MATHEMATICAL MODELING AND NUMERICAL ANALYSIS II

CSE 7870: Lecture Notes

Lectures by Prof. EunJae Park

YOONKU HWANG, DONSUB RIM AND DONGWOOK SHIN

FALL 2011

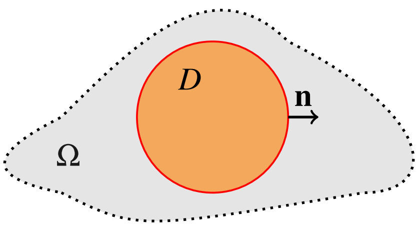
TUE SEP 6, 2011

1 Model of Incompressible Fluid Flow

Consider a fluid of density ρ moving in $\Omega \subset \mathbb{R}^3$.

Definition 1.1 (Basic Definitions)**.**

1. Position $\mathbf{x}$.
2. Velocity.
$$\mathbf{u} = (u_x, u_y, u_z) = \lim_{\delta t \to 0} \frac{\delta \mathbf{x}}{\delta t}, \qquad \delta \mathbf{x} = \mathbf{x}(t + \delta t) - \mathbf{x}(t).$$
3. The total mass of fluid in $D \subset \Omega$.
$$\int_D \rho \, d\Omega.$$



4. The amount of fluid flowing out of D across ∂D.
$$\int_{\partial D} \rho \mathbf{u} \cdot \mathbf{n} \, dS, \qquad \mathbf{n}\text{: unit outward normal to } \partial D.$$

5. Conservation of Mass.

$$\text{(the rate of change of mass in } D) = \text{(the amount of fluid flowing into } D \text{ across } \partial D)$$

$$\frac{d}{dt}\int_D \rho \, d\Omega = -\int_{\partial D} \rho \mathbf{u}\cdot\mathbf{n}\, dS \underset{|}{=} -\int_D \nabla\cdot(\rho\mathbf{u})\, d\Omega$$

$$\text{by Div-THM: } \int_{\partial D} \mathbf{v}\cdot\mathbf{n}\, dS = \int_D \nabla\cdot\mathbf{v}\, d\Omega$$

$$\therefore \ \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \quad \text{in } \Omega.$$

For incompressible and homogeneous fluid, the density $\rho =$ constant, $\forall x, t$. $\Rightarrow \nabla\cdot\mathbf{u} = 0$ in Ω. (incompressible)

Proposition 1.2 (Fluid acceleration)**.** *Let* $\mathbf{u} = \mathbf{q}(\mathbf{x}, t)$: *function of position* $\mathbf{x}$, *time* t. *Then,*

$$\begin{aligned}
&\mathbf{u} + \delta\mathbf{u} = \mathbf{q}(\mathbf{x}+\delta\mathbf{x}, t+\delta t)\\
&\Longrightarrow \delta\mathbf{u} = \mathbf{q}(\mathbf{x}+\delta\mathbf{x}, t+\delta t) = \underbrace{\mathbf{q}(\mathbf{x}+\delta\mathbf{x}, t+\delta t) - \mathbf{q}(\mathbf{x}, t+\delta t)}_{(\delta\mathbf{x}\cdot\nabla)\mathbf{q}(\mathbf{x},t+\delta t)+\mathcal{O}(\|\delta\mathbf{x}\|^2)} + \underbrace{\mathbf{q}(\mathbf{x}, t+\delta t) - \mathbf{q}(\mathbf{x}, t)}_{\delta t\frac{\partial}{\partial t}\mathbf{q}(x,t)+\mathcal{O}(\delta t^2)}\\
&\Longrightarrow \frac{d\mathbf{u}}{dt} = \lim_{\delta t\to 0}\frac{\partial\mathbf{u}}{\partial t} = (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{\partial\mathbf{u}}{\partial t} \quad \textit{(convective derivative, natural derivative)}
\end{aligned}$$

In general,

$$\underset{\substack{|\\ \text{rate of change of a scalar quantity } v \text{ that is following the fluid}}}{\frac{dv}{dt}} = \frac{\partial}{\partial t}v + (\mathbf{u}\cdot\nabla)v$$

when fluid is moving and the position of a particle changes with time.

Remark 1.3. Let $\mathbf{x}(t) = (x(t), y(t), z(t))$: path followed by a fluid particle, so that

$$\mathbf{u}(\mathbf{x}(t), t) = \frac{d\mathbf{x}}{dt}(t).$$

The acceleration of a fluid article:

$$\begin{aligned}
\mathbf{a}(t) &= \frac{d^2}{dt^2}\mathbf{x}(t) = \frac{d}{dt}\mathbf{u}(x(t), y(t), z(t), t)\\
&= \frac{\partial\mathbf{u}}{\partial x}\underbrace{\frac{dx}{dt}}_{u_x} + \frac{\partial\mathbf{u}}{\partial y}\underbrace{\frac{dy}{dt}}_{u_y} + \frac{\partial\mathbf{u}}{\partial z}\underbrace{\frac{dy}{dt}}_{u_z} + \frac{dy}{dt}\\
&= \left(u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}\right)\mathbf{u} + \frac{\partial\mathbf{u}}{\partial t}\\
&= (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{\partial\mathbf{u}}{\partial t} = \frac{d}{dt}\mathbf{u}(\mathbf{x}, t)\left(= \frac{D}{Dt}\mathbf{u}(\mathbf{x}, t).\right)
\end{aligned}$$

For the volume D,

$$\text{(the rate of change of momentum)} \underset{\substack{|\\ f=ma}}{=} \text{(mass)} \times \text{(acceleration)} = \int_D \rho\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) d\Omega.$$

Definition 1.4 (Ideal fluid)**.** Incompressible and homogeneous fluid that has no viscosity.

$$\text{Force:} \begin{cases} \text{due to pressure} \\ \text{external body force } \mathbf{f} \text{ (due to gravity)} \end{cases}$$

The pressure of the surrounding fluid

$$\int_{\partial D} p(-\mathbf{n})\, dS$$

The effect of the body force

$$\int_D \rho \underset{\substack{| \\ \text{given body force per unit mass}}}{\mathbf{f}}\, d\Omega$$

<u>Note</u>

$$\int_{\partial D} p\mathbf{n}\, dS = \int_D \nabla p\, d\Omega$$

Proposition 1.5 (Newton's 2nd Law of Motion)**.**

(the rate of change of momentum of fluid in D) = (the sum of external forces)

For arbitrary D,

$$\int_D \rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} \right) d\Omega = \int_{\partial D} p(-\mathbf{n})\, dS + \int_D \rho \mathbf{f}\, d\Omega$$
$$\Longrightarrow \int_D \left[\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} \right) + \nabla p - \rho \mathbf{f} \right] d\Omega = 0. \qquad \text{(balance of momentum)}$$

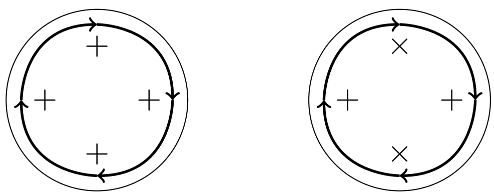
Proposition 1.6 (Euler Equation for an ideal incompressible fluid)**.**

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{f} \quad \text{in } \Omega$$
$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega$$

Proposition 1.7 (Special Cases)**.** *vorticity:* $\omega = \nabla \times \mathbf{u}$.
Assume $\underset{\substack{| \\ \text{irrotational: see example}}}{\omega = 0}$. *(e.g. motion of small amplitude water wave)*
Then $\exists$ *fluid potential* ϕ *s.t.* $\mathbf{u} = -\nabla\phi$ *such that* $\mathbf{u} = -\nabla\phi$.

Example 1.8.

$$\mathbf{v} = \left(\frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2} \right), \quad \mathbf{x} \neq 0$$



$+$ does not rotate on the left: irrotational. $+$ does rotate on the right: rotational.
Petals spin in reverse direction in different hemispheres. (Coriolis Effect)

Proposition 1.9 (Incompressibility condition)**.**

$$\operatorname{div}\mathbf{u} = 0. \Longrightarrow -\Delta\phi = 0(\ or\ -\nabla^2\phi = 0)$$

More generally, potential flow problem (inviscid and irrotational)

$$-\nabla^2\phi = f \ \mathit{in}\ \Omega \quad \mathit{with\ BC} \quad \mathit{(Poisson\ Equation)}$$

Recovery of velocity $\mathbf{u}$ and pressure p.

(1) $\mathbf{u} = -\nabla\phi$

(2) for pressure, assume the body forces are conservative so that $\mathbf{f} = -\nabla\xi$, for some scalar potential ξ.

THU SEP 8, 2011

Recall

- Conservation of Mass (Continuity equation) ;

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \qquad \text{in } \Omega.$$

- Homogeneous and incompressible fluid;

$$\nabla\cdot\mathbf{u} = 0, \qquad \text{in } \Omega.$$

- Ideal and incompressible fluid (Newton's second law of motion);

$$\begin{cases} \dfrac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\dfrac{1}{\rho}\nabla p + \mathbf{f}, & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega \end{cases}$$

  , called Euler equation.

- Irrotational fluid

  If vorticity, $\mathbf{w} = \nabla\times\mathbf{u} = 0, \exists$ potential ϕ $s.t.$ $\mathbf{u} = -\nabla\phi$.

- Incompressiblity ($\nabla\cdot\mathbf{u} = 0$) yields

$$\nabla\cdot\mathbf{u} = \nabla\cdot(-\nabla\phi) = -\nabla^2\phi = -\Delta\phi = 0$$

- Recovery of velocity $\mathbf{u}$ and pressure p.

  i) $\mathbf{u} = -\nabla\phi$

ii) pressure? assume that the body forces are conservative so that $\mathbf{f} = -\nabla\xi$ for some scalar potential ξ.

$$\text{e.g. } \mathbf{f} = \rho\mathbf{g} \to \xi = -\rho\mathbf{g}\cdot\mathbf{x}.$$

□

,Next, remind $(\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times(\nabla\times\mathbf{u})$. Since, $\nabla\times\mathbf{u} = 0$, and $\mathbf{f} = -\nabla\xi$,

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{f}$$

$$\frac{\partial\mathbf{u}}{\partial t} + \frac{1}{\rho}\nabla\left(p + \rho\xi + \frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}\right) = 0$$

Let total pressure, $p_T = p + \rho\xi + \dfrac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}$.
With $\mathbf{u} = -\nabla\phi$, we get

$$\nabla\left(-\frac{\partial\phi}{\partial t} + \frac{1}{\rho}p_T\right) = 0, \qquad \text{in } \Omega.$$

It means that there exists $h(t)$ that is independent of space $\mathbf{x}$ such that

$$-\frac{\partial\phi}{\partial t} + \frac{1}{\rho}p_T = h(t).$$

May assume $h(t) = 0$, then, $p_T = \rho\dfrac{\partial\phi}{\partial t}$. Therefore,

$$\begin{cases} p = p_T - \rho\xi - \dfrac{\rho}{2}\mathbf{u}\cdot\mathbf{u} \\ \quad = \rho\left(\dfrac{\partial\phi}{\partial t} - \dfrac{1}{2}\mathbf{u}\cdot\mathbf{u} - \xi\right) \\ \mathbf{u} = -\nabla\phi \end{cases}$$

where ϕ is solution of Laplace equation with B.C.

2 Viscous fluid

Each small volume of fluid is not only acted on by pressure forces (normal stresses) but also by tangential stresses (shear stresses).

- Pressure forces (normal stresses)

$$\int_{\partial D} p(-\mathbf{n})ds = \int_{\partial D} -pI\mathbf{n}dS$$

where $I \in \mathbb{R}^{3\times 3}$ is identity matrix.

