Characterizing Markov-Switching Rational Expectations Models *

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Abstract

We develop a solution method and solution refinement for a class of general Markov-switching rational expectations (MSRE) models. Then we derive very tractable conditions for determinacy in mean-square stability sense. Our methodology enables us to completely characterize the set of economically relevant rational expectations equilibria under determinacy and indeterminacy. Introducing Markov-switching property to an otherwise standard linear RE model can expand the parameter space over which a model has a unique stable equilibrium. Stochastic regime changes also inject additional volatility to the system. We apply our methodology to a standard New-Keynesian model subject to regime-switching in monetary policy and show that determinacy under regime-switching is in general neither necessary nor sufficient for determinacy under fixed regimes.

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1 Introduction

The main purpose of this paper is two-fold. First, we develop a solution method and solution refinement for a class of general Markov-switching rational expectations (MSRE) models. Second, we derive very tractable conditions for determinacy in mean-square stability sense. Then we completely characterize the set of rational expectations equilibria (REE) to both determinate and indeterminate MSRE models.

Only recently has Markov-switching - one of the most productive modeling techniques in time-series econometrics - been applied to structural rational expectations (RE) macroeconomic models. For instance, Davig and Leeper (2007) consider a regime-switching monetary policy in an otherwise canonical New-Keynesian model and provide a new perspective for understanding U.S. historical monetary policy. They show that the possibility of regime-switching between passive and active policy stances can expand the parameter space over which the unique bounded equilibrium exists. MSRE models can also be used to deal with uncertainty in important structural parameters governing optimal behavior of private agents. Liu, Waggoner and Zha (2009), for instance, consider regime-switching in the optimal price-setting behavior of firms and derive a regime-switching New-Keynesian Phillips curve à la Calvo (1983). The optimal monetary policy of the central bank facing regime-shifting behaviors of private agents is also of great importance, as emphasized by Davig (2007). This new class of macroeconomic models has already been estimated by Bianchi (2011), Liu, Waggoner and Zha (forthcoming) and Davig and Doh (2009).

Due to the inherent non-linearities arising from regime-dependency, standard solution techniques or determinacy conditions for linear RE models such as Blanchard and Kahn (1980)'s cannot be directly applied to MSRE models. Farmer, Waggoner and Zha (2009) derive a set of determinacy conditions. However, their methodology works only for the models without predetermined variables, and the conditions are quite demanding to completely examine. Farmer et al. (forthcoming) made an important contribution by proposing a numerical algorithm to find minimum state variable solutions. Unfortunately, little information regarding determinacy has been available to the models with predetermined variables. Another important class of MSRE models, such as the model with regime-switching behaviors of private agents in Davig (2007), cannot be represented in their form either because parameters depend on future regimes, not just the current regime.

This paper's main purpose is to provide a necessary tool for the analysis of the general MSRE models. The first contribution is to derive a tractable solution method for a fundamental solution. We extend the forward method of Cho and Moreno (2011) developed for the standard linear RE models to the class MSRE models. The forward method is essentially to "solve forward" the model recursively and if there is a stable relation between the current endogenous variables and the state variables as the forward recursion continues, the relation is defined as the well-known forward (forward-looking) solution in the sense of Blanchard (1979), which is a minimum state variable (MSV) solution.

While the method proposed by Farmer et al. (forthcoming) may be used to solve for the MSV solutions, we adopt the forward method mainly because the information embedded in the forward method plays a decisive role for identifying determinacy conditions, which we will elaborate extensively in this paper. The forward method also provides a powerful solution refinement criterion, called no-bubble condition. When a model is solved forward, the expectational term is typically assumed to die out as the forward recursion continues. However, we show that the expectational term goes to zero if and only if it is computed with the forward solution and all other MSV, often called bubble-free solutions fail to satisfy no-bubble condition. Hence, no-bubble condition can be used to refine away all other MSV solutions.

Second and the most important contribution is to derive tractable determinacy conditions. To proceed, it is necessary to have a relevant concept of stability as different stability criteria, essentially identical for the linear models, do not coincide with each other for MSRE models. While there may be several candidates, we use mean-square stability adopted by Farmer et al. (2009) because it is the most relevant from a econometric point of view and it works with a weak covariance-stationarity restriction on both the structural and bubble shocks. Unlike the case for linear RE models, stability of a REE in the first moment does not imply stability in the second moment under a Markov-switching environment. This is because the regime-switching possibility injects additional volatility to a REE so that its variance *can* be unbounded. If all shock processes are assumed to be bounded, then boundedness adopted by Benhabib (2009) and Davig and Leeper (2007) can also be a relevant stability concept but we focus on mean-square stability mainly because of its tractability.

The essential idea underlying our methodology can be illustrated as follows. As Lubik and Schorfheide (2004) show, any REE to a MSRE model can be decomposed into a fundamental, MSV solution and a non-fundamental component. Determinacy is therefore the case where there is a unique (mean-square) stable fundamental solution and there is no stable non-fundamental component associated with that MSV solution. In this regard, Cho and McCallum (2011) derive a tight relation between forward method and determinacy for linear RE models. The forward method provides the forward solution and the unique matrix F governing stability of the non-fundamental (bubble) component. They show that (1) if all the eigenvalues of the matrix F associated with the forward solution lie inside or on the unit circle, there is no stable bubble component. (2) If in addition, the forward solution is stable, then there is no other stable MSV solution. Therefore, the model at hand becomes determinate under these two conditions and the forward solution is the determinate equilibrium. It is this idea that we utilize from the forward method. We extend their work to the class of MSRE models and derive a similar condition except that F is replaced with an appropriate probability weighted matrix of F for all regimes.

The most powerful consequence of identifying determinacy in this alternative way is that one does not need to solve for all the non-fundamental components. Farmer et al. (2009) derive the closed form processes for non-fundamental components. However, the class of such processes is extremely large, thus it is quite a daunting task to find all of them and examine mean-square stability. To summarize, our determinacy conditions for MSRE models are as simple as those for the standard linear models. One has only to examine the existence of the forward solution, check stability of that solution and the matrix pertaining to the non-fundamental components. No-bubble condition, the solution refinement scheme of the forward method also enables us to completely characterize the set of economically relevant equilibria under determinacy and indeterminacy.

We apply our methodology to a standard New-Keynesian model subject to regime-switching in monetary policy stance. Davig and Leeper (2007) has shown that the central bank switching between a passive and active policy stances can be an admissible determinate equilibrium when shocks are bounded. We show that such an equilibrium can also be determinate in mean-square stability sense when shocks are unbounded, covariance-stationary. However, we also find that if one regime is too active relative to the other, the additional volatility induced by regime-switching can actually lead the economy to indeterminacy. We also apply our methodology to a model with a Markov-switching elasticity of intertemporal substitution parameter of the households. The model can help us understand the optimal behavior of agents whose preferences are subject to regime changes and the reaction of the central bank in such an environment. Our impulse response analysis also exhibits quantitatively and qualitatively rich dynamics. Finally, we examine determinacy of other examples in the literature and show that not all the solutions are economically relevant based on our solution refinement.

This paper is organized as follows. Section 2 presents two illustrative MSRE models. In section 3, we present a class of general MSRE models and the classes of REEs. Section 4 develops our forward method and the solution refinement criterion. Section 5 derives determinacy conditions under mean-square stability and characterizes the complete set of the relevant REEs under determinacy and indeterminacy. In section 6, we apply our methodology to several examples. Section 7 concludes.[1]

2 Illustrative Examples

We present two types of Markov-switching models in this section and analyze them after we develop our determinacy conditions and solution methodology for MSRE models.

Example A (Markov-switching monetary policy): Consider a Markov-switching monetary policy in a canonical New-Keynesian model studied by Davig and Leeper (2007), and Farmer et al. (2009):

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t + z_{S,t}, \tag{1a}$$

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}) + z_{D,t}, \tag{1b}$$

$$i_t = (1-\rho)(\phi_\pi(s_t)\pi_t + \phi_y y_t) + \rho i_{t-1} + z_{MP,t}, \tag{1c}$$

where π_t, y_t and i_t are inflation, the output gap and the short-term interest rate, respectively. s_t is a Markov chain that switches over two states. $\phi_\pi(s_t)$ represents the monetary policy stance against inflation, which is active (passive) if $\phi_\pi(s_t) > 1$ ($\phi_\pi(s_t) < 1$). ρ captures the interest rate smoothing behavior of the central bank. $z_{i,t}$ for $i = S, D$ and MP represents respectively the exogenous, regime-independent aggregate supply, demand and monetary policy shocks. All these shocks are assumed to be covariance-stationary. $E_t[\cdot] \equiv E[\cdot|\mathcal{I}_t]$ is the mathematical expectation operator conditional on the information set $\mathcal{I}_t$ which includes the current regime s_t.

[1] Matlab codes for our methodology and the numerical examples in this paper are provided at http://web.yonsei.ac.kr/sc719.

It is well-known that in the case of $\phi_y = 0$ and $\rho = 0$, if an active (passive) regime prevails forever, the model is determinate (indeterminate) for regime-independent models. Davig and Leeper (2007) show that the parameter space over which the model is unique and bounded is larger than that of its fixed regime counterpart. This is expressed in their long-run Taylor principle (LRTP): even though monetary policy can be temporarily passive, if there is a sufficiently large probability that the policy shifts to the active regime, the model can have a unique bounded equilibrium. When the shock processes are not bounded, however, the model where the LRTP holds can be indeterminate, as Farmer et al. (2009) show. While the determinacy conditions of Farmer et al. (2009) cannot be applied to this model with predetermined variables, our methodology easily quantifies the determinacy area and shows that the condition analogous to the LRTP governs stability in the first, but not the second moment of the system, hence it is not sufficient for determinacy in the sense of mean-square stability. In particular, the model (1) may not be determinate even when both regimes are active if one regime is *too* active relative to the other, and the interest rate smoothing behavior is strong.

From a theoretical perspective, this model with an ad-hoc specification of regime-switching in monetary policy may not be entirely satisfactory. The policy rule is not optimal by construction: it is not easy to justify why the monetary authority shifts its policy stances while forward-looking rational private agents are not regime-switching. This observation leads us to consider an alternative example in which private agents are subject to regime-shifting. For illustrative purposes, we consider the optimal behavior of households when their elasticity of substitution shifts stochastically.

Example B (Markov-switching elasticity of intertemporal substitution): Consider a standard power utility function of a representative household where the inverse of elasticity of intertemporal substitution σ depends on a regime variable s_t:

$$U(C_t) = \frac{H_t {C_t}^{1-\sigma(s_t)} - 1}{1-\sigma(s_t)}, \tag{2}$$

where C_t is consumption and H_t is an exogenous preference shifter. Just as policy stances can change over time, preferences of households may fluctuate as well. σ is an important parameter in both economics and finance, but its empirical estimate varies over a wide range. While the time variation of σ due to the history-independent Markov chain is not very satisfactory, this kind of model can shed light on explaining the parameter instability of σ reported in the literature. The intertemporal optimality condition is given by:

$$E\left[\exp(-\psi - \sigma(s_{t+1})\ln C_{t+1} + \sigma(s_t)\ln C_t + \ln(H_{t+1}/H_t) + \ln R_{t+1})|\mathcal{I}_t\right] = 1, \qquad (3)$$

where $\exp(-\psi)$ is the time discount factor and R_{t+1} is the gross real interest rate. Now we log-linearize the Euler equation (3) around the steady states with respect to C_{t+1}, H_{t+1}/H_t and R_{t+1} while preserving non-linearity of the regime-switching parameters. Combining this equation with the same aggregate supply curve, (1a) and a regime-independent policy rule yields:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t + z_{S,t}, \qquad (4a)$$

$$y_t = E_t\left[\frac{\sigma(s_{t+1})}{\sigma(s_t)} y_{t+1}\right] - \frac{1}{\sigma(s_t)}(i_t - E_t\pi_{t+1}) + \frac{1}{\sigma(s_t)} z_{D,t}, \qquad (4b)$$

$$i_t = (1-\rho)(\phi_\pi \pi_t + \phi_y y_t) + \rho i_{t-1} + z_{MP,t}, \qquad (4c)$$

where we have assumed that consumption equals output Y_t in equilibrium. $y_t = \ln Y_t - \ln Y$, i_t and π_{t+1} are the output gap, the nominal interest rate and inflation net of their steady states, and $z_{D,t} = -E_t[\ln(H_{t+1}/H_t)]$.

This model differs from the Markov-switching monetary policy model in an important aspect. Conditional on the current regime, agents choose their optimal consumption by taking into account the possibility that their risk appetites may shift in the future. Current consumption depends on the expectations of future risk aversion and future consumption in a non-linear fashion; hence, $\sigma(s_{t+1})$ enter inside the expectational term. This implies that assigning regime-switching on the structural parameter in a linearized behavioral equation such as replacing (1b) with $y_t = E_t[y_{t+1}] - \frac{1}{\sigma(s_t)}(i_t - E_t\pi_{t+1}) + z_{D,t}$ may not represent the optimal behavior of agents.

3 Markov-Switching Rational Expectations Models and the Class of Equilibria

In this section, we present the class of general MSRE models and the full set of equilibria.

Consider the following class of MSRE models:

$$x_t = E_t[A(s_t, s_{t+1})x_{t+1}] + B(s_t)x_{t-1} + C(s_t)z_t, \quad (5)$$
$$z_t = Rz_{t-1} + \epsilon_t, \quad (6)$$

where x_t is an $n \times 1$ vector of endogenous variables, measured from its steady state, which is assumed to be known to all agents in the model. z_t is an $m \times 1$ vector of exogenous variables and ϵ_t is an $m \times 1$ asymptotically covariance-stationary vector with $\epsilon_t \sim (0, D)$.[2] s_t is a $S-$states Markov chain with the transition matrix P where its (i, j)-th element is $p_{ij} = \Pr(s_{t+1} = j | s_t = i)$ for all $i, j \in \{1, 2, .., S\}$. The information set being conditioned in the expectation at time t is given by $\mathcal{I}_t = \{x_t, x_{t-1}, .., s_t, s_{t-1}, ...\}$. A, B and C at each state s_t and s_{t+1} are $n \times n$, $n \times n$ and $n \times m$ matrices of the structural parameters. R is an $m \times m$ matrix.[3] The model is quite general in that A may depend on both current and future regimes and it can be singular for all states. The model also allows the presence of predetermined variables. Note that (5) nests a standard linear RE model as a special case with $S = 1$ and $P = 1$. One can easily see that the aforementioned models in the previous section are nested in (5).

3.1 Rational Expectations Equilibria to MSRE Models

While there can be many alternative but equivalent ways of characterizing the complete set of the REEs, we characterize it as a sum of the two parts: the fundamental solution, also known as minimum state variables (MSV) solution following McCallum (1983), and the non-fundamental component that depends on arbitrary bubble or sunspot shocks. Our characterization of the REEs to (5) is summarized in the following proposition.

Proposition 1 *Any Rational Expectations solution to model (5) with (6) can be written as a sum of a MSV (fundamental) solution and a non-fundamental component, w_t as:*

$$x_t = [\Omega(s_t)x_{t-1} + \Gamma(s_t)z_t] + w_t. \quad (7)$$

The first two components of the right-hand side constitute a MSV solution, where

[2] Andolfatto and Gomme (2003) analyze a model with regime-dependent variances such that $D = D(s_{t-1})$. While our methodology can be easily extended to incorporate this specification, we do not explicitly consider regime-dependent variance for simplicity.

[3] The model with regime switching R can also be represented in the form of (5) by redefining x_t to include z_t.

