundamental solution to the model that satisfies the No-Bubble Condition among all the Rational Expectations equilibria.*

Proof. *See Appendix B.* ■

There is one final condition to be examined. The forward convergence does not imply the dynamic stability of the forward solution. If the eigenvalues of Ω^* are all inside the unit circle, then the forward solution is stationary. If not, the model does not have a dynamically stable fundamental REE satisfying the NBC. This completes the forward method.

4.3 NBC as a Solution Refinement

As in the univariate models, this step is not actually required as part of the forward method. The purpose here is to support our claim that the forward solution is the most economically relevant REE to any linear RE model within the class of fundamental solutions. Suppose that there are J real-valued fundamental solutions, $(\Omega(j), \Gamma(j))$ in $\mathcal{S}$ for $j = 1, 2, ..., J$:

$$x_t = \Omega(j)x_{t-1} + \Gamma(j)z_t, \tag{44}$$

where $J \leq {}_{2n}C_n$. The expectational term, $M_k E_t x_{t+k}$, in the forward representation can be directly computed with each solution as follows.

Proposition 2. *Consider the model, (29) and (30):*

1. Suppose that the model satisfies the FCC and therefore, the forward solution exists. If there exist other fundamental solutions, $(\Omega(j), \Gamma(j)) \neq (\Omega^*, \Gamma^*)$ *for* $1 \leq j \leq J$, *and expectations are formed with any one of those solutions, then* $M_k E_t x_{t+k}$ *converges for a given set of state variables, but not to zeros. Specifically,*

$$\lim_{k\to\infty} M_k E_t x_{t+k} = L^x(j)x_t + L^z(j)z_t \neq 0_{n\times 1},$$

where $L^x(j) = \lim\limits_{k\to\infty} M_k \Omega(j)^k \neq 0_{n\times n}$ *and* $L^z(j) = \lim\limits_{k\to\infty} M_k \sum_{i=1}^{k} \Omega(j)^{k-i}\Gamma(j)R^i$.

2. Suppose that the FCC does not hold. If there exist fundamental REEs, then the expectational term $M_k E_t x_{t+k}$ does not converge when expectations are formed with those fundamental solutions.

Proof. *See Appendix C.* ■

All arguments made for the univariate models carry over to the multivariate framework. Recall that the fundamental solutions may well exist independently of the forward convergence. Part 1 of the proposition shows that when the model satisfies the FCC and expectations are formed with other solutions, the expectational terms depend on x_t or z_t or both. It is not easy to interpret that such a terminal condition, the non-zero time-varying "bubble term" is associated with any fundamental solution in infinite horizon models. Part 2 states that the expectational term must explode or oscillate as k goes to infinity for any solution to the models that are not forward-convergent. These non-convergent terminal conditions are once again hard to justify economically.

Proposition 1 states that the forward solution is an economically sensible solution. Proposition 2 states that all other fundamental solutions are hard to justify as REEs of a given model. Putting together, the forward solution is the most plausible equilibrium path to any linear RE model. As a solution technique, the forward method is straightforward to implement. One simply constructs sequences of matrices from the model coefficients (Ω_k, Γ_k) and checks the FCC. But one necessarily needs to employ the forward method in order to avoid accepting fundamental REEs that actually violate the NBC. In fact, even the unique stable equilibrium to a determinate model may not be acceptable as a relevant REE. Cho and McCallum (2009) show such an example. In their example, the forward solution differs from the determinate REE associated with the smallest eigenvalues, which indeed violates the NBC.

5 Illustrative Examples

In this section, we show how our methodology can be applied to popular economic models. Several examples are provided where the solutions obtained by popular selection criteria can actually violate the NBC. We consider the MSV, MOD and E-stability criteria. The corresponding solutions selected by these criteria are denoted by the MSV, MOD and E-stable solutions, respectively. Example 1 considers a determinate model whereas example 2 features an indeterminate model where multiple stationary solutions exist. In both examples, the model satisfies the FCC and the solution chosen by all three criteria coincide with the forward solution, and consequently it satisfies the NBC. Example 3 is an indeterminate model but the model does not satisfy the FCC. Here the MSV (and the MOD) solution exists but this solution violates the NBC. Example 4 is an indeterminate model, where, similarly to example 2, the FCC holds, but it has two E-stable solutions and one of them violates the NBC. This example reproduces a version of the model in Dornbusch (1976) laid out by Evans and Honkapohja (2001).

