"Refining Linear Rational Expectations Models and Equilibria"

Preliminary and Incomplete

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ABSTRACT: This paper proposes forward convergence as a model refinement scheme for linear rational expectations models and no-bubble condition as a solution selection criterion. We relate these two concepts to determinacy and characterize the complete set of economically relevant Rational Expectations equilibria to the LRE models under determinacy and indeterminacy. Our result shows that (1) why the determinate equilibrium is economically meaningful mostly, and (2) those models that are not forward-convergent have no relevant equilibria.

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1. Introduction

Macroeconomic literature has utilized the concept of determinacy/indeterminacy as a primary guidance to characterize the economic property of a given rational expectations (RE) model and its equilibria. While some researchers argue that determinacy is necessary and sufficient for that model to be economically relevant, others also argue that multiple rational expectations equilibria (REEs) can be admissible to indeterminate models as well. However, determinacy alone does not automatically guarantee economic plausibility of a given model and its determinate equilibrium as argued by Bullard and Mitra (2002), Cho and McCallum (2009). Nor are the models and their REEs dismissed as implausible simply because they are indeterminate. Several solution selection criteria have been proposed to narrow down the set of relevant equilibria to indeterminate models, for instance, the MSV criterion of McCallum (1983) and E-stability criterion of Evans and Honkapohja (2001).

This kind of disagreement over determinacy may have been a consequence of lacking a step for refining "models" to begin with. We propose a model and solution refinement schemes for general linear models and characterize the complete set of economically relevant equilibria under determinacy and indeterminacy. To do so, we adopt the forward convergence and no-bubble conditions proposed by Cho and Moreno (2011), and link those refinement schemes to determinacy by an alternative characterization theorem proposed by McCallum (2008).

In their study of monetary reform, Flood and Garber (1980) introduce "process consistency," an essential characteristic of anything which pretends to serve as money. They argue that any "process inconsistent" money supply will be rejected by the public as it does not provide a finite solution for price in a Cagan-type hyperinflationary money market. Process consistency simply amounts to the case that a RE model can be solved forward and they argue that any reasonable model should pass this minimal economic characteristic. Their model is a proto-type model that can be solved forward recursively as it has no lagged variables. However, this kind of way to solve RE models forward has not been developed and applied to general models with lagged variables before Cho and Moreno (2011) provided the forward method for those models. Their forward convergence, which we propose as a model refinement, is a generalization of process consistency and we show that indeed, any model that fails to satisfy the forward convergence condition has no meaningful equilibria. Moreover, whenever a model is forward-convergent, their methodology provides a well-known forward (forward-looking) solution in Blanchard (1979)'s sense.

Second, when a model is solved forward, there remains a term involving expectation of future endogenous variables, often called "bubbles". Seeking for fundamental, minimum state variable (MSV) solutions in the sense of McCallum (1983), researchers often assume this expectational term to disappear. Blanchard and Kahn (1980) in fact impose a restriction to a given RE model that expectations of the endogenous and predetermined and non-predetermined variables do not explode , which *rules out "bubbles" of the sort considered by Flood and Garber (1980)*. While their "bubbles" are not exactly the same as the expectational term, we interpret their restriction of no-bubble as a requirement for the fundamental solutions to satisfy. Cho and Moreno (2011) show that no-bubble condition holds only for the forward solution and all other MSV – often called "bubble-free" – solutions fail to satisfy it. Hence no-bubble condition is a relevant solution refinement scheme.

Cho and Moreno (2008), however, did not relate their refinement schemes to determinacy. It is our task here to derive the relation following McCallum (2008) and characterize all the REEs to determinate and indeterminate models.

This paper is organized as follows. Section 2 presents a general class of linear Rational Expectations (RE) models and characterizes the set of REEs à la Lubik and Schorfheide (2004). In Section 3, necessary and sufficient conditions for determinacy are stated following McCallum (2008). In section 4, we formally define the concept of forward convergence and no-bubble conditions and study the relation between determinacy and forward convergence. Section 5 classifies the RE models with these two properties and characterizes the full set of REEs. In section 6, we apply our methodology to a standard New-Keynesian model. Section 7 concludes.