$(\Omega(s_t), \Gamma(s_t))$ *must satisfy the following conditions for all* s_t *and* $s_{t+1} = 1, 2, ..., S$*:*

$$\Omega(s_t) = \Xi(s_t)^{-1} B(s_t), \tag{8a}$$

$$\Gamma(s_t) = \Xi(s_t)^{-1} C(s_t) + E_t[F(s_t, s_{t+1})\Gamma(s_{t+1})]R, \tag{8b}$$

where $\Xi(s_t)$ *and* $F(s_t, s_{t+1})$ *are defined as:*

$$\Xi(s_t) = (I - E_t[A(s_t, s_{t+1})\Omega(s_{t+1})]), \tag{9}$$

$$F(s_t, s_{t+1}) = \Xi(s_t)^{-1} A(s_t, s_{t+1}), \tag{10}$$

under the regularity condition that $\Xi(s_t)$ *is non-singular for all* s_t*. The non-fundamental component* w_t *must satisfy the following:*

$$w_t = E_t[F(s_t, s_{t+1})w_{t+1}]. \tag{11}$$

Proof. *It is straightforward to show that plugging (7) into the model and rearranging them yield the formula (8a) through (11). Since* w_t *represents any process that is left other than MSV component, the set of equilibria of the form (7) is exhaustive. Q.E.D.* ■

The fundamental component of the general solution (7),

$$x_t = \Omega(s_t)x_{t-1} + \Gamma(s_t)z_t \tag{12}$$

is by itself a RE solution because it solves the model (5). Analogously to the linear RE models, we refer to this type of solution as the MSV solution because it depends on the minimal set of state variables, x_{t-1}, z_t, and the regime variable s_t. While this solution is inherently non-linear, it is linear in the first two state variables and the Markov chain enters as regime-switching coefficients. These coefficient matrices depend only on s_t and are independent of x_{t-1} and z_t. The non-fundamental component w_t represents any other process that cannot be written in the form of the MSV solutions and its functional form will be given in Section 5.

3.2 Road Map toward Defining Determinacy and Deriving Determinacy Conditions

Our characterization of the REEs to a MSRE model enables us to define determinacy as the case where there is a unique stable fundamental solution and there is no stable process w_t subject to (11), regardless of which concept of stability is utilized. We adopt mean-square stability as a main device, but discuss several other plausible concepts of stability in Section 5. To the best of our knowledge, there is no known way of identifying the full set of either fundamental solutions or non-fundamental components. Farmer et al. (forthcoming) propose a numerical algorithm to find multiple MSV solutions to a MSRE model. It is however not clear how many MSV solutions a model has in general. Moreover, their method is not applicable to MSRE models with $A(s_t, s_{t+1})$.[4] Farmer et al. (2009) contribute to the literature by showing the explicit functional form of the whole w_t processes in the case of $F(s_t, s_{t+1}) = F(s_t)$. However, even in this simpler case, the class of such processes is too huge to verify stability of all w_t processes.

Our methodology circumvents these technical obstacles and provides a very efficient way to identify determinacy and solve for the determinate solution. To complete our mission, we follow the four steps explained below.

1. In Section 4, we develop a simple method to solve for a particular fundamental equilibrium, referred to as forward solution, $x_t = \Omega^*(s_t)x_{t-1} + \Gamma^*(s_t)z_t$ together with $F^*(s_t, s_{t+1})$, which does not require any information of stability and determinacy.

2. In Second 5.1 we introduce the concept of mean-square stability and its properties. Section 5.2.1 shows that the forward solution is mean-square stable if all the eigenvalues of a probability weighted matrix $\bar{\Psi}_{\Omega^* \otimes \Omega^*}$ (to be defined later) lie inside the unit circle.

3. In Section 5.2.2, it is proved that if a probability weighted matrix $\Psi_{F^* \otimes F^*}$ has all

[4]Solving for $\Omega(s_t)$ is equivalent to solving for the MSV solutions because once $\Omega(s_t)$ is solved for, the rest of the matrices can be obtained. Unfortunately, the complete set of solutions for $\Omega(s_t)$ is difficult to obtain, since the problem is non-homogenous in the sense that (8a) involves cross-products of $\Omega(s_t = i)$ and $\Omega(s_t = j)$ for all $i, j = 1, ..., S$. One may expand (8a) as a regime-independent matrix quadratic form such that $\tilde{A}\tilde{\Omega}^2 + \tilde{B}\tilde{\Omega} + \tilde{C} = 0_{nS \times nS}$ where $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are functions of the original matrices, A, B and C at all states. One may then attempt to employ the generalized Schur decomposition theorem to solve for $\tilde{\Omega}$ and then back Ω out. Unfortunately, $\tilde{B}$ is always singular and rank conditions are violated. To our knowledge, the existence and the number of solutions to such rank-deficient matrix quadratic equations are not yet known in mathematics literature. This may explain why Farmer et al. (forthcoming) propose a numerical procedure for finding Ω.

its eigenvalues lie inside or on the unit circle, there is no other mean-square stable process w_t. A key implication is that one does not need to obtain the full set of solutions of w_t and examine stability of them.

4. Section 5.3 presents the main proposition that if $r_\sigma(\bar{\Psi}_{\Omega^*\otimes\Omega^*}) < 1$ and $r_\sigma(\Psi_{F^*\otimes F^*}) \leq 1$ where $r_\sigma(\cdot)$ represent the maximum absolute eigenvalue of the argument matrix, then there is no other mean-square stable MSV solutions. This property of the forward method together with non-existence of mean-square stability derived in step 3 finally enables us to derive determinacy conditions: the model (5) is determinate in mean-square sense under the two conditions above (including the existence of the forward solution), and the forward solution is the determinate equilibrium.

4 The Forward Method

The forward method we develop here for MSRE models is essentially identical to the one that Cho and Moreno (2011) developed for linear RE models. We first present the solution methodology, followed by the solution refinement scheme.

4.1 Solution Methodology

The first step is to derive a forward representation *implied* by the model in which the vector of current endogenous variables is related to the expectations of future endogenous variables and the current state variables recursively using the law of iterative expectations and additionally, the Markov chain property. The forward method simply amounts to textbook-style "solving the model forward" as stated in the following.

Proposition 2 *Consider model (5) together with (6). For any initial regime s_t, x_t, x_{t-1} and z_t, there exists a unique sequence of real-valued matrices $(\Omega_k(s_t), \Gamma_k(s_t))$, $k = 1, 2, 3, ...$ such that:*

$$x_t = E_t[M_k(s_t, s_{t+1}, ..., s_{t+k})x_{t+k}] + \Omega_k(s_t)x_{t-1} + \Gamma_k(s_t)z_t, \tag{13}$$

where $\Omega_1(s_t) = B(s_t)$, $\Gamma_1(s_t) = C(s_t)$ and for $k = 2, 3, ...$,

$$\Omega_k(s_t) = \Xi_{k-1}(s_t)^{-1}B(s_t), \tag{14a}$$

$$\Gamma_k(s_t) = \Xi_{k-1}(s_t)^{-1}C(s_t) + E_t[F_{k-1}(s_t, s_{t+1})\Gamma_{k-1}(s_{t+1})]R, \tag{14b}$$

with $\Xi_{k-1}(s_t)$ and $F_{k-1}(s_t, s_{t+1})$ given by:

$$\Xi_{k-1}(s_t) = (I_n - E_t[A(s_t, s_{t+1})\Omega_{k-1}(s_{t+1})]), \tag{15}$$

$$F_{k-1}(s_t, s_{t+1}) = \Xi_{k-1}(s_t)^{-1} A(s_t, s_{t+1}), \tag{16}$$

if the following regularity condition is satisfied for all $k > 1$ and $s_t = 1, 2, ..., S$:

$$|\Xi_{k-1}(s_t)| \neq 0. \tag{17}$$

Proof. *See Appendix A.*[5] ■

The conditional expectations in (14a), (14b) and (15) can be easily computed. For instance, $E[A(s_t, s_{t+1})\Omega_{k-1}(s_{t+1})|s_t = i] = \sum_{j=1}^{S} p_{ij} A(i,j)\Omega_{k-1}(j)$ for all s_t. Note that if the sequence of the matrices defined in (14a) through (16) converge as the forward recursion continues, their limiting expressions fulfill (8a) through (10). This hints that the limiting values of these sequences will define a MSV solution. Formally, the convergence property of these sequences and the forward solution are defined as follows.

Definition 1 *The MSRE model (5) is said to satisfy the forward convergence condition (FCC) if the coefficients of the state variables, $(\Omega_k(s_t), \Gamma_k(s_t))$ in the forward representation of the model (13) converge for every regime s_t as k tends to infinity. Under the FCC, the model implies:*

$$x_t = \lim_{k \to \infty} E_t[M_k(s_t, s_{t+1}, ..., s_{t+k})x_{t+k}] + \Omega^*(s_t)x_{t-1} + \Gamma^*(s_t)z_t, \tag{18}$$

where $\Omega^*(s_t) = \lim_{k \to \infty} \Omega_k(s_t)$, $\Gamma^*(s_t) = \lim_{k \to \infty} \Gamma_k(s_t)$ and $F^*(s_t, s_{t+1}) = \lim_{k \to \infty} F_k(s_t, s_{t+1})$ for every s_t and s_{t+1}.

Definition 2 *When the MSRE model (5) satisfies the FCC, the forward solution is defined as the model-implied forward representation of the model in the limit without the expectational term in (18):*

$$x_t = \Omega^*(s_t)x_{t-1} + \Gamma^*(s_t)z_t. \tag{19}$$

[5] We show in the appendix that there exists a unique sequence for $M_k(s_t, s_{t+1}, ..., s_{t+k})$ as well, but our methodology does not require us to compute this complicated term, including $E_t[M_k(s_t, s_{t+1}, ..., s_{t+k})x_{t+k}]$.

Note that $\Omega^*(s_t)$ and $F^*(s_t, s_{t+1})$ exist if and only if $\Xi^*(s_t)$ exists. By construction, the forward solution is a MSV solution because $(\Omega^*(s_t), \Gamma^*(s_t))$ and $\Xi^*(s_t)$, $F^*(s_t, s_{t+1})$ satisfy (8), (9) and (10) from Proposition 1. Moreover, the forward solution is unique for every single model if it exists because the sequence of $(\Omega_k(s_t),\ \Gamma_k(s_t))$ is uniquely constructed by the initial coefficient matrices given by the model. Therefore, Proposition 2 is a solution algorithm to compute the forward solution. This solution is by itself a very natural fundamental equilibrium to any model, regardless of determinacy, because it is the relation uniquely *implied* by the underlying model that is absent from the expectational effect involving the endogenous variables far in the future on the current variables. This completes the technical aspect of the forward method.

4.2 No-Bubble Condition as a Solution Refinement

We emphasize the uniqueness of the forward solution and this should be an advantage of the forward method. One may think that our method is unsatisfactory because it does not solve for all MSV solutions. However, the forward method provides a very intuitive solution refinement criterion and the forward solution is the unique equilibrium that passes the criterion, which is the following no-bubble condition. Formally,

Definition 3 *A rational expectations solution to the MSRE model (5) is said to satisfy the no-bubble Condition (NBC) if the expectational term involving future endogenous variables converges to zero in the forward representation of model (13) for every s_t when expectations are formed with that solution:*

$$\lim_{k\to\infty} E_t[M_k(s_t, s_{t+1}, ..., s_{t+k})x_{t+k}] = 0_{n\times 1}. \tag{20}$$

It should be emphasized that the NBC is a solution property; that is, it depends on a particular solution with which expectations are formed. The forward solution must satisfy the NBC by construction. Hence, when expectations are formed with this solution, the expectations of endogenous variables far in the future should not affect the current endogenous variables. A standard linear model without lagged variables is often solved forward and this condition is typically *assumed* to hold as there is a unique MSV solution to such a model. Even in this simple case, we show that such a seemingly innocuous assumption can be misleading. When a model includes predetermined variables, multiple MSV solutions exist and all those solutions other than the forward solution must violate the NBC because, otherwise, it is a contradiction to the fact that the MSV solution differs from the forward solution. This property of the NBC is first observed by Cho and Moreno (2011) and it is preserved for MSRE models as well, as the the following proposition show.

Proposition 3 *The forward solution (19) to the MSRE model (5) with (6) exists if and only if the model satisfies the FCC, and it is the unique MSV solution that satisfies the no-bubble condition.*

Proof. *See Appendix B.* ■

Note that we have not discussed stability of REEs. Nor did we use any information about determinacy throughout the forward method in this section. Nevertheless, the NBC becomes a very powerful solution refinement within the class of MSV solutions. The NBC eliminates two kinds of equilibria. First, there may well exist MSV solutions to the models that are not forward convergent, but all of them fail to satisfy the NBC. Specifically, note that any solution of such kind must satisfy the forward representation of the model (13). If that solution satisfies the NBC, i.e, the expectational term converge to zeros, then the forward representation becomes $x_t = \lim_{k\to\infty} (\Omega_k(s_t)x_{t-1} + \Gamma_k(s_t)z_t)$ and must converge to that solution, but it is a contradiction to the supposition that at least one of $\Omega_k(s_t)$ and $\Gamma_k(s_t)$ is not convergent.

To build up intuition behind the logic of the NBC, it is sufficient to think about a simple linear model. For instance, consider the Fisher equation $i_t = E_t\pi_{t+1} + r_t$. where i_t and π_t are the nominal interest rate and inflation respectively. The real interest rate r_t is assumed to be 0 for simplicity. Suppose that the monetary policy rule is given by $i_t = \alpha\pi_t + z_t$ where the monetary policy shock z_t follows an AR(1) process $z_t = \rho z_{t-1} + \epsilon_t$ with $0 < \rho < 1$ and an *i.i.d.* shock ϵ_t. Substituting out the nominal rate yields $\pi_t = aE_t\pi_{t+1} - az_t$ where $a = 1/\alpha$. The model is determinate if $a \leq 1$ and indeterminate if $a > 1$. Suppose that the policy rule is passive: $0 < \alpha < 1$, implying $a > 1$. When solved forward, the model becomes $\pi_t = M_k E_t\pi_{t+k} + \Gamma_k z_t$ where $M_k = a^k$ and $\Gamma_k = -a\sum_{i=1}^{k}(a\rho)^{k-1}$. If $a\rho < 1$, the forward solution exists and is given by $\pi_t = \Gamma^* z_t = -az_t/(1 - a\rho)$.[6] Therefore, inflation is stabilized following a contractionary monetary policy shock as inflation reacts negatively to z_t. However, if

[6]Of course in the latter case, the full set of solution is given by $\pi_t = \Gamma^* z_t + w_t$, where $w_t = (1/a)w_{t-1} + \xi_t$ with an arbitrary white noise ξ_t.

$a\rho > 1$ under indeterminacy, Γ_k goes to *negative* infinity and the model fails to satisfy the FCC. Nevertheless, there does exist a stable solution $\pi_t = -az_t/(1-a\rho)$, which has the same functional form as the forward solution. But when the expectational term is computed with this solution, it goes to *positive* infinity. This MSV solution just obtained (by other method, say, the method of undetermined coefficients) is obviously implausible as a relevant inflation process, because it implies that inflation actually increases following a contractionary (positive) monetary policy shock. This is precisely in line with the process consistency of Flood and Garber (1980). For further discussion, refer to Cho and McCallum (2011) who generalize the concept of process consistency to linear RE models.[7] Therefore, the NBC is a powerful refinement scheme to rule out all the MSV solutions to the models that fail to satisfy the FCC. Farmer et al. (forthcoming) recently propose a solution method for the MSV solutions and solve for a MSV solution to a model similar to the above. We show that it is in fact an MSV solution to a non-forward convergent MSRE model. We will come to this issue in Section 6.1 after we derive determinacy conditions for regime-switching models.

Second, suppose that the FCC holds for a model with predetermined variables. Then, there may well exist multiple equilibria, stable or not. Proposition 3 states that if any other MSV solution exists, different from the forward solution, then it - often called a bubble-free solution - must violate the no-bubble condition. These solutions are hence as implausible as the example of the first kind above. While non-fundamental bubble solutions would violate the NBC as they should by definition, violation of the NBC is hard to justify economically for any fundamental (MSV) solution, as argued by Cho and Moreno (2011). Therefore, the NBC plays the role of solution refinement within the class of fundamental MSV solutions.

Now the remaining tasks are to define determinacy under a suitable choice of stability, and relate it to the forward method to derive determinacy conditions.

[7] In their study of monetary reform, Flood and Garber (1980) solve their model forward to express the price level as the discounted sum of the current and future exogenous monetary aggregates. They require the money supply process to be "process consistent" so that the discounted sum of money supplies will be finite, and consequently, the price level will be well-defined.