- Shear stresses can act in any direction at different points on ∂D.

  Let T_{jk} be the kth component of the stress vector acting on the face j. For example, T_{xz} is the third component of the stress vector acting on the face x.

$$T = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}$$

and $\overrightarrow{T}_x = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \end{bmatrix}$, $\overrightarrow{T}_y = \begin{bmatrix} T_{yx} & T_{yy} & T_{yz} \end{bmatrix}$, and $\overrightarrow{T}_z = \begin{bmatrix} T_{zx} & T_{zy} & T_{zz} \end{bmatrix}$.

$$\int_{\partial D} T\mathbf{n}ds = \int_D \nabla \cdot T d\Omega,$$

where $\nabla \cdot T$ is rowwise divergence, i.e., $\nabla \cdot T = \begin{bmatrix} \nabla \cdot \overrightarrow{T}_x \\ \nabla \cdot \overrightarrow{T}_y \\ \nabla \cdot \overrightarrow{T}_z \end{bmatrix}$.

<u>Newtonian fluid</u> : shear stress is linear function of the rate of strain tensor,

$$\begin{aligned} \varepsilon &= \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) \\ &= \frac{1}{2}\begin{bmatrix} 2\frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial z} & \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \\ & 2\frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \\ sym & & 2\frac{\partial u_z}{\partial z} \end{bmatrix} \end{aligned}$$

where $\mathbf{u} = [u_x, u_y, u_z]^T$ and $\nabla\mathbf{u} = \begin{bmatrix} - & \nabla u_x & - \\ - & \nabla u_y & - \\ - & \nabla u_z & - \end{bmatrix}$.

<u>Newtonian</u> : $T = 2\mu\varepsilon + \lambda(\mathrm{tr}(\varepsilon))I$ where μ is molecular viscosity.

Note that

$$\begin{aligned} \nabla \cdot T &= 2\mu\nabla\cdot\varepsilon \qquad (\because \mathrm{tr}(\varepsilon) = \nabla\cdot\mathbf{u} = 0) \\ &= \mu\begin{bmatrix} \Delta u_x + \frac{\partial}{\partial x}\nabla\cdot\mathbf{u} \\ \Delta u_y + \frac{\partial}{\partial y}\nabla\cdot\mathbf{u} \\ \Delta u_z + \frac{\partial}{\partial z}\nabla\cdot\mathbf{u} \end{bmatrix} \qquad (\because \nabla\cdot\mathbf{u} = 0) \\ &= \mu\Delta\mathbf{u} \end{aligned}$$

Remark.

In ε, $\nabla\mathbf{u}$ is row-wise gradient and, naturally, $\nabla\mathbf{u}^T$ is column-wise gradient.
In $\nabla\cdot\varepsilon = \frac{1}{2}[\Delta\mathbf{u} + \nabla(\nabla\cdot\mathbf{u})]$, $\nabla(\nabla\cdot\mathbf{u})$ is column-wise.

Thus, by Newton's second law of motion, we have ;

$$\int_D \rho\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right)d\Omega = \int_{\partial D} -pI\mathbf{n}ds + \int_D \mu\Delta\mathbf{u}d\Omega + \int_D \rho\mathbf{f}d\Omega$$

, that gives Navier-Stokes equation ;

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\dfrac{1}{\rho}\nabla p + \nu\Delta\mathbf{u} + \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \end{cases}$$

where $\nu = \dfrac{\mu}{\rho}$ is kinematic viscosity.

<u>Steady state Navier-Stokes</u> : (with $p \leftarrow \dfrac{p}{\rho}$), incompressible viscous fluid ;

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega. \end{cases}$$

Dropping quadratic nonlinear therm, we have Stokes equation (with $\mathbf{u} \leftarrow \nu\mathbf{u}$) ;

$$\begin{cases} -\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega. \end{cases}$$

Next, replace $(\mathbf{u} \cdot \nabla)\mathbf{u}$ by $(\mathbf{w} \cdot \nabla)\mathbf{u}$ with $\mathbf{w}$ known, providing Oseen equation,

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{w} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

and drop pressure term to obtain *convection-diffusion equation*;

$$-\nu\Delta\mathbf{u} + (\mathbf{w} \cdot \nabla)\mathbf{u} = \mathbf{f} \qquad \text{(vector convection-diffusion equation)}$$

and

$$-\nu\Delta u + \mathbf{w} \cdot \nabla u = f \qquad \text{(scalar convection-diffusion equation).}$$

THU SEP 15, 2011

3 Advection-diffusion Equations and Boundary Layers

Consider

$$\begin{aligned} -\varepsilon\Delta u + \beta\nabla u &= f \quad \text{in } \Omega \subset \mathbb{R}^2 \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Assume $\varepsilon > 0$ constant.

$\beta = (\beta_1, \beta_2)$ constant vector field.

Ω: polygonal domain.

$\mathscr{T}_h = \{K\}$: family of triangulations.

$\mathscr{T}_h$: shape-regular.

$h = \max_{K\in\mathscr{T}_h} h_k$, $h_k = \operatorname{diam}(K)$.

Definition 3.1 (Bilinear Form)**.** Bilinear Form Associated with (1):

$$\begin{aligned} &\forall u, v \in H_0^1(\Omega), \\ &a(u,v) = 2\int_\Omega \nabla u\cdot\nabla v + \int_\Omega (\beta\cdot\nabla u)v \\ &F(v) = \int_\Omega fv. \end{aligned}$$

Definition 3.2 (Variational Formulation of Problem (1))**.** Find $u \in H_0^1(\Omega)$ s.t

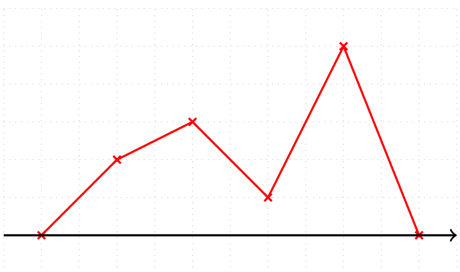
$$a(u,v) = F(v) \quad \forall v \in H_0^1(\Omega).$$

Finite element approximation

$$V_h = \left\{ v_h \in H_0^1(\Omega) : v_h|_K \text{ linear for } \forall K \in \mathscr{T}_h \right\}$$

Conforming piecewise linear (C^0-elt): need continuity.

Example 3.3 (1D-case)**.** Typical elt in V_h.



Find $u_h \in V_h \subset H_0^1(\Omega)$ such that $a(u_h, v_h) = F(v_h)$ for $\forall v_h \in V_h$.

Remark 3.4 (Goal)**.** u_h; close to u.

A priori estimates:

$$\overbrace{\|u - u_h\|_1}^{\text{error in } H^1(\Omega)\text{-norm}} \leq c h \overbrace{|u|_{2,\Omega}}^{H^2(\Omega)\text{-semi-norm}}$$

$|u|_{2,\Omega}$: unknown

c: constant independent of h: (depends on shape-regularity of the mesh and on the coeffs of the eqn)

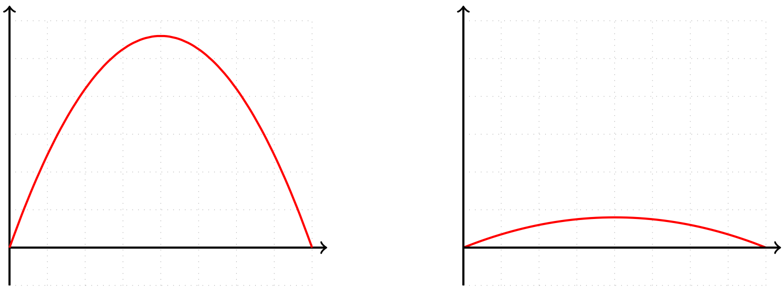
When possible, $\|u - u_h\|_0 \leq c h^2 |u|_{2,\Omega}$.

Remark 3.5. "a posteriori" error estimates produce constructive measure of the error.

1D-example: pure diffusion equation:

$$-\varepsilon u'' = 1 \quad 0 < x < 1$$
$$u(0) = u(1) = 0.$$

Exact solution: $-\frac{1}{2\varepsilon} x(x-1)$.



ε small $\qquad$ ε big

Let us see what happens when convection term comes into play (bad guy).

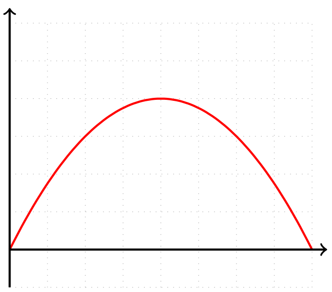
Convective bad guy: $\beta \cdot \nabla u = \beta u'$ in 1D.

We will consider

$$-\varepsilon u'' + u' = 1 \quad 0 < x < 1 \qquad (2)$$
$$u(0) = u(1) = 0$$

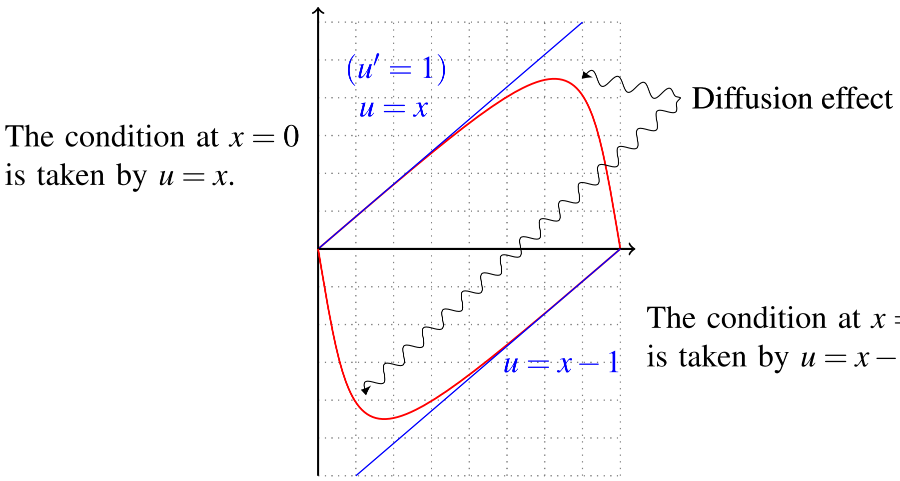
We could easily write down the exact solution of (2) but it's not worthwhile.

If ε is big, then u' is negligible w.r.t $-\varepsilon u''$ and the soln will be similar to the pure diffusion case:



ε big ($\gg 1 = \beta$)

If ε is small, the eqn "reduces" to $u' = 1$, but the two boundary values cannot be taken simultaneously. The term $-\varepsilon u''$ will help to reach the other BC.



Q. Which is the right one?

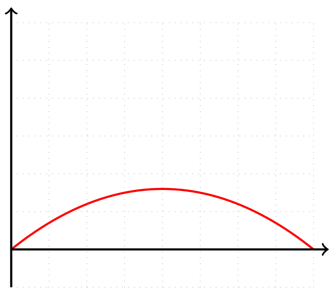
A. The positive one.

by continuity with the case ε big.

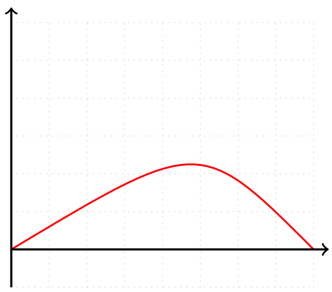
by Maximum principle.

Remark 3.6. Roughly speaking, "first order PDE can have spurious solution (or spurious oscillation.)" "adding a 2^{ND} order term with the right sign selects the true (physical) solution."
e.g. Burgers eqn.

