Caratheodory Theorem

Definition. (2.2.1; Outer measure)

- *Let $(X, \mathcal{M}, \mu)$ be a measure space.*
    - *Recall*
        (i) *X is a set.*
        (ii) *$\mathcal{M}$ is a $\sigma-$algebra, that is, closed under a countable union and complementations.*
        (iii) *μ is a measure on $\mathcal{M}$, non-negative & countably additive .*
    - *A* **null set** *is a set N s.t. $\mu(N) = 0$*
    - *If $\sigma-$algebra $\mathcal{M}$ includes all null set, then μ is said to be* **complete.**
- *An outer measure on a non-empty set X is a set function μ^* defined on $\mathcal{P}(X)$ which is* **non-negative, monotone and countably subadditive.**

Why introduce the outer measure? Want to **describe a general constructive procedure for obtaining complete measure.**

Example of outer measure in $X = \mathbb{R}^2$

- $X = \mathbb{R}^2$, $\mathcal{E}$=the σ-algebra generated by the set of all open rectangles in $\mathbb{R}^2$, and define

$$\rho(E) = \text{ the area of } E, \qquad E \in \mathcal{E}$$

- $(X, \mathcal{E}, \rho)$ is a measure space but it may not be complete.
- This ρ is called **pre-measure**.
- For $A \subset X$, we define

$$\mu^*(A) = \inf\{\rho(E) \ : \ A \subset E, \ E \in \mathcal{E}\}.$$

Then μ^* is an outer measure.

Proposition. (2.2.2: Construction of outer measure μ^* on $\mathcal{P}(X)$)

Let $\mathcal{E} \subset \mathcal{P}(X)$ be an algebra of sets and $\rho : \mathcal{E} \to \mathbb{R}^+ \cup \{0\}$ an set valued function such that $\rho(\emptyset) = 0$. For $A \subset X$, we define

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \rho(E_j) \ : \ A \subset \cup_{j=1}^{\infty} E_j, \ E_j \in \mathcal{E} \right\}.$$

Then μ^* is an outer measure.

Proof.

1. **Non-negative.** By its definition, $\mu^*(\emptyset) = 0$ and $\mu(A) \geq 0$ for $A \subset X$.
2. **Monotone.** If $A \subset B$ and $B \subset \cup_{j=1}^{\infty} E_j$, then $A \subset \cup_{j=1}^{\infty} E_j$ and $\mu^*(A) \leq \mu^*(B)$.
3. See the next page.

Proposition. (Continue...)

$\mu^*(A) = \inf\left\{\sum_{j=1}^{\infty} \rho(E_j) \ : \ A \subset \cup_{j=1}^{\infty} E_j, \ E_j \in \mathcal{E}\right\}$: *outer measure.*

Continue...

3. It remains to prove **countable subadditivity.**
   Let $A = \cup_{j=1}^{\infty} A_j$. Let $\epsilon > 0$ be given.
   - For each $j = 1, 2, \cdots,$ $\exists E_{jk} \in \mathcal{E}$ s .t.

$$A_j \subset \cup_{k=1}^{\infty} E_{jk} \ \ \& \ \ \sum_{k=1}^{\infty} \mu^*(E_{jk}) \leq \mu^*(A_j) + \epsilon 2^{-j}$$

   and therefore

$$\mu^*(A) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mu^*(E_{jk}) \leq \sum_{j=1}^{\infty} \left(\mu^*(A_j) + \epsilon 2^{-j}\right) \leq \sum_{j=1}^{\infty} \mu^*(A_j) + \epsilon$$

   Since ϵ is arbitrary small, $\mu^*(A) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$.

Definition. (2.2.3: μ^*-measurable by Caratheodory)

Let μ^* be an outer measure on a set X. A subset $A \subset X$ is said to be **μ^*-measurable** *if*

$$\forall E \subset X, \quad \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$$

- This is SUPER CLEVER defintion!
- This definition provides a method of constructing a complete measure space $(X, \mathcal{M}, \mu^*)$ where $\mathcal{M}$ is the collection of all measurable sets.
- **Example.** The Lebesgue measure on $X = \mathbb{R}$ is an extension of the pre-measure defined by $\rho((a, b]) = b - a$.
  1. Let $X = \mathbb{R}$. Let $\mathcal{E}$ be the smallest $\sigma-$algebra generated by half-open intervals $(a, b]$. Then $(\mathbb{R}, \mathcal{E}, \rho)$ is an measure space.
  2. Define the outer measure μ^* as in Prop 2.2.2.
  3. Denote by $\mathcal{M}$ the collection of all measurable sets.
  4. Then $(\mathbb{R}, \mathcal{M}, \mu^*)$ is a complete measure space.