The first three examples are based on a standard New-Keynesian macro model. Consider a New-Keynesian model consisting of aggregate supply (AS) and aggregate demand (IS) equations and a Taylor-type monetary policy rule proposed by Woodford (2003):

$$
\begin{aligned}
\pi_t &= \delta E_t \pi_{t+1} + \kappa y_t, && (45)\\
y_t &= \mu E_t y_{t+1} + (1-\mu) y_{t-1} - \phi(i_t - E_t \pi_{t+1} - r_t^n), && (46)\\
i_t &= \beta E_t \pi_{t+1} + \lambda y_t, && (47)
\end{aligned}
$$

where π_t is inflation, y_t is the output gap and i_t is the nominal short-term interest rate. r_t^n is the natural real interest rate and we assume that it follows an AR(1) process such that:

$$r_t^n = \rho r_{t-1}^n + \epsilon_t \tag{48}$$

where $0 \leq \rho < 1$ and ϵ_t is white noise.[17] All the parameters are assumed to be positive by theory. The model can be reduced to the following two-variable, two-equation model, by substituting the policy rule into the IS equation:[18]

$$\begin{aligned} \pi_t &= \delta E_t \pi_{t+1} + \kappa y_t, & (49) \\ y_t &= \mu_1' E_t y_{t+1} + \mu_2' y_{t-1} - \beta' E_t \pi_{t+1} + \phi' r_t^n, & (50) \end{aligned}$$

where $\mu_1' = \frac{\mu}{1+\phi\lambda}$, $\mu_2' = \frac{1-\mu}{1+\phi\lambda}$, $\beta' = \frac{\phi(\beta-1)}{1+\phi\lambda}$, $\phi' = \frac{\phi}{1+\phi\lambda}$. In matrix form,

$$x_t = A E_t x_{t+1} + B x_{t-1} + C z_t, \tag{51}$$

where $x_t = (\pi_t \; y_t)'$ and $z_t = r_t^n$. $A = B_1^{-1} A_1$, $B = B_1^{-1} B_2$ and $C = B_1^{-1} C_1$ where B_1, A_1, B_2 and C_1 are defined as:

$$B_1 = \begin{bmatrix} 1 & -\kappa \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} \delta & 0 \\ -\beta' & \mu_1' \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 0 & \mu_2' \end{bmatrix}, C_1 = \begin{bmatrix} 0 \\ \phi' \end{bmatrix}.$$

If a real-valued stationary fundamental solution exists, it must be of the following form:

$$x_t = \Omega x_{t-1} + \Gamma z_t, \tag{52}$$

where (Ω, Γ) must obey the relations, $\Omega = (I_2 - A\Omega)^{-1} B$ and $\Gamma = (I_2 - A\Omega)^{-1}(C + A\Gamma\rho)$.

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[17] We also examined an alternative specification where r_t^n is assumed to be zero but the policy rule has a monetary policy shock that follows an AR(1) process, just like equation (48). All the results remain unaltered.

[18] Of course it is possible to directly use a three-variable system, but the exposition is more complicated.

Standard solution methods can completely characterize the solution set by employing, for instance, the QZ method. Since there are 4 generalized eigenvalues in this case, there is a maximum ${}_4C_2$ number of solutions. If there are exactly two generalized eigenvalues less than unity in absolute value, then the model is determinate and there is a unique stationary fundamental solution. If there are more than two generalized eigenvalues less than unity, then the model becomes indeterminate and there may exist multiple fundamental solutions as well as an infinite number of non-fundamental REEs. It is now well-known that determinacy of the model critically hinges on the Taylor principle, i.e., whether β is greater than 1 or not.

Example 1: This example illustrates a case where there exists a unique stationary fundamental solution. Suppose that $\beta = 1.5$, so that the Taylor principle holds. The remaining parameters are $\delta = 0.99$, $\kappa = 0.3$, $\mu = 0.55$, $\phi = 1$, $\lambda = 0.1$ and $\rho = 0.8$. The generalized eigenvalues are given by $[g_1\ g_2\ g_3\ g_4] = [0\ \ 0.4622\ \ 1.1362 + 0.7051i\ \ 1.1362 - 0.7051i]$ and therefore, the solution associated with g_1 and g_2 is the unique stationary fundamental solution, which is given by:

$$\Omega(g_1, g_2) = \begin{bmatrix} 0 & 0.2556 \\ 0 & 0.4622 \end{bmatrix}, \ \Gamma(g_1, g_2) = \begin{bmatrix} 1.6648 \\ 0.6261 \end{bmatrix}.$$