2. Linear Rational Expectations Models and the Rational Expectations Equilibria

2.1 The Model

(1) $$x_t = AE_t x_{t+1} + Bx_{t-1} + Cz_t, \quad z_t = Rz_{t-1} + e_t,$$

where x_t is an n×1 vector of endogenous variables and z_t is an m×1 vector of exogenous stationary variables. e_t is an m×1 vector of *i.i.d* shock processes. $E_t(\cdot)$ is mathematical expectation operator conditional on the information set available at time t. The linear model (1) we have in mind is that it is the local linear approximation of the underlying dynamic stochastic general equilibrium model around steady states. Hence, we assume that the steady state of $x^{ss} \equiv E[x_t]$ is well-defined and known to all the economic agents.

2.2 Classes of Solutions

Any process x_t, which is consistent with the model (1), is a solution to the model. We decompose it into two components following Lubik and Schorfheide (2004):

(2) $$x_t = x_t^{FUN} + w_t,$$

where x_t^{FUN} is a fundamental (also known as minimum state variables or MSV) solution and w_t is a non-fundamental component. Note that for any given x_t^{FUN}, there is the corresponding class of w_t. That is, the process of w_t is restricted by a particular x_t^{FUN}. We discuss two classes one by one in detail.

A. Fundamental Solutions

The fundamental solution has the following form.

(3) $$x_t = \Omega x_{t-1} + \Gamma z_t,$$

where (Ω, Γ) must satisfy the following conditions:

(4a) $$\Omega = (I - A\Omega)^{-1} B,$$

(4b) $$\Gamma = (I - A\Omega)^{-1} C + F\Gamma R,$$

where F is given by:

(5) $$F = (I - A\Omega)^{-1} A.$$

There are in general multiple solutions for Ω (thus Γ as well).[1] But the number of Ω, thus fundamental solutions is at most ${}_{2n}C_n$, hence finite.

B. Non-fundamental Solutions

The class of non-fundamental solutions has the following form:

(6) $$x_t = \Omega x_{t-1} + \Gamma z_t + w_t,$$

where w_t is an arbitrary process satisfying

(7) $$w_t = F E_t w_{t+1}.$$

It is important to note that F is restricted by a particular Ω. Given F, w_t has the following explicit functional form:

(7') $$w_t = \Lambda w_{t-1} + \eta_t,$$

where η_t is an appropriately defined $n \times 1$ vector of white noises.

While the class of non-fundamental components is large, Λ has a simple structure: non-zero eigenvalues of Λ are the inverses of some or all of non-zero eigenvalues of F. This implies that we have only to study F in order to deduce stability of w_t and determinacy of the model without solving for (7') directly. For a compact exposition of stability and determinacy, we define the spectral radius operator.

[1] When there is no predetermined variables (B=0), Ω=0 and F=A, implying that the fundamental solution is at most one if it exists.

Definition (Spectral Radius): $r(X) = \max_{\{1 \le i \le n\}} |\xi_i| : R^{n \times n} \to R^{+}$ *where* ξ_i *is the i-th eigenvalue of an n by n matrix X.*

Now we can state that a fundamental solution (3) is dynamically stable if $\mathrm{r}(\Omega) < 1$. There exist stationary processes w_t if $\mathrm{r(F)} > 1$ because Λ in equation (7') can always be constructed such that it contains an eigenvalue $1/\mathrm{r(F)} < 1$. Therefore, $\mathrm{r(F)} \le 1$ is the condition under which there is no stationary stochastic process of w_t.

3. Determinacy

A model is said to be determinate if the model has a unique stable RE solution. Different researchers use different representations of the underlying model, but they use essentially the same matrix decomposition theorem to derive the conditions for determinacy, e.g., Blanchard and Kahn (1980). But their method works only when A is non-singular. Thus, one has to reformulate a model into a canonical form of Blanchard and Kahn (1980) if it has a singular A, following the procedure proposed by King and Watson (1998) for instance. Instead, one can use a simpler way to identify determinacy without transformation of (1) following Klein (2000) or McCallum (2008), who utilize the generalized Schur decomposition theorem. To do so, define $\tilde{\mathrm{B}} = \begin{bmatrix} \mathrm{I} & \mathrm{-B} \\ \mathrm{I} & 0 \end{bmatrix}$ and $\tilde{\mathrm{A}} = \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{I} \end{bmatrix}$.