Theorem. (2.2.4: Caratheodory extension theorem)

Let μ^* is an outer measure on X. Let $\mathcal{M}$ be the collection of all measurable sets. Then $\mathcal{M}$ is $\sigma-$algebra and **the restriction of μ^* to $\mathcal{M}$ is a complete measure.**

Proof.

- Prove that $\mathcal{M}$ is $\sigma-$algebra. Easy.
- Prove that $(X, \mathcal{M}, \mu^*)$ is a measure space. Easy.
- Prove that μ^* is complete measure.
  Proof. If $\mu^*(A) = 0$, then for any $E \subset X$

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \setminus A) \leq \mu^*(A) + \mu^*(E) = \mu^*(E)$$

  Hence, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$ for any $E \subset X$.
  Hence, $A \in \mathcal{M}$.

Lebesgue-Stieltjes measure

- **Example: Lebesgue-Stieltjes measure on $X = \mathbb{R}$.**
  - Let $\mathcal{E}$ be the algebra containing half open intervals $(a, b]$.
  - Define $\rho_F((a, b]) = F(b) - F(a)$ where $F : \mathbb{R} \to \mathbb{R}$ is a monotone increasing function.
  - ρ_F is a pre-measure measure on $\mathcal{E}$ but ρ_F is not complete.
  - Let μ^* be the outer measure defined as before.
  - Denote by $\mathcal{M}$ the collection of all measurable sets.
  - Denote by $\mu = \mu^*|_{\mathcal{M}}$ the restriction of μ^* on $\mathcal{M}$.
  - This μ is called a Lebesgue-Stieltjes measure generated by F.
- **Example: Lebesgue-Stieltjes measure on $X = \mathbb{R}^n$ or metric space.** The corresponding outer measure of Lebesgue measure μ is

$$\mu^*(A) = \inf \{\rho(U) \ : \ A \subset U, \ U \text{ open } \}$$

where ρ is a pre-measure defined on open sets in X. For example in $X = \mathbb{R}^2$, $\rho(U) =$ the volume of U.

Definition. (Metric Space (X, d) equipped with d =distance)

A metric space (M, d) is a set M and a function $d : M \times M \to R$ such that

1. $d(x, y) \geq 0$ *for all* $x, y \in X$.
2. $d(x, y) = 0$ *iff* $x = y$.
3. $d(x, y) = d(y, x)$ *for all* $x, y \in X$.
4. $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y \in X$.

Example [Fingerprint Recognition] Let X be a data set of fingerprints in Seoul city police department.

- Motivation: Design an efficient access system to find a target.
- We need to define a **dissimilarity** function stating the distance between the data. The distance $d(x, y)$ between two data x and y must satisfy the above four rules.
- **Similarity queries.** For a given target $x^* \in X$ and $\epsilon > 0$, arrest all having finger print $y \in X$ such that $d(y, x^*) < \epsilon$.

Chapter 3. Measurable functions

- A function $f : \mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable if $f^{-1}(U)$ is Lebesgue measurable for every open set U.
- Let X be a metric space and let $(X, \mathcal{M}, \mu)$ be a measure space. **A function $f : X \to \mathbb{R}$ is measurable if $f^{-1}(U) \in \mathcal{M}$ whenever U is an open or closed interval, or open ray (a, ∞).** It is a simple exercise to show the followings:
  - $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$.
  - $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$.
  - $f^{-1}(E^c) = \left[f^{-1}(E)\right]^c$.
- In particular, **$f : X \to \mathbb{R}$ is measurable if $\{x \in X \;:\; f(x) > a\} \in \mathcal{M}$ for all $a \in \mathbb{R}$.**
- Given two function f and g we define

$$f \vee g = \max\{f, g\} \qquad f \wedge g = \min\{f, g\}$$
$$f^+ = f \vee 0 \qquad f^- = (-f) \vee 0$$

Proposition. (3.1.2)

If f and g are measurable, then so are
$f+g,\ fg,\ f\vee g,\ f\wedge g,\ f^+,\ f^-$, *and* $|f|$.