5 Determinacy

Defining determinacy requires us to formally choose a suitable concept of stability for MSRE models. Davig and Leeper (2007) and Benhabib (2009) adopt bounded stability, requiring both the MSV solutions and the non-fundamental components - that would emerge under indeterminacy - to be all bounded, or have finite moments. On the other hand, Farmer et al. (2009) propose an alternative concept called mean-square stability, requiring these stochastic processes to have finite first and second moments. As Farmer et al. (2009) show, while bounded stability is equivalent to mean-square stability in linear models under the assumption of the bounded shocks, they are different in a Markov-switching environment. While both concepts of stability are well admissible as pinning down an economically relevant equilibrium, we adopt mean-square stability for several reasons. First, we want to characterize determinacy for a larger set of stochastic processes that are covariance-stationary, not necessarily bounded. Second, most of theoretical DSGE models and empirical time series models assume covariance-stationary, unbounded processes such as normally distributed shocks. As such, we are more or less in line with Farmer et al. (2009) in adopting mean-square stability as a practical guidance to judge determinacy of MSRE models. However, we do not adopt boundedness mainly because of technical difficulty, not because it is implausible. It would be an important task to identify the conditions for determinacy under bounded stability. We hope our method can shed light on defining determinacy under bounded stability, which we leave as a future research agenda.

We now formally define determinacy in mean-square stability sense.

Definition 4 *The MSRE model (5) is determinate in mean-square stability sense if there exists a unique mean-square stable MSV solution, and there is no mean-square stable stochastic process for w_t associated with that MSV solution.*

The remaining subsections discuss mean-square stability and derive the conditions stated in the definition of determinacy above.

5.1 Mean-Square Stability

This subsection introduces the concept of mean-square stability (MSS) and briefly summarizes some of its important properties that we utilize to derive determinacy conditions.

Consider the following $n \times 1$ process y_{t+1}:

$$y_{t+1} = G(s_t, s_{t+1})y_t + H(s_{t+1})\eta_{t+1}, \tag{21}$$

where $G(s_t, s_{t+1})$ and $H(s_{t+1})$ are $n \times n$, $n \times m$ matrices, respectively. η_t is an arbitrary $m \times 1$ covariance-stationary (wide-sense stationary) process, independent of s_t. s_t is an ergodic Markov chain and the transition probability is given by p_{ij} from $s_t = i$ to $s_{t+1} = j$ for $i, j = 1, 2, ..., S$. Let $G_{ij} = G(s_t = i, s_{t+1} = j)$. Mean-square stability amounts to existence of mean and variance of y_t. Formally,

Definition 5 *The process (21) is mean-square stable if there exist* $\bar{y}$ *and and* Q *such that* $\lim_{t\to\infty}(E(y_t) - \bar{y}) = 0_{n\times 1}$ *and* $\lim_{t\to\infty}(E(y_t y_t') - Q) = 0_{n\times n}$.

We closely follow Costa et al. (2005) to derive a simple condition to identify MSS of (21), but we need to adjust their results as (21) has a more general form than their models in that both G and H depends on s_{t+1} and η is measured at time $t+1$. While it is straightforward to redefine the variables to represent (21) in their form, one can also directly work with (21), following the approach taken by Petreczky and Vidal (2007). In Appendix C, we show that both approaches yield identical results for MSS and hence, we can apply some important results of Costa et al. (2005) directly to the model (21). Theorem 3.33 of Costa et al. (2005) states that under the assumption on H and η_t, we have only to analyze the homogenous part (21), $y_{t+1} = G(s_t, s_{t+1})y_t$ in order to study its mean-square stability. Now, we define the following matrices:

$$\begin{aligned} \Psi_G &= [p_{ij}G_{ij}] = \begin{bmatrix} p_{11}G_{11} & ... & p_{1S}G_{1S} \\ ... & ... & ... \\ p_{S1}G_{S1} & ... & p_{SS}G_{SS} \end{bmatrix}, \\ \bar{\Psi}_G &= [p_{ji}G_{ji}] = \begin{bmatrix} p_{11}G_{11} & ... & p_{S1}G_{S1} \\ ... & ... & ... \\ p_{1S}G_{1S} & ... & p_{SS}G_{SS} \end{bmatrix}, \end{aligned}$$

$$\Psi_{G\otimes G} = [p_{ij}G_{ij} \otimes G_{ij}] = \begin{bmatrix} p_{11}G_{11} \otimes G_{11} & ... & p_{1S}G_{1S} \otimes G_{1S} \\ ... & ... & ... \\ p_{S1}G_{S1} \otimes G_{S1} & ... & p_{SS}G_{SS} \otimes G_{SS} \end{bmatrix},$$

$$\bar{\Psi}_{G\otimes G} = [p_{ji}G_{ji} \otimes G_{ji}] = \begin{bmatrix} p_{11}G_{11} \otimes G_{11} & ... & p_{S1}G_{S1} \otimes G_{S1} \\ ... & ... & ... \\ p_{1S}G_{1S} \otimes G_{1S} & ... & p_{SS}G_{SS} \otimes G_{SS} \end{bmatrix},$$

where $\otimes$ denotes the Kronecker product. Define $m^y_{i,t} = E[y_t(s_t)1_{\{s_t=i\}}]$ and $m^y_t = [(m^y_{1,t})' \ ... \ (m^y_{S,t})']'$ where $1_{\{s_t=i\}}$ is an indicator function that yields 1 when $s_t = i$ and 0 otherwise. The second moment of y_t can also be defined as $Q^y_{i,t} = E[y_t(s_t)y_t(s_t)'1_{\{s_t=i\}}]$ and $Q^y_t = [Q^y_{1,t} \ ... \ Q^y_{S,t}]$. By stacking $m^y_{i,t}$ and $vec(Q^y_{i,t})$ over all $i = 1, ..., S$, one can show that the first and second moments of the homogenous part of (21) have the following representations:

$$m^y_{t+1} = \bar{\Psi}_G m^y_t, \tag{24}$$

$$v^y_{t+1} = \bar{\Psi}_{G\otimes G} v^y_t, \tag{25}$$

where $v^y_t = vec(Q^y_t)$. (See chapter 3 of Costa et al. (2005) for details.) Note that this transformation enables us to write the model in terms of familiar first-order vector autoregressive form for m^y_t and v^y_t, whose coefficient matrices are regime-independent. Using m^y_t and v^y_t we can define the first and second moments of y_t as $E(y_t) = \sum_{i=1}^S m^y_{i,t}$ and $E(y_t y_t') = \sum_{i=1}^S Q^y_{i,t}$. It is convenient to define a spectral radius operator, $r_\sigma(M) = \max_{1\le i\le n}\{|\lambda_i|\}$ where λ_i is an eigenvalue of an $n \times n$ matrix M. Then, the existence of $\bar{y}$ and Q can be expressed as $r_\sigma(\bar{\Psi}_G) < 1$ and $r_\sigma(\bar{\Psi}_{G\otimes G}) < 1$. For the fixed regime models where G is regime-independent, $r_\sigma(G) < 1$ if and only if $r_\sigma(G \otimes G) < 1$. Therefore, stability of y_t in mean is equivalent to stability in variance as $r_\sigma(G \otimes G) = [r_\sigma(G)]^2$. However, this is not true any more for MSRE models, as the following theorem states:

Theorem 1 *For the process (21), if* $r_\sigma(\bar{\Psi}_{G\otimes G}) < 1$*, then* $r_\sigma(\bar{\Psi}_G) < 1$.

Proof. *See Proposition 3.6 of Costa et al. (2005).* ■

It is important to note that the converse is not true: $r_\sigma(\bar{\Psi}_G) < 1$ does not imply $r_\sigma(\bar{\Psi}_{G\otimes G}) < 1$. This implies that the volatility of y_{t+1} in equation (21) can be amplified by regime-switching of $G(s_t, s_{t+1})$ such that while y_{t+1} is bounded in mean, it may be unbounded in variance. Therefore, we need to examine $r_\sigma(\bar{\Psi}_{G\otimes G}) < 1$ for MSS of a process y_t. The following, which is a simpler version of Theorem 3.9 of Costa et al. (2005), is an important result in order to establish determinacy for MSRE models.

Theorem 2 *The process (21) is mean-square stable if and only if* $r_\sigma(\bar{\Psi}_{G\otimes G}) < 1$.

Proof. *See Proposition 3.9 of Costa et al. (2005).* ■

As is clear from equations (24) and (25), $r_\sigma(\bar{\Psi}_G)$ and $r_\sigma(\bar{\Psi}_{G\otimes G})$ are related to identifying stability in mean, and MSS respectively. As we explore in Section 5.3, these theorems will be used to deduce the relation in depth in order to study the LRTP and determinacy conditions in mean-square stability sense.

5.2 Mean-Square Stability of the REEs

5.2.1 Fundamental solutions

We now formally state mean-square stability of the fundamental solution.

Proposition 4 *A fundamental solution* $x_t = \Omega(s_t)x_{t-1} + \Gamma(s_t)z_t$ *in equation (12) is mean-square stable if*

$$r_\sigma(\bar{\Psi}_{\Omega\otimes\Omega}) < 1 \tag{26}$$

where

$$\bar{\Psi}_{\Omega\otimes\Omega} = [p_{ji}\Omega_j \otimes \Omega_j] = \begin{bmatrix} p_{11}\Omega_1 \otimes \Omega_1 & ... & p_{S1}\Omega_S \otimes \Omega_S \\ ... & ... & ... \\ p_{1S}\Omega_1 \otimes \Omega_1 & ... & p_{SS}\Omega_S \otimes \Omega_S \end{bmatrix}$$

Proof. *(12) has the same form as (21), thus* $\bar{\Psi}_{\Omega\otimes\Omega}$ *is the relevant matrix to determine mean-square stability from Theorem 2. Note that* Ω *at time* t *depends only on* s_t*. Hence,* $\Omega_j = \Omega(s_t = j)$ *for all* $s_{t-1} = i = 1, ..., S$*. Q.E.D.* ■

5.2.2 Non-fundamental Components

For ease of exposition we reproduce the restriction for any non-fundamental component of the REE to the model (5) to obey:

$$w_t = E_t[F(s_t, s_{t+1})w_{t+1}]. \tag{27}$$

We seek for the condition under which there is no MSS process w_t subject to (27). To do so, one may obtain the complete set of solutions for w_t and verifying that all those solutions are not MSS. As we hinted in Section 3.2, such a task can be completed by simply examining $r_\sigma(\Psi_{F\otimes F}) \leq 1$ where $\Psi_{F\otimes F} = [p_{ij}F_{ij} \otimes F_{ij}]$. To formally prove this claim, however, we need to derive the functional form of w_t subject to (27). The following proposition is essentially identical to Theorem 1 of Farmer et al. (2009) but extends it to the case where F may depend not just the current state s_t, but also the future state s_{t+1}, and F may be singular for all states (s_t, s_{t+1}).[8]

Proposition 5 *Any non-fundamental component w_t in (27) can be written as:*

$$w_{t+1} = \Lambda(s_t, s_{t+1})w_t + V(s_{t+1})V(s_{t+1})'\eta_{t+1} \tag{28}$$

where $V(s_t)$ is an $n \times k(s_t)$ matrix with orthonormal columns, $0 \leq k(s_t) \leq n$ and $k(s_t) > 0$ for some s_t. η_t is an arbitrary $n \times 1$ covariance-stationary innovations such that $E_t[V(s_{t+1})V(s_{t+1})'\eta_{t+1}] = 0_{n\times 1}$, $\Lambda(s_t, s_{t+1}) = V(s_{t+1})\Phi(s_t, s_{t+1})V(s_t)'$ for some $k(s_{t+1}) \times k(s_t)$ matrix $\Phi(s_t, s_{t+1})$ such that

$$\sum_{j=1}^{S} p_{ij}F_{ij}V_j\Phi_{ij} = V_i, \quad for \quad 1 \leq i \leq S. \tag{29}$$

where $V_i = V(s_t = i)$, $\Phi_{ij} = \Phi(s_t = i, s_{t+1} = j)$, $F_{ij} = F(s_t = i, s_{t+1} = j)$.

Proof. *See Appendix D.* ■

This proposition can be used to construct some non-fundamental components by searching for V_i and Φ_{ij}, (hence Λ_{ij}) given p_{ij} and F_{ij} for all i, j subject to the constraints (29). Therefore it is straightforward to check whether the obtained w_t process (28) is MSS by computing $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda})$. However, determinacy requires $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) \geq 1$ for all the possible solutions of the form (28). This is a daunting task. For instance, consider a model (27) with $S = n = 2$ and F_{ij} is non-singular for all $i, j = 1, 2$. For the choice of $k_i = k(s_t = i) = 2$, the restriction (29) for $i = 1$ is $p_{11}F_{11}\Lambda_{11} + p_{12}F_{12}\Lambda_{12} = I_2$, which implies that there are 4 free parameters in Λ_{11} and Λ_{12}. Thus, there are 8 free parameters to construct $\Lambda(s_t, s_{t+1})$. Moreover, we should also do the same tasks for all the cases with $0 \leq k_1, k_2 \leq 2$, except for $k_1 = k_2 = 0$.

Now we present two key lemmas in deriving a condition for non-existence of MSS non-fundamental components.

[8]Note that our $F(s_t, s_{t+1})$ collapses to $A(s_t, s_{t+1})$ when there is no predetermined variable in the model (i.e., $\Omega(s_t) = 0_{n\times n}$ for all s_t). When $A(s_t, s_{t+1}) = A(s_t)$ and this is invertible for all s_t, their $\Gamma(s_t)$ corresponds to our $A(s_t)^{-1}$.

Lemma 3 *Consider two processes $w_{t+1} = \Lambda(s_t, s_{t+1})w_t$ and $u_{t+1} = F'(s_t, s_{t+1})u_t$ of the form (21). The following holds.*

1. If $r_\sigma(\bar{\Psi}_{F' \otimes F'}) < 1$ and $r_\sigma(\bar{\Psi}_{\Lambda \otimes \Lambda}) < 1$, then $r_\sigma(\bar{\Psi}_{\Lambda \otimes F'}) < 1$ and $w_{t+1} + u_{t+1}$ is mean-square stable.

2. If $r_\sigma(\bar{\Psi}_{F' \otimes F'}) \leq 1$ and $r_\sigma(\bar{\Psi}_{\Lambda \otimes F'}) \geq 1$, then $r_\sigma(\bar{\Psi}_{\Lambda \otimes \Lambda}) \geq 1$.

Proof. *See Appendix E.* ■

This lemma simply states that if two processes are mean-square stable, then their sum is also mean-square stable. Hence, if $w_{t+1} + u_{t+1}$ is not mean-square stable, then at least one of the two processes is not mean-square stable. It should be emphasized that the first condition in Assertion 2 includes the case $r_\sigma(\bar{\Psi}_{F' \otimes F'}) = 1$, which we prove using the property of the spectral radius in the appendix. Note that $\bar{\Psi}_{F' \otimes F'} = (\Psi_{F \otimes F})'$ and hence, $r_\sigma(\Psi_{F \otimes F}) = r_\sigma(\bar{\Psi}_{F' \otimes F'})$.

Lemma 4 *For any process (28) subject to (29), the following holds.*

1. $\bar{\Psi}_{\Lambda \otimes F'}$ contains at least one root of 1, hence, $r_\sigma(\bar{\Psi}_{\Lambda \otimes F'}) \geq 1$.

2. $r_\sigma(\bar{\Psi}_{\Lambda \otimes \Lambda}) \geq 1/[r_\sigma(\Psi_{F \otimes F})]$ for all $\Lambda(s_t, s_{t+1})$.

Proof. *See Appendix F.* ■

Assertion 2 states that $1/[r_\sigma(\Psi_{F \otimes F})]$ is the lower bound of $r_\sigma(\bar{\Psi}_{\Lambda \otimes \Lambda})$ for all Λ subject to (29). Notice that Lemma 4 holds for any positive number, $r_\sigma(\Psi_{F \otimes F})$. Now we are ready to determine non-existence of mean-square stable non-fundamental components.

Proposition 6 *Consider equation (27). Suppose that the following condition holds:*

$$r_\sigma(\Psi_{F \otimes F}) \leq 1 \tag{30}$$

Then there is no stochastic MSS process w_t satisfying (27).