Solving for real-valued Ω amounts to choosing n roots out of the 2n generalized eigenvalues of the matrix pencil $\lambda(\tilde{\mathrm{B}}, \tilde{\mathrm{A}}) = \{\lambda_1, \lambda_2, \ldots, \lambda_{2n}\}$ where $|\lambda_1| \le \ldots \le |\lambda_{2n}|$.[2]

[2] If a complex root is included, the conjugate member should also be included for Ω to be real-valued. Equality holds when two eigenvalues are a complex conjugate.

Following McCallum (2008), determinacy can be stated in terms of the matrices Ω and F in equation (3) and (5), which govern the stability of fundamental and non-fundamental components of the REEs, respectively. His definition enables us to relate determinacy and the forward convergence.

To proceed, we introduce an important property regarding the eigenvalues of Ω and F: for any Ω associated with n eigenvalues in $\lambda(\tilde{B}, \tilde{A})$, the eigenvalues of the corresponding F are the inverses of the *remaining* eigenvalues in $\lambda(\tilde{B}, \tilde{A})$. For instance, let Ω^{MOD} be the solution associated with n smallest eigenvalues, $(\lambda_1, \lambda_2, ..., \lambda_n)$. Then the eigenvalues of $F^{MOD} = (1/\lambda_{n+1}, 1/\lambda_{n+2}, ..., 1/\lambda_{2n})$. This implies that if a model is determinate, the determinate solution must be the MOD solution. Following this idea of McCallum (2008), determinacy (indeterminacy) conditions can be stated in the following way.

Proposition 1. *Linear RE models of the form (1) can be classified as follows.*

1. The model (1) is determinate if and only if $r(\Omega^{MOD}) < 1$ *and* $r(F^{MOD}) \leq 1$.

2. The model (1) is indeterminate if and only if $r(\Omega^{MOD}) < 1$ *and* $r(F^{MOD}) > 1$.

3. The model (1) has no stable REEs if and only if $r(\Omega^{MOD}) > 1$.

Proof*: See McCallum (2008).* [3]

[3] A subtle issue may arise in the case of $r(F) = 1$. For instance, consider a univariate model. When $r(F) = 1$, $w_t = a$ solves (7) for any arbitrary constant a. There seems no general guidance whether this case is considered as indeterminate or determinate. However, since such a solution is non-stochastic, we include $r(F) = 1$ as determinacy. If one alternatively treats this case as indeterminate, then

4 The Forward Convergence and No Bubble Conditions

Nothing in determinacy/indeterminacy conditions endorses economic relevance to the determinate or indeterminate solutions except for their dynamic stability. The forward method of Cho and Moreno (2011) provide model and solution refinement schemes in another dimension. Following their method, we solve the model (1) forward to derive the forward representation of the model:

(8) $x_t = M_k E_t x_{t+k} + \Omega_k x_{t-1} + \Gamma_k z_t$,

where $(M_k, \Omega_k, \Gamma_k)$ is given by: $M_1 = A$, $\Omega_1 = B$, $\Gamma_1 = C$, and for k>1,

(9) $M_k = F_{k-1} M_{k-1}$,

(10a) $\Omega_k = (I - A\Omega_{k-1})^{-1} B$,

(10b) $\Gamma_k = (I - A\Omega_{k-1})^{-1} C + F_{k-1} \Gamma_{k-1} R$,

where F_{k-1} is given by:

(11) $F_k = (I - A\Omega_k)^{-1} A$,

if the regularity condition, $|I - A\Omega_k| \neq 0$ is satisfied for all k=1,2,3,....

Definition (Forward Convergence Condition, FCC) The model (1) is said to satisfy the forward convergence condition if (Ω_k, Γ_k) defined in (10) converge.