Proof. We will denote $\{f > a\} := \{x \in X :\ f(x) > a\}$

- $f+g$ is measurable because
  $\forall a \in \mathbb{R},\quad \{f+g > a\} = \cup_{t\in\mathbb{Q}}(\{f > t\} \cap \{g > a-t\})$.
  $\mathbb{Q} :=$ the set of rational numbers.
- f^2 is measurable since $\{f^2 > a\} = X$ if $a < 0$ and
  $\forall a \geq 0,\quad \{f^2 > a\} = \{f > \sqrt{a}\} \cup \{f < -\sqrt{a}\}$.
- fg is measurable because $fg = \frac{(f+g)^2 - f^2 - g^2}{2}$.
- f^+ is measurable because $\{f^+ > a\} = X$ if $a < 0$ &
  $\{f^+ > a\} = \{f > a\}$ if $a \geq 0$.
- $|f|$ is measurable because $|f| = f^+ + f^-$.
- $f \vee g,\ f \wedge g$ are measurable because
  $f \vee g = \frac{f+g+|f-g|}{2},\quad f \wedge g = \frac{f+g-|f-g|}{2}$.

Theorem. (3.1.3)

If $\{f_j\}$ is a sequence of measurable functions, then $\limsup_j f_j$, $\limsup_j f_j$ *are measurable.*

Proof. Denote $\phi := \limsup_j f_j$.

1. Recall $\phi := \limsup_j f_j = \lim_{n\to\infty} g_n$ where $g_n = \sup_{j\geq n} f_j$.
2. $\{g_n > a\} = \cup_{j\geq n}\{f_j > a\}$. Hence, g_n is measurable.
3. Since $g_n \searrow$, $\limsup_j f_j = \inf_{n\geq 0}\ g_n$.
4. Hence, $\{\phi > a\} = \cap_{n=1}^{\infty}\{g_n > a\}$.
5. Therefore ϕ is measurable.
6. A similar proof shows that $\liminf_j f_j$ is measurable.

3.2 Integration of non-negative functions

Let $(X, \mathcal{M}, \mu)$ be measure space where X is a metric space. **If your mathematical background is poor, you regard X as $X = \mathbb{R}^2$ and μ as the standard Lebesgue measure, that is, $\mu(A) =$ the area of A. Throughout this lecture, E, E_j are a measurable set.**

- The **characteristic function of** E denoted by χ_E is the function defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

- A **simple function** is a finite linear combination of characteristic functions

$$\phi = \sum_{j=1}^{n} c_j \chi_{E_j}$$

Hence, $E_j = \{\phi = c_j\}$.

Theorem. (3.2.1)

Let $f : X \to \mathbb{R}$ be measurable and $f \geq 0$. Then

$$\phi_n = \sum_{k=0}^{2^{2n}-1} k2^{-n}\chi_{E_{n,k}} + 2^n\chi_{F_n} \nearrow f$$

where $E_{n,k} = f^{-1}((k2^{-n}, (k+1)2^{-n}])$, $F_n = f^{-1}((2^n, \infty])$.
Moreover, each ϕ_n *satisfies*

$$\phi_n \leq \phi_{n+1} \quad \& \quad 0 \leq f(x) - \phi_n(x) \leq 2^{-n} \qquad \text{for } x \in X \setminus F_n$$

Proof. Straightforward.
From the above theorem, we can prove that for any measurable function f there is a sequence of simple functions ϕ_n such that $\phi_n \to f$ on any set on which f is bounded.
Why? $f = f^+ - f^-$ where $f = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$.

Let $(X, \mathcal{M}, \mu)$ be a measure space.

- **Definition of Lebesgue Integral for simple functions** The integral of a measurable simple function $\phi = \sum_{j=1}^{n} c_j \chi_{E_j}$ is defined to be
$$\int \phi d\mu = \sum_{j=1}^{n} c_j \ \mu(E_j)$$
- We use the convention that $0 \cdot \infty = 0$.
- If ϕ is a simple function, then $\phi \geq 0 \implies \int \phi d\mu \geq 0$.
- **Let $\mathcal{S}_{imple}$ be a vector space of measurable simple functions.** Then the integral $\int \Box d\mu$ can be viewed as a linear functional on $\mathcal{S}_{imple}$, that is, $\int \Box \, d\mu : \mathcal{S}_{imple} \to \mathbb{R}$ is linear.

Lemma. (3.2.2)

Let $(X, \mathcal{M}, \mu)$ *be a measure space. Given a non-negative, measurable simple function* ϕ *and* $A \in \mathcal{M}$*, define*

$$\nu(A) = \int_A \phi \, d\mu = \int_X \phi \chi_A \, d\mu$$

Then $(X, \mathcal{M}, \nu)$ *is also a measure space.*

Proof. Let $\phi = \sum_{k=1}^n c_k \chi_{E_k}$ where $E_k \in \mathcal{M}$. Assume $A = \cup_j A_j$ where $A_j \in \mathcal{M}$ are mutually disjoint. Then

$$\begin{aligned} \nu(A) &= \int \phi \chi_A \, d\mu = \sum_{k=1}^n \int c_k \chi_{E_k} \chi_A d\mu = \sum_{k=1}^n \int c_k \chi_{E_k \cap A} d\mu \\ &= \sum_{k=1}^n c_k \mu(E_k \cap A) = \sum_{k=1}^n \sum_{j=1}^\infty c_k \mu(E_k \cap A_j) \\ &= \sum_{j=1}^\infty \sum_{k=1}^n c_k \mu(E_k \cap A_j) = \sum_{j=1}^\infty \int_{A_j} \phi d\mu = \sum_j \nu(A_j) \end{aligned}$$