Proof. *From Proposition 5, any solution w_t can be written in the form of (28) and Assertion 1 of Lemma 4 implies $r_\sigma(\bar{\Psi}_{\Lambda \otimes F'}) \geq 1$. From Assertion 2 of Lemma 4, if $r_\sigma(\Psi_{F \otimes F}) \leq 1$, $r_\sigma(\bar{\Psi}_{\Lambda \otimes \Lambda}) \geq 1$. Q.E.D.* ■

To build up further intuition behind this result, consider (27) without regime switching: $w_t = FE_t[w_{t+1}]$. Any solution to this equation has the form of $w_{t+1} = \Lambda w_t + \xi_{t+1}$ where

ξ_{t+1} is an appropriately chosen vector of white noises and Λ must be such that $F\Lambda w_t = w_t$ for any non-zero w_t. This restriction implies that $F\Lambda$ has a unit root. Also, note that any Λ contains the inverse of some or all of the eigenvalues of F. Therefore if $r_\sigma(F) \leq 1$, this model is determinate and there is no stochastic stable process w_t, that is, $r_\sigma(\Lambda) \geq 1$. For MSRE models however, the constraint for Λ is $w_t = E_t[F(s_t, s_{t+1})\Lambda(s_t, s_{t+1})]w_t$. The full set of Λ subject to this constraint is huge and it is hard to identify all of them. Moreover the eigenvalues of Λ_{ij} are not directly related to those of F_{ij} for each state $s_t = i, s_{t+1} = j$. Fortunately, the structure of the solutions for w_{t+1} - first observed by Farmer et al. (2009) - enables us to develop our Lemma 4, without solving for all Λ: Finally, this observation together with Lemma 3 and 4 leads to Proposition 6.

Proposition 6 differs from the linear model counterpart in two aspects. First, the linear models can be handled in our setup as a special case where the number of regimes is 1 and $P = 1$. Thus (30) can be written as $r_\sigma(F \otimes F) \leq 1$. In this case, $r_\sigma(F \otimes F) \leq 1$ if and only if $r_\sigma(F) \leq 1$. However, this is not true for MSRE models. In section 6, we show that for a wide range of the parameters in F, $r_\sigma(\Psi_F) \leq 1$ but $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) < 1$. Hence, $r_\sigma(\Psi_F) \leq 1$ is not sufficient for non-existence of MSS non-fundamental components.[9]

Second, note that (30) is, technically speaking, sufficient but not necessary. Let τ_2 be the minimum value of $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda})$ over all $\Lambda(s_t, s_{t+1})$ in (28). If $\tau_2 = 1/[r_\sigma(\Psi_{F\otimes F})]$, then (30) becomes necessary as well as sufficient for non-existence of MSS w_t.[10] For linear models, we can always construct Λ such that $\tau_2 = 1/[r_\sigma(F \otimes F)]$. Will there be a case in the MSRE framework where there is no MSS process w_t when $r_\sigma(\Psi_{F\otimes F}) > 1$? Such case can happen only when $\tau_2 > 1 > 1/[r_\sigma(\Psi_{F\otimes F})]$. We will explain what is meant by this condition in Section 5.4. In practice, even if one has a model such that $r_\sigma(\Psi_{F\otimes F}) > 1$, one can search for Λ and verify the existence of MSS w_t. Since we know the lower bound of τ_2, finding Λ to minimize $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda})$ subject to (29) is a much easier constrained optimum problem than finding all Λ. Appendix G provides a detailed procedure. If there exists Λ such that $\tau_2 < 1$, then the model is indeed indeterminate.

The last step for deriving determinacy is to find a condition under which there is a unique mean-square stable MSV solution with $\Omega(s_t)$ and $r_\sigma(\Psi_{F\otimes F}) \leq 1$ where $F(s_t, s_{t+1})$

[9] One may think that $r_\sigma(\Psi_F) \leq 1$ is sufficient for non-existence of mean stable w_t, i.e., $r_\sigma(\bar{\Psi}_\Lambda) \geq 1$ for all Λ subject to (29). But this is not true either. The condition $r_\sigma(\Psi_{F\otimes F}) \leq 1$ implies that $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) \geq 1$, but there may well exist $r_\sigma(\bar{\Psi}_\Lambda) < 1$. In fact, when there exists Λ such that $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) = 1/[r_\sigma(\Psi_{F\otimes F})]$, one can show that the minimum of $r_\sigma(\bar{\Psi}_\Lambda) = r_\sigma(\Psi_F)/r_\sigma(\Psi_{F\otimes F})$, hence the condition for non-existence of mean stable non-fundamental components is $r_\sigma(\Psi_{F\otimes F}) \leq r_\sigma(\Psi_F) \leq 1$, which is even stronger than $r_\sigma(\Psi_{F\otimes F}) \leq 1$.

[10] We find that there exist uncountably many Λ yielding the same τ_2 for multivariate models.

is associated with that MSV solution. This task is performed in the following subsection. (30) is by itself a readily applicable determinacy condition for those models without predetermined variables as there is a unique MSV solution with $\Omega = 0_{n\times n}$, and $F(s_t, s_{t+1}) = A(s_t, s_{t+1})$.

5.3 Conditions for Determinacy in MSS sense

Recall that we propose the forward method that does not solve for all MSV solutions, but yields just one MSV equilibrium, the forward solution: $x_t = \Omega^*(s_t)x_{t-1} + \Gamma^*(s_t)z_t$, with the corresponding matrix $F^*(s_t, s_{t+1})$. Hence, the condition $r_\sigma(\bar{\Psi}_{\Omega^*\otimes\Omega^*}) < 1$ alone does not guarantee that the forward solution is the only stable MSV solution. However, when $r_\sigma(\Psi_{F^*\otimes F^*}) \leq 1$ as well, the following proposition shows that there is no other MSS fundamental solution. Now we present our main finding in Proposition 7.

Proposition 7 *Suppose that the MSRE model (5) satisfies the following properties.*

1. *The forward solution (19) exists.*
2. $r_\sigma(\bar{\Psi}_{\Omega^*\otimes\Omega^*}) < 1$
3. $r_\sigma(\Psi_{F^*\otimes F^*}) \leq 1$,

Then, there is no other mean-square stable MSV solution. Therefore, the is determinate in MSS sense and the forward solution is the determinate equilibrium.

Proof. *See Appendix H* ■

Our Proposition 7 is very compact and easy to apply to any general MSRE model in our framework. One has only to examine the existence of the forward solution and compute the conditions 2 and 3 for determinacy. Proposition 3 of Cho and McCallum (2011) is essentially identical to Proposition 7 here except that the two conditions are $r_\sigma(\Omega^*) < 1$ and $r_\sigma(F^*) \leq 1$, which are obviously a special case of ours. The proof strategy of ours is, however, quite different and much more involved.

While the formal proof of Proposition 7 is given in the appendix, it is instructive to think about why conditions $r_\sigma(\bar{\Psi}_{\Omega^*\otimes\Omega^*}) < 1$ and $r_\sigma(\Psi_{F^*\otimes F^*}) \leq 1$ rule out the possibility of existence of another MSS fundamental solution. For simplicity, assume away the exogenous variables z_t. Then the forward representation becomes $x_t = E_t[M_k(s_t, s_{t+1}, ..., s_{t+k})x_{t+k}] + \Omega_{k-1}(s_t)x_{t-1}$. When there exists another MSV solution, say, $x_t = \tilde{\Omega}(s_t)x_{t-1}$, then the expectational term evaluated recursively with this solution becomes $L_k(s_t : \tilde{\Omega})x_t$ where the coefficient matrix is defined as $L_k(s_t : \tilde{\Omega}) =$

$E_t[F_{k-1}(s_t, s_{t+1})L_{k-1}(s_{t+1})\tilde{\Omega}(s_{t+1})]$. As the recursion goes to infinity, this term should converge to a non-zero matrix $L(s_t : \tilde{\Omega})$ because otherwise the forward representation converges to the forward solution.[11] Therefore, the recursion formula in the limit becomes

$$L(s_t : \tilde{\Omega}) = E_t[F^*(s_t, s_{t+1})L(s_{t+1} : \tilde{\Omega})\tilde{\Omega}(s_{t+1})]. \tag{31}$$

By vectorizing this relation and stacking them over $s_t = 1, 2, ... S$, one can show that the matrix $\Psi_{\tilde{\Omega}' \otimes F^*}$ has a root of 1 and hence, $r_\sigma(\Psi_{\tilde{\Omega}' \otimes F^*}) = r_\sigma(\bar{\Psi}_{\tilde{\Omega} \otimes (F^*)'}) \geq 1$. (A similar logic was utilized for proving Assertion 1 of Lemma 4.) Since $r_\sigma(\bar{\Psi}_{(F^*)' \otimes (F^*)'}) = r_\sigma(\Psi_{F^* \otimes F^*}) \leq 1$, this implies that $r_\sigma(\bar{\Psi}_{\tilde{\Omega} \otimes \tilde{\Omega}}) \geq 1$ from Assertion 2 of Lemma 3.

Now we completely characterize the set of relevant REEs to a forward convergent model under determinacy and indeterminacy, and show that there is no economically relevant REE in all other cases according to no-bubble condition.

5.4 Characterization of the REEs

We can classify the entire family of MSRE models into two groups: one that satisfies the FCC and the other that does not. Based on our solution refinement scheme, NBC, we identify two subsets of MSRE models that have economically relevant equilibria: one under determinacy and the other one under indeterminacy, both belong to the first group of forward-convergent models. For ease of exposition, let τ_2^* be the minimum of $r_\sigma(\bar{\Psi}_{\Lambda \otimes \Lambda})$ where Λ is subject to (29) with $F = F^*$. Thus, $\tau_2^* \geq 1$ implies that there is no mean-square stable w_t. We present these two cases with the identifying conditions and the set of relevant solutions.

Case 1. The forward solution exists and $r_\sigma(\bar{\Psi}_{\Omega^* \otimes \Omega^*}) < 1$ and $r_\sigma(\Psi_{F^* \otimes F^*}) \leq 1$: These are the conditions of Proposition 7. The model is determinate and the unique mean-square stable solution is given by the forward solution, which passes the NBC.

$$x_t = \Omega^*(s_t)x_{t-1} + \Gamma^*(s_t)z_t.$$

Case 2. The forward solution exists and $r_\sigma(\bar{\Psi}_{\Omega^* \otimes \Omega^*}) < 1$, $r_\sigma(\Psi_{F^* \otimes F^*}) > 1$ and $\tau_2^* < 1$. The model becomes indeterminate. While there may exist multiple mean-square stable MSV solutions, the forward solution is the only one that passes the NBC. Hence,

[11] Specifically, the following holds for any MSV solution $(I_n - L(s_t : \tilde{\Omega}))\tilde{\Omega}(s_t) = \Omega^*(s_t)$ when the FCC holds.

the set of economically relevant equilibria is the forward solution itself and the non-fundamental solutions associated with the forward solution. The last condition, $\tau_2^* < 1$ should be checked numerically following the procedure explained in Appendix G. The entire class of the economically relevant indeterminate equilibria is then given by

$$x_t = \Omega^*(s_t)x_{t-1} + \Gamma^*(s_t)z_t + w_t,$$

where the functional form of w_t associated with $F = F^*$ is in (28), and any Λ is admissible as long as $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) \in [\tau_2^*, 1)$. Since all other mean-square stable MSV solutions, if they exist, violate the NBC, the non-fundamental solutions associated with these MSV solutions are not considered as relevant equilibria for the underlying model.

What would be the most plausible equilibrium path under indeterminacy? For linear models, one strand of researchers advocate for fundamental solutions as relevant equilibria, for instance, Bullard and Mitra (2002), Bullard (2006) and McCallum (2007). In this case, the forward solution without bubble terms is perfectly acceptable. Others such as Lubik and Schorfheide (2004) treat non-fundamental equilibria as equally plausible equilibrium paths. For the MSRE models, however, picking up a certain type of non-fundamental equilibria is a non-trivial task. Recall that the roots of Λ are the reciprocals of some or all of the eigenvalues of F^* for linear RE models. Hence, apart from the bubble shocks, there is at most a finite number of Λ exhibiting similar dynamic paths. For the MSRE models, however, the eigenvalues of Λ are not directly related to those of F^* and there are uncountably many Λ having different sets of eigenvalues from those with $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) = \tau_2^*$ to others with $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda})$ being arbitrarily close to 1. Moreover, there are in general uncountably many different types of Λ yielding the same $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda})$. This implies that virtually any type of dynamic equilibrium path is possible under indeterminacy.

Remark. We have found cases where the forward solution exists, $r_\sigma(\bar{\Psi}_{\Omega^*\otimes\Omega^*}) < 1$, $r_\sigma(\Psi_{F^*\otimes F^*}) > 1$ and $\tau_2^* \geq 1$. In this case, the model may or may not be determinate, but the model has a unique MSS solution that passes the NBC, which is the forward solution. Such cases can arise when for given j, $p_{ij}F_{ij}^* = 0_{n\times n}$ for some $i \neq j$. This implies that at least one state is absorbing, i.e., $p_{ii} = 1$ for some i if $F_{ij}^* \neq 0_{n\times n}$ for all i and j. All the examples we examine consider the cases $p_{ii} < 1$ for all i and as we show in the subsequent section, $\tau_2 = 1/[r_\sigma(\Psi_{F^*\otimes F^*})]$ holds. Nevertheless, it would be an important extension of our research to derive conditions under which $\tau_2 = 1/[r_\sigma(\Psi_{F^*\otimes F^*})]$.

Now we complete our classification of the models and the REEs by showing that all the remaining classes of the models have no relevant equilibria.

Case 3. The forward solution does not exist. All the MSV solutions to this class of the models fail to satisfy the NBC. Hence, the entire class of the solutions including the non-fundamental components are dismissed.

Remark We have shown that $r_\sigma(\Psi_{F^*}) \leq 1$ is necessary, not sufficient for determinacy. Suppose that $\Omega_k(s_t)$ converges.[12] We now emphasize that $r_\sigma(\Psi_{F^*}) \leq 1$ is an important sufficient condition for a model to satisfy the FCC. Alternatively, if a model fails to satisfy the FCC, then $r_\sigma(\Psi_{F^*}) > 1$. We can solve out the expectational term in (14b) and get a closed form expression for $\Gamma_k(s_t)$:

$$\begin{bmatrix} \Gamma_k(1) \\ \ldots \\ \Gamma_k(S) \end{bmatrix} = \begin{bmatrix} \Xi_{k-1}(1)^{-1}C(1) \\ \ldots \\ \Xi_{k-1}(S)^{-1}C(S) \end{bmatrix} + \Psi_{R' \otimes F_{k-1}} \begin{bmatrix} \Gamma_{k-1}(1) \\ \ldots \\ \Gamma_{k-1}(S) \end{bmatrix} \tag{32}$$

It can be shown that $r_\sigma(\Psi_{R' \otimes F^*}) = r_\sigma(\Psi_{F^*}) r_\sigma(R)$ as R is regime-independent. Therefore, if $r_\sigma(\Psi_{F^*}) \leq 1$, then $r_\sigma(\Psi_{R' \otimes F^*}) < 1$ as long as $r_\sigma(R) < 1$. Hence, $r_\sigma(\Psi_{R' \otimes F^*}) \leq 1$ becomes sufficient for the FCC. However, if $r_\sigma(\Psi_{F^*}) > 1$, there is a chance that a model fails to satisfy the FCC. Specifically, the condition for a model not to be forward-convergent is given by $r_\sigma(\Psi_{R' \otimes F^*}) > 1$. That is, if $r_\sigma(\Psi_{F^*}) > 1$ and $r_\sigma(R)$ is sufficiently high, then the FCC fails to hold. The Fisherian model in Section 4.2 is exactly of this kind and we will show a similar example under regime-switching in the following section.

Proposition 2 of Davig and Leeper (2007) states, the LRTP together with some positivity restrictions holds if and only if $r_\sigma(M) < 1$ where their M matrix is in their equation (10). In our framework, Ψ_{F^*} corresponds to their M matrix. In particular, when there is no predetermined variable and $A(s_t, s_{t+1}) = A(s_t)$, Ψ_{F^*} is identical to their M. In this sense $r_\sigma(\Psi_{F^*}) \leq 1$ may be interpreted as a generalized LRTP, although it is not clear what the relevant positivity restriction is, and the way they reformulate a MSRE model into what they call a linear model counterpart of the MSRE model is quite different from ours.[13]

[12] $\Omega_k(s_t)$ seems to converge in any case under determinacy, or indeterminacy, although there is no closed form expression for this condition.