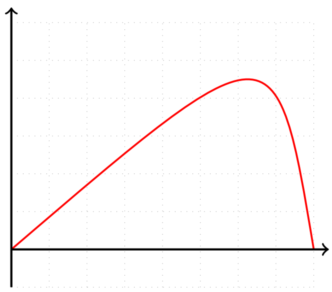
$$\overbrace{-\varepsilon u_{xx}}^{\text{this term selects the physical solution}} + u_t + \underbrace{u u_x}_{=\left(\frac{1}{2}u^2\right)_x} = 0$$



ε big ≈ 10



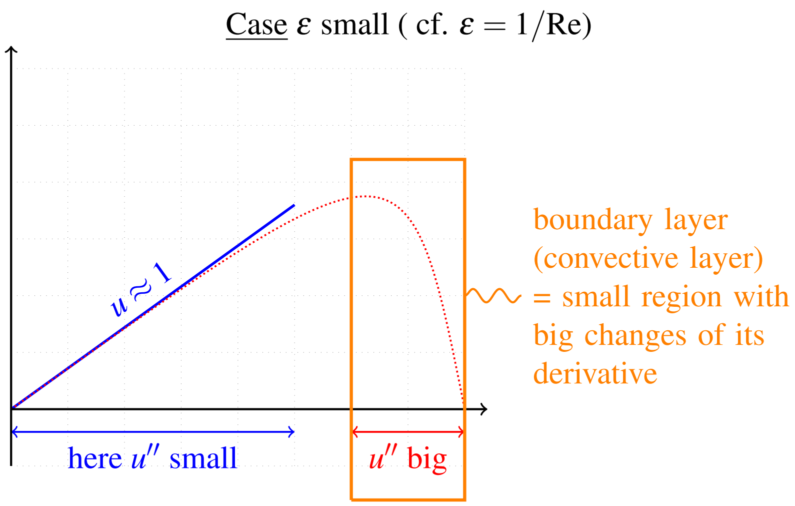
ε intermediate $\approx 10^0$



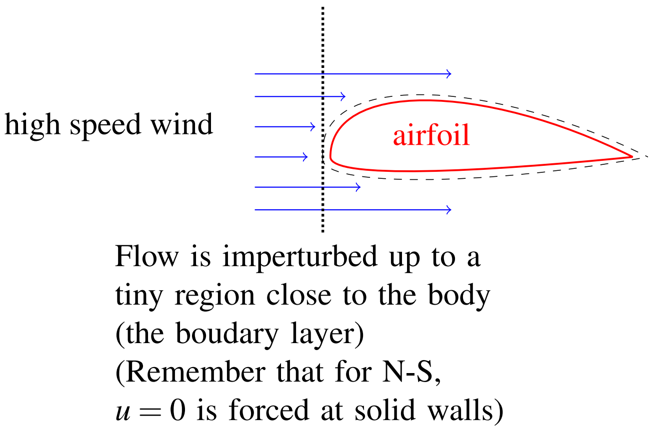
ε small $\approx 10^{-2}$

<u>Solution of the BVP</u>

$$-\varepsilon u'' + u' = 1 \quad 0 < x < 1$$
$$u(0) = u(1) = 0$$



Example 3.7 (Real life example: Navier-Stokes Equation)**.**

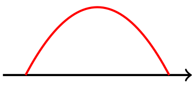


$$
\begin{aligned}
-\frac{1}{\mathrm{Re}}\nabla^2 u + (u\cdot\nabla)u + \nabla p = f \quad &\text{in } \Omega\\
\operatorname{div} u = 0 \quad &\text{in } \Omega\\
u = 0 \quad &\text{on } \Omega
\end{aligned}
$$

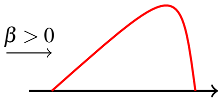
TUE SEP 20, 2011

Recall 3.8 (1D model)**.**

$$\begin{cases} -\varepsilon u'' + u' = 1 & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$



($\beta = 0$)



($\beta = 1$) ($\varepsilon << \beta$)

❇

Discretization

$\beta = 0$ (pure diffusion case) ;

$$u(x) = \frac{1}{2\varepsilon} x(1 - x).$$

❇

FE approximation

Take a uniform grid on [0, 1] with $h = \frac{1}{3}$ consisting of 3 elements.
In this very particular case,

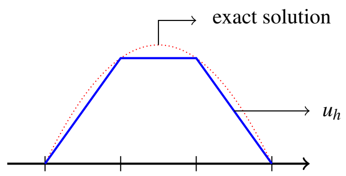
FE = Centered Finite Difference (exercise).

Resulting scheme

$$\begin{cases} -\varepsilon \dfrac{u_0 - 2u_1 + u_2}{h^2} = 1 & (\text{node } x_1) \\ \\ -\varepsilon \dfrac{u_1 - 2u_2 + u_3}{h^2} = 1 & (\text{node } x_2) \end{cases}$$

We know $u_0 = u_3 = 0$ (B.C.) and $h = \frac{1}{3}$.
Then, we have $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\dfrac{1}{9\varepsilon} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\Rightarrow u_1 = u_2 = \frac{1}{9\varepsilon}$

❇



Note We have exact values:

$$u\left(\frac{1}{3}\right) = \frac{1}{9\varepsilon} = u_1, \quad u\left(\frac{2}{3}\right) = \frac{1}{9\varepsilon} = u_2,$$

Hence, for all values of ε, our very economical approximation gives a reasonably good answer.

Remark 3.9. It can be (easily) shown that the P_1 approximation of the 1-D problem

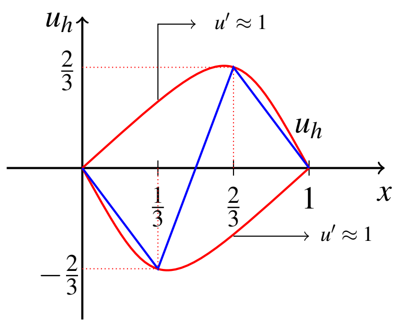
$$\begin{cases} -u'' = f, & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

is always exact at nodes, even for a non-uniform subdivision of $[0,1]$.
In 2-D (with $-\triangle u = f$), this is no longer true, but few points are enough to give an idea of the solution. □

Finite Difference ($\beta = 1$)

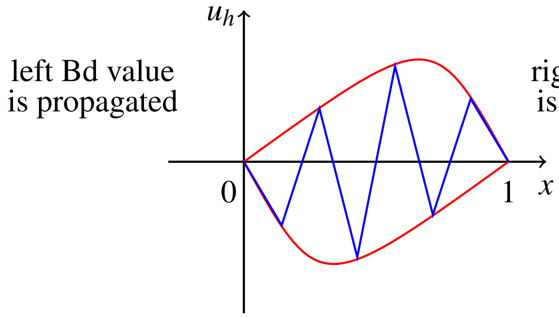
$$\begin{cases} u''(x_i) \approx \dfrac{u_{i-1} - 2u_i + u_{i+1}}{h^2} \\ u'(x_i) \approx \dfrac{u_{i+1} - u_{i-1}}{2h} \end{cases} \Rightarrow \begin{cases} -\varepsilon \dfrac{u_0 - 2u_1 + u_2}{h^2} + \dfrac{u_2 - u_0}{2h} = 1 & (\text{node } x_1) \\ -\varepsilon \dfrac{u_1 - 2u_2 + u_3}{h^2} + \dfrac{u_3 - u_1}{2h} = 1 & (\text{node } x_2) \end{cases}$$

$$h = \frac{1}{3}, \quad u_0 = u_3 = 0 (\text{B.C.})$$



$$\Rightarrow \text{If } \varepsilon \text{ is small, the system reduces to } \begin{cases} \dfrac{u_2}{2h} \approx 1 \\ -\dfrac{u_1}{2h} \approx 1 \end{cases} \text{ or } \begin{cases} u_2 \approx \dfrac{2}{3} \\ u_1 \approx -\dfrac{2}{3} \end{cases}$$

The approximation solution has nothing to do with the true one. If ε is small, the centered scheme implies that u_2 sees only u_0 and u_1 sees only u_3.
In general (with more points), we have a solution like this.



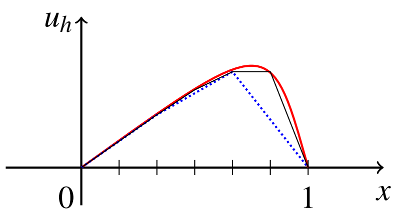
$\sim$ combination of the two solution for $u' = 1$.

Remark 3.10. The boundary value at right is propagated to the odd-numbered nodes. The boundary value at left is propagated to the even-numbered nodes. Thus, big oscillations occur!
Solution
For the discretization of u' in

$$-\varepsilon u'' + u'$$

We want that u_i is determined by the value at the node "before" i.e. u_{i-1}. Hence, substitute the centered difference $\frac{u_{i+1} - u_{i-1}}{2h}$ with the one-sided difference $\frac{u_i - u_{i-1}}{h}$ (upwind). □

Note
If we have $-\varepsilon u'' + u'$, then $\frac{u_{i+1} - u_{i-1}}{2h}$ must be substituted with $\frac{u_i - u_{i-1}}{h}$. We select the UPWIND node
Now it works, the result we get is given in Figure 3



of course, we can not hope to do better with only 2 degrees of freedom! The concept of "upwinding" is fundamental in CFD. In high speed flows, it *must* be present ; otherwise, the solution will oscillate when upwind is really necessary?
It depends on the size of the viscosity ε, the mesh size h, and the transport field β.

Example 3.11.

$$\begin{cases} -\varepsilon u'' + \beta u' = 1 & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

Mesh
β = positive constant
Mesh Peclet number $= \frac{|\beta| h}{2\varepsilon}$

$$\begin{cases} \text{If } \frac{|\beta|h}{2\varepsilon} > 1, & \text{upwind} \\ \text{If } \frac{|\beta|h}{2\varepsilon} \le 1, & \text{upwind is NOT necessary} \end{cases}$$

The Peclet number measures how good my mesh is w.r.t the transport term.
We will see later that the Peclet number is *local* concept.

- a mesh can be OK in one region of the domain but bad in another one
- $\begin{cases} h \text{ may vary} \\ \beta \text{ may vary (Recall } \beta = u \text{ in N-S)} \end{cases}$
- Upwind must be used only in regions where the Peclet number is "big"

Remark 3.12. Accuracy If we use upwind up to the limit $h \to 0$, our solution is less accurate i.e. the error in L^2 is $\mathscr{O}(h)$ NOT $\mathscr{O}(h^2)$
Why?

- $u(x_{i+1}) = u(x_i) + u'(x_i)h + \mathscr{O}(h^2) \Rightarrow u'(x_i) = \dfrac{u_{i+1} - u_i}{h} + \mathscr{O}(h)$,
- $u'(x_i) = \dfrac{u_{i+1} - u_{i-1}}{2h} + \mathscr{O}(h^2)$.