Definition. (Lebesgue integral of non-negative measurable function)

The integral of a non-negative measurable function f is defined by

$$\int f \; d\mu = \sup \left\{ \int \phi d\mu \; : \; \phi \le f \;\; \& \;\; \phi \in \mathcal{S}_{imple} \right\}$$

Recall that the integral of a measurable simple function $\phi = \sum_{j=1}^{n} c_j \chi_{E_j}$ is defined to be

$$\int \phi d\mu = \sum_{j=1}^{n} c_j \; \mu(E_j)$$

From the definition, we obtain

$$f \le g \quad \Longrightarrow \quad \int f d\mu \le \int g d\mu$$

Theorem. (3.2.3: MCT(Monotone Convergence Thm))

If $\{f_n\}$ is a nondecreasing sequence of non-negative measurable functions, then

$$\int \lim_n f_n \, d\mu \quad = \quad \lim_n \int f_n \, d\mu$$

- Since $f_n \nearrow$, $\lim_n f_n = \ ^{\exists}f$ and is measurable. Note that it is possible that $f(x) = \infty$ at some x.
- Since $\int f_n d\mu \nearrow$ and $f_n \leq f$, $\int f_n \leq \int f$ and therefore

$$\lim_n \int f_n d\mu \quad \leq \quad \int f \quad d\mu$$

- It remains to prove $\lim_n \int f_n d\mu \quad \geq \quad \int f \quad d\mu$.

Theorem. (3.2.3: Continue ......MCT)

If $f_n \nearrow$, *then* $\int \lim_n f_n \ d\mu \quad = \quad \lim_n \int f_n \ d\mu$

Continue... Aim to prove $\lim_n \int f_n d\mu \quad \geq \quad \int f \quad d\mu$.

- Since $\int f \ d\mu = \sup\{\int \phi d\mu \ : \ \phi \leq f, \quad \phi \in \mathcal{S}_{imple}\}$, it suffices to prove that for any α, $0 < \alpha < 1$ and any $\phi \in \mathcal{S}_{imple}$ with $\phi \leq f$,

$$\lim_n \int f_n d\mu \quad \geq \quad \alpha \int \phi \quad d\mu$$

- Let $E_n = \{f_n \geq \alpha\phi\}$. Then

$$\int f_n d\mu \ \geq \ \int_{E_n} f_n d\mu \ \geq \ \int_{E_n} \alpha\phi \ d\mu \ \stackrel{define}{:=} \ \alpha \ \nu(E_n)$$

- Since ν is a measure and $E_n \nearrow X$, $\lim_n \nu(E_n) = \nu(X) = \int \phi d\mu$. Thus, $\lim_n \int f_n d\mu \geq \alpha \int \phi d\mu$.

Corollary. (3.2.4: $\phi_n \nearrow f$)

Let $\mathcal{M}^+_{able}$ be the set of non-negative measurable functions.

- *If $\phi_n \in \mathcal{S}_{imple}$ and $\phi_n \nearrow f$ for some $f \in \mathcal{M}_{able}$, then*

$$\lim_n \int \phi_n d\mu \quad = \quad \int f d\mu$$

- *The map $\int \Box d\mu \ : \ \mathcal{M}^+_{able} \to \mathbb{R}$ is linear.*

Proof. Let $f, g \in \mathcal{M}^+_{able}$, $\quad \phi \in \mathcal{S}^+_{imple} \nearrow f$, and $\psi \in \mathcal{S}^+_{imple} \nearrow g$. Then

- $\int f + g \ d\mu = \int \lim_n(\phi_n + \psi_n) \ d\mu$
  $= \lim_n \int (\phi_n + \psi_n) \ d\mu = \int f \ d\mu + \int g \ d\mu$
- $\int \alpha f \ d\mu = \int \lim_n \alpha\phi_n d\mu = \alpha \lim_n \int \phi_n \ d\mu = \int f \ d\mu$.

Proposition. (3.2.7: $f = 0$ almost everywhere)

Let $f \in \mathcal{M}^+_{able}$. Then

$$\int f d\mu = 0 \iff f = 0 \ a.e.$$

Proof.