Note that if (Ω_k, Γ_k) converges to (Ω^*, Γ^*), (10) fulfill the conditions (4). Note that F_k in (11) converges to F^* if and only if Ω_k converges. Hence, under the FCC, and (11)

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one may define determinacy that excludes $r(F) = 1$ in Assertion 1 and includes it in Assertion 2.

fulfills (5) as well. Therefore, the following forward solution,

(12) $$x_t = \Omega^* x_{t-1} + \Gamma^* z_t$$

is a fundamental solution and it exists if and only if the model satisfies the FCC. Hence the FCC and the existence of the forward solution are equivalent.

(Ω_k, Γ_k) is unique and "implied" by the model. Therefore, the forward solution is the model-implied relation and consequently, it is economically sensible by itself. Forward convergence is exactly the same concept to general LRE models as process consistency of Flood and Garber (1980), which had been appreciated less in the literature.

A key implication of the forward method is that the expectational term $M_k E_t x_{t+k}$ depends on a particular solution with which expectations are formed. In principle, the limiting behavior of this "bubble" term should be verified for each REE, instead of assuming its behavior. Formally we define no-bubble condition.

Definition (No bubble condition, NBC) A solution to the model (1) is said to satisfy NBC if $\lim_{k\to\infty} M_k E_t x_{t+k} = 0_{n\times 1}$ in (8) when expectations are formed with that solution.

The following proposition provides a central result of the forward method.

Proposition 2 *The forward solution is the unique REE that satisfies the NBC.*

Proof*: See Cho and Moreno (2011).*

The NBC is the unique feature that differentiates the forward solution from all other fundamental REEs. The NBC removes two kinds of equilibria. First, it refines away all the equilibria to the models that fail to satisfy the FCC. Suppose that a stable MSV solution $x_t = \Omega x_{t-1} + \Gamma z_t$ exists to a model where one or both of (Ω_k, Γ_k) explodes as k tends to infinity. Since this solution must satisfy the forward representation (8) for all k, this implies that the expectational term evaluated with this solution must explode as well. This would be the major implication underlying the restriction in Blanchard and Kahn (1980). In this sense, the FCC can be interpreted as a model refinement scheme and the NBC refine away all those MSV solutions. The role of the NBC is however, not confined to the models that are not forward-convergent. The NBC also refines away all the stable MSV solutions, different from the forward solution to the models that are convergent, which would arise in the case of indeterminacy. Fundamental (MSV) solutions are often referred to as bubble-free solutions. Thus, bubble-free solutions violating no-bubble condition do not seem to make a lot of sense. This is a new phenomenon that arises to the models with predetermined variables.

In both cases, since any REE can be written as sum of a fundamental solution and the associated bubble term, if the fundamental solutions violate the NBC, the whole set of REEs associated with those solutions does not make economic sense either.

5 Complete Characterization of the REEs under FCC

Now we bridge the relation between the FCC and determinacy, which is absent in Cho and Moreno (2011). Under the FCC, $F^* = \lim_{k\to\infty} F_k$ exists from equation (11). The following proposition shows that under FCC, determinacy (indeterminacy) corresponds to $r(F^*) \leq 1$ ($r(F^*) > 1$), and characterize the full set of REEs.

Proposition 3 *Suppose that the RE model (1) satisfies the forward convergence condition.*

1. If $r(\Omega^*) < 1$ and $r(F^*) \leq 1$, *then model (1) is determinate and the unique stationary solution is given by the forward solution:*

(13) $$x_t = \Omega^* x_{t-1} + \Gamma^* z_t .$$

2. If $r(\Omega^*) < 1$ and $r(F^*) > 1$, *then (1) is indeterminate and the set of stable REEs of which fundamental solution satisfying the NBC is given by:*

(14) $$x_t = \Omega^* x_{t-1} + \Gamma^* z_t + w_t ,$$

where w_t *is an arbitrary stationary process such that* $w_t = F^* E_t w_{t+1}$.

Proof: See Appendix.