Interpretation of upwind

$$\underbrace{\frac{u_i - u_{i-1}}{h}}_{\text{upwind } u'} - \underbrace{\frac{u_{i+1} - u_{i-1}}{2h}}_{\text{centered } u'} = -\frac{u_{i-1} - 2u_i + u_{i+1}}{2h} \quad \to \left(= \text{Centered } u'' \text{ with } \frac{h}{2} = -\frac{h}{2}\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} \right)$$

Using upwind means to add a term $-\frac{h}{2}\left(\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}\right)$ to the equation. i.e.,

$$\underbrace{\left(-\varepsilon + \frac{h}{2}\right)}_{\text{"artificial viscosity"}} \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \frac{u_{i+1} - u_{i-1}}{2h} = 1.$$

Remark 3.13. the sign is always $+$ even for $-\varepsilon u'' - u'$

THU SEP 22, 2011

Recall 3.14.

$$\begin{cases} -\varepsilon u'' + \beta u' = 1 & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}.$$

(i) $\beta > 0$,

$$\beta \frac{u_i - u_{i-1}}{h} - \beta \frac{u_{i+1} - u_{i-1}}{2h} = \underbrace{-\beta \frac{u_{i-1} - 2u_i + u_{i+1}}{2h}}_{-\frac{\beta h}{2} \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} \left(\approx -\frac{\beta h}{2} u''\right)}.$$

(Seen as a diffusion)

(ii) $\beta < 0$,

$$\overbrace{\beta \frac{u_{i+1} - u_i}{h}}^{\text{upwind}} - \overbrace{\beta \frac{u_{i+1} - u_{i-1}}{2h}}^{\text{centered difference}} = \underbrace{\beta \frac{u_{i-1} - 2u_i + u_{i+1}}{2h}}_{-\left(-\frac{\beta h}{2}\right) \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} \left(\approx -\left(-\frac{\beta h}{2}\right) u''\right)}.$$

The added artificial viscocity is > 0.

Summary 3.15 (Modified Problem)**.** For our toy problem, introducing upwind in the scheme means to solve

$$-\Bigg(\varepsilon + \underbrace{\frac{|\beta| h}{2}}_{\text{artificial diffusion}}\Bigg) \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \beta \underbrace{\frac{u_{i+1} - u_{i-1}}{2h}}_{\text{centered difference}} = 1$$

In other words, we have applied a centered scheme to a modified problem:

$$-\left(\varepsilon + \frac{|\beta| h}{2}\right) u'' + \beta u' = 1$$

The modified problem has a *larger* "visocity" so that the centered scheme works.

Remark 3.16 (Loss of accuracy)**.** We see also here that we cannot expect to have an L^2-error of order h^2, we have in fact *modified* the original problem with an $\mathscr{O}(h)$ term:

$$\text{from: } -\varepsilon u'' + \beta u' = 1 \tag{3}$$

$$\text{to: } -\left(\varepsilon + \frac{|\beta| h}{2}\right) u'' + \beta u' = 1 \tag{4}$$

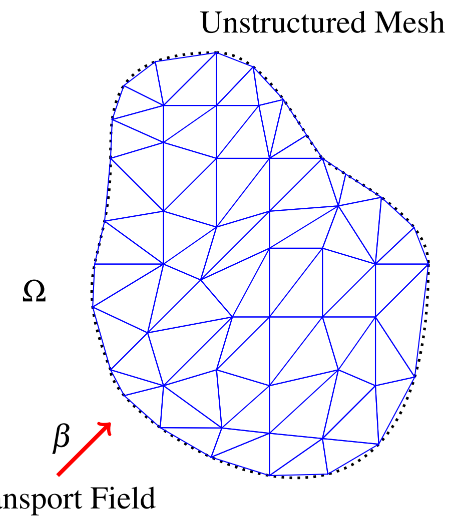
We can approximate problem (3) with any method, but we cannot do better than $\mathscr{O}(h)$ (even with the exact solution.)

2D - cross diffusion, h_x, h_y, etc. $\rightarrow$ harder.
Back to Finite Elements

Toy Problem:

$$\begin{cases} -\varepsilon\Delta u + \beta\cdot\nabla u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $\mathscr{T}_h$ be a family of unstructured triangulation of Ω. Assume β constant, $|\beta| \gg \varepsilon > 0$.
We already know that w have to do somethig more than the usual P_1-element. We have to introduce upwind in the scheme.
The question is *how*?



The notion of upwind is very artificial to apply in the FE context.
There is no notion of *upwind* or *downwind* triangulation here. $\Rightarrow$ This is one of the reasons why it is so difficult to use the FEM in CFD; cf. other methods (e.g. finite difference, finite volume can define upwind naturally.)
But, we have shown that *upwind* means *artificial diffusion.*

Remark 3.17 (Idea)**.** Change our problem to

$$-\Big(\varepsilon + \underbrace{\frac{|\beta|h}{2}}_{\text{or something similar (when } \varepsilon\gg|\beta|)}\Big)\Delta u + \beta\cdot\nabla u = f$$

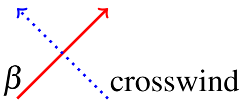
Is this compatible with FE?
We have to check that a bilinear form can be written

$$\underset{\text{“Artificial Diffusion”}}{a_{\text{AD}}(u,v)} = \left(\varepsilon + \frac{|\beta|h}{2}\right)\int_\Omega \nabla u\cdot\nabla v + \int_\Omega (\beta\cdot\nabla u)v$$

Even if β and h vary (assume they are piecewise constant)

$$a_{\mathrm{AD}}(u,v) = \underbrace{\varepsilon \int_\Omega \nabla u \cdot \nabla v + \int_\Omega (\beta \cdot \nabla u) v}_{a(u,v)} + \underbrace{\sum_K \frac{|\beta_K| h_K}{2} \int_K \nabla u \cdot \nabla v}_{\text{modification}}$$

Unfortunately, it does **NOT** work. $\rightarrow$ crosswind artificial diffusion is also added.



Artificial diffusion must be used **ONLY** in the direction of streamlines.
We must **NOT** add $-\dfrac{|\beta|h}{2}\Delta u$.

But $-\dfrac{|\beta|h}{2} \underbrace{\overbrace{\frac{1}{|\beta|^2}}^{\text{normalization}} (\beta \cdot \nabla) \overbrace{(\beta \cdot \nabla u)}^{\text{directional derivative of } u}}_{\text{2nd derivative of } u \text{ in the direction of } \beta}$

Since if v is a test function, we have ($\beta =$ constant)

$$-\frac{1}{|\beta|^2} \int_\Omega (\beta \cdot \nabla)(\beta \cdot \nabla u) v \underset{\text{integration by parts}}{=} \frac{1}{|\beta|^2} \int_\Omega (\beta \cdot \nabla u)(\beta \cdot \nabla v).$$

Thus we have the following variational form:

$$\underset{\text{“Stabilize”}}{a^{\mathrm{s}}}_{\mathrm{AD}} = a(u,v) + \frac{h}{2|\beta|} \int_\Omega (\beta \cdot \nabla u)(\beta \cdot \nabla v)$$

or in “localized” form,

$$a^{\mathrm{s}}_{\mathrm{AD}} = a(u,v) + \sum_K \frac{h_K}{2|\beta_K|} \int_K (\beta \cdot \nabla u)(\beta \cdot \nabla v)$$

We have learned from the 1D case that upwind is NOT always necessary. Only if the mesh is too coarse.
We define a (local) mesh Péclet number (as in 1D).

$$\mathrm{Pe}_k = \frac{|\beta_K| h_K}{2\varepsilon}.$$

and paramter τ_k

$$\tau_k = \begin{cases} \dfrac{h_k}{2|\beta|_K} & \text{if } \mathrm{Pe}_k > 1 \\ 0 & \text{if } \mathrm{Pe}_k \leq 1 \end{cases}$$

The method now reads as:
Find $u_h \in V_h$ such that

$$\underbrace{\varepsilon \int_\Omega \nabla u_h \cdot \nabla v_h + \int_\Omega (\beta \cdot \nabla u_h) v_h}_{\text{original bilinear from}} + \underbrace{\sum_K \tau_K \int_K (\beta \cdot \nabla u_h)(\beta \cdot \nabla v_h)}_{\text{modification}} = \int_\Omega f v_h \quad \forall v_h \in V_h$$

TUE SEP 27, 2011

Recall 3.18. We have
To find $u_h \in V_h$ s.t.

$$a_{\mathrm{AD}}^{\mathrm{S}}(u_h, v_h) = \underbrace{\varepsilon \cdot \int_\Omega \nabla u_h \cdot \nabla v_u + \int_\Omega (\beta \cdot \nabla u_h)\, v_h}_{=a(u_h,v_h) \text{ original bilinear form}} + \underbrace{\sum_K \tau_K \int_K (\beta \cdot \nabla u_h)(\beta \cdot \nabla v_h)}_{\text{modification}}$$

$$= \int_\Omega f v_h \, dx \quad \forall v_h \in V_h$$

where

$$\tau_K = \begin{cases} \frac{h_K}{2|\beta_K|} & \text{if } \mathrm{Pe}_K > 1 \\ 0 & \text{if } \mathrm{Pe}_K \le 1 \end{cases}$$

(Recall, $\mathrm{Pe}_K = \frac{|\beta_K| h_K}{2\varepsilon}$.)

3.1 Refinements

As it is, the method is NOT consistent.
In the FE terminology, "consistency" means:

The exact solution also solves the variational approximation

Example 3.19. The usual approximation is consistent.

$$u: \text{ exact solution of } a(u,v) = F(v) \quad \forall v \in V u_h: \text{ approx solution of } a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$$

Since $V_h \subset V$ (conforming), we have also

$$\boxed{a(u, v_h) \underset{\text{consistency}}{=} F(v_h) \quad \forall v_h \in V_h.}$$

In our case, we have changed the bilinear form of the continuous problem. So consistency does NOT hold anymore.

Remark 3.20 (Idea)**.** Use residual. We had

$$\tau_K \int_K \underbrace{(\beta \cdot \nabla u_h)}_{-\varepsilon \underbrace{\Delta u_h}_{\text{identically zero for linear elements}} + \beta \cdot \nabla u_h - f} (\beta \cdot \nabla v_h)$$

It can be proved that consistency enhances the method $\to$ not easy to prove!

3.2 Further Refinements

τ_K changes to abruptly. Several reasons (not elementary) lead to different choice.

$$\tau_K = \begin{cases} \frac{h_K}{2|\beta|} & \text{if } \mathrm{Pe}_K > 1 \\ \frac{h_K^2}{2\varepsilon} & \text{if } \mathrm{Pe}_K \leq 1 \end{cases}$$

Later we will derive and "optimal" way of choosing the parameter τ_K using $\underbrace{\text{Variational Multiscale approach (VMS).}}_{\text{or Residual free bubble}}$

This completely defines the SUPG method (without shock capturing)

Find $u_h \in V_h$ such that

$$\varepsilon \int_\Omega \nabla u_h \cdot \nabla v_h + \int_\Omega (\beta \nabla u_h) v_h + \sum_K \tau_K \int_K (\overbrace{-\varepsilon \Delta u_h}^{\text{consistency}} \underbrace{+\beta \nabla u_h}_{\text{upwind}} \overbrace{-f}^{\text{consistency}}) \underbrace{(\beta \cdot \nabla v_h)}_{\text{upwind}}$$
$$= \int_\Omega f v_h \quad \forall v_h \in V_h$$

Summary 3.21.

- naïve Galerkin method does NOT work for convection-diffusion problems.
- a form of upwind (= good treatment of 1^{ST} order term) must be used.
- it is difficult to define upwind in a finite element context.
- in 1-D upwind is equivalent to adding artifical viscosity.
- we add the right amount of artificial viscosity in the direction of streamlines.
- modification to have consistency.
- slightly different definition of the parameter τ (artificial diffusion).

$$\Downarrow$$

SUPG

The SUPG method has been invented in '84 by TJR HUghes and successively improved and analyzed by many people.

SUPG means: $\underbrace{\text{Streamline Upwind}}_{\text{clear from the above discussion}}$ $\underbrace{\text{Petrov-Galerkin}}_{\text{another way of introducing it}}$

Recall 3.22 (Galerkin)**.** Find $u_h \in V_h$ such that

$$a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h.$$

Recall 3.23. Petrov-Galerkin Find $u_h \in V_h$ such that

$$a(u_h, w_h) = F(w_h) \quad w_h \in W_h$$
$$(\dim V_h = \dim W_h).$$

SUPG is PG:

$$RHS = \int_\Omega f \underbrace{(v_h + \beta \nabla v_h)}_{=: w_h} dx$$

Further Topics

- Mathematical analysis of the method (convergence, ...)
- Extension to more complicated problems
- Definition of a shock-capturing term to reduce the remaining small oscillation.