- If $f \in \mathcal{S}^+_{imple}$, then the statement is immediate.
- If $f = 0$ $a.e.$ and $\phi \leq f$, then $\phi = 0$ $a.e.$, and hence $\int f = \sup\{\int \phi \ : \ \phi \leq f, \ \ \phi \in \mathcal{S}^+_{imple}\} = 0$.
- Conversely, let $\int f \ d\mu = 0$. Then

$$0 = \int f \ d\mu \ \ \geq \ \ \frac{1}{n}\mu(\{f > \frac{1}{n}\}), \quad n = 1, 2, \cdots$$

$\therefore$ $f = 0$ $a.e.$ Why?

$$\mu(\{f > 0\}) = \mu\left(\cup_{n=1}^{\infty}\{f > \frac{1}{n}\}\right) \leq \cup_{n=1}^{\infty}\mu(\{f > \frac{1}{n}\}) = 0.$$

Lemma. (3.2.9: Fatou's Lemma)

For any sequence $f_n \in \mathcal{M}^+_{able}$, *we have*

$$\int \liminf_n f_n \, d\mu \quad \leq \quad \liminf_n \int f_n \, d\mu$$

Proof.

$$\liminf_n \int f_n \, d\mu = \sup_{k\geq 1} \inf_{j\geq k} \int f_j \, d\mu \geq \sup_{k\geq 1} \int \inf_{j\geq k} f_j \, d\mu$$

Since $g_k = \inf_{j\geq k} f_j \nearrow$,

$$\sup_{k\geq 1} \int \inf_{j\geq k} f_j \, d\mu = \lim_{k\to\infty} \int \inf_{j\geq k} f_j \, d\mu = \int \lim_{k\to\infty} \inf_{j\geq k} f_j \, d\mu$$

$L^1(X, d\mu)$: Complete metric space

Definition. (Integrability)

Let $(X, \mathcal{M}, \mu)$ be measure space. A function $f : X \to \mathbb{R}$ is integrable if $f \in \mathcal{M}_{able}$ & $\int |f| d\mu < \infty$. We denote by $L^1(X, d\mu)$ the class of all integrable functions. For $f \in L^1(X, d\mu)$, we define

$$\int f \; d\mu \;\; = \;\; \int f^+ \; d\mu \; - \; \int f^- \; d\mu$$

Question: Prove that $L^1(X, d\mu)$ is **a complete normed space** (or Banach space) when it is equipped with the norm

$$\|f\| = \int |f| d\mu \qquad (\textit{its metric}: \; d(f, g) = \|f - g\| \;)$$

To answer this question, we need to study several convergence theorems.

Proposition.

$L^1(X, d\mu)$ is a normed space equipped with the norm

$$\|f\| = \int |f| d\mu$$

This means that $L^1(X, d\mu)$ is a vector space satisfying

1. $\|f\| \geq 0, \qquad \forall f \in L^1$
2. $\|f\| = 0$ *iff* $f = 0$ *a.e.*.
3. $\|\lambda f\| = |\lambda| \|f\|, \qquad \forall f \in L^1$ *and every scaler* λ.
4. $\|f + g\| \leq \|f\| + \|g\|, \qquad \forall f, g \in L^1$

.

The proof is the straightforward.

Theorem. (Lebesgue Dominate Convergence Theorem)

Assume $\{f_n\} \subset L^1$ *such* $f_n \to f$ *a.e. and* $\exists g \in L^1$ *so that* $|f_n| \leq g$ *a.e. for all* n. *Then*

$$f \in L^1 \quad \& \quad \int f \, d\mu = \lim_n \int f_n \, d\mu$$

Proof. Since $g + f_n \geq 0$,

$$\int \liminf_n \, (g + f_n) \, d\mu \overset{Fatou's \ Lemma}{\leq} \liminf_n \int (g + f_n) \, d\mu$$

Hence, $\int f \, d\mu \leq \lim_n \inf \int f_n \, d\mu$.
Applying the same argument to the sequence $g - f_n \geq 0$, we obtain

$$-\int f \, d\mu \leq \liminf_n \int (-f_n) \, d\mu = -\limsup_n \int f_n \, d\mu$$

Example. (Gaussian function)

The fundamental solution of the heat equation in 1-D is

$$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.$$

Then

$$\int_{\mathbb{R}} G(x,t)\, dx = 1 \quad \textit{for all } t > 0 \quad \& \lim_{t \to 0^+} G(x,t) = 0 \ \textit{a.e.}$$

- Let $f_n(x) = G(x, 1/n)$. Then $f_n \to f = 0$ *a.e.* and

$$\int f\, d\mu = 0 \ \neq \ 1 = \lim_n \int f_n\, d\mu$$

This is the reason why LDC requires the assumption that $\{f_n\}$ is **dominated** by a fixed L^1-function g.