The conditions in Assertion 1 of Proposition 3 are sufficient for determinacy, but not necessary. This implies that there can be the cases in which a model is determinate but it fails to satisfy FCC. It can happen only when $\Omega^* \neq \Omega^{MOD}$ from Proposition 1. An example of this kind is shown by Cho and McCallum (2009). Assertion 1 provides a general way to rule out such models as economically plausible ones. Nevertheless, Assertion 1 justifies why the determinate equilibrium is economically sensible because $\Omega^* = \Omega^{MOD}$ for almost all well-formulated models.

Now we consider the indeterminate case in which the model satisfies the FCC, but $r(F^*) > 1$. In this case, there exists a continuum of non-fundamental REEs associated with the forward solution. But there may well exist fundamental solutions associated with $\tilde{\Omega}$ different from Ω^* and the corresponding $\tilde{F}$ as long as $r(\tilde{\Omega}) < 1$ and $r(\tilde{F}) > 1$.[4] Such a solution must violate the NBC from Proposition 2. Therefore, the set of REEs that are consistent with the FCC and NBC are the ones associated with the forward solution.

Proposition 3 excludes the following cases as they have no relevant REEs.
First, if $r(\Omega^*) \geq 1$, then the model has no stationary solution of which fundamental component satisfies the NBC. Second, those models that are not forward convergent have no fundamental solutions satisfying the NBC. This latter case is the one that the NBC is the most important as a solution refinement: without verifying the FCC, one may find solutions to a model that fails to satisfy the FCC as we show an example of this kind in the following section.

Our results show that one necessarily examines the FCC of RE models. Moreover, our methodology is sufficient to identify determinate and indeterminate cases and provides a complete set of REEs to any LRE models satisfying the FCC. Another important feature of our methodology is that the solution method and the solution refinements are obtained using only the rationality assumption and the recursive structure of the underlying macroeconomic model, without solving for all the mathematical solutions using matrix decomposition techniques.

[4] There cannot be a fundamental solution such that $r(\tilde{\Omega}) < 1$ and $r(\tilde{F}) \leq 1$ because it implies that the model is determinate.

6. Example

In this section, we present a New-Keynesian model similar to the one considered by Cho and Moreno (2011), but detect determinacy and indeterminacy using Proposition 3 and further investigate why the MSV solutions to the model does not make sense when it is not forward convergent. The model is given by the three equations.

(15a) $\pi_t = \delta E_t \pi_{t+1} + \kappa y_t$,

(15b) $y_t = \mu E_t y_{t+1} + (1-\mu) y_{t-1} - \phi(i_t - E_t \pi_{t+1}) + z_t$,

(15c) $i_t = \beta E_t \pi_{t+1} + \lambda y_t$,

where π_t, y_t, i_t are respectively inflation, the output gap and the nominal interest rate. z_t is an aggregate demand shock that follows an AR(1) process: $z_t = \rho z_{t-1} + \varepsilon_t$ where $0 \le \rho < 1$ and ε_t is an *i.i.d*. shock. By substituting i_t out, the model (16) can be cast into a bivariate system as $x_t = AE_t x_{t+1} + Bx_{t-1} + Cz_t$ where $x_t = [\pi_t \;\; y_t]'$ and $A = B_1^{-1} A_1$ $B = B_1^{-1} B_2$, $C = B_1^{-1} C_1$ with

$$B_1 = \begin{bmatrix} 1 & -\kappa \\ 0 & 1+\phi\lambda \end{bmatrix},\; A_1 = \begin{bmatrix} \delta & 0 \\ -\phi(\beta-1) & \mu \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1-\mu \end{bmatrix},\; C_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We set the parameter values as $\delta = 0.99,\; \kappa = 0.3,\; \mu = 0.55,\; \phi = 1,\; \lambda = 0.1,\; \rho = 0.8$.

Case 1. When $\beta = 1.5$, the FCC holds and $r(\Omega^*) = 0.46$ and $r(F^*) = 0.75$. Therefore, the model is determinate and the determinate forward solution is given by:

$$x_t = \begin{bmatrix} 0 & 0.26 \\ 0 & 0.46 \end{bmatrix} x_{t-1} + \begin{bmatrix} 1.66 \\ 0.62 \end{bmatrix} z_t.$$

This solution implies that when there is a rise in aggregate demand shock of size 1, the output and inflation increase, which is perfectly consistent with economic theory.