We now apply the forward method. Ω_k and Γ_k converge at $k = 25$ so that the model satisfies the FCC: $\lim_{k\to\infty} \Omega_k = \Omega^*$ and $\lim_{k\to\infty} \Gamma_k = \Gamma^*$. The following table shows the values of the elements of Ω_k and Γ_k:

k	1	10	20	25	k	1	10	20	25
$\omega_{k,11}$	0	0	0	0	$\gamma_{k,11}$	0.2727	1.6621	1.6648	1.6648
$\omega_{k,12}$	0	0	0	0	$\gamma_{k,21}$	0.9091	0.6033	0.6260	0.6261
$\omega_{k,21}$	0.1227	0.2556	0.2556	0.2556					
$\omega_{k,22}$	0.4091	0.4622	0.4622	0.4622					

where $\omega_{k,ij}$ and $\gamma_{k,ij}$ are the (i, j)-th elements of Ω_k and Γ_k, respectively. Therefore, the forward solution, $x_t = \Omega^* x_{t-1} + \Gamma^* z_t$, coincides with the unique stationary fundamental solution. This is the MOD solution by definition. We also apply the MSV and E-stability criteria and all three criteria yield the same solution as the forward solution, implying that it satisfies the NBC.[19]

Example 2. This example shows a case where multiple stationary fundamental solutions exist and one of them is the forward solution. Suppose that $\beta = 0.95$ so that the Taylor principle does not hold. The generalized eigenvalues are given by $[g_1\ g_2\ g_3\ g_4] = [0\quad 0.6036\quad 0.8824\quad 1.5516]$. Since there are three stable roots, there may be three stationary fundamental solutions associated with (g_1, g_2), (g_1, g_3) and (g_2, g_3). However, the solution associated with (g_2, g_3) violates the rank condition in McCallum's (1983) sense and therefore, there are only two stationary fundamental solutions.

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[19] The MSV criterion consists of selecting the solution associated with the generalized eigenvalues converging to zeros when a constant, α, is multiplied by B and the generalized eigenvalues associated with $(A, \alpha B)$ are computed as α goes from one to zero.

For the E-stability criterion, (52) with unrestricted (Ω^u, Γ^u) serves as the perceived law of motion (PLM). When expectations are formed with the PLM in (51), the mapping from the PLM to the actual law of motion (ALM) is given by $T(\Omega^u, \Gamma^u) = ((I_2 - A\Omega^u)^{-1}B, (I_2 - A\Omega^u)^{-1}(C + A\Gamma^u\rho))$. Let DT_Ω and DT_Γ be the derivatives of $T(\Omega^u, \Gamma^u)$ with respect to $vec(\Omega^u)$ and $vec(\Gamma^u)$. If DT_Ω and DT_Γ are evaluated with an REE, $(\Omega^u, \Gamma^u) = (\Omega, \Gamma) \in \mathcal{S}$ and all their eigenvalues have real parts less than 1, it is said that the solution (Ω, Γ) is E-stable.

Now, let us apply the forward method to the model. The model satisfies the FCC and, consequently, the forward solution exists and is given by $\Omega^* = \begin{bmatrix} 0 & 0.4500 \\ 0 & 0.6036 \end{bmatrix}$ and $\Gamma^* = \begin{bmatrix} 22.1021 \\ 6.1669 \end{bmatrix}$. Again $(\Omega^*, \Gamma^*) \in \mathcal{S}$ and this forward solution coincides with the fundamental solution with $\Omega(g_1, g_2)$ and $\Gamma(g_1, g_2)$.

Now we show that the solution associated with (g_1, g_3) can be eliminated by our refinement scheme based on the NBC. From Proposition 2, the solution associated with (g_1, g_3) must violate the NBC. Indeed, if one directly computes the expectational term $M_k E_t x_{t+k}$ with $x_t = \Omega(g_1, g_3)x_{t-1} + \Gamma(g_1, g_3)z_t$, then, $\lim_{k\to\infty} M_k E_t x_{t+k} = L^x(g_1, g_3)x_t + L^z(g_1, g_3)z_t$ where

$$L^x(g_1, g_3) = \begin{bmatrix} 0 & 1.8633 \\ 0 & 0.3160 \end{bmatrix}, \quad L^z(g_1, g_3) = \begin{bmatrix} -46.8068 \\ -7.9371 \end{bmatrix},$$

thus violating the NBC. As argued above, an arbitrary terminal condition for $\lim_{k\to\infty} M_k E_t x_{t+k}$ is very restrictive and hard to interpret. In this example, the solution chosen by the MSV, MOD and E-stability criteria coincides with the forward solution. Therefore, all three REE refinement schemes eliminate the solution associated with (g_1, g_3) which violates the NBC.[20]

Example 3. This example illustrates a situation where there are multiple stationary fundamental solutions but the forward solution does not exist. However, the MSV and

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[20] When the FCC holds, does the solution chosen by the MSV criterion always satisfy the NBC? The answer is not necessarily. McCallum (2004) shows an example where the solution chosen by the MSV criterion is not the unique stationary fundamental solution, which is the MOD solution even in case of determinacy. We verified that the model satisfies the FCC and the forward solution is the MOD solution and consequently, the MSV solution violates the NBC. However, the example is not an economically motivated model, and the MSV solution is highly likely to satisfy the NBC under the FCC of well formulated economic models.