Corollary. (3.3.2)

Let $\{f_j\} \subset L^1$ *s.t.* $\sum_j \int |f_j| \, d\mu < \infty$. *Then* $\exists\, f \in L^1$ *such that*

$$\lim_{n\to\infty} \sum_{j=1}^{n} f_j = f \ a.e. \quad \& \quad \int f \, d\mu = \sum_j \int f_j \, d\mu$$

(※ *Denote* $f = \sum_{j=1}^{\infty} f_j$.)

1. Let $g_n = \sum_{j=1}^{n} |f_j|$ and $g = \sum_{j=1}^{\infty} |f_j|$.
2. Since $g_n \nearrow g$, it follows from the monotone convergence theorem that

$$\int g d\mu = \lim_n \int g_n \, d\mu = \sum_j \int |f_j| \, d\mu < \infty$$

   Hence, $g \in L^1$ and $g < \infty$ *a.e.*
3. Since $\left|\sum_{j=1}^{n} f_j\right| < g$ *a.e* and $g \in L^1$, the result follows by the Dominate Convergence Theorem.

Theorem. (3.3.3: $\mathcal{S}_{imple}$ is dense in L^1)

- $\mathcal{S}_{imple}$ *is* **dense** *in* L^1, *i.e., every element in* L^1 *is a* L^1-*limit of a sequence of elements in* $\mathcal{S}_{imple}$.
- $C_0(\mathbb{R})$ *is* **dense** *in* $L^1(\mathbb{R}, \mathcal{M}, \mu)$, *where* μ *is any Borel measure on* $\mathbb{R}$. *Here, the definition of* $C_0(\mathbb{R})$ *is* $C_0(\mathbb{R}) := \{f \in C(\mathbb{R}) \ : \ \exists N \ \ s.t. \ f(x) = 0 \ \ for \ |x| > N\}$.

Proof of the first statement: $\mathcal{S}_{imple}$ is dense in L^1.

- Let $f \in L^1$. By Thm 3.2.1,

$$\exists \ \phi_n \in \mathcal{S}_{imple} \ \ s.t. \ \ \phi_n \to f \ a.e. \ \ \& \ \ |\phi_n| < |f| \ a.e.$$

- By LDCT(Lebesgue Dominate Convergence Theorem), $\|\phi_n - f\| = \int |\phi_n - f| \ d\mu \to 0$. This completes the proof.

Proof of the second statement: $C_0(\mathbb{R})$ is **dense** in $L^1(\mathbb{R}, \mathcal{M}, \mu)$.

1. Since $\mathcal{S}_{imple}$ is **dense** in L^1, it suffices to prove that any $\phi \in \mathcal{S}_{imple} \cap L^1$ can be approximated by a sequence $\{f_n\} \subset C_0(\mathbb{R})$.
2. If $\phi = \chi_{(0,1)}$, then a sequence of continuous functions

$$f_n(x) := \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{if } 0 < -1/n \\ 0 & \text{if } x > 1 + 1/n \\ \text{linear} & \text{otherwise} \end{cases} \quad \rightarrow \quad \phi = \chi_{(0,1)} \text{ in } L^1\text{-sense.}$$

   Indeed, $\|f_n - \phi\| = 1/n \rightarrow 0$.
3. Similarly, if A is a finite union of bounded open intervals, then $\phi = \chi_A$ can be approximated by a sequence $\{f_n\} \subset C_0(\mathbb{R})$.

Continue....

4. Let E be a Borel measurable set with $\mu(E) = \|\chi_E\| < \infty$. That is,

$$\mu(E) = \inf \left\{ \sum_j \mu(I_j) \ : \ E \subset \cup I_j, \quad I_j = (a_j, b_j) \right\} \ < \ \infty$$

5. Hence, for any $\epsilon > 0$, $\exists$ a finite union of open intervals $A = \cup_{j=1}^{N} I_j$ such that

$$\|\chi_E - \chi_A\| = \mu(E \triangle A) < \epsilon$$

   where $E \triangle A = (E \setminus A) \cup (A \setminus E)$.