[13] It should be stressed that they use a different stability criterion, boundedness. Thus, we cannot directly compare determinacy or the LRTP under bounded shock processes with ours under unbounded covariance-stationary processes. The point of discussion here is to show the role of the LRTP in our framework. Also, our condition slightly differs from the LRTP as it includes equality.

There is yet another case, which would not likely to arise for reasonable economic models.

Case 4. The forward solution exists and $r_\sigma(\bar{\Psi}_{\Omega^*\otimes\Omega^*}) \geq 1$. The model has no relevant equilibria because the forward solution is not MSS and other MSV solutions fail to satisfy the NBC. Even if there is a unique mean-square stable MSV solution and hence technically it is a determinate solution, it fails to satisfy the NBC. Cho and McCallum (2011) provide such an example in the linear RE models and argue that determinacy *per se* does not guarantee that the determinate equilibrium is economically relevant.[14]

5.5 Implementation of the Forward Method

We summarize our methodology as the following flowchart. The first step is to formulate a model in the form of (5) and (6), and construct sequences $\Omega_k(s_t)$, $F_k(s_t, s_{t+1})$ and $\Gamma_k(s_t)$ following Proposition 2.

1. If FCC holds, the forward solution (19) exists.

   (a) If $r_\sigma(\Psi_{\Omega^*\otimes\Omega^*}) < 1$, $r_\sigma(\Psi_{F^*\otimes F^*}) \leq 1$, The model is determinate and the unique MSS solution is the forward solution.

   (b) If $r_\sigma(\Psi_{\Omega^*\otimes\Omega^*}) < 1$, $r_\sigma(\Psi_{F^*\otimes F^*}) > 1$, then search for Λ that minimizes $\tau_2 = r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda})$.

      i. If $\tau_2^* = \min r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) < 1$, the model is indeterminate. The class of the relevant MSV solutions is given by the forward solution plus the non-fundamental component w_t, which can be constructed by Proposition 4.

      ii. If $\tau_2^* \geq 1$, the forward solution is the unique relevant MSS solution. The model may or may not be determinate technically, but all other MSV solutions violate the NBC.

   (c) If $r_\sigma(\Psi_{\Omega^*\otimes\Omega^*}) \geq 1$, the model has no MSS solution satisfying the NBC.

2. If the FCC fails, there is no relevant equilibria because any MSV solution violates the NBC.

[14]Cho and McCallum (2009) presents such a model in linear RE model framework: two completely independent univariate equations constitute a bivariate model, where one sector has two stable roots and the other has unstable roots. Then the determinate equilibrium is the one associated with the roots of the first equation, which has no economic sense. In our setup, it is the case $r_\sigma(\Omega^* \otimes \Omega^*) > 1$ and $r_\sigma(F^* \otimes F^*) > 1$, thus we rule out the determinate solution, different from the forward solution, as it violates the NBC.

6 Applications

We now examine the existence of the mean-square stable forward solution and determinacy for several examples including the two models introduced in Section 2. All examples belong to one of the first three cases: Determinacy, Indeterminacy and No FCC (The forward solution does not exist.). For all the models we examine, we find Λ such that $\tau_2^* = r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) = 1/[r_\sigma(\Psi_{F^*\otimes F^*})] < 1$. Therefore, the FCC, $r_\sigma(\bar{\Psi}_{\Omega^*\otimes\Omega^*}) < 1$ and $r_\sigma(\Psi_{F^*\otimes F^*}) > 1$ are the conditions for indeterminacy. We first consider the Fisherian model introduced in Section 4.1, followed by New-Keynesian models with regime-switching monetary policy and then models with regime-switching in the private sector. At the end, we compare our methodology with that proposed by Farmer et al. (forthcoming).

6.1 The Fisherian Model

We examine the inflation process given by the Fisher equation $i_t = E_t\pi_{t+1} + r_t$ in Section 4.1, together with $r_t = 0$ and regime-switching monetary policy rule $i_t = \alpha(s_t)\pi_t + z_t$. where $\alpha(s_t)$ represents the policy stance for two regimes $s_t = 1$ and 2. Substituting out the interest rate yields a equation for π_t:

$$\pi_t = a(s_t)E_t\pi_{t+1} - a(s_t)z_t,\ z_t = \rho z_{t-1} + \epsilon_t,$$

where $a(s_t) = 1/\alpha(s_t)$. Let $\alpha_i = \alpha(s_t = i)$ for $i = 1, 2$. Since there is no predetermined variable, $\Omega(s_t) = 0$, $F^*(s_t) = a(s_t)$ and the forward solution is given by $\pi_t = \Gamma^*(s_t)z_t$. Let $p_{11} = 0.8$, $p_{22} = 0.95$, $\rho = 0.95$ and $\alpha_2 = 1.5$. Table 1 shows the three cases.

Table 1: **Fisherian Model with Persistent Monetary Policy Shock**

Case	Determinacy	Indeterminacy	No FCC
$\alpha(s_t)$	$\alpha_1 = .95, \alpha_2 = 1.5$	$\alpha_1 = .9, \alpha_2 = 1.5$	$\alpha_1 = .8, \alpha_2 = 1.5$
$r_\sigma(\Psi_{F^*\otimes F^*})$	0.95	1.06	1.33
The MSV solution	$\Gamma^*(1) = -9.44$ $\Gamma^*(2) = -2.42$	$\Gamma^*(1) = -15.26$ $\Gamma^*(2) = -2.89$	$\Gamma(1) = 65.77$ $\Gamma(2) = 3.55$

When $\alpha_1 = 0.95$, the forward solution exists and the model is determinate by Proposition 7. Indeed, even when the policy is passive in regime 1, the probability of switching from the passive to active regime is sufficiently high, thus the model yields a unique MSS solution. If $\alpha_1 = 0.9$, however, $r_\sigma(\Psi_{F^* \otimes F^*}) = 1.06$ and there exists a continuum of Λ such that $1/1.06 \leq r_\sigma(\bar{\Psi}_{\Lambda \otimes \Lambda}) < 1$. Hence, the model becomes indeterminate and the class of indeterminate solutions is $x_t = \Gamma^*(s_t) z_t + w_t$ such that $w_t = a(s_t) E_t w_{t+1}$.[15] We also find that $r_\sigma(\Psi_{R' \otimes F^*}) = 0.91 < 1$ where $R = \rho$. These correspond to the indeterminate case ($a > 1$ and $0 < a\rho < 1$) for the fixed regime model in Section 4.1. In both cases $\Gamma^*(s_t) < 0$ for all regimes: inflation decreases following a contractionary monetary policy shock.

However, when $\alpha_1 = 0.8$, $r_\sigma(\Psi_{R' \otimes F^*}) = 1.02 > 1$ and therefore the FCC fails to hold: $\Gamma_k(s_t)$ explodes (goes to negative infinity). This again corresponds to the linear model that fails to satisfy the FCC ($a\rho > 1$). Nevertheless, the unique MSV solution still exists and can be obtained using the formula analogous to (32), which is shown in Table 1. In contrast to the determinacy/indeterminacy cases, $\Gamma(s_t)$ implies that inflation rises following a contractionary monetary policy. We rule out this solution, plus any potentially MSS non-fundamental process, by the NBC, just as we did for the linear model. Note that we have only to check forward convergence in order to rule out such a MSV solution without solving for this MSV solution and check the NBC.

Now we examine the model in section IV.1 of Farmer et al. (forthcoming). The model is composed of the same Fisher equation with an AR(1) process of the real interest rate and the policy rule $i_t = \alpha(s_t)\pi_t$. It can be written as

$$\alpha(s_t)\pi_t = E_t \pi_{t+1} + r_t, \quad r_t = \rho r_{t-1} + \epsilon_t$$

$\alpha(s_t)$ is equal to $\delta(s_t)$ in their model. We take their parameter values, $p_{11} = 0.8$, $p_{22} = 0.9$, $\rho = 0.9$. The forward solution is again given by $\pi_t = \Gamma^*(s_t) r_t$ under determinacy and indeterminacy, and inflation rises following a real rate shock as they should. We again consider three cases in Table 2.

The third case is the one considered by Farmer et al. (forthcoming). The policy stance is quite passive in both regimes and it is no wonder why the model fails to satisfy the FCC. Again the MSV solution does exist but it implies that inflation falls following a rise in real rate when the initial regime is 1, which is again counterintuitive. While $\Gamma(2)$ is positive, it is because the regime 2 is also passive. Indeed $\Gamma(2)$ becomes negative when

[15] While $F^*(s_t) = a(s_t)$ does not depend on s_{t+1}, the functional form of w_t is still given by (28) with $\Lambda(s_t, s_{t+1})$.

Table 2: **Fisherian Model with Persistent Real Interest Rate**

Case	Determinacy	Indeterminacy	No FCC
$\alpha(s_t)$	$\alpha_1 = .95, \alpha_2 = 1.5$	$\alpha_1 = .9, \alpha_2 = 1.5$	$\alpha_1 = .5, \alpha_2 = .8$
$r_\sigma(\Psi_{F^* \otimes F^*})$	0.91	1.06	3.27
The MSV solution	$\Gamma^*(1) = 6.11$ $\Gamma^*(2) = 2.25$	$\Gamma^*(1) = 8.01$ $\Gamma^*(2) = 2.50$	$\Gamma(1) = -12.14$ $\Gamma(2) = 9.29$

$\alpha(2) = 2$. In either case, the FCC does not hold and this MSV solution fails to satisfy the NBC.

6.2 Regime-Switching Monetary Policy

We apply our methodology to a standard New-Keynesian framework and show that the main results are virtually the same as those of the Fisherian model with a persistent monetary policy shock. The Markov-switching monetary policy model, (1) can be cast into the form of (5) with $x_t = [\pi \ y_t \ i_t]'$ and $z_t = [z_{S,t} \ z_{D,t} \ z_{MP,t}]'$ and and $\epsilon_t = [\epsilon_t^S \ \epsilon_t^D \ \epsilon_t^{MP}]'$.

$$B_1(s_t)x_t = A_1(s_t)E_t[x_{t+1}] + B_2 x_{t-1} + z_t, \ \ z_t = Rz_{t-1} + \epsilon_t$$

where

$$B_1(s_t) = \begin{bmatrix} 1 & -\kappa & 0 \\ 0 & 1 & 1/\sigma \\ -(1-\rho)\phi_\pi(s_t) & 0 & 1 \end{bmatrix}, \ A_1(s_t) = \begin{bmatrix} \beta & 0 & 0 \\ 1/\sigma & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix}.$$

The matrix R is a diagonal matrix with the diagonal elements, ρ_S, ρ_D and ρ_M. Then the model is given by

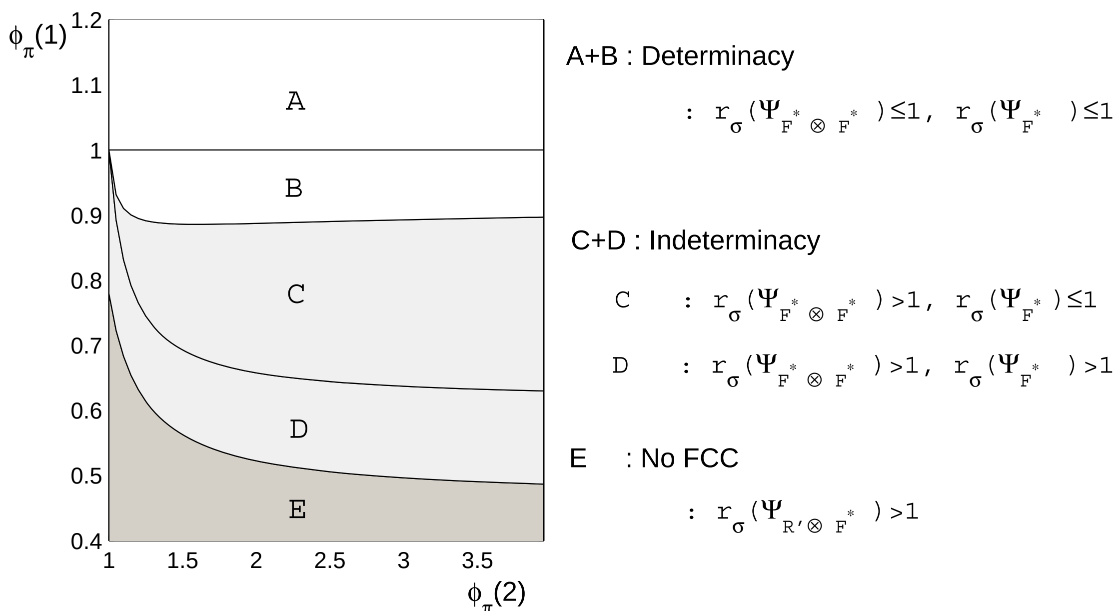
$$x_t = A(s_t)E_t[x_{t+1}] + B(s_t)x_{t-1} + C(s_t)z_t,$$

where $A(s_t) = B_1(s_t)^{-1}A_1(s_t)$, $B(s_t) = B_1(s_t)^{-1}B_2$ and $C(s_t) = B_1(s_t)^{-1}$.

6.2.1 Regime-Switching Monetary Policy without Predetermined Variables

We first consider the case without a predetermined variable ($\rho = 0$). For simplicity, we set $\rho_S = \rho_D = 0$ but $\rho_{MP} = 0.95$. The other parameter values are similar to those of Farmer et al. (2009), $\beta = 0.99$, $\kappa = 0.132$, $\sigma = 1$, $p_{11} = 0.85$ and $p_{22} = 0.95$. For a range of $\phi_\pi(2) \in [1, 4]$, we compute the value of $\phi_\pi(1)$ such that $r_\sigma(\Psi_{F^* \otimes F^*}) = 1$, $r_\sigma(\Psi_{F^*}) = 1$ and $r_\sigma(\Psi_{R' \otimes F^*}) = 1$. These three curves are plotted in Figure 1.

Figure 1: Determinacy Regions for Markov-Switching Monetary Policy Model Without Predetermined Variables



The area A and B represents the determinacy area as F^* exists and $r_\sigma(\Psi_{F^* \otimes F^*}) \leq 1$. The region C and D represents indeterminacy as $r_\sigma(\Psi_{F^* \otimes F^*}) > 1$. Region C represents the area $r_\sigma(\Psi_{F^*}) \leq 1 < r_\sigma(\Psi_{F^* \otimes F^*})$. Therefore, the region with $r_\sigma(\Psi_{F^*}) \leq 1$ (A, B and C) analogous to the LRTP is not quite sharp enough to ensure determinacy under MSS. In region D, $r_\sigma(\Psi_{F^*}) > 1$ but $r_\sigma(\Psi_{R' \otimes F^*}) \leq 1$. Hence $\Gamma^*(s_t)$ still exists and the model is indeterminate. The region E is the area where the model fails to satisfy the FCC as $r_\sigma(\Psi_{R' \otimes F^*}) \geq 1$.

The forward solution under determinacy and indeterminacy is given by $x_t = \Gamma^*(s_t) z_t$. For the case of No FCC, the MSV solution $x_t = \Gamma(s_t) z_t$ again exists and it can be obtained using (32) without k, but exhibits a counterintuitive dynamics. The following table shows the forward solution $\Gamma^*(s_t)$ when $\phi_\pi(1) = 0.9$ and $\phi_\pi(2) = 1.5$ under determinacy and the MSV solution in region E when $\phi_\pi(1) = 0.5$ and $\phi_\pi(2) = 1.5$. (the forward solution under indeterminacy with $\phi_\pi(1) = 0.8$ for example, is qualitatively similar to the determinate solution and thus not reported to save space.)

$$\begin{array}{c}\text{Forward Solution}\\ \text{(Determinacy)}\end{array} : \Gamma^*(1) = \begin{bmatrix} 0.89 & 0.12 & -4.85 \\ -0.80 & 0.89 & -4.80 \\ 0.80 & 0.11 & -3.37 \end{bmatrix}, \Gamma^*(2) = \begin{bmatrix} 0.84 & 0.11 & -2.41 \\ -1.25 & 0.84 & -0.21 \\ 1.25 & 0.17 & -2.61 \end{bmatrix}$$

$$\begin{array}{c}\text{MSV Solution}\\ \text{(No FCC)}\end{array} : \Gamma(1) = \begin{bmatrix} 0.94 & 0.11 & 26.14 \\ -0.47 & 0.94 & 35.26 \\ 0.47 & 0.06 & 14.07 \end{bmatrix}, \Gamma(2) = \begin{bmatrix} 0.84 & 0.11 & 4.18 \\ -1.25 & 0.84 & -5.94 \\ 1.25 & 0.17 & 7.27 \end{bmatrix}$$

The third column of Γ^* for each regime represents the initial responses of inflation and the output gap and the interest rate to a contractionary monetary policy shock of size 1. Regardless of the initial regime, a contractionary monetary policy lowers inflation and the interest rate under determinacy. However, when the model fails to satisfy the FCC, the MSV solution implies that inflation and the interest rate *rise* following the same shock. We rule out this solution as it fails to satisfy the NBC.