THU SEP 29, 2011

Recall 3.24. Error estimate

$$\|u-u_h\|_1 \le \frac{C}{\varepsilon}\|u\|_2 \tag{5}$$

conv-diffusion eqn

$$-\varepsilon\Delta u + \beta \cdot \nabla u = f$$

$$\varepsilon\|u\|_1^2 \le a(u,u)$$

3.3 Multiscale Problem

$$\begin{cases} \mathbf{u} = -\kappa\nabla p & \text{(constitutive equation or Darcy's law)} \\ \operatorname{div}\mathbf{u} = f & \text{(conservation of mass)} \end{cases}$$

Example 3.25 (An example of Darcy's law)**.**

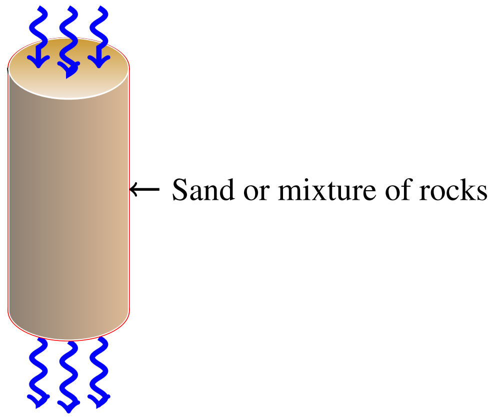


Figure 1: An Example of Darcy's Law: Porous Media

Also used in elasticity. (constants are also derived from experiments) Was at first difference now a differentiation.
If κ is constant it's easy but if κ varies as in Figure 1 then it is very hard. In practice you dig and look at the cross section etc. to determine κ (sometimes stochastic (cf. Dongbin Xiu))

$$\Rightarrow -\operatorname{div}(\kappa\nabla p) = f$$
$$\kappa : \text{(diffusion / conductivity, permeability)}$$

Remark 3.26 (Problem of Scale)**.** Fine scale variation in κ (the permeability) leads to fine scale variation in the solution (u, p).

The problem of scale

Suppose κ varies on the scale ε. Then $|\nabla p| = \mathscr{O}(\varepsilon^{-1})$ and $|D^k p| = \mathscr{O}(\varepsilon^{-k})$.
Typical error estimates: from polynomial approximation theory the best approximation on a finite element partition of $\mathscr{T}_h$ is

$$\inf_{q \in P_{k-1}(\mathscr{T}_h)} \|p - q\|_0 \leq C\|p\|_k h^k = C\left(\frac{h}{\varepsilon}\right)^k$$

if the error is measured in L^2-norm. If measured in H^1-norm, we have 1 reduced order as in (5).

- If $h > \varepsilon$, this is NOT small. So the estimate is useless. Then use small h? very big problem.
- To resolve p we need a spatial discretization of $h < \varepsilon$. That is, we must resolve κ in the same way! The question is *how*. How to resolve this problem with fine-scale feature while still having $h > \varepsilon$?

Remark 3.27 (The Brute Force Approach)**.** The computational load is excessive for a fully resolved $h < \varepsilon$ and fully coupled problem.

3.4 Volume Averaging

Effective properties: We want to solve the problem on a coarse grid.

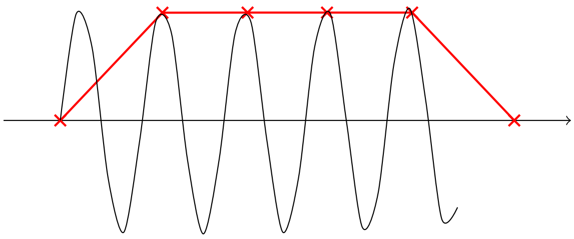


Figure 2: Coarse Grid when ε small.

Upscaling: The system is represented on a larger scale by defining average or *effective* or *macroscopic* parameters in place of the true parameter (in our case, κ.)

Naïve averaging: consider 1-D. Select $\varepsilon > 0$ as an averaging window and define the averages

$$\overline{u}(x) = \frac{1}{\varepsilon}\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} u(\xi)\mathrm{d}\xi, \qquad \overline{f}(x) = \frac{1}{\varepsilon}\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} f(\xi)\mathrm{d}\xi,$$

$$\overline{p}(x) = \frac{1}{\varepsilon}\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} \rho(\xi)\mathrm{d}\xi, \qquad \overline{\kappa}(x) = \frac{1}{\varepsilon}\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} \kappa(\xi)\mathrm{d}\xi,$$

and upscale the micromodel to the macromodel.

$$\begin{cases} u = -\kappa p' \\ u' = f \end{cases} \qquad \Rightarrow \begin{cases} \overline{u} = -\frac{1}{\varepsilon}\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} \kappa p' \overset{?}{=} -\overline{\kappa}\,\overline{p'} \\ \overline{u'} = \overline{f} \end{cases}$$

<u>Fundamental Problem</u> in upscaling: Nonlinearities.

$$\text{average of } f(x) \neq f(\text{average of } x)$$

<u>Simple averaging of the data I.</u> Replace κ by some local average $\overline{\kappa}$, which varies on a larger scale, so the coarse grid sol$\underline{\text{n}}$ is accurate.

$$\overline{\kappa}(x) = \frac{1}{\varepsilon}\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} \kappa(\xi)\mathrm{d}\xi.$$

The macromodel :

$$\begin{cases} \overline{u} = -\overline{\kappa}\nabla\overline{p} & \text{in } \Omega \\ \operatorname{div}\overline{u} = f & \text{in } \Omega \end{cases}$$

(Boundary Condition is ommitted.)

The questions are:

1. Is this the “right” way to average?
2. Is $(\overline{u},\overline{p}) \approx (u,p)$? i.e. $\overline{\kappa\cdot\nabla p} \approx \overline{\kappa}\nabla\overline{p}$?

CFD people do this often. There is “closure problem”.

<u>Simple averaging of the data II.</u>

What is the correct average?

We simply did the volume average, but there are different averages. Which ones are better and which ones are not?

– Arithmatic averaging

$$\overline{\kappa} = \frac{1}{n}\sum_{i=1}^{n}\kappa_i$$

– Harmonic averaging

$$\overline{k} = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\kappa_i}\right)^{-1}.$$

The reciprocal of the average of the reciprocals. *Emphasizes the small values.*

Example 3.28 (1D solution along layers, “1.5D.”)**.**

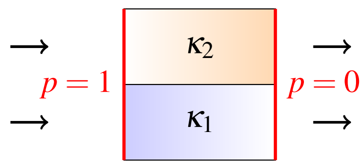


Figure 3: 1D Layer Problem.

Let κ take on 2 values as above and consider the problem

$$\begin{cases} \operatorname{div} u = 0 & 0 < x < 1, \quad 0 < y < 1 \\ u = -\kappa \nabla p \end{cases}$$
$$p(0,y) = 1 \text{ and } p(1,y) = 0.$$

The solution is $p = 1 - x$.
$u_2 = 0$ (vertical direction) $u_1 = \begin{cases} \kappa_2 & y > \frac{1}{2} \\ \kappa_1 & y < \frac{1}{2} \end{cases}$.
The average flux is $\overline{u}_1 = \overline{\kappa} = \frac{\kappa_1+\kappa_2}{2}$.

Equivalent solution: p and $\overline{u}$ solve the same problem with $\overline{\kappa}$ in place of κ.
The arithmetic average is therefore the correct average along layers!
We change κ_1, κ_2 to $\overline{\kappa}$.

$$\overline{u} = -\overline{\kappa}(-1) = \overline{\kappa}.$$

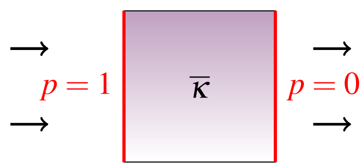


Figure 4: 1D Layer Problem.

From mathematical point of view we look at the conductivity/permeability field, and we like to do some volume average process to understand macroscopic behavior.

Recall 3.29 (1-D along layers)**.**

$$\begin{cases} \operatorname{div} u = 0 & 0 < x < 1, 0 < y < 1 \\ u = -\kappa \nabla p \end{cases}$$

$$p(0,y) = 1 \text{ and } p(1,y) = 0.$$

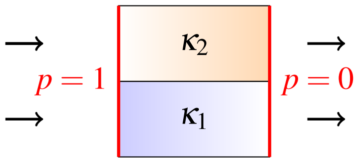


Figure 5: 1D Problem Along Layers.

<u>Solution</u>:

$$p(x,y) = 1 - x,$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \begin{cases} k_2 & y > \frac{1}{2} \\ k_2 & y < \frac{1}{2} \end{cases} \\ 0 \end{bmatrix}.$$

The average flux $\overline{u}_1 = \overline{k} = \frac{k_1 + k_2}{2}$.

<u>Equivalent solution</u>: p and $\overline{u}$ solve the same problem with $\overline{k}$ in place of k. The arithmetic average is therefore the correct average along layers.

Example 3.30 (1-D Solution across layers)**.**

$$\begin{cases} \operatorname{div} u = 0 & 0 < x < 1, 0 < y < 1 \\ u = -\kappa \nabla p \end{cases}$$

$$p(0,y) = 1 \text{ and } p(1,y) = 0. \text{ (B.C)}$$

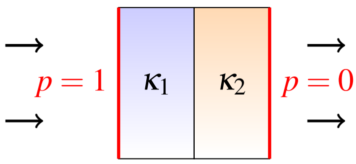
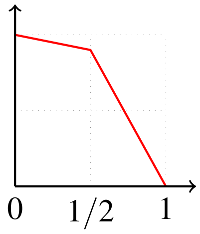


Figure 6: 1D Problem Across Layers.

The Solution: We can guess. Take $\kappa_1 \gg \kappa_2$. Then for example κ_1 big implies conductivity is high so the temperature does not change much. So we expect a solution that looks like:



$$p(x,y) \overset{\text{check!}}{=} \begin{cases} 1 - x\frac{\overline{k}}{k_1} & x < \frac{1}{2} \\ (1-x)\frac{\overline{k}}{k_2} & x > \frac{1}{2} \end{cases} \tag{6}$$

$$u_1 = \overline{k} = 2/\left(\frac{1}{k_1} + \frac{1}{k_2}\right) : \text{ (Harmonic average)} \tag{7}$$

$$u_2 = 0.$$

The average flow is fixed $\overline{u}_1 = \overline{k}$.

Equivalent solution: $\overline{p} = 1 - x$ and $\overline{u}$ solve the same problem with $\overline{k}$ in place of k.
Replacing κ with constant $\overline{\kappa}$ we have a Laplace equation in place of (7).

$$\begin{cases} u = -\overline{\kappa}\nabla\overline{p} \\ \Delta\overline{p} = 0 \end{cases} .$$

Over a small region (which will become a grid element), solve the problem of unit flow in each direction (horizontal flow and vertical flow).