6. Since $\epsilon > 0$ is arbitrary, χ_E can be approximated by a sequence $\{f_n\} \subset C_0(\mathbb{R})$.

Theorem. (Riemann $\int$ v.s. Lebesgue $\int$)

Let $f : [0,1] \to \mathbb{R}$ be a bounded and **Riemann integrable**. *Then*

- $f \in L^1([0,1], d\mu)$ *where* $\mu((a,b]) = b - a$.
- *Lebesgue and Riemann integrals agrees.*

1. Let $\mathcal{P}_n = \{j2^{-n} \ : \ j = 1, \cdots, 2^n\}$, a partition of $[a,b]$.
2. Denote $E_{n,j} := (j2^{-n}, (j+1)2^{-n}]$ and

$$m_{n,j} := \inf_{x \in E_{n,j}} f(x) \quad M_{n,j} := \sup_{x \in E_{n,j}} f(x)$$
$$\phi_n = \sum_{j=1}^{2^n} m_{n,j} \ \chi_{E_{n,j}} \quad \psi_n = \sum_{j=1}^{2^n} M_{n,j} \chi_{E_{n,j}}$$

3. Therefore $\phi_n \leq \phi_{n+1} \leq f \leq \psi_{n+1} \leq \psi_n$.
4. Hence, $\exists \psi = \lim_n \psi_n$ and $\exists \psi = \lim_n \psi_n$.
5. By def'n, $L(\mathcal{P}_n, f) = \int_0^1 \phi_n(x) dx \leq U(\mathcal{P}_n, f) = \int_0^1 \psi_n(x) dx$

Continue....

6. By definition of the Lebesgue integral for simple functions,

$$L(\mathcal{P}_n, f) = \int \phi_n d\mu \;\; \& \;\; U(\mathcal{P}_n, f) = \int \psi_n d\mu$$

7. From Riemann integrability of f,
   $\inf_n U(\mathcal{P}_n, f) = \sup_n L(\mathcal{P}_n, f) \;\; = \int_0^1 f(x)\, dx$
8. By LDCT,

$$\begin{aligned} \int \phi d\mu &= \lim_n \int \phi_n \, d\mu = \lim_n U(\mathcal{P}_n, f) \\ &= \inf_n U(\mathcal{P}_n, f) = \sup_n L(\mathcal{P}_n, f) = \lim_n L(\mathcal{P}_n, f) \\ &= \lim_n \int \psi_n \, d\mu = \int \psi d\mu \end{aligned}$$

9. Hence, $\int \phi d\mu = \int_0^1 f dx = \int \phi d\mu$
10. Therefore, $\psi = f = \phi$ *a.e.*.

Theorem. (3.4.3)

Let $f(x,t): X \times [a,b] \to \mathbb{R}$ be a mapping. Suppose that f is differentiable with respect to t and that

$$g(x) := \sup_{t \in [a,b]} \left| \frac{\partial}{\partial t} f(x,t) \right| \in L^1(X, d\mu)$$

Then $F(t) = \int f(x,t) d\mu$ is differentiable on $a \leq t \leq b$ and

$$\frac{\partial}{\partial t} \int f(x,t) d\mu = \int \frac{\partial}{\partial t} f(x,t) d\mu$$

For each $t \in (a,b)$, we can apply LDCT to the sequence

$$h_n(x) = \frac{f(x,t_n) - f(x,t)}{t_n - t}, \quad t_n \to t$$

($\because \quad |h_n| \leq g$ from the mean value theorem.)

3.5 Pointwise, Uniform, Norm convergence, & Convergence in measure

Four important examples.

- Consider $\phi_n \in L^1(\mathbb{R}, d\mu)$ s.t. $\|\phi_n\| = \int_{\mathbb{R}} |\phi_n| d\mu = 1 \nrightarrow 0$ but $\phi_n \to 0$ in some sense.
  - If $\heartsuit$ $\phi_n = \frac{1}{n}\chi_{(0,n)}$, then $\phi_n \to 0$ uniformly.
  - If $\diamondsuit$ $\phi_n = \chi_{(n,n+1)}$, then $\phi_n \to 0$ pointwise.
  - If $\spadesuit$ $\phi_n = n\chi_{(0,1/n)}$, then $\phi_n \to 0$ a.e.
- For each $k = 0, 1, 2, \cdots$, define a sequence $\clubsuit$ $\psi_{k,j} = \chi_{(j2^{-k},(j+1)2^{-k})}$ for $j = 0, \cdots, 2^k - 1$.
  1. Denote $\phi_1 = \psi_{0,0}$, $\phi_1 = \psi_{1,0}$, $\phi_2 = \psi_{1,1}$, $\phi_3 = \psi_{2,0}$, $\phi_4 = \psi_{2,1}$, $\phi_3 = \psi_{2,2}$, $\phi_3 = \psi_{2,3}$, $\phi_4 = \psi_{3,0}$.
  2. Then $\phi_1 = \chi_{(0,1)}$, $\phi_2 = \chi_{(0,2^{-1})}$, $\phi_3 = \chi_{(2^{-1},1)}, \cdots$
  3. $\|\psi_{k,j} - 0\| = 2^{-k} \to 0$, while $\phi_n(x) \nrightarrow 0$ for any x.