6.2.2 Regime-Switching Monetary Policy with Predetermined Variables

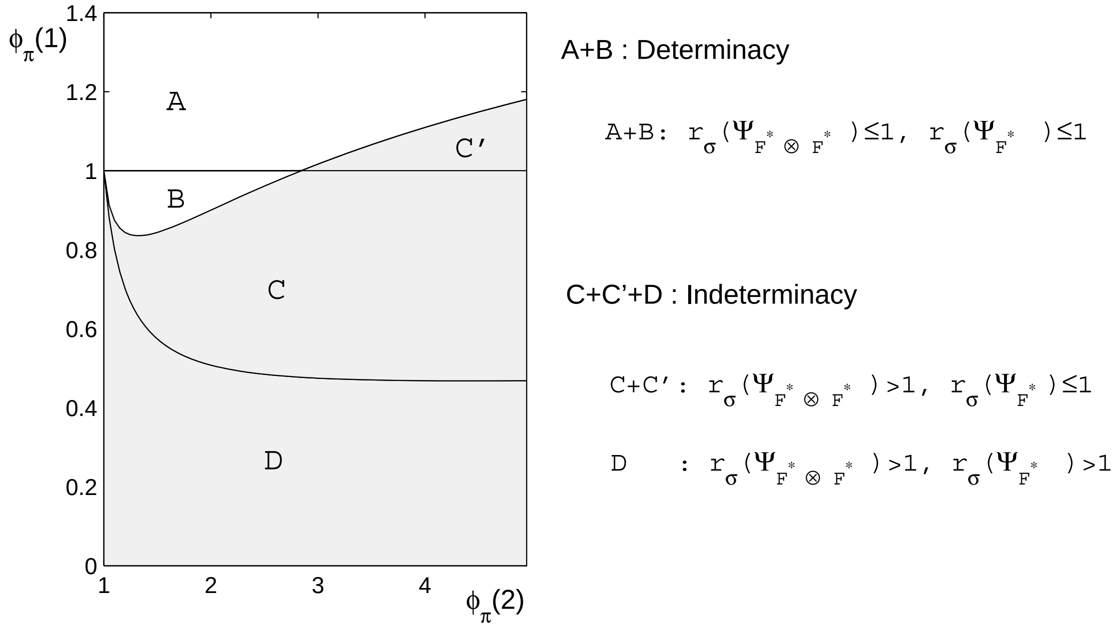
We re-examine the same New-Keynesian model with interest rate smoothing. Assume that all three shocks in z_t are white noises and take the parameter values $p_{11} = 0.85$, $p_{22} = 0.95$, $\rho = 0.95$, $\beta = 0.99$. $\kappa = 0.132$, and $\sigma = 1$. Figure 2 depicts the determinacy and indeterminacy regions in terms of the parameter values of $\phi_\pi(1)$ and $\phi_\pi(2)$. Since there is no persistent shock process, the forward solution exists for all regions.

The first thing to note is that the determinacy for the regime-switching model is neither necessary nor sufficient for determinacy for the fixed regime counterpart: on the one hand, there does exist a determinacy region B similar to the example above. For instance, when $\phi_\pi(1) = 0.9$ and $\phi_\pi(2) = 1.5$, we have $r_\sigma(\bar{\Psi}_{\Omega^*\otimes\Omega^*}) = 0.458 < 1$ and $r_\sigma(\Psi_{F^*\otimes F^*}) = 0.99 < 1$. In this case, the determinate forward solution is given by:

$$\begin{bmatrix} \pi_t \\ y_t \\ i_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & -5.42 \\ 0 & 0 & -13.56 \\ 0 & 0 & 0.71 \end{bmatrix} \begin{bmatrix} \pi_{t-1} \\ y_{t-1} \\ i_{t-1} \end{bmatrix} + \begin{bmatrix} 0.74 & 0.10 & -5.70 \\ -0.64 & 0.92 & -14.27 \\ 0.04 & 0.00 & 0.74 \end{bmatrix} \begin{bmatrix} z_{S,t} \\ z_{D,t} \\ z_{MP,t} \end{bmatrix}, \text{ if } s_t = 1,$$

$$\begin{bmatrix} \pi_t \\ y_t \\ i_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3.89 \\ 0 & 0 & -9.91 \\ 0 & 0 & 0.66 \end{bmatrix} \begin{bmatrix} \pi_{t-1} \\ y_{t-1} \\ i_{t-1} \end{bmatrix} + \begin{bmatrix} 0.69 & 0.09 & -4.10 \\ -0.78 & 0.90 & -10.43 \\ 0.05 & 0.01 & 0.69 \end{bmatrix} \begin{bmatrix} z_{S,t} \\ z_{D,t} \\ z_{MP,t} \end{bmatrix}, \text{ if } s_t = 2.$$

Figure 2: Determinacy Regions for Markov-switching Monetary Policy Model with Predetermined Variables



This implies that a contractionary monetary policy shock stabilizes inflation and the output gap in both regimes. On the other hand, there is a region C' where both regimes are active but the economy is indeterminate under regime switching: for a given weakly *active* monetary policy stance at regime 1, the economy can enter this indeterminacy region if the policy becomes more and more active at regime 2. For instance, when $\phi_\pi(1)$ is 1.05 in the less active regime and $\phi_\pi(2) = 3.5$ in the more active regime, $r_\sigma(\bar{\Psi}_{\Omega^*\otimes\Omega^*}) = 0.464$ but $r_\sigma(\Psi_{F^*\otimes F^*}) = 1.003 > 1$, and hence the model becomes indeterminate. The presence of the predetermined interest rate is responsible for generating the parameter region C'. Note that in this model $F^*(s_t) = (I_n - A(s_t)E_t[\Omega^*(s_{t+1})])^{-1}A(s_t)$ from equation (15) and (16). A large swing of the policy stance between less and more active regimes injects additional volatility into the model through a fluctuation in $\Omega^*(s_{t+1})$, which depends on the parameter ρ. This fluctuation of $\Omega^*(s_{t+1})$ in turn induces $F^*(1)$ to have a root greater than 1, which can never happen under each fixed regime, and makes $r_\sigma(\Psi_{F^*\otimes F^*}) > 1$ ultimately. While indeterminate, the forward solution is qualitatively similar to the one under determinacy and the general solution is given by $x_t = \Omega^*(s_t)x_{t-1} + \Gamma^*(s_t)z_t + w_t$

where w_t is an arbitrary MSS non-fundamental component of the form (28) subject to (29). This indeterminate solution is much more volatile than the forward solution as there are continuum of Λ such that $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda})$ is arbitrarily close to 1. An immediate implication is that in order to ensure a determinate equilibrium under regime-switching, one regime should not be too active relative to the other, even if the policy is active in both regimes.

In this model, since the structural shocks are assumed to be *i.i.d*, the whole region D is indeterminate. If a persistent shock is present, however, there will be a region E analogous to the example above and the model may still fail to satisfy the FCC. For example, with $\phi_\pi(1) = 0.25$, $\phi_\pi(2) = 1.25$ and $\rho_D = 0.98$, $\Omega_k(s_t)$ converges to $\Omega^*(s_t)$ but $\Gamma_k(s_t)$ fails to converge as $r_\sigma(\Psi_{R'\otimes F^*}) = 1.002$, the case we refine away any MSV solution.

6.3 A Model of Markov-Switching Elasticity of Intertemporal Substitution

In this subsection, we analyze Markov-switching elasticity of intertemporal substitution in a standard New-Keynesian framework, (4) in section 2. The model can be written in matrix form:

$$\begin{aligned} B_1(s_t)x_t &= E_t[A_1(s_t, s_{t+1})x_{t+1}] + B_2 x_{t-1} + C_1(s_t)z_t, \\ z_t &= Rz_{t-1} + \epsilon_t, \qquad E_{t-1}(\epsilon_t) = 0_{n\times 1}, \end{aligned}$$

where the coefficient matrices are defined as:

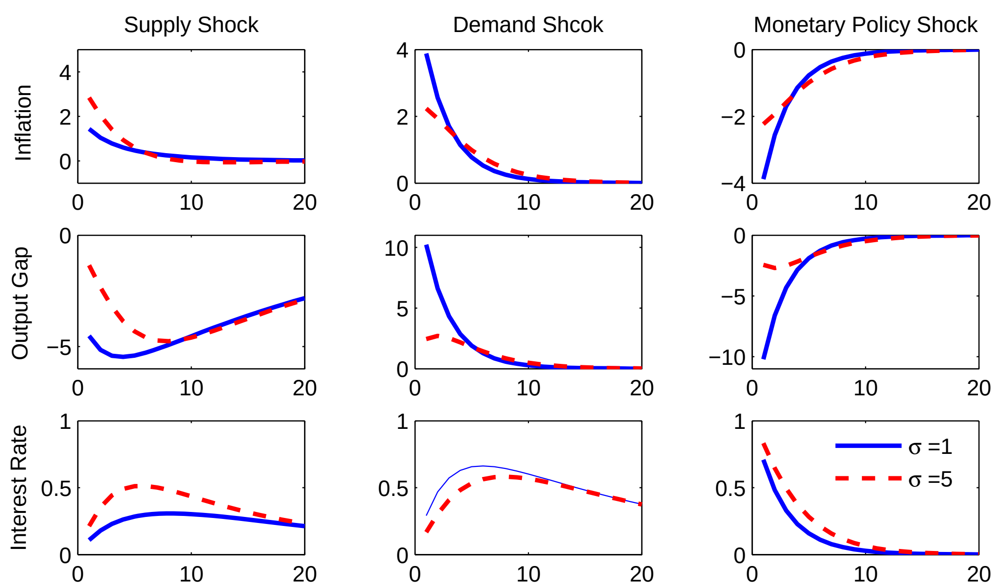
$$\begin{aligned} B_1(s_t) &= \begin{bmatrix} 1 & -\kappa & 0 \\ 0 & 1 & \frac{1}{\sigma(s_t)} \\ -(1-\rho)\phi_\pi & -(1-\rho)\phi_y & 1 \end{bmatrix}, \; A_1(s_t, s_{t+1}) = \begin{bmatrix} \beta & 0 & 0 \\ \frac{1}{\sigma(s_t)} & \frac{\sigma(s_{t+1})}{\sigma(s_t)} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix}, \; C_1(s_t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sigma(s_t)} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; R = \begin{bmatrix} \rho_S & 0 & 0 \\ 0 & \rho_D & 0 \\ 0 & 0 & \rho_{MP} \end{bmatrix}. \end{aligned}$$

The model can be written in the form of (5) with $A(s_t, s_{t+1}) = B_1^{-1}(s_t)A_1(s_t, s_{t+1})$, $B(s_t) = B_1^{-1}(s_t)B_2$ and $C(s_t) = B_1^{-1}(s_t)C_1(s_t)$. The forward solution will have the following form if it exists:

$$x_t = \Omega^*(s_t)x_{t-1} + \Gamma^*(s_t)z_t.$$

As a numerical example, consider the parameter values $\sigma(1) = 1$, $\sigma(2) = 5$. It is not very unreasonable for agents to be highly risk-averse for some periods like a severe recession. In the asset pricing literature, even a higher value of σ is often required to account for the so-called equity premium puzzle. We specify $p_{11} = 0.95$ and $p_{22} = 0.875$. This means that the average durations of the regimes of high and low elasticity of intertemporal substitution are 5 years and 2 year, respectively. The remaining parameter values are specified as $\beta = 0.99$, $\kappa = 0.132$, $\phi_\pi = 1.5$, $\phi_y = 0$, $\rho_D = \rho_S = 0.95$, $\rho_{MP} = 0$ and $\rho = 0.95$. Since the policy stances against expected inflation are active and the same in both regimes, the model would be determinate. Indeed, we confirm that this model is determinate as the forward solution exists and $r_\sigma(\bar{\Psi}_{\Omega^* \otimes \Omega^*}) = 0.561$ and $r_\sigma(\Psi_{F^* \otimes F^*}) = 0.952$. Figure 3 shows the impulse response functions to each shock of size 1 starting at different regimes.

Figure 3: Impulse Responses under Regime-switching Risk Aversion in a New-Keynesian Model



The regime-switching elasticity of intertemporal substitution leads to quantitatively sizable differences in the responses of all variables following any shock. When σ is high in the initial regime, the initial responses of all variables to all shocks are much smaller than those under low risk aversion. σ is also an important parameter governing the monetary policy transmission channel. When σ is high, the stabilizing effect of monetary policy is quite small. This implies that monetary policy may need to take into account the present regime and the possibility of regime shifts in the private sector in its optimal policy design. The responses under different initial regimes converge to each other, and the rate of convergence becomes faster as the probability of regime shift becomes higher.

6.4 Examples in the Literature

We finally compare our methodology to that of Farmer et al. (forthcoming). They provide a numerical procedure of solving the MSV solutions, which is indeed an important contribution to the literature. In addition, they suggest a likelihood-based solution selection criterion when multiple mean-square stable MSV solutions are found. Our methodology differs in several aspects. First of all, our methodology provides very tractable determinacy, indeterminacy conditions. They provide four examples similar to our New-keynesian models here. Our methodology enables us to find that their first two examples based on the monetary policy switching over a passive and active stances are indeterminate and the last two based on regime-switching in private sector or shock variance are determinate. Indeed, they find two MSS fundamental solutions for the first two examples, and thus they are indeterminate as well, but they did not clearly show whether the last two examples are determinate.

Second, our solution selection criterion can identify the economic relevance of any MSV solution, whereas the likelihood-based criterion is a relative measure. For instance, when a model has no predetermined solution, thus has only one MSV solution, the likelihood approach does not tell whether the unique solution is itself economically sensible as there is no other solution to compare the likelihood with. As such, they did not apply the criterion to the Fisherian model with the persistent real interest rate. In contrast, recall that the model fails to satisfy the FCC and therefore we rule out the MSV solution by the NBC.

Third, when there are multiple stationary MSV solutions, they show that the likelihood-based criterion may well select a solution with a larger $r_\sigma(\bar{\Psi}_{\Omega\otimes\Omega})$, other than the MOD solution using the Fisherian model with a predetermined variable using simulated data.[16] Since the linear models are nested in the class of MSRE models, their criterion may not pick up the MOD solution for linear models as well. This is in stark contrast to the

[16] It is not clear which solution they select between the two solutions for the first two examples of their New-Keynesian model because they did not perform simulation exercise and compare the likelihood.

convention that the MSV solution with the smallest eigenvalues, also known as the MOD solution, is chosen to almost all linear RE models. In fact, most of existing selection criteria such as the MSV criterion of McCallum (1983), E-stability criterion of Evans and Honkapohja (2001) pick up the MOD solution almost surely. Our NBC criterion carries over this principle to MSRE models: we verified that the MSV solution with the smallest $r_\sigma(\bar{\Psi}_{\Omega\otimes\Omega})$ for all of their indeterminate examples coincides with the forward solution. Hence, according to our NBC criterion, their solutions different from the forward solution violate the NBC.

7 Conclusion

The main results of this paper can be summarized as follows. First, our methodology provides very tractable determinacy conditions for the MSRE models in mean-square stability sense. Second, it is accompanied by the solution refinement criterion as well as the solution method. Third, it characterizes the complete set of relevant REEs to MSRE models under determinacy and indeterminacy. Our methodology is straightforward to implement. One only has to examine the existence of the forward solution and the conditions for determinacy and indeterminacy under mean-square stability.

It is surprising that until very recently, Markov-switching has not been actively applied to the dynamic stochastic general equilibrium models under rational expectations, the workhorse of modern macroeconomics. Davig and Leeper (2007) show a novel and important perspective of monetary policy in this framework. Regime-switching monetary policy against regime-independent private sectors is not however, optimal by construction. Davig (2007) and Farmer et al. (forthcoming) witness the importance of potential regime-switching behavior of private agents. In particular, Davig (2007) analyzes the optimal reaction of the central bank in such environments.

