# Another Weakness of "Determinacy" as a Selection Criterion for Rational Expectations Models

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Preliminary

February 9, 2009

ABSTRACT: It is well-known that dynamic linear rational expectations (RE) models typically have multiple solutions and that much of the literature approaches resulting issues by establishing whether a solution is, or is not, "determinate," i.e., is the only solution that is dynamically stable. This paper argues against the use of determinacy as a guide, deemed necessary and sufficient, to interpretation of outcomes implied by a RE model. The argument proceeds by means of an example in which a determinate solution exists but differs sharply in dynamic behavior from that implied by the model's specification, when considered on a sector-by-sector basis.

JEL codes: C62, D84
Keywords: Determinacy, stability, rational expectations

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## 1. Introduction

It is well-known that dynamic linear rational expectations (RE) models often have multiple solutions, in the sense of expressions that generate endogenous variables as functions of predetermined and exogenous variables while satisfying all of the model's equations. It is also well-known that much of the literature, especially in monetary economics, approaches issues concerning such multiplicities by establishing whether a solution is, or is not, "determinate" in the sense of being the only solution that is dynamically stable. Often, cases featuring "indeterminacy," defined as the existence of more than one stable solution, are regarded as problematic and to be avoided (by means of policy) if possible.[1] On the other hand, several authors, including Bullard (2006), Cho and Moreno (2008), Evans and Honkapohja (2001), and McCallum (2003, 2007) have—implicitly, in some cases—questioned this practice on various grounds. For example, determinate solutions may not be learnable (Bullard (2006), Bullard and Mitra (2002)) whereas cases with indeterminacy may possess only one "plausible" solution (McCallum (2003, 2007)). In the present paper we present another argument against the use of determinacy as a guide, deemed necessary and sufficient, to interpretation of outcomes implied by a RE model.[2] This argument proceeds by means of an example in which a determinate solution, according to the usual determinacy criterion, exists but differs sharply in behavior from that implied by the model's specification, when carefully considered on a sector-by-sector basis.

[1] One could argue that use of the term "determinacy" to refer to a case in which a model has a single stable solution begs the issue (of whether this model possesses a unique equilibrium) by ruling out explosive solutions. This would be inappropriate, arguably, because transversality conditions necessary for individual optimality rule out many, but not all, explosive solutions. We shall use the term in this common manner, nevertheless, in the present paper.

[2] We limit our discussion to linear models but suggest that such a limitation is not inherent in the argument.

## 2. Example

For our basic example consider the following bivariate linear model for the determination of the variables $y_{1t}$ and $y_{2t}$:

(1) $$y_t = AE_t y_{t+1} + Cy_{t-1}$$

where $y_t = \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix}$ with $A = \begin{bmatrix} -1.5 & 0.0 \\ 0.5 & 0.3 \end{bmatrix}$ and $C = \begin{bmatrix} 0.4 & 0.0 \\ 0.0 & -1.4 \end{bmatrix}$.

We consider fundamental solutions of the form

(2) $$y_t = \Omega y_{t-1}.$$

This implies that $\Omega$ must satisfy the matrix quadratic

(3) $$A\Omega^2 - \Omega + C = 0\,,^3$$

which can be written in first-order form as

$$\begin{bmatrix} \Omega^2 \\ \Omega \end{bmatrix} = \begin{bmatrix} A^{-1} & -A^{-1}C \\ I & 0 \end{bmatrix} \begin{bmatrix} \Omega \\ I \end{bmatrix}.$$

Then the system's eigenvalues, i.e., the eigenvalues of $\begin{bmatrix} A^{-1} & -A^{-1}C \\ I & 0 \end{bmatrix}$, are as follows:

4.3951, –1.0618, –0.9480, and 0.2813.[4] Accordingly, with two stable eigenvalues and two predetermined variables, the system is determinate; there is exactly one dynamically stable solution (see, e.g., Blanchard and Kahn (1980)). Thus the usual "determinacy" criterion indicates that this bivariate system is nicely behaved and can in principle serve as a guide to actual economic behavior.