MOD criteria select a solution that violates the NBC. Suppose that $\beta = 0.9$ instead of 0.95 so that once again, the Taylor principle does not hold. In contrast to example 2, however, the model does not satisfy the FCC and consequently, if a solution is obtained by the existing REE refinement schemes, it must violate the NBC. The generalized eigenvalues are given by $[g_1\ g_2\ g_3\ g_4] = [0\ \ 0.6692\ \ 0.7506\ \ 1.6455]$. There exist two fundamental solutions associated with (g_1, g_2), (g_1, g_3). The solution associated with (g_2, g_3) again violates the rank condition.

Now let us apply the forward method to the model. While Ω_k converges to $\Omega(g_1, g_2)$, Γ_k explodes. Specifically, Γ_k is given by:

k	1	25	50	75	100
$\gamma_{k,11}$	0.2727	85.8805	555.5786	2881.3	14332
$\gamma_{k,21}$	0.9091	26.9752	162.7462	835.0	4145

for $k = 1,\ 25,\ 50,\ 75,\ 100$. Therefore, the FCC does not hold and consequently the two solutions obtained above violate the NBC.

In contrast, one can show that the solution, $(\Omega(g_1, g_2), \Gamma(g_1, g_2))$ is the MSV solution, which is also the MOD solution. To implement the MSV criterion, we multiply a constant α to the matrix B in equation (51) and then compute the generalized eigenvalues by changing α from 1 to 0. The following table shows the generalized eigenvalues:

α	1	0.9	0.7	0.5	0.3	0.1	0
g_1	0	0	0	0	0	0	0
g_2	0.6692	0.5164	0.3545	0.2338	0.1320	0.0418	0
g_3	0.7506	0.8458	0.9027	0.9289	0.9447	0.9552	0.9593
g_4	1.6455	1.7030	1.8081	1.9024	1.9885	2.0681	2.1059

Since g_1 and g_2 converge to zero as α goes to zero, $\Omega(g_1, g_2)$ is the MSV solution, which violates the NBC. We apply the E-stability criterion and none of the two solutions passes the criterion. Hence, in this instance the E-stability criterion also eliminates the solutions which violate the NBC.

This example is analogous to Cagan's (1956) model in section 2 when $a\rho > 1$. It was easy to show that the solution obtained by the method of undetermined coefficients fails to satisfy the NBC, so that it is dismissed as a legitimate REE. This fact is well-known in the literature. Intuitively, the model does not satisfy the FCC because the exogenous term in the forward representation does not go to zero at a sufficiently fast rate when ρ is high. However, in more general models with predetermined variables, it is hard to build up such an intuition. Indeed, just a small change in β from 0.95 in example 2 to 0.9 in example 3, makes a critical difference on the FCC of the model. This example confirms that one needs to apply the forward method to the model at hand in order to avoid selecting a solution that violates the NBC.

Example 4. The final example illustrates a case where there are multiple stationary fundamental solutions, the forward solution exists and several solutions pass the E-stability condition. Since there can only be one solution satisfying the NBC in any RE model, one can conjecture that at least one of the E-stable solutions must violate the NBC. Evans and Honkapohja (2001) consider a Dornbusch-type model consisting of a Phillips curve, an open-economy IS curve, an LM curve and an open-economy interest rate parity condition. The model is reproduced as follows:

$$\begin{aligned} p_t &= p_{t-1} + \pi d_t, & (53) \\ d_t &= -\gamma(r_t - E_t p_{t+1} + p_t) + \eta(e_t - p_t), & (54) \\ r_t &= \lambda^{-1}(p_t - \vartheta p_{t-1}), & (55) \\ e_t &= E_t e_{t+1} - r_t, & (56) \end{aligned}$$

where p_t is the (log) price level, d_t is (log) aggregate demand, r_t is the nominal interest rate and e_t is the (log) nominal exchange rate.[21] The model can be reduced to a univariate representation in terms of p_t as:

$$p_t = \beta_1 E_t p_{t+1} + \beta_2 E_t p_{t+2} + \delta p_{t-1}, \tag{57}$$

where $\beta_0 = (2+\pi(\gamma+\eta+\gamma/\lambda+\eta/\lambda+\gamma\vartheta/\lambda))$, $\beta_1 = (1+\pi(2\gamma+\eta+\gamma/\lambda))/\beta_0$, $\beta_2 = -\pi\gamma/\beta_0$, $\delta = (1+\pi\vartheta(\gamma+\eta)/\lambda)/\beta_0$. Evans and Honkapohja (2001) use the parameter values $\pi = 1.5$, $\gamma = 1.5$, $\lambda = 10$, $\vartheta = 1.1$ and $\eta = -0.1$. The fundamental solution (and the perceived law of motion) is of the form:

$$p_t = \omega p_{t-1}. \tag{58}$$

There are three stationary solutions for ω: 0.7160, 07721 and 0.9897. One can show that the solutions $\omega = 0.7160$ and $\omega = 0.9897$ are E-stable.[22]