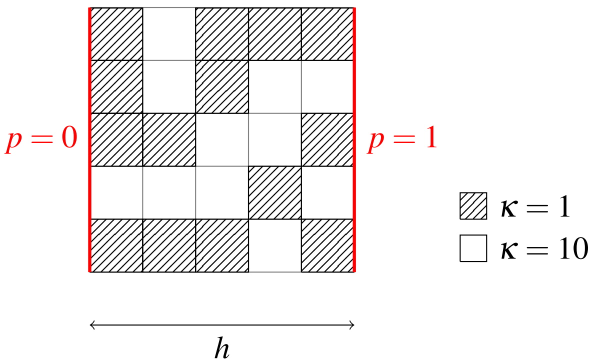


Figure 7: Anisotropic Medium

Compute the average velocity $\overline{u}_1$ in the direction of flow,

$$\overline{u} = \int_{\text{right face}} u_1\, dy = \int_{\text{left face}} u_1\, dy$$

(Note that $\bar{\cdot}$ is an integral process which turns out sometimes to be arithmetic average or harmonic average.)
and define $\overline{\kappa}_{11}$ by $u = -\kappa \nabla p$.

$$\begin{aligned} \overline{u}_1 &= \overline{\kappa}_{11} \frac{1}{h} \\ \Rightarrow \quad \overline{\kappa}_{11} &= h \overline{u}_1 \end{aligned}$$

Do this for both directions.
We get a tensor for $\overline{\kappa}$:

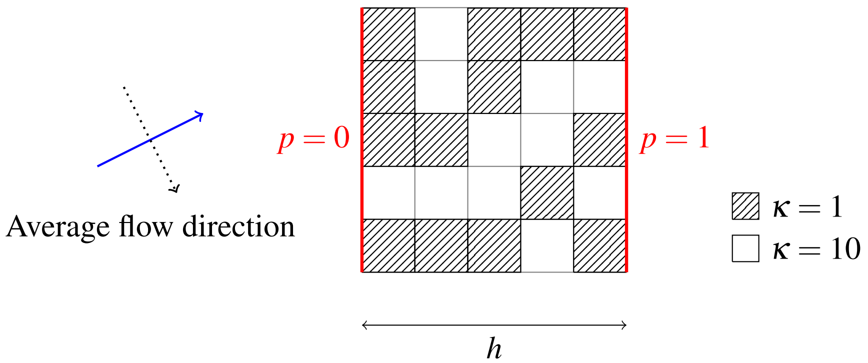
$$\overline{k} = \begin{bmatrix} \kappa_{11} & 0 \\ 0 & \kappa_{22} \end{bmatrix}$$

Definition 3.31 (Closure assumption)**.** To get the local solution, we impose a unit pressure drop in a certain direction. That is, we imposed boundary conditions, which are not seen in the true flow. We call this assumption a closure assumption. It is the source of our upscaling error.
In the three problems we have seen we imposed boundary conditions (unit condition drop from $p = 1$ to $p = 0$) which is not natural in real physical systems. i.e., this is not what we see in true flow. However we can compute something with it. In many multiscale methods we impose closure assumption (CFD people do this all the time) and we would like to reduce the error due to this assumption.

Definition 3.32 (Anisotropy)**.** Locally the medium is isotropic (i.e. the same in all direction).
However $\overline{\kappa}$ should be

$$\overline{\kappa} = \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix}$$

That is, $\overline{\kappa}$ is anisotropic.



The average flow direction indicates some kind of *eigen* direction.

Remark 3.33. It is not so easy to quantify this anisotropy computationally. What BCs should we impose?

Tom Hou (Caltech) Oversampling: You solve the problem on a larger domain and "carve-out" the original domain from the solution. See Figure 8.

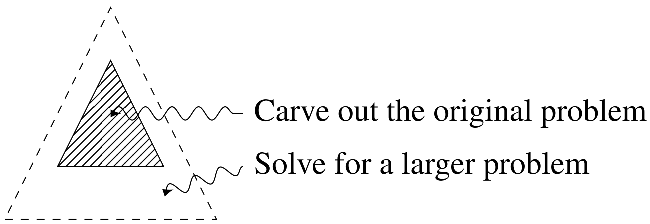


Figure 8: Oversampling

3.5 Multiscale Numerics

Model

$$\begin{cases} -\operatorname{div}(\kappa\nabla p) = f & \text{in } \Omega \\ -\kappa\nabla p\cdot\nu = 0 & \text{on } \partial\Omega \end{cases}$$

Function space

$$X = H^1(\Omega)/\mathbb{R} = \left\{\omega \in L^2(\Omega) : \nabla\omega \in L^2(\Omega)^2, \int_\Omega \omega\, dx = 0\right\}$$

$$(\psi,\phi) = \int_\Omega \psi(x)\cdot\phi(x)\, dx \quad \text{(inner product)}$$

A variational problem
Find $p \in X$ s.t.

$$a(p,\omega) \equiv (\kappa\nabla p, \nabla\omega) = (f,\omega) \quad \forall\omega\in X.$$

Theorem 3.34. *The problems are equivalent and there exists a unique solution.*

Galerkin's method
Let $X_h \subset X$ be finite diml̲ subspace. The approximate variational problem:

$$\text{Find} \quad p_h \in X_h \text{ s.t. } a(p_h,\omega_h)(f,\omega_h) \quad \forall\omega_h\in X_h.$$

Theorem 3.35. $\exists C > 0$ *s.t.*

$$\|p-p_h\| \le C \min_{\omega_h\in X_h} \|p-\omega_h\|_1$$

where $\|\omega\|_1 = \left\{\int_\Omega \left(|\omega|^2 + |\nabla\omega|^2\right) dx\right\}^{1/2}$

Remark 3.36 (Optimality)**.** The approximation is optimal up to constant C.

Construction of X_h: Define a grid over Ω over each grid element E. Let $\omega_h \in X_h$ be a polynomial. Piece them together so they are continuous.

Theorem 3.37. *For polynomials of degree k,*

$$\min_{\omega_h \in X_h} \|p - \omega_h\|_1 \leq C\|p\|_{k+1} h^k$$

Corollary 3.38. $p_h \to p$ *as* $h \to 0$. *In fact,*

$$\|p - p_h\|_1 \leq C\|p\|_{k+1} h^k = \mathscr{O}(h^k).$$

Heterogenity & problems of scale

Difficulty: Fine-scale variation in κ (permeability in dispersive media) leads to fine-scale variation in the soln (u, p).

THU OCT 6, 2011

Recall 3.39.

$$\|p-p_h\| \leq C\|p\|_{k+1}h^k = \mathscr{O}(h^k).$$

Suppose κ varies on the scale ε (e.g. $\kappa = 1+\sin^2\frac{\pi x}{\varepsilon}$). Then

$$|\nabla p| = \mathscr{O}(\varepsilon^{-1})$$
$$|D^\kappa p| = \mathscr{O}(\varepsilon^{-k})$$

Typical error estimates

$$\underbrace{\inf_{q\in P_{k-1}(\mathscr{T}_h)}}_{\text{mesh)}} \|p-q\|_0 \leq \underbrace{C}_{\text{Interpolation error}} \|p\|_k h^k \sim C\left(\frac{h}{\varepsilon}\right)^k$$

- If $h > \varepsilon$, this is NOT small!
- To resolve p, we need a spatial discretization. That is, we must resolve κ in some way.

Multiscale Approachs

Objective: To solve the problem in a way that

- does NOT fully incorporate the problem dynamics (i.e. solves some global coarse scale problem to resolution $H > \varepsilon$ (H emphasizes *coarseness*)
- yet capture significcant features of the soln, by taking into account the micro-structure (to resolution $h < \varepsilon$.)

The Variational Multiscale Method (VMS)

(Hughes et al '95, '98, Brezzi '99)

Goal: Find the part of the solution that is unresolved in the standard finite element approximation.

Problem: Find $u \in X$ s.t.

$$a(u,v) = f(v) \quad \forall v \in X.$$

Direct sum decomposition : Define coarse and fine (i.e. subgrid) scales

$$X = \overline{X} \oplus X'.$$

Separating Scales: Find $\overline{u} \in \overline{X}$ and $u' \in X'$ such that

$$a(\overline{u}+u',\overline{v}) = f(\overline{v}) \quad \forall \overline{v} \in \overline{X} \text{ (coarse scale)}$$
$$a(\overline{u}+u',v') = f(v') \quad \forall v' \in X' \text{ (subgrid scale)}$$

Closure Operators (due to T. Arbogast)

We can define u': $\overline{X} \to X'$ by

$$a(\overline{v}+u'(\overline{v}),v') = f(v') \quad \forall v' \in X'$$

Affine representation: Define the linear operator $\hat{u}' : \overline{X} \to X'$ by

$$a(\overline{v}+\hat{u}'(\overline{v}),v') = 0 \quad \forall v' \in X'$$

and constant term $\tilde{u}' \in X'$ by

$$a(\tilde{u}',v') = f(v') \quad \forall v' \in X',$$

Then

$$u' = u'(\overline{u}) = \hat{u}'(\overline{u}) + \tilde{u}'.$$

Remark 3.40. Given the coarse scale, we recover the fine scale. In upscaling theory closure operators are often assumed rather than being derived. Hence the term supgrid upscaling.

Upscaling the Problem

upscaled problem: Find $\overline{u} \in \overline{X}$ such that

$$a(\overline{u}+\hat{u}'(\overline{u}),\overline{v}) = f(\overline{v}) - a(\tilde{u}',\overline{v}) \quad \forall \overline{v} \in \overline{X}.$$

or, in symmetric form,

$$a\big(\overline{u}+\hat{u}'(\overline{u}),\overline{v}+\underbrace{\hat{u}'(\overline{v})}_{a(\overline{u}+\hat{u}'(\overline{u}),\hat{u}'(\overline{v}))=}\big) = f(\overline{v}) - a(\tilde{u}',\overline{v}) \quad \forall v \in \overline{X}.$$

Change of scale results in modifying both a and f where

$$\begin{aligned} \mathscr{A} :& \overline{X} \times \overline{X} \to \mathbb{R} \\ & (\overline{u},\overline{v}) \mapsto \mathscr{A}(\overline{u},\overline{v}) = a\left(\overline{u}+\hat{u}'(\overline{u}),\overline{v}+\hat{u}'(\overline{v})\right) \\ F :& \overline{X} \to \mathbb{R} \\ & \overline{v} \mapsto F(v) = f(\overline{v}) - a(\tilde{u}',\overline{v}) \end{aligned}$$

Full two-scale solution

$$u = \overline{u} + u'(\overline{u}) = \overline{u} + \hat{u}'(\overline{u}) + \tilde{u}'.$$

Upscaled Problem: Find $\overline{u} \in \overline{X}$ such that

$$a(\overline{u}+\hat{u}'(\overline{u}),\overline{v}+\hat{u}'(\overline{v})) = f(\overline{v}) - a(\tilde{u}',\overline{v}) \quad \forall \overline{v} \in \overline{X}.$$

Finite Element Approximation: Find $\overline{u}_h \in \overline{X}_h \subset \overline{X}$ such that

$$a(\overline{u}_h+\hat{u}'(\overline{u}_h),\overline{v}_h+\hat{u}'(\overline{v}_h)) = f(\overline{v}_h) - a(\tilde{u}',\overline{v}_h) \quad \forall \overline{v} \in \overline{X}.$$

in practice this is discretized
e.g. quadrature

Multiscale FE Spaces Let

$$\widehat{X}_h = \left\{ \overline{v}_h + \hat{u}'(\overline{v}_h) : \overline{v}_h \in \overline{X}_h \right\}$$

Note $\dim \overline{X}_h = \dim \widehat{X}_h$.

Equivalent form: To find

$$u_h \in \widehat{X}_h + \tilde{u}' \quad \text{such that}$$
$$a(u_h, \hat{v}_h) = f(\hat{v}_h) \quad \forall \hat{v}_h \in \widehat{X}_h$$

Remark 3.41. The Key is to find a decomposition $X = \overline{X} \oplus X'$ so that we can efficiently compute the upscaling operator $\hat{u}'$ on $\overline{X}_h$.