[3] If A and C are m×m matrices, in a m-variable setup, there are $(2m)!/(m!)^2$ distinct solutions provided by the different combinations of 2m eigenvalues taken m at a time.

[4] Similar results obtain in cases in which the counterpart of A is singular; see Section 4 below. Invertibility is assumed here only for expositional simplicity.

But inspection of the system specified above indicates that it is recursive. In particular, variable 1 (i.e., $y_{1t}$) is autonomous, with its behavior determined without reference to the generation or behavior of variable 2. By contrast, the behavior of the latter is dependent upon the path of variable 1. Specifically, $y_{2t}$ is generated by the equation

(4) $$y_{2t} = 0.3E_t y_{2t+1} - 1.4 y_{2t-1} + 0.5 E_t y_{1t+1}.$$

With $y_{1t}$ exogenous with respect to $y_{2t}$, the relevant eigenvalues for this univariate system for $y_{2t}$ are 4.3951 and −1.0618. So the behavior of variable 2 is explosive—it explodes away from the path implied by the behavior of variable 1.

What is the behavior of variable 1, considered in isolation? The roots of the relevant quadratic are found to be –0.9480, and 0.2813 (the same as the two smallest eigenvalues of the bivariate system). There are (at least) two distinct positions taken by different analysts as to the behavior of this variable. Analysis of the least-squares learnability of the two implied solutions indicates that the one with $\omega_{11}$ value 0.2813 is learnable whereas the other one is not.[5] This analysis, based on conditions reported by Evans and Honkapohja (2001, p. 238), is of the type promoted in McCallum (2007). Thus analysts such as Bullard (2006), McCallum (2007), and presumably Evans and Honkapohja (2001), would argue that $y_{1t}$ would behave according to

(5) $$y_{1t} = 0.2813 y_{1t-1}.$$

Supporters of the standard determinacy strategy would, by contrast, argue that the behavior of this variable is indeterminate; one cannot know what its process will be.

It should be stressed, however, that for present purposes it is not necessary to

[5] Here, and in what follows, $\omega_{ij}$ denotes the i,j th element of $\Omega$.

choose between these two positions. For the proponents of both would argue that, whatever the behavior of $y_{1t}$, the second variable $y_{2t}$ will be explosive. Accordingly, both positions conflict with the determinacy finding based on the bivariate system, namely, that both variables will have unique stable solutions and that these are the solutions that should be expected to prevail (i.e., are predicted by the model). Thus we see that consideration of the components of the bivariate model indicates that, if applied to the bivariate system (1), standard determinacy analysis produces a fundamentally misleading conclusion.[6]

It is of interest, in this context, to ask what would be predicted in the bivariate case at hand by other selection criteria, ones that differ from the determinacy criterion. First, the minimum-state-variable (MSV) criterion discussed by McCallum (2003) would call for a solution in which the eigenvalues of $\Omega$ would approach zeros if all elements of the C matrix were to approach zeros. As a consequence of eigenvalue continuity with respect to the elements of A and C it can be shown that in the case in hand this criterion yields the solution in which the eigenvalues of $\Omega$ are 0.2813 and –1.0618, thereby indicating the same overall explosive behavior as that suggested by consideration of the individual sectors. As a second alternative, we apply the forward-solution approach of Cho and Moreno (2008), which yields either a unique solution or none. In fact, the Cho and Moreno algorithm converges to a solution in which $\Omega$ is the same as that found by the MSV and individual-sector analyses, again disagreeing with the determinacy suggestion.

[6] It might be noted that if $a_{22}$ in the example were changed to ).45, the bivariate system would become indeterminate, the eigenvalues being 3.1957, –0.9735, –0.9480, and 0.2813. In this case, too, reliance on the bivariate system could be potentially misleading, for example, if the analyst were to adopt as relevant the solution that results from arrangement of the eigenvalues in order of decreasing modulus.