Now, let us apply the forward method. We rewrite the model (57) in order to make

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[21] Here we use d_t in the Phillips curve to simplify the analysis while Evans and Honkapohja (2001) use $E_{t-1}d_t$. When lagged expectations are used, the model can be reformulated following Binder and Pesaran (1997) so that the model belongs to the class of (51). We verified that in this instance the same results are obtained.

[22] To see this, note that the mapping from the perceived law of motion to the actual law of motion is given by $T(\omega) = (1 - \beta_1\omega - \beta_2\omega^2)^{-1}\delta$. A straightforward computation of $DT(\omega) = (1 - \beta_1\omega - \beta_2\omega^2)^{-2}\delta(\beta_1 + 2\beta_2\omega)$ yields 0.9799, 1.0172 and 0.8923 for the three ω solutions. Therefore, the solutions associated with $\omega = 0.7160$ and $\omega = 0.9897$ are E-stable.

it belong to the class of (51) (it is also possible to use the original four-variable system):

$$x_t = AE_t x_{t+1} + Bx_{t-1}, \tag{59}$$

where $f_t = E_t p_{t+1}$, $x_t = (p_t \ f_t)'$, $A = \begin{bmatrix} \beta_1 & \beta_2 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix}$. The forward solution exists and it is given by:

$$x_t = \Omega^* x_{t-1} = \begin{bmatrix} 0.7160 & 0 \\ 0.7160^2 & 0 \end{bmatrix} x_{t-1}. \tag{60}$$

That is, $p_t = 0.7160p_{t-1}$ is the forward solution whereas the other E-stable solution $p_t = 0.9897p_{t-1}$ violates the NBC. We verified that the MSV criterion chooses the solution with $p_t = 0.7160p_{t-1}$, which is the MOD solution as well.

6 Conclusion

This paper generalizes the forward method of recursive substitution to multivariate RE models with predetermined variables. We show that the forward solution exists under the FCC of a given model and is unique by construction. Moreover, it is the only REE which satisfies the no-bubble condition. We also show that any other REE violating that condition is hard to justify economically. Therefore, the NBC can be used as an economically sensible refinement scheme for fundamental RE solutions. We show through several economic examples that REEs obtained by the existing solution methods and refined by solution selection criteria can actually violate the no-bubble condition. From a practical perspective, one does not need to compute all the REEs and examine the no-bubble condition. Our forward method detects automatically the existence of the forward solution by simply verifying the forward convergence condition, and it yields the forward solution when the model at hand satisfies this condition.

There is a number of important issues remaining for future work. First, while our methodology is by itself a complete solution technique and refinement scheme, it would also be of a great importance to relate the forward method and determinacy. We have assumed that economic agents know the value of the steady state as well as the structural parameters so that any model can be written in mean-deviation form without constants. It is also possible to directly work with the model by taking explicit account of the steady states, which requires an additional forward convergence condition for the constant term. Our tentative conjecture, based on our current work, is that the determinacy condition is very closely related to this additional condition for constants. Second, as we pointed out at the outset of the paper, it is not yet resolved which class of REEs –fundamental or non-fundamental solutions– is more relevant to a given RE model when both classes of solutions exist. An exploration of the relation between our methodology and determinacy may shed some light on this issue as well. Third, while we presented several examples regarding the relation between the forward method (and the no-bubble condition) and other solution selection criteria, it would be interesting to explore the general theoretical relations, such as necessity and sufficiency among them. Fourth, we analyzed E-stability as a solution refinement scheme in one of our examples, but in the literature it is also used as a practical guidance for Least-Squares learnability of REEs. While there is a close relation between E-stability and learnability –known as the E-stability principle– it would be of interest to directly examine the relation between the forward solution and its learnability.