TUE OCT 11, 2011

Recall 3.42 (VMS)**.** Find $u \in X$ such that $a(u,v) = f(v)$, $\forall v \in X$. Write,

$$X = \underbrace{\overline{X}}_{\substack{\text{coarse}\\ \text{average}\\ \text{(resolved)}}} \oplus \underbrace{X'}_{\substack{\text{fine}\\ \text{average}\\ \text{unresolved}}} \qquad \text{(direct sum decomposition)}$$

which gives $u = \overline{u} + u'$.
Find $\overline{u} \in \overline{X}$ and $u' \in X'$ s.t

$$\begin{aligned} a(\overline{u}+u', \overline{v}) &= f(\overline{v}), \quad \forall \overline{v} \in \overline{X} \\ a(\overline{u}+u', v') &= f(v'), \quad \forall v' \in X'. \end{aligned}$$

❇

<u>Closure Operation</u>
$u' : \overline{X} \to X'$, $\overline{v} \mapsto u'(\overline{v})$.

$$u' = u'(\overline{u}) = \hat{u}'(\overline{u}) + \tilde{u}'$$

where

$$a(\overline{v} + \underbrace{\hat{u}'(\overline{v})}_{\substack{\in X'\\ \text{linear operator}}}, v') = 0 \quad \forall v' \in X'$$

$$a(\underbrace{\tilde{u}'}_{\substack{\in X'\\ \text{constant term}}}, v') = f(v') \quad \forall v' \in X'$$

❇

<u>Symmetric Form</u>
unscaled problem.

$$\begin{aligned} a(\overline{u} + \hat{u}'(\overline{u}), \overline{v}) &= f(\overline{v}) - a(\tilde{u}', \overline{v}) \quad \forall \overline{v} \in \overline{X} \\ \Rightarrow \underbrace{a(\overline{u} + \hat{u}'(\overline{u}), \overline{v} + \hat{u}'(\overline{v}))}_{\text{symm if } a \text{ is symm}} &= f(\overline{v}) - a(\tilde{u}', \overline{v}) \quad \forall v \in \overline{X} \end{aligned}$$

❇

<u>FE</u> Approx.: Find $\overline{u}_h \in \overline{X}_h \subset \overline{X}$.

$$a(\overline{u}_h + \hat{u}'(\overline{u}_h), \overline{v}_h + \hat{u}'(\overline{v}_h)) = f(\overline{v}_h) - a(\tilde{u}', \overline{v}_h) \quad \forall v_h \in \overline{X}_h$$

❇

Multiscale FE Space

Let

$$\widehat{X}_h = \{\overline{v}_h + \underbrace{\hat{u}'(\overline{v}_h)}_{\text{non-polynomial}} : \overline{v}_h \in \overline{X}_h\}$$

Note $\dim \widehat{X}_h = \dim \overline{X}_h$.

❇

Equivalent form:

Find $u_h \in \widehat{X}_h + \tilde{u}'$ s.t

$$a(u_h, \hat{v}_h) = f(\hat{v}_h) \quad \forall \hat{v}_h \in \widehat{X}_h. \qquad \left(u_h = \overbrace{\underbrace{\overline{u}_h}_{\text{coarse}} + \underbrace{\hat{u}'(\overline{u}_h)}_{\text{lin}}}^{=\hat{u}_h} + \underbrace{\tilde{u}'}_{\text{const.}} \right)$$

❇

Remark 3.43. The key is the find a decomposition $X = \overline{X} \oplus X'$ so that we can efficiently compute the upscale operator $\hat{u}'$ such that on $\overline{X}_h$.

Example 3.44 (Babuska & Osborn '83, Hou & Wu '97)**.** Differential problem:

$$\begin{cases} -\frac{\mathrm{d}}{dx}\left(\kappa \frac{\mathrm{d}p}{dx}\right) = 0, & 0 < x < 1 \\ p(0) = 0, & p(1) = 1 \end{cases}$$

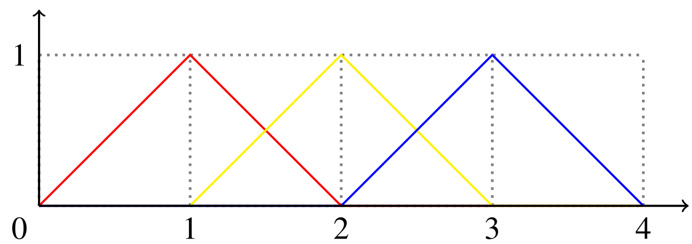


Figure 9: Nodal Basis Functions

The soln $p(x) \in X + x$,

$$X = \left\{\omega \in H^1(I) : \omega(0) = \omega(1) = 0\right\}$$

satisfies

$$a(p, \omega) = (\kappa p_x, \omega_x) = 0 \quad \forall \omega \in X$$

Constructing $\overline{X}_h$ Choose a uniform grid of five points

$$x_i = \frac{i}{4}, i = 0, 1, 2, 3, 4.$$

Two-scale decomposition $X = \overline{X} \oplus X'$.

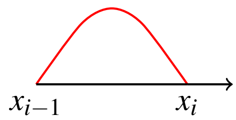


Figure 10: Bubble Function

Let X' be the "bubble functions" over the grid

$$X' = \{\omega \in H^1 : \omega(x_i) = 0, i = 0,1,2,3,4\}.$$

❄

Localization
$\hat{u}'(\overline{v})$ breaks in to 4 small or localized problems

$$a(\overline{v} + \hat{u}'(\overline{v}), v') \qquad \forall v' \in X' \text{ on } (x_{i-1}, x_i), \quad i = 1, \ldots, 4.$$

❄

Constructing X_h

$$\psi = \overline{\omega} + \hat{u}'(\overline{\omega}).$$

$$\begin{cases} -\frac{\mathrm{d}}{dx}\left(\kappa \frac{\mathrm{d}\psi}{dx} = 0\right) & 0 < x < \frac{1}{4} \\ \psi(0) = 0, \psi(\frac{1}{4}) = 1 \end{cases}$$
$$\begin{cases} -\frac{\mathrm{d}}{dx}\left(\kappa \frac{\mathrm{d}\psi}{dx} = 0\right) & 0 < x < \frac{1}{4} \\ \psi(\frac{1}{4}) = 1, \psi(\frac{1}{2}) = 0 \end{cases}$$

Note that if $\kappa = 1$ then the solution would be the basis of the first basis (red). Otherwise it will look like the function in Figure 11.
Thomas Hou (Oversampling method in 2D)

❄

2D

$$\begin{cases} -\operatorname{div}(\kappa \nabla p) = 0 & T \in \mathscr{T}_h \\ p = 0 & \text{on } \partial T. \end{cases}$$

Hou, instead of doing this did oversampling

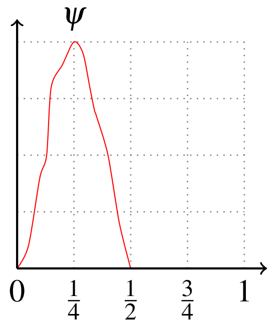


Figure 11: Multiscale Basis

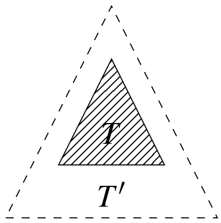


Figure 12: Oversampling

TUE NOV 1, 2011

Recall 3.45.

$$\begin{cases} -\varepsilon\Delta u + \beta\cdot\nabla u + u = f, & \Omega \\ u = 0, & \Gamma \end{cases}$$

$0 < \varepsilon \ll 1$

$$a(u,v) = \int_\Omega \varepsilon\nabla u\cdot\nabla v + \int_\Omega (\beta\cdot\nabla u + u)v\,dx$$

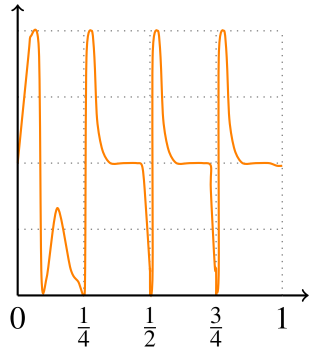
$$a(u,u) = \int_\Omega \varepsilon|\nabla u|^2 + \int_\Omega u^2\,dx \geq \varepsilon\|u\|^2_{H^1(\Omega)}$$

$$\left(\text{Since } \int_\Omega (\beta\cdot\nabla u)u\,dx \underset{\operatorname{div}\beta=0}{=} \int_\Omega \operatorname{div}(\beta u)u = -\int_\Omega \beta u\cdot\nabla u + \int_{\partial\Omega} u^2\beta\cdot n\,ds\right)$$

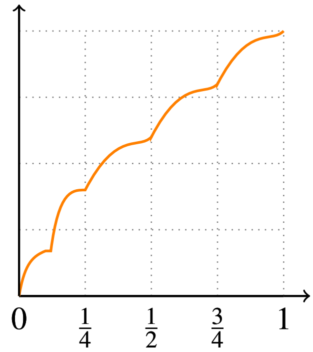
$$\ell(f) = \int_\Omega f v\,dx \quad \Rightarrow \quad |\ell(f)| \leq \gamma\|v\|.$$

Therefore, we have

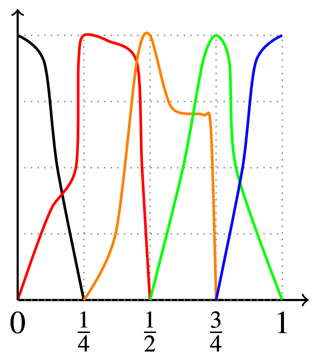
$$\|u-u_h\|_{H^1(\Omega)} \leq \frac{\gamma}{\varepsilon}\|u-v_h\|_{H^1(\Omega)} \quad \forall v_h \in V_h$$



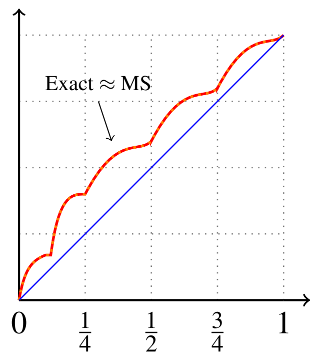
True Solution ∇p



True Solution p



Multiscale Basis



Multiscale (r) vs. Standard (b)

Table 1: A Sketch of a Multiscale Solution

where $\underset{\text{piecewise linear}}{V_h} \subset V$,

$$\begin{aligned}
\min_{v_h \in V_h} \|u - uh\|_{H^1(\Omega)} &\le \|u - \prod_h u\|_{H^1(\Omega)} \\
&\le Ch|u|_{H^2(\Omega)} \\
\Rightarrow \quad \|u - u_h\|_{H^1(\Omega)} &\le \underbrace{C\frac{h}{\varepsilon}|u|_{H^2(\Omega)}}_{\text{can be big}}
\end{aligned}$$

$$a(u,u) = (u_\beta + u, u) = \int_\Omega u_\beta u\,dx + \int_\Omega u^2 d\,dx \ge \int_\Omega u^2\,dx = \underset{\text{NOT full norm}}{\|u\|_{L^2(\Omega)}}$$

since

$$\int_\Omega u_\beta u\, dx \underset{\operatorname{div}\beta=0}{=} -\int_\Omega u u_\beta\, dx + \underbrace{\int_\Gamma u^2 \beta\cdot n\, ds}_{=\int_{\Gamma_+} u^2\beta\cdot n\,ds + \underbrace{\int_{\Gamma_-} u^2\beta\cdot n\, ds}_{=0}} \Rightarrow \int_\Omega u_\beta u\, dx = \frac{1}{2}\int_{\Gamma_+} u^2\beta\cdot n\, ds \geq 0$$

Standard Galerkin with strongly imposed boundary condition.
Find $u^h \in V_h$, $u^h = g$ on the nodes of Γ_-.

$$(u^h_\beta + u^h, v^h) = (f, v^h) \quad \forall v^h \in V_h \text{ with } v^h = 0 \text{ on } \Gamma_-. \tag{*}$$

Standard Galerkin with weakly imposed boundary condition
Find $u^h \in V_h$ s.t.

$$(u^h_\beta + u^h, v^h) - \langle u^h, v^h\rangle = (f, v^h) - \langle g, v^h\rangle \quad \forall v^h \in V_h. \tag{**}$$

(This is hyperbolic equation so we cannot give boundary condition everywhere, only at the inflow boundary.)
Where $\langle v, w\rangle_- = \int_{\Gamma_-} vw\beta\cdot n\, ds$
Define bilinear form $b(\omega, v) := (\omega_\beta + w, v) - \langle \omega, v\rangle_-$ and $\ell(v) = (f, v) - \langle g, v\rangle_-$.
Find $u^h \in V - H$ such that $b(u^h, v^h) = \ell(v^h) \quad \forall v^h \in V_h$.
Let u be a solution of the exact problem

$$b(u, v^h) = \ell(v^h) \quad \forall v^h \in V_h \subset \underset{\text{solution space for} u}{V}.$$

Remark 3.46. regularity In the hyperbolic case $u \in H^1(\Omega)$ may not be guaranteed (cf. elliptic case). The solution may be discontinuous if g is not continuous. We need to have some notion from hyperbolic PDEs. we only need the existence of the direction derivative $u_\beta = \beta\cdot\nabla u$. So it is basically ODE along the streamline.

$$e := u^h - u.$$

Then $b(e, v^h) = 0$, $\forall v^h \in V_h$.