In sum, MSRE models provide a flexible and promising way to theoretically model the optimal behavior of economic agents and the central bank facing structural parameter uncertainty, and to examine those models empirically. We believe that our work provides a technical foundation for future research in the field of MSRE models.

Appendix

A Proof of Proposition 2

The MSRE model is given by:

$$x_t = E_t[A(s_t, s_{t+1})x_{t+1}] + B(s_t)x_{t-1} + C(s_t)z_t.$$

Let $M_1(s_t, s_{t+1}) = A(s_t, s_{t+1})$, $\Omega_1(s_t) = B(s_t)$ and $\Gamma_1(s_t) = C(s_t)$. Suppose that there exists a set of sequences of matrices $\{M_{k-1}(s_t, s_{t+1}, ..., s_{t+k-1}),\ \Omega_{k-1}(s_t),\ \Gamma_{k-1}(s_t)\}$ for $k > 1$ such that:

$$x_t = E_t[M_{k-1}(s_t, s_{t+1}, ..., s_{t+k-1})x_{t+k-1}] + \Omega_{k-1}(s_t)x_{t-1} + \Gamma_{k-1}(s_t)z_t.$$

Shift this equation forward one period and pre-multiply $A(s_t, s_{t+1})$ to both sides. Then, by taking conditional expectations of $A(s_t, s_{t+1})x_{t+1}$, we have:

$$\begin{aligned} E_t[A(s_t, s_{t+1})x_{t+1}] &= E_t[A(s_t, s_{t+1})M_{k-1}(s_{t+1}, s_{t+2}, ..., s_{t+k})x_{t+k}] \\ &\quad + E_t[A(s_t, s_{t+1})\Omega_{k-1}(s_{t+1})]x_t + E_t[A(s_t, s_{t+1})\Gamma_{k-1}(s_{t+1})]Rz_t, \end{aligned}$$

by the law of iterative expectations. Substitute this into the model and rearrange it to yield:

$$\begin{aligned} x_t &= E_t[F_{k-1}(s_t, s_{t+1})M_{k-1}(s_{t+1}, s_{t+2}, ..., s_{t+k})x_{t+k}] \\ &\quad + \Xi_{k-1}(s_t)^{-1}B(s_t)x_{t-1} \\ &\quad + (\Xi_{k-1}(s_t)^{-1}C(s_t) + E_t[F_{k-1}(s_t, s_{t+1})\Gamma_{k-1}(s_{t+1})]R)z_t, \end{aligned}$$

where $\Xi_{k-1}(s_t) = I - E_t[A(s_t, s_{t+1})\Omega_{k-1}(s_{t+1})]$ and $F_{k-1}(s_t, s_{t+1}) = \Xi_{k-1}(s_t)^{-1}A(s_t, s_{t+1})$, provided that $\Xi_{k-1}(s_t)$ is invertible for all s_t. Note that Ξ_{k-1} depends only on s_t as s_{t+1} is integrated out in the expectational term. Thus it can enter inside the expectations on the right-hand side. Therefore, there exists a sequence of matrices $M_k(s_t, s_{t+1}, ..., s_{t+k})$, $\Omega_k(s_t)$ and $\Gamma_k(s_t)$ such that:

$$x_t = E_t[M_k(s_t, s_{t+1}, ..., s_{t+k})x_{t+k}] + \Omega_k(s_t)x_{t-1} + \Gamma_k(s_t)z_t, \tag{33}$$

where,

$$\begin{aligned} M_k(s_t, s_{t+1}, ..., s_{t+k}) &= F_{k-1}(s_t, s_{t+1}) M_{k-1}(s_{t+1}, s_{t+2}, ..., s_{t+k}), \\ \Omega_k(s_t) &= \Xi_{k-1}(s_t)^{-1} B(s_t), \\ \Gamma_k(s_t) &= \Xi_{k-1}(s_t)^{-1} C(s_t) + E[F_{k-1}(s_t, s_{t+1}) \Gamma_{k-1}(s_{t+1})] R. \end{aligned}$$

If $\Xi_{k-1}(s_t)$ is invertible for all $k_t = 2, 3, ...$ and for all $s_t = 1, 2, ..., S$, the sequences, $\{M_k(s_t, s_{t+1}, ..., s_{t+k}),\ \Omega_k(s_t),\ \Gamma_k(s_t)\}$ are well-defined. Since the initial values of these sequences are real-valued and given by the model for all s_t, they are unique and real-valued if they exist. *Q.E.D.*

B Proof of Proposition 3

When the FCC does not hold, the forward solution does not exist by definition. Therefore, at least one of the elements of $(\Omega_k(s_t), \Gamma_k(s_t))$ in (13) is either not well-defined if the regularity condition is violated or does not converge when the regularity condition is met. Now suppose that there exists a MSV solution. If this satisfies the NBC, it is a contradiction to the supposition that it is a MSV solution.

Suppose now that the FCC holds. From Proposition 2, since $(\Omega_k(s_t), \Gamma_k(s_t))$ is unique and real-valued given the initial state s_t, the limiting values $(\Omega^*(s_t), \Gamma^*(s_t))$ are also unique and real-valued. Since $(\Omega^*(s_t), \Gamma^*(s_t))$ solves equation (8), the forward solution (19) is unique and a MSV solution to the model by Proposition 1 and therefore, it must solve the forward representation of the model (18) as k goes to infinity:

$$x_t = \lim_{k \to \infty} E_t[M_k(s_t, s_{t+1}, ..., s_{t+k}) x_{t+k}] + \Omega^*(s_t) x_{t-1} + \Gamma^*(s_t) z_t. \tag{35}$$

Therefore, it must be true that $\lim_{k\to\infty} E_t[M_k(s_t, s_{t+1}, ..., s_{t+k}) x_{t+k}] = 0_{n\times 1}$ when expectations are formed with the forward solution, implying that the forward solution satisfies the NBC.

Now suppose that the NBC holds for a MSV solution, different from the forward solution. Since the solution must solve (35), (35) becomes the forward solution under the NBC, which is a contradiction to the supposition that this solution differs from the forward solution. *Q.E.D.*

C Extended State Space

For ease of exposition, we reproduce equation (21) at time t:

$$y_t = G(s_{t-1}, s_t)y_{t-1} + H(s_t)\eta_t \tag{36}$$

Define an extended state $\hat{s}_t = (s_{t-1}, s_t)$ as:

$\hat{s}_t$	1	2	...	S	$S+1$	$S+2$	...	$2S$	...	$(S-1)S+1$	$(S-1)S+2$	...	S^2
s_{t-1}	1	1	...	1	2	2	...	2	...	S	S	...	S
s_t	1	2	...	S	1	2	...	S	...	1	2	...	S

(37)

The corresponding transition matrix is defined as $\hat{P} = (i_S \otimes I_S \otimes i'_S)diag(vec(P'))$ where i_S is an $S \times 1$ column vector of ones. For instance, when $S = 2$, $\hat{P}$ is given by:

$$\hat{P} = [\hat{p}_{ij}] = \begin{bmatrix} p_{11} & p_{12} & 0 & 0 \\ 0 & 0 & p_{21} & p_{22} \\ p_{11} & p_{12} & 0 & 0 \\ 0 & 0 & p_{21} & p_{22} \end{bmatrix}. \tag{38}$$

Now define $\hat{y}_{t+1} = y_t$, $\hat{G}(\hat{s}_t) = G(s_{t-1}, s_t)$, $\hat{H}(\hat{s}_t) = H(., s_t) = H(s_t)$. Then, (36) can be written as:

$$\hat{y}_{t+1} = G(\hat{s}_t)\hat{y}_t + H(\hat{s}_t)\eta_t \tag{39}$$

Therefore, we can represent equation (36) into a canonical form of (3.27) of Costa et al. (2005). It is straightforward to show that the nonzero eigenvalues of $\bar{\Psi}_{\hat{G}}$ ($\bar{\Psi}_{\hat{G}\otimes\hat{G}}$) and $\bar{\Psi}_G$ ($\bar{\Psi}_{G\otimes G}$) are equivalent, as they should because (39) and (36) are in fact identical. Therefore, we can safely apply important results in Costa et al. (2005) directly to the process (36).

D Proof of Proposition 5

The proof closely follows that of Theorem 1 of Farmer et al. (2009). Let V_i be n by k_i matrix where the columns are orthonormal, spanning the support of $w_t 1\{s_t = i\}$ for all t and $1\{s_t = i\} = 1$ when $s_t = i$ and 0 otherwise. Thus, for any $w_t = w$ and $s_t = i$, it must be true that $w = E[w_t|w_t = w, s_t = i] \in Col(V_i)$ and $E[w_{t+1}|w_t = w, s_t = i, s_{t+1} =$

$j] \in Col(V_j)$ almost surely. Moreover, $w = \sum_{j=1}^{S} p_{ij} F_{ij} E[w_{t+1}|w_t = w, s_t = i, s_{t+1} = j]$. Hence, there must exist a $k_j \times k_i$ matrix Φ_{ij} such that $\sum_{j=1}^{S} p_{ij} F_{ij} V_j \Phi_{ij} = V_i$. It should be noted that V_i can be the full matrix even when F_{ij} has a rank less than n. Since $w_{t+1} \in Col(V(s_{t+1}))$, $V(s_{t+1})V(s_{t+1})'\eta_{t+1} \in Col(V(s_{t+1}))$ almost surely. Finally, $E_t[F(s_t = i, s_{t+1})V(s_{t+1})V(s_{t+1})'\eta_{t+1}] = E_t[F(s_t = i, s_{t+1})w_{t+1} - F(s_t = i, s_{t+1})\Lambda(s_t, s_{t+1})w_t] = w_t - E_t[F(s_t = i, s_{t+1})V(s_{t+1})\Phi(s_t = i, s_{t+1})V_i']w_t = w_t - \sum_{j=1}^{S} p_{ij} F_{ij} V_j \Phi_{ij} V_i' w_t = w_t - V_i V_i' w_t = 0_{n\times 1}$ almost surely as $w_t \in Col(V_i)$.

E Proof of Lemma 3

Proof of Assertion 1. Consider two processes $w_{t+1} = \Lambda(s_t, s_{t+1})w_t$ and $u_{t+1} = F'(s_t, s_{t+1})u_t$ of the form (21). To prove Lemma, we use the extended regime variable $\hat{s}_t = (s_{t-1}, s_t)$ and the corresponding extended transition probability matrix $\hat{P}$ as in Appendix C so that both Λ and F' depend on the same regime variable $\hat{s}_t$. Then w_{t+1} and u_{t+1} can be written as $\hat{w}_{t+1} = \hat{\Lambda}(\hat{s}_{t+1})\hat{w}_t$ and $\hat{u}_{t+1} = \hat{F}'(\hat{s}_{t+1})\hat{u}_t$. Hereafter, variable or matrix with a hat denotes those associated with the extended regime variable $\hat{s}_t$ and the corresponding $\hat{P}$. Define $\bar{\Psi}_{\hat{F}'\otimes\hat{F}'} = [\hat{p}_{ji}\hat{F}'(\hat{s}_t = i) \otimes \hat{F}'(\hat{s}_t = i)]$ for $i, j = 1, 2, ..., S^2$. Then, it can be easily verified that $r_\sigma(\bar{\Psi}_{\hat{F}'\otimes\hat{F}'}) = r_\sigma(\bar{\Psi}_{F'\otimes F'})$, and thus, w_t is MSS if and only $\hat{w}_t$ is MSS. Similarly, we have $r_\sigma(\bar{\Psi}_{\hat{\Lambda}\otimes\hat{\Lambda}}) = r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda})$. Therefore, we need to prove that if $r_\sigma(\bar{\Psi}_{\hat{F}'\otimes\hat{F}'}) < 1$ and $r_\sigma(\bar{\Psi}_{\hat{\Lambda}\otimes\hat{\Lambda}}) < 1$, then $r_\sigma(\bar{\Psi}_{\hat{\Lambda}\otimes\hat{F}'}) < 1$.

Let $\hat{z}_t = \hat{u}_t + \hat{w}_t$. *From theorem 3.9 of Costa et al. (2005), $r_\sigma(\bar{\Psi}_{\hat{F}'\otimes\hat{F}'}) < 1$ if and only if $\sum_{t=0}^{\infty} E(||\hat{u}_t||^2) < \infty$ for all $\hat{u}_0$* and $\hat{s}_0$ *where $||\cdot||$ is a standard vector norm such that $||\hat{u}_t||^2 = trace(\hat{u}_t\hat{u}_t')$. Similarly, $r_\sigma(\bar{\Psi}_{\hat{\Lambda}\otimes\hat{\Lambda}}) < 1$ if and only if $\sum_{t=0}^{\infty} E(||\hat{w}_t||^2) < \infty$. Then, $0 \le ||\hat{z}_t||^2 = ||\hat{u}_t + \hat{w}_t||^2 \le 2(||\hat{u}_t||^2 + ||\hat{w}_t||^2)$. Hence,*

$$0 \le \sum_{t=0}^{\infty} E(||\hat{z}_t||^2) \le 2\left(\sum_{t=0}^{\infty} E(||\hat{u}_t||^2) + \sum_{t=0}^{\infty} E(||\hat{w}_t||^2)\right) < \infty.$$

This implies that $\hat{z}_t$ is also mean-square stable. Now $\hat{z}_t$ can be written as:

$$\hat{z}_t = \hat{u}_t + \hat{w}_t = F'(\hat{s}_t)\hat{u}_{t-1} + \hat{\Lambda}(\hat{s}_t)\hat{w}_{t-1}.$$

Define $Q^{\hat{u}}_{i,t} = E[\hat{u}_t\hat{u}_t' 1_{\{\hat{s}(t)=i\}}]$. $Q^{\hat{w}}_{i,t}$ and $Q^{\hat{z}}_{i,t}$ are defined analogously. Define $Q^{\hat{u}\hat{w}}_{i,t} =$

$E[\hat{u}_t\hat{w}_t' 1_{\{\hat{s}(t)=i\}}]$ and $Q^{\hat{w}\hat{u}}_{i,t} = E[\hat{w}_t\hat{u}_t' 1_{\{\hat{s}(t)=i\}}]$. Then,

$$\begin{aligned}
Q^{\hat{z}}_{i,t} &= Q^{\hat{u}}_{i,t} + Q^{\hat{w}}_{i,t} + Q^{\hat{u}\hat{w}}_{i,t} + Q^{\hat{w}\hat{u}}_{i,t} \\
&= \sum_{j=1}^{S^2} \hat{p}_{ji}\left(\hat{F}'(i)Q^{\hat{u}}_{j,t-1}\hat{F}(i) + \hat{\Lambda}(i)Q^{\hat{w}}_{j,t-1}\hat{\Lambda}'(i) + \hat{F}'(i)Q^{\hat{u}\hat{w}}_{j,t-1}\hat{\Lambda}'(i) + \hat{\Lambda}(i)Q^{\hat{w}\hat{u}}_{j,t-1}\hat{F}(i)\right). \\
v^{\hat{z}}_{i,t} &= v^{\hat{u}}_{i,t} + v^{\hat{w}}_{i,t} + v^{\hat{u}\hat{w}}_{i,t} + v^{\hat{w}\hat{u}}_{i,t} \\
&= \sum_{j=1}^{S^2} \hat{p}_{ji}\left(\hat{F}(i)' \otimes \hat{F}(i)' v^{\hat{u}}_{j,t-1} + \hat{\Lambda}(i) \otimes \hat{\Lambda}(i) v^{\hat{w}}_{j,t-1} + \hat{\Lambda}(i) \otimes \hat{F}(i)' v^{\hat{u}\hat{w}}_{j,t-1} + \hat{F}(i)' \otimes \hat{\Lambda}(i) v^{\hat{w}\hat{u}}_{j,t-1}\right)
\end{aligned}$$

where $v^{\hat{u}}_{i,t} = vec(Q^{\hat{u}}_{i,t})$ $v^{\hat{w}}_{i,t} = vec(Q^{\hat{w}}_{i,t})$and $v^{\hat{u}\hat{w}}_{i,t} = vec(Q^{\hat{u}\hat{w}}_{i,t})$. By stacking the vectors for all $\hat{s}_t = 1, ..., S^2$, we have the following:

$$v^{\hat{z}}_t = v^{\hat{u}}_t + v^{\hat{w}}_t + v^{\hat{u}\hat{w}}_t + v^{\hat{w}\hat{u}}_t = \bar{\Psi}_{\hat{F}'\otimes\hat{F}'} v^{\hat{u}}_{t-1} + \bar{\Psi}_{\hat{\Lambda}\otimes\hat{\Lambda}} v^{\hat{w}}_{t-1} + \bar{\Psi}_{\hat{F}'\otimes\hat{\Lambda}} v^{\hat{u}\hat{w}}_{t-1} + \bar{\Psi}_{\hat{\Lambda}\otimes\hat{F}'} v^{\hat{w}\hat{u}}_{t-1}.$$