Appendix

A Proof of the Claim

The model is given by:

$$x_t = AE_t x_{t+1} + Bx_{t-1} + Cz_t \tag{61}$$

$$z_t = Rz_{t-1} + \epsilon_t, \quad E_t \epsilon_{t+1} = 0_{m\times 1}. \tag{62}$$

Let $M_1 = A$, $\Omega_1 = B$ and $\Gamma_1 = C$. Suppose that there exist a set of matrices, $\{M_{k-1}, \Omega_{k-1}, \Gamma_{k-1}\}$ for some $k-1 > 0$ such that:

$$x_t = M_{k-1}E_t x_{t+k-1} + \Omega_{k-1}x_{t-1} + \Gamma_{k-1}z_t. \tag{63}$$

Shifting this equation forward one period and taking conditional expectations yields:

$$E_t x_{t+1} = M_{k-1}E_t x_{t+k} + \Omega_{k-1}x_t + \Gamma_{k-1}Rz_t \tag{64}$$

by the law of iterative expectations and from equation (62). Substituting (64) into (61), we have:

$$(I_n - A\Omega_{k-1})x_t = AM_{k-1}E_t x_{t+k} + Bx_{t-1} + (C + A\Gamma_{k-1}R)z_t. \tag{65}$$

Provided that $(I_n - A\Omega_{k-1})$ is non-singular, the set $\{M_k, \Omega_k, \Gamma_k\}$ is well-defined for k where these matrices are given by (37) through (39). Therefore, if $(I_n - A\Omega_{k-1})$ is invertible for all k, the sequences of $\{M_k, \Omega_k, \Gamma_k\}$ are well defined for $k = 1, 2, ...$ *Q.E.D.*

B Proof of Proposition 1

Suppose that the FCC holds and the forward solution is given by:

$$x_t = \Omega^* x_{t-1} + \Gamma^* z_t. \tag{66}$$

Since the pair (Ω_k, Γ_k) is unique and real-valued given the structural parameters in A, B, C and R, the limiting values (Ω^*, Γ^*) are also unique and real-valued. Since (Ω^*, Γ^*) solves equations (33) and (34), $(\Omega^*, \Gamma^*) \in \mathcal{S}$. This implies that the forward solution is a fundamental solution to the model and therefore, it must solve the forward representation of the model (41) when k goes to infinity:

$$x_t = \lim_{k\to\infty} M_k E_t x_{t+k} + \Omega^* x_{t-1} + \Gamma^* z_t. \tag{67}$$

Therefore, it must be true that $\lim\limits_{k\to\infty} M_k E_t x_{t+k} = 0_{n\times 1}$ when expectations are formed with the forward solution, implying that the forward solution satisfies the NBC.

When the FCC holds, suppose that the NBC holds for a (fundamental or non-fundamental) solution, different from the forward solution. Since the solution must solve (67), (67) becomes the forward solution under the NBC, i.e., $\lim\limits_{k\to\infty} M_k E_t x_{t+k} = 0_{n\times 1}$, which is a contradiction to the fact that this solution is different from the forward solution. When the FCC does not hold, the pair (Ω_k, Γ_k) in (36) is either not well-defined if the regularity condition is violated or does not converge even if the regularity condition is met. Consequently, there is no forward solution and for any other fundamental or non-fundamental solution, $\lim\limits_{k\to\infty} M_k E_t x_{t+k}$ is not well-defined, implying the violation of the NBC. *Q.E.D.*

C Proof of Proposition 2

1. Since the FCC holds, the forward solution, (Ω^*, Γ^*) exists and it must be a member of $\mathcal{S}$, say, $(\Omega(1), \Gamma(1))$. Suppose that there are other solutions:

$$x_t = \Omega(j) x_{t-1} + \Gamma(j) z_t. \tag{68}$$

where $(\Omega(j), \Gamma(j))$ belongs to S for $j = 2, 3, .., J$. Then each solution must solve (36) for all $k = 1, 2, ..$ and for all $j = 2, 3, ..., J$. Recall that while M_k is uniquely determined by the model, different solutions lead to different values of $E_t x_{t+k}$. The expectational term associated with (68) can be directly computed as $M_k E_t x_{t+k} = L_k^x(j) x_t + L_k^z(j) z_t$ where $L_k^x(j) = M_k \Omega(j)^k$ and $L_k^z(j) = M_k \sum_{i=1}^{k} \Omega(j)^{k-i} \Gamma(j) R^i z_t$. Substituting this expression into equation (36) and matching the coefficients of x_{t-1} and z_t yields:

$$\begin{aligned} (I_n - L_k^x(j))\Omega(j) &= \Omega_k && (69) \\ (I_n - L_k^x(j))\Gamma(j) &= \Gamma_k + L_k^z(j) && (70) \end{aligned}$$

for all j and k. In the limit, $\lim_{k\to\infty} \Omega_k = \Omega(1)$ and $\lim_{k\to\infty} \Gamma_k = \Gamma(1)$. This implies that from equation (69), $L_k^x(j)$ converges for all j. Therefore, the left-hand-side of (69) and (70) must converge, implying that $L_k^z(j)$ must converge as well. However, for all $j \neq 1$, $L_k^x(j)$ in equation (69), cannot converge to zeros because $\Omega(j) \neq \Omega^* = \Omega(1)$. $L_k^z(j)$ may be zeros when z_t is not present, but the whole expectational term, $M_k E_t x_{t+k}$ in (36) cannot converge to zeros, because otherwise, it is a contradiction to $(\Omega(j), \Gamma(j)) \neq (\Omega^*, \Gamma^*)$. Therefore, $\lim_{k\to\infty} M_k E_t x_{t+k} \neq 0_{n\times 1}$ for all $j \neq 1$.