Case 2. When $\beta = 0.95$, the FCC holds and $r(\Omega^*) = 0.60$ and $r(F^*) = 1.13$. Hence, the model is indeterminate. The forward solution exists and it is qualitatively similar to the one under determinacy. There exists, however, another stable MSV solution:

$$x_t = \begin{bmatrix} 0 & 2.10 \\ 0 & 0.88 \end{bmatrix} x_{t-1} + \begin{bmatrix} -29.53 \\ -2.59 \end{bmatrix} z_t .$$

In contrast to the forward solution, both coefficients in Γ are negative. This is clearly counterintuitive because an exogenous increase in aggregate demand decreases both inflation and the output gap. We rule out this solution on the ground of the NBC. Cho and Moreno (2011) show that other solution selection criteria picks up the forward solution as well. However, it is not the case in the following model that fails to satisfy the FCC.

Case 3. When $\beta = 0.9$, the FCC does not hold. Whereas holds Ω^* (and F^*) still exists, Γ_k explodes. From equation (11b), one can see that the matrix governing convergence of Γ_k is ρF^*. In this example, $r(F^*) = 1.33$ and $r(\rho F^*) = 1.20 > 1$, implying that Γ_k grows without bound. Nevertheless, there are two MSV solutions and hence technically, the model is indeterminate. Unlike Case 2, however, none of the solutions is economically sensible. The MOD solution is given by:

$$x_t = \begin{bmatrix} 0 & 0.59 \\ 0 & 0.67 \end{bmatrix} x_{t-1} + \begin{bmatrix} -39.08 \\ -9.15 \end{bmatrix} r_t^n .$$

The other MSV solution is also qualitatively similar to the MOD solution. Just like the MSV solution in Case 2 other than the forward solution, a rise in aggregate demand decreases inflation and the output gap. Surprisingly, the existing solution selection criteria such as E-stability do not distinguish Case 2 and Case 3, and fail to refine away the MOD solution as it turns out to be E-stable. This example illustrates the importance of the forward convergence.

7. Conclusion

The forward convergence and no-bubble conditions generalize and modify process consistency of Flood and Garber (1980) and the restrictions on the behavior of the variables in Blanchard and Kahn (1980), respectively, both of which have been often assumed and neglected in the literature. We show the importance of the forward convergence as a powerful model refinement scheme for linear RE models. Within the class of forward convergent models, we completely characterize the set of economically sensible equilibria to a given model at hand based on the solution refinement scheme. Our result shows that the FCC holds for almost all economic models and hence, it provides an economic justification for the determinate equilibrium. In case of indeterminacy, however, the forward convergence and no-bubble condition detect whether a given model is by itself economically reasonable, and if so, the forward method provides the set of relevant equilibria. We show that there is a non-trivial probability for one to accept a (set) of equilibria to non-forward convergent models. Therefore, the FCC must be verified, not assumed, and would be the most useful and important in practice as a model refinement in such a case.

Appendix

Proof of Proposition 3. Assertion 1. Note that the eigenvalues of Ω^* and the inverses of the eigenvalues of F^* constitute the generalized eigenvalues of the model. Therefore, if $r(\Omega^*) < 1$ and $r(F^*) \leq 1$, then there are exactly n generalized eigenvalues inside the unit circle. Therefore, Ω^* is Ω^{MOD} and the model is determinate from Proposition 1.

Assertion 2. Suppose that $r(\Omega^*) < 1$ and $r(F^*) > 1$. From Proposition 2, the forward solution component of (14) satisfies the NBC and all other fundamental solutions violate the NBC from Proposition 2. Note that F^* must contain at least one root outside the unit circle. Let m be the number of some or all of such unstable roots $(1 \leq m \leq n)$. It can be shown that there exist uncountably many processes such that

$w^*_t = \Lambda E_t w^*_{t+1} + VV'u_t$ where the non-zero eigenvalues of Λ are the inverses of the chosen unstable roots of F^* and V is an n by m matrix of which columns are orthonormal bases corresponding to those roots. u_t is an n by 1 vector of arbitrary write noises. *Q.E.D.*

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