Lemma 3.47. *coarsivity*

$$b(v, v) = \|v\|_{L^2(\Omega)} + \frac{1}{2}|v|^2$$

where $|v| = \left(\int_\Gamma v^2 |n\cdot\beta|\, ds\right)^{1/2}$

❄

Proof.

$$(v_\beta, v) = \langle v, v\rangle - (v, v_\beta)$$

$$\Rightarrow \quad (v_\beta, v) = \frac{1}{2}\langle v, v\rangle \left(= \frac{1}{2}\int_\Gamma v^2 \underbrace{\beta \cdot n}\, ds \right)$$
$$= \frac{1}{2}\langle v, v\rangle_+ \frac{1}{2}\langle v, v\rangle_-.$$

$$\begin{aligned} b(v,v) &= (v_\beta + v, v) - \langle v, v\rangle_- \\ &= \|v\|^2_{L^2(\Omega)} + \underbrace{\frac{1}{2}\langle v, v\rangle_+}_{v_h} - \underbrace{\frac{1}{2}\langle v, v\rangle_-}_{v_0}. \\ &= \|v\|_{L^2(\Omega)} + \frac{1}{2}\int_\Gamma v^2 |\beta \cdot n|\, ds \end{aligned}$$

□

Remark 3.48.

Lemma $\Rightarrow$ unique solvability of

(square system: existence iff uniqueness)

Theorem 3.49. *$\exists$ a constant C s.t. if u is the solution of (*) and $U^h \in V_h$ is in the solution of (**) then*

$$\left\| u - u^h \right\|_{L^2(\Omega)} + |u - u^h| \le Ch^r \|u\|_{H^{r+1}(\Omega)}.$$

Proof. Let $\tilde{u}^h$ be that interpolant of u.

$$\left\| u - \tilde{u}^h \right\|_{L^2(\Omega)} \le Ch^{r+1}\|u\|_{H^{r+1}(\Omega)} \tag{1}$$

$$\|u - \tilde{u}^h\|_{H^1(\Omega)} \le Ch^{r+1}\|u\|_{H^{r+1}(\Omega)} \tag{2}$$

$$|u - \tilde{u}| \le Ch^{r+\frac{1}{2}}\|u\|_{H^{r+1}(\Omega)} \tag{3}$$

(3) follows from "trace inequality".

$$\|\tilde{e}\|_{L^2(\partial\Omega)} \le C\|\tilde{e}\|_{L^2(\Omega)}{}^{\frac{1}{2}}\|\nabla e\|_{L^2(\Omega)}{}^{\frac{1}{2}}. \quad \text{(cf. Brenner and Scott)}$$
$$\left(\le C\underbrace{h^{\frac{r+1}{2}} h^{\frac{r}{2}}}_{h^{r+\frac{1}{2}}}\|u\|_{H^{r+1}(\Omega)} \right)$$

□

TUE NOV 3, 2011

Recall 3.50. We have

$$\begin{cases} u_\beta + u = f & \Omega \\ u = g & \Gamma_- = \{x \in \partial\Omega : \beta \cdot n < 0\} \end{cases}$$

FEM (weakly imposed BC) Find $u^h \in V_h \subset V = H^1(\Omega)$ s.t.

$$b(u^h, v^h) = (u^h_\beta + u^h, v^h) - \langle u^h, v^h \rangle_- = (f, v^h) - \langle g, v^h \rangle_-$$

where $\langle u, v \rangle_- = \int_{\Gamma_-} uv\beta \cdot n\, ds$. We have

$$b(v,v) = \|v\|_{L^2(\Omega)} + \frac{1}{2}|v|^2$$

with $|v|^2 \int_\Gamma v^2 |\beta \cdot n|\, ds$
We have interpolation properties

$$\left\| u - \tilde{u}^h \right\|_{L^2(\Omega)} \leq Ch^{r+1} \|u\|_{H^{r+1}(\Omega)}$$
$$\|u\|_{H^1(\Omega)} \leq Ch^{r+1} \|u\|_{H^{r+1}(\Omega)}$$
$$|u - \underbrace{\tilde{u}^h}_{\text{interpolation of } u}| \leq Ch^{r+1} \|u\|_{H^{r+1}(\Omega)}$$

Theorem 3.51.

$$\|u - u^h\|_{L^2(\Omega)} + |u - u^h| \leq Ch^r \|u\|_{H^{r+1}(\Omega)}$$

Proof. Define $\eta^h = u - \tilde{u}^h, e^h = u^h - \tilde{u}^h$. Then $e = u - u^h = \eta^h - e^h$. We have error equation:

$$b(e, v^h) = 0 \quad \forall v^h \in V^h.$$

$$
\begin{aligned}
\|e^h\|^2_{L^2(\Omega)} + \frac{1}{2}|e^h|^2 &= b(e^h, e^h) \\
&= b(\eta^h - e, e^h) \\
&= b(\eta^h, e^h) - \underbrace{b(e, e^h)}_{=0} \\
&= (\eta^h_\beta, e^h) + (\eta^h, e^h) - \langle \eta^h, e^h \rangle_- \\
&\underset{\substack{ab \le \frac{\varepsilon}{2} a^h + \frac{1}{2\varepsilon} b^2 \\ ab \le a^2 + \frac{1}{4} b^2}}{\le} \|\eta^h_\beta\| \|e^h\| + \|\eta^h\| \|e^h\| + |\eta^h||e^h| \\
&\le \|\eta^h_\beta\|^2 + \frac{1}{4}\|e^h\|^2 + \|\eta^h\|^2 + \frac{1}{4}\|e^h\|^2 + \|\eta^h\|^2 + |\eta^h|^2 + \frac{1}{4}|e^h|^2 \\
&\Rightarrow \underbrace{\frac{1}{2}\|e^h\|^2 + \frac{1}{4}|e^h|^2}_{\frac{1}{2}(\|e^h\| + |e^h|)^2 \le} \le \|\eta^h_\beta\|^2 + \|\eta^h\|^2 + |\eta^h|^2 \\
&\le \left[ch^r \|u\|_{H^{r+1}(\Omega)}\right]^2 \\
&\Rightarrow \|e^h\| + |e^h| \le ch^r \|u\|_{H^{r+1}(\Omega)}.
\end{aligned}
$$

Therefore,

$$
\underbrace{\|e\|}_{\cdots + |e|} = \underbrace{\|u - u^h\|}_{\cdots + |u - u^h|} \le \|\eta^h\| + |\eta^h| + \|e^h\| + |e^h| \le C\left(\|\eta^h\| + |\eta^h|\right)
$$

□

Try to look at the bilinear form and see that

$$
b(v, v) \underset{\text{or } \ge}{=} \|v\|^2 + \frac{1}{2}|v|^2.
$$

Remark 3.52. Our scheme (**) is NOT optimal.

Streamline Diffusion Method (SDM)

Find $u^h \in V^h$ s.t.

$$
\begin{aligned}
&(u^h_\beta + u^h, v^h + h v^h_\beta) - (1 + h)\langle u^h, v^h \rangle_- \\
&= (f, v^h + h v^h_\beta) - (1 + h)\langle g, v^h \rangle_- \quad \forall v^h \in V_h
\end{aligned}
$$

Remark 3.53. SU $\underbrace{\text{PG}}_{\text{Petrov-Galerkin}}$

1. Consistency: replace u^h by $\underset{\text{exact solution of } (*)}{u}$

$$
\begin{aligned}
&(u_\beta + u, v^h + h v^h_\beta) - (1 + h)\langle v, v^h \rangle_- \\
&\quad = (f, v^h + h v^h_\beta) - (1 + h)\langle g, v^h \rangle_-
\end{aligned}
$$

2. Stability:

$$
\begin{aligned}
b(w,v) &= (w_\beta + w, v + hv_\beta) - (1+h)\langle w,v\rangle_- \\
\ell(v) &= (f, v + hv_\beta) - (1+h)\langle g,v\rangle_-
\end{aligned}
$$

$$
b(v,v) = (v_\beta + v, v + hv_\beta) - (1+h)\langle v,v\rangle_-
$$

Lemma 3.54.

$$
B(v,v) = \|v\|_\beta^2 \quad \forall v \in H^1(\Omega)
$$

where $\|v\|_\beta^2 := h\|v_\beta\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} + \frac{1+h}{2}|v|^2$.

Remark 3.55.

$$
(**) \Rightarrow b(v,v) = \|v\|^2_{L^2(\Omega)} + \frac{1}{2}|v|^2.
$$

Proof. Recall that

$$
(v_\beta, v) \underset{\text{Green's THM}}{=} \frac{1}{2}\langle v,v\rangle \left(= \int_\Gamma v^2(\beta\cdot n)\,ds \right)
$$

$$
\begin{aligned}
B(v,v) &= (v_\beta + v, v + hv_\beta) - (1+h)\langle v,v\rangle_- \\
&= \underbrace{(v_\beta, v)}_{\frac{1}{2}\langle v,v\rangle + h\|v_\beta\|^2} + \|v\|^2 + \underbrace{h(v, v_\beta)}_{\frac{h}{2}\langle v,v\rangle} - (1+h)\langle v,v\rangle_- \\
&= \frac{1+h}{2}\left(\langle v,v\rangle_+ - \langle v,v\rangle_-\right) + \|v\|^2 + h\|v_\beta\|^2 \\
&= \frac{1+h}{2}|v|^2 + \|v\|^2 + h\|v_\beta\|^2 \\
&= \|v\|_\beta^2
\end{aligned}
$$

$$
\left(|v|^2 = \int_\Gamma v^2 |\beta\cdot|\,ds, \quad \frac{1}{2}\int_\Gamma = \frac{1}{2}\int_{\Gamma_+} + \frac{1}{2}\int_{\Gamma_-} \right)
$$

□

Remark 3.56. $B(v,v)$ has enhanced stability compared with $b(v,v)$.

This is a standard method but will not work on nonlinear problems. Shock capturing.
Error equation : $B(u,v^h) = \ell(v^h)$ (Consistency) or $B(u-u^h, v^h) = 0 \quad \forall v^h \in V^h$.