2. This is an immediate consequence of Proposition 1. Suppose that the FCC does not hold. Then, either Ω_k or Γ_k or both do not converge. If $\mathcal{S}$ is not-empty, then $(\Omega(j), \Gamma(j))$

are constants for all $j = 1, 2, .., J$, and consequently, either $L_k^x(j)$ or $L_k^z(j)$ or both do not converge from (69) and (70), implying that $M_k E_t x_{t+k}$ explodes or oscillates for any $(\Omega(j), \Gamma(j))$ in $\mathcal{S}$. $Q.E.D.$

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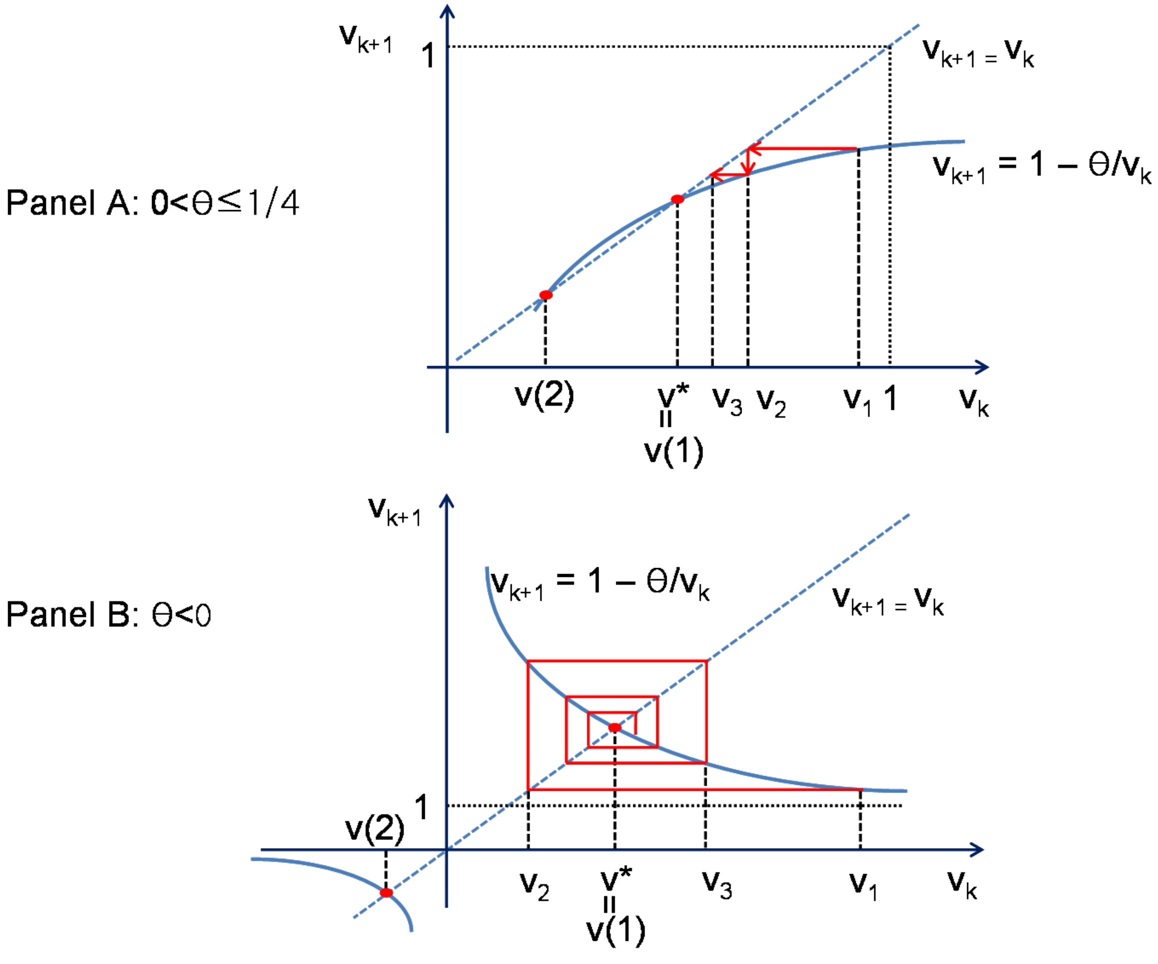
Figure 1: Existence of the Forward Solution