Since $\hat{z}_t$ is MSS and homogenous, $v^{\hat{z}}_t$ must converge to the vector of zeros as t goes to infinity. Since $\hat{u}_t$ and $\hat{w}_t$ are also MSS, $v^{\hat{u}}_t$ and $v^{\hat{w}}_t$ converge to zeros as well because $r_\sigma(\hat{\Psi}_{\Omega\otimes\Omega}) < 1$ and $r_\sigma(\hat{\Psi}_{F'\otimes F'}) < 1$. This implies that $v^{\hat{u}\hat{w}}_{i,t} + v^{\hat{w}\hat{u}}_t$ must converge to zeros. Note that $v^{\hat{u}\hat{w}}_{i,t}$ is the vectorized version of a positive semi-definite matrices, $\sum_{i=1}^{S^2} Q^{\hat{u}\hat{w}}_{i,t}$. Same is true for $v^{\hat{w}\hat{u}}_t$. Therefore, for $v^{\hat{u}\hat{w}}_{i,t} + v^{\hat{w}\hat{u}}_t$ to converge to zeros, both vectors must converge to zeros individually, implying $r_\sigma(\bar{\Psi}_{\hat{F}'\otimes\hat{\Lambda}}) < 1$ and $r_\sigma(\bar{\Psi}_{\hat{\Lambda}\otimes\hat{F}'}) < 1$. (*In fact, the eigenvalues of $\bar{\Psi}_{\hat{F}'\otimes\hat{\Lambda}}$* are the same as those of $\bar{\Psi}_{\hat{\Lambda}\otimes\hat{F}'}$. *Finally, it is easy to see that $r_\sigma(\bar{\Psi}_{\hat{\Lambda}\otimes\hat{F}'}) = r_\sigma(\bar{\Psi}_{\Lambda\otimes F'})$.*

Proof of Assertion 2. Assertion 1 implies that if $r_\sigma(\bar{\Psi}_{F'\otimes F'}) < 1$ and $r_\sigma(\bar{\Psi}_{\Lambda\otimes F'}) \geq 1$, then $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) \geq 1$. Now, we need to prove that the first condition is replaced with equality. $r_\sigma(\bar{\Psi}_{F'\otimes F'}) = 1$. Suppose that $r_\sigma(\bar{\Psi}_{F'\otimes F'}) = 1$ and $r_\sigma(\bar{\Psi}_{\Lambda\otimes F'}) \geq 1$, but $\tau_2 = r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) < 1$. Then, consider two new processes, $\hat{u}_{t+1} = \sqrt{\alpha}\hat{F}'(\hat{s}_{t+1})\hat{u}_t$ and $\hat{w}_{t+1} = \left(\frac{1}{\sqrt{\alpha}}\right)\hat{\Lambda}(\hat{s}_{t+1})\hat{w}_t$ for any $\alpha \in (\tau_2, 1)$. Then, $r_\sigma(\bar{\Psi}_{\sqrt{\alpha}F'\otimes\sqrt{\alpha}F'}) = r_\sigma(\alpha\bar{\Psi}_{F'\otimes F'}) = \alpha r_\sigma(\bar{\Psi}_{F'\otimes F'}) < 1$. Likewise $r_\sigma\left(\bar{\Psi}_{\left(\frac{1}{\sqrt{\alpha}}\right)\Lambda\otimes\left(\frac{1}{\sqrt{\alpha}}\right)\Lambda}\right) = \left(\frac{1}{\alpha}\right) r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) = \left(\frac{\tau_2}{\alpha}\right) < 1$. But $r_\sigma\left(\bar{\Psi}_{\left(\frac{1}{\sqrt{\alpha}}\right)\Lambda\otimes\sqrt{\alpha}F'}\right) = r_\sigma(\bar{\Psi}_{\Lambda\otimes F'}) \geq 1$. This is a contradiction to Assertion 1. *Q.E.D.*

F Proof of Lemma 4

Proof of Assertion 1. Multiply V_i' to both sides of (29) to replace $V_j \Phi_{ij} V_i'$ with Λ_{ij}. Since $\Lambda_{ij} = V_j V_j' \Lambda_{ij}$, (29) can be written as:

$$\sum_{j=1}^{S} p_{ij} F_{ij} \Lambda_{ij} = \sum_{j=1}^{S} p_{ij} F_{ij} V_i V_i' \Lambda_{ij} = V_i V_i' \tag{40}$$

We first vectorize equation (29) for each i and stack them over $i = 1$ through S to yield:

$$\begin{bmatrix} p_{11}\Lambda_{11}' \otimes F_{11} & p_{12}\Lambda_{12}' \otimes F_{12} & ... & p_{1S}'\Lambda_{1S} \otimes F_{1S} \\ p_{21}\Lambda_{21}' \otimes F_{21} & p_{22}\Lambda_{22}' \otimes F_{22} & ... & p_{2S}\Lambda_{2S}' \otimes F_{2S} \\ ... & ... & ... & ... \\ p_{S1}\Lambda_{S1}' \otimes F_{S1} & p_{S2}\Lambda_{S2}' \otimes F_{S2} & ... & p_{SS}\Lambda_{SS}' \otimes F_{SS} \end{bmatrix} \begin{bmatrix} vec(V_1 V_1') \\ vec(V_2 V_2') \\ ... \\ vec(V_S V_S') \end{bmatrix} = \begin{bmatrix} vec(V_1 V_1') \\ vec(V_2 V_2') \\ ... \\ vec(V_S V_S') \end{bmatrix} \tag{41}$$

Note that the matrix of the left-hand side is $\Psi_{\Lambda' \otimes F}$ and the vector in this equation, if normalized, becomes an eigenvector. This implies that $\Psi_{\Lambda' \otimes F}$ has an eigenvalue 1 for any Λ subject to (40). Note that $\bar{\Psi}_{\Lambda \otimes F'} = \Psi_{\Lambda' \otimes F}$. Therefore, $r_\sigma(\bar{\Psi}_{\Lambda \otimes F'}) \geq 1$. *Q.E.D.*

Proof of Assertion 2. Equation (40) is invariant to multiplying an arbitrary positive scalar to F_{ij} and its reciprocal to Λ_{ij}.

$$\sum_{j=1}^{S} p_{ij} (\sqrt{\alpha} F_{ij}) \left(\frac{1}{\sqrt{\alpha}} \Lambda_{ij} \right) = V_i V_i' \tag{42}$$

Let $\tilde{F}_{ij} = \sqrt{\alpha} F_{ij}$ and $\tilde{\Lambda}_{ij} = \Lambda_{ij}/\sqrt{\alpha}$ for an arbitrary positive scalar α. From Assertion 1, for any pair of F and Λ subject to (40), $r_\sigma(\bar{\Psi}_{\Lambda \otimes F'}) \geq 1$. Hence, $r_\sigma(\bar{\Psi}_{\tilde{\Lambda} \otimes \tilde{F}'}) \geq 1$ for $\tilde{F}$ and $\tilde{\Lambda}$ as well. Moreover, $r_\sigma(\Psi_{\tilde{F} \otimes \tilde{F}}) = \alpha r_\sigma(\Psi_{F \otimes F})$, $r_\sigma(\bar{\Psi}_{\tilde{\Lambda} \otimes \tilde{\Lambda}}) = \left(\frac{1}{\alpha}\right) r_\sigma(\bar{\Psi}_{\Lambda \otimes \Lambda})$, and $r_\sigma(\bar{\Psi}_{\tilde{\Lambda} \otimes \tilde{F}'}) = r_\sigma(\bar{\Psi}_{\Lambda \otimes F'})$. α can be arbitrarily chosen such that $r_\sigma(\Psi_{\tilde{F} \otimes \tilde{F}}) < 1$, and for such $\tilde{F}$, $r_\sigma(\bar{\Psi}_{\tilde{\Lambda} \otimes \tilde{\Lambda}}) \geq 1$ from Assertion 2 of Lemma 3. Now suppose that there exists Λ subject to (40), $r_\sigma(\Psi_{F \otimes F}) = \xi_2$, $r_\sigma(\bar{\Psi}_{\Lambda \otimes \Lambda}) = \tau_2$ and $\xi_2 \tau_2 < 1$. Then by setting $\alpha = \sqrt{\tau_2/\xi_2}$, we have $\tilde{F}$ and $\tilde{\Lambda}$ such that $r_\sigma(\Psi_{\tilde{F} \otimes \tilde{F}}) = r_\sigma(\bar{\Psi}_{\tilde{\Lambda} \otimes \tilde{\Lambda}}) = \sqrt{\xi_2 \tau_2} < 1$. This implies that $r_\sigma(\bar{\Psi}_{\tilde{\Lambda} \otimes \tilde{F}'}) < 1$ from Assertion 1 of Lemma 3, which contradicts Assertion 1 here. Therefore, $\xi_2 \tau_2 \geq 1$ for all Λ subject to (40). *Q.E.D.*

G Finding Λ that minimizes $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda})$

Suppose that $r_\sigma(\Psi_{F\otimes F}) > 1$. The problem is to choose Λ to minimize the objective function $r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda})$. Since $\Lambda_{ij} = V_j \Phi_{ij} V_i'$, we choose a set of arbitrary $n \times k_i$ matrices V_i of which columns are orthonormal and a set of k_j by k_i arbitrary matrices Φ_{ij} for $0 \leq k_i \leq n$, $i, j = 1, ..., S$, subject to (29). One can start with $k_i = n$ for all i and if Λ is found such that $\tau_2 \equiv \min r_\sigma(\bar{\Psi}_{\Lambda\otimes\Lambda}) = 1/[r_\sigma(\Psi_{F\otimes F})]$, then the process w_t with this Λ is MSS as $\tau_2 < 1$. Note that the procedure starts with randomized initial values, thus each minimization problem will produce different Λ for multivariate models, but yield the same τ_2. As we showed, it is possible that $\tau_2 \geq 1 > 1/[r_\sigma(\Psi_{F\otimes F})]$, but this is not likely to arise for the models without absorbing states. One may find Matlab codes implementing this procedure at http://web.yonsei.ac.kr/sc719.

H Proof of Proposition 6

Suppose that the forward solution is MSS ($r_\sigma(\bar{\Psi}_{\Omega^*\otimes\Omega^*}) < 1$) and $r_\sigma(\Psi_{F^*\otimes F^*}) \leq 1$. Then the task is to show that there is no other mean-square stable MSV solution. Since the proof does not hinge on the presence of exogenous variables, we assume them away for simplicity. Any MSV solution must satisfy the forward representation of the model:

$$x_t = E_t[M_k(s_t, s_{t+1}, ..., s_{t+k})x_{t+k}] + \Omega_k(s_t)x_{t-1}. \tag{43}$$

at all $k = 1, 2, ...$ First, we show that $E_t[M_k(s_t, s_{t+1}, ..., s_{t+k})x_{t+k}]$ evaluated with a particular MSV solution $x_t = \Omega(s_t)x_{t-1}$ can be expressed as $L_k(s_t : \Omega)x_t$. For $k = 1$, $M_1(s_t, s_{t+1}) = A(s_t, s_{t+1}) = F_1(s_t, s_{t+1})$. Therefore,

$$E_t[M_1(s_t, s_{t+1})x_{t+1}] = L_1(s_t; \Omega)x_t,$$

where $L_1(s_t; \Omega) = E_t[F_1(s_t, s_{t+1})\Omega(s_{t+1})]$. For $k = 2$,

$$\begin{aligned} E_t M_2(s_t, s_{t+1}, s_{t+2})x_{t+2} &= E_t[F_1(s_t, s_{t+1})E_{t+1}[M_1(s_{t+1}, s_{t+2})x_{t+2}]] \\ &= E_t[F_1(s_t, s_{t+1})L_1(s_{t+1}; \Omega)x_{t+1}] \\ &= E_t[F_1(s_t, s_{t+1})L_1(s_{t+1}; \Omega)\Omega(s_{t+1})]x_t \\ &= L_2(s_t; \Omega)x_t. \end{aligned}$$

In this way, one can construct a sequence $L_k(s_t;\Omega)$ such that:

$$L_k(s_t;\Omega) = E_t[F_{k-1}(s_t, s_{t+1})L_{k-1}(s_{t+1};\Omega)\Omega(s_{t+1})]. \tag{44}$$

Therefore, when expectations are formed with a solution $x_t = \Omega(s_t)x_{t-1}$, the forward representation of the model (43) can be written as $(I - L_k(s_t;\Omega))x_t = \Omega_k(s_t)x_{t-1}$. Hence, it must be true that:

$$(I - L_k(s_t;\Omega))\Omega(s_t) = \Omega_k(s_t),$$

for any MSV solution. Since the model satisfies the FCC, $\lim_{k\to\infty}\Omega_k(s_t) = \Omega^*(s_t)$ and $\lim_{k\to\infty} F_k(s_t, s_{t+1}) = F^*(s_t, s_{t+1})$. Therefore, $L_k(s_t;\Omega)$ must converge to a solution-dependent matrix, $L(s_t;\Omega)$. From proposition 3, the forward solution must satisfy the NBC and all the other solutions violate it. Hence, when $\Omega = \Omega^*$, $L(s_t;\Omega^*) = 0_{n\times n}$. Now suppose that there exists another MSS solution $x_t = \tilde{\Omega}(s_t)x_{t-1}$ different from the forward solution. Then $L(s_t;\tilde{\Omega}) \neq 0_{n\times n}$.

Next, we express equation (44) as:

$$L_k(i;\Omega) = \sum_{j=1}^{S} p_{ij}F_{k-1}(i,j)L_{k-1}(j;\Omega)\Omega(j).$$

By vectorizing this equation and stacking them over $i = 1, 2, .., S$, we have the following:

$$\begin{bmatrix} vec(L_k(1;\Omega)) \\ ... \\ vec(L_k(S;\Omega)) \end{bmatrix} = \Psi_{\Omega'\otimes F_{k-1}} \begin{bmatrix} vec(L_{k-1}(1;\Omega)) \\ ... \\ vec(L_{k-1}(S;\Omega)) \end{bmatrix}.$$

For any MSV solution $x_t = \tilde{\Omega}(s_t)x_{t-1}$ with $\tilde{\Omega}(s_t) \neq \Omega^*(s_t)$, we showed that $\lim_{k\to\infty} L_k(s_t;\tilde{\Omega})) = L(s_t;\tilde{\Omega})) \neq 0_{n\times n}$. This implies the vector on both sides becomes an eigenvector if normalized and $r_\sigma(\Psi_{\tilde{\Omega}'\otimes F^*})$ has a root of 1. Therefore, $r_\sigma(\Psi_{\tilde{\Omega}'\otimes F^*}) \geq 1$. From the definition of Ψ and $\bar{\Psi}$, $\bar{\Psi}_A = (\Psi_{A'})'$. Hence, $r_\sigma(\Psi_{\tilde{\Omega}'\otimes F^*}) = r_\sigma(\bar{\Psi}_{\tilde{\Omega}\otimes(F^*)'})$. Therefore, if $r_\sigma(\bar{\Psi}_{\tilde{\Omega}\otimes(F^*)'}) \geq 1$ and $r_\sigma(\bar{\Psi}_{(F^*)'\otimes(F^*)'}) = r_\sigma(\Psi_{F^*\otimes F^*}) \leq 1$, then $r_\sigma(\bar{\Psi}_{\tilde{\Omega}\otimes\tilde{\Omega}}) \geq 1$ by Assertion 2 of Lemma 3. *Q.E.D.*

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