## 3. Contrasting Example

In the preceding example, the conclusions would apparently continue to prevail for other numerical values so long as there is an autonomous sector that has two stable solutions and a second with no stable solutions. A quite different conclusion regarding determinacy would be obtained, however, if the autonomous variable was explosive and the other variable had two stable solutions. We can specify such a case by changing $a_{21}$ in system (1) to zero and making $a_{12}$ nonzero. Let us then set $a_{12} = 0.5$ and $a_{21} = 0$. Now variable 2 is autonomous and explosive, with variable 1 (i.e., $y_{1t}$) dependent upon its explosive path. In this case the standard determinacy analysis for the bivariate system would conclude that there is no stable solution. That fact is demonstrated by finding that under this specification a necessary rank condition discussed by King and Watson (1998, p. 1022) and Klein (2000, p. 1413) is not satisfied.[7] Accordingly, application of Klein's solution algorithm results in the message "Rank condition not satisfied" and a halt to the calculations. If, however, the eigenvalues and eigenvectors are rearranged so that the two explosive solutions are ordered after the two stable solutions, one can obtain a solution that agrees with the univariate analyses.[8]

What, then, is the appropriate conclusion about how a system of the type under discussion would behave in practice? Analysis of Evans and Honkapohja (2001, pp. 219) suggests that just one of the two (explosive) univariate solutions for $y_{2t}$ is learnable. Furthermore, only one of the two (stable) solutions for $y_{1t}$ is learnable. Accordingly, the position of McCallum (2007) would be that the model at hand has one learnable solution;

[7] This condition is mentioned in footnote 12 of McCallum (2007), where it is stated that the system can be analyzed by "simpler methods." These methods are, in the case at hand, analysis of the two sectors separately.

[8] There are two such solutions, depending upon the order selected for the explosive and stable eigenvalues.

which features a unique explosive path for $y_{2t}$ and unique stable behavior for $y_{1t}$ relative to that (explosive) path. (This argument, it must be admitted, does not consider the possibility of non-fundamental sunspot solutions.) Since learnability is arguably a necessary condition for plausibility of a RE equilibrium, there is then some basis for the position that, despite the absence of dynamically stable solutions, the model under discussion does have a single well-specified candidate for equilibrium. If the implied path violates a transversality condition, this candidate will not represent a RE equilibrium, however, even though it implies a unique RE solution path. In any event, this case, like the one discussed in Section 2, is not favorable to the notion that the existence of a single stable solution—i.e., a situation of "determinacy"—is a satisfactory criterion for plausibility of a RE solution.

## 4. Extensions

We conclude with two brief extensions of the foregoing argument. First, if the A matrix in expression (1) is not invertible, the analysis differs only in that it writes the quadratic (3) as

$$\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Omega^2 \\ \Omega \end{bmatrix} = \begin{bmatrix} I & -C \\ I & 0 \end{bmatrix} \begin{bmatrix} \Omega \\ I \end{bmatrix}$$

and therefore involves, as system eigenvalues, the generalized eigenvalues of the matrix $\begin{bmatrix} I & -C \\ I & 0 \end{bmatrix}$ with respect to $\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$. In this case, there will be some "infinite" generalized eigenvalues, that is, ones defined as ratios in which the denominators are zeros.

Another obvious matter is whether the argument developed by means of the example in Section 2 can be generalized. For the two-variable case it seems clear that the result will pertain whenever the system is recursive, the autonomous variable has two stable solutions, and the non-autonomous variable has two explosive solutions. But what about extension to setups with more variables?

Suppose that each of the two sectors includes two variables. Then the counterpart of the basic example in Section 2 has sector 1 autonomous and with multiple stable solutions (either three or four stable eigenvalues) while sector 2 is dependent and explosive (either one or zero stable eigenvalues). With sector 1 autonomous, the system is recursive and the eigenvalues of the overall system are the same as the eigenvalues of the two sectors, as in Section 2. Then the overall system may have four stable eigenvalues and therefore be regarded as determinate. So the argument is the same as in Section 2. Furthermore, this argument can be extended to more variables per sector and also more than two sectors.

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