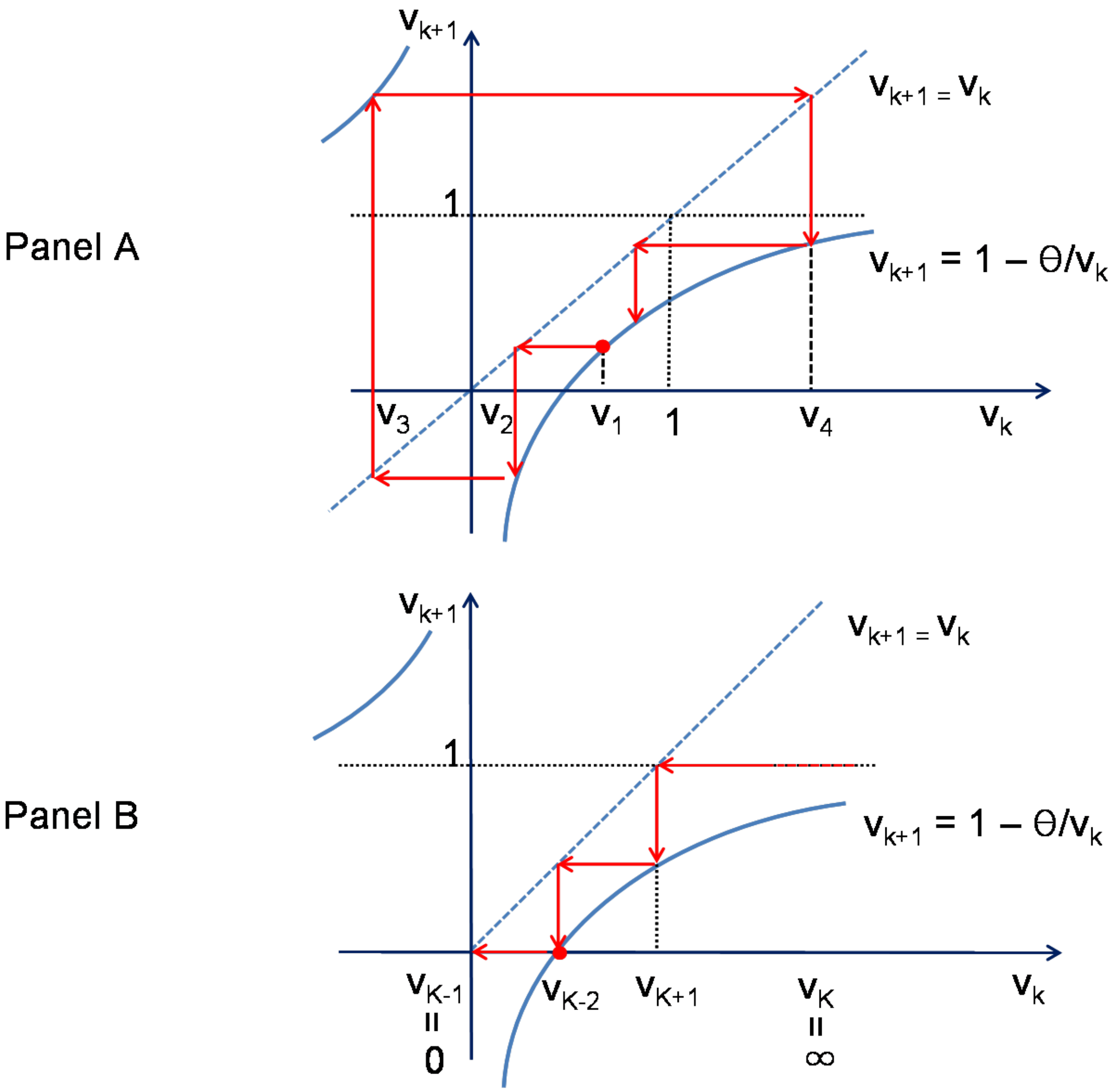


This figure illustrates the forward method graphically when the following model satisfies the forward convergence condition, and therefore the forward solution exists:

$$x_t = aE_t x_{t+1} + bx_{t-1}$$

Panels A and B describe the convergence path of v_k defined in equation (27) when $0 < \theta \leq \frac{1}{4}$, and $\theta < 0$, respectively, where $\theta = ab$.

Figure 2: Non-Existence of the Forward Solution



This figure illustrates the forward method graphically when the following model does not satisfy the forward convergence condition, and therefore the forward solution does not exist:

$$x_t = aE_t x_{t+1} + bx_{t-1}$$

In both panels $\theta > \frac{1}{4}$ where $\theta = ab$. Panel A illustrates the case where the regularity condition holds, but v_k explodes with oscillation. Panel B illustrates the case where the regularity condition is violated.