Definition. (3.5.1: $f_n \to f$ (meas))

- $\{f_n\}$ *is said to* **converge in measure to** f *if*

$$\forall \epsilon > 0, \quad \lim_{n\to\infty} \mu(\{|f_n - f| \geq \epsilon\}) = 0$$

- $\{f_n\}$ *is said to be* **Cauchy sequence in measure** *if*

$$\forall \epsilon > 0, \quad \lim_{n,m\to\infty} \mu(\{|f_n - f_m| \geq \epsilon\}) = 0$$

- The sequences ♠ ♣, ♡ converge to 0 in measure. That is, $\phi_n = \frac{1}{n}\chi_{(0,n)}, \ \chi_{(0,1/n)}, \ \chi_{(j2^{-k},(j+1)2^{-k})} \to 0$ [meas].
- The sequence ♢ $\phi_n = \chi_{(n,n+1)}$ does not converge to 0 in measure.

Theorem. (3.5.2: f_n: Cauchy in meas $\Rightarrow f_n \to^{\exists} f$ in meas)

Suppose $\{f_n\}$ is a Cauchy seq in measure. Then

- $\exists\ f \in \mathcal{M}_{able}$ *s.t.* $f_n \to f$ *in measure.*
- $\exists\ f_{n_k}$ *s.t.* $f_{n_k} \to f$ *a.e.*
- f *is uniquely determined a.e.*

Proof.

1. Choose a subsequence n_k such that $g_k = f_{n_k}$,

$$\mu(E_k) \leq 2^{-k}, \qquad E_k = \{|g_k - g_{k+1}| \geq 2^{-k}\}$$

2. Let $Z_k := \cup_{j=k+1}^{\infty} E_j$. Then $\mu(Z_k) \leq 2^{-k}$.
3. Let $Z = \cap_{k=1}^{\infty} Z_k$. Then $\mu(Z) = 0$.
4. Prove that $\lim_{k\to\infty} g_k(x) = {}^{\exists} f(x)$ for all $x \in X \setminus Z$
5. Prove that g_k converges to f uniformly on $X \setminus Z_N$ (N=1,2,...).

- Prove that $\lim_{k\to\infty} g_k(x) = {}^{\exists} f(x)$ for all $x \in X \setminus Z$

Proof.

1. For $x \in X \setminus Z_N$ and $j > i$, we have

$$
\begin{aligned}
|g_{N+i}(x) - g_{N+j}(x)| &\le \sum_{k=i}^{j-1} |g_{N+k}(x) - g_{N+k+1}(x)| \\
&\le \sum_{k=i}^{j-1} 2^{-N-k} \le 2^{-N-i+1}
\end{aligned}
$$

2. $\therefore$ $g_k(x)$ is Cauchy sequence if $x \in X \setminus Z_N$, $N = 1, 2, \cdots$.
3. $\therefore$ $\lim_{k\to\infty} g_k(x)$ exists if $x \in X \setminus Z$.
4. Define $f : X \to \mathbb{R}$ by $f(x) = \lim_{k\to\infty} g_k(x)$ for $x \in X \setminus Z$ and $f(x) = 0$ for $x \in Z$.
5. Since $\mu(Z) = 0$, $g_k \to f$ *a.e.*.

- Prove that g_k converges to f uniformly on $X \setminus Z_N$ (N=1,2,...).

Proof. For $x \in X \setminus Z_N$, we have
$|g_{N+k}(x) - f(x)| = \lim_j |g_{N+k}(x) - g_{N+j}(x)| \leq 2^{-N-k+1}$,
that is,

$$\sup_{X \setminus Z_N} |g_{N+k} - f| \leq 2^{-N-k+1}$$

- Prove that f_n converges to f in measure.

Proof. For any $\epsilon > 0$, we have

$$\begin{aligned} \mu(\{|f - f_n| \geq \epsilon\}) &= \mu(\{|f - g_k| \geq \frac{\epsilon}{2}\}) + \mu(\{|g_k - f_n| \geq \frac{\epsilon}{2}\}) \\ &\to 0 \quad \text{as} \quad k, n \to \infty \end{aligned}$$

Corollary. (3.5.3)

If $f_n \to f$ in L^1, then there is a subsequence f_{n_k} such that $f_{n_k} \to f$ a.e.

Proof. For any $\epsilon > 0$,

$$\mu(\{|f_n - f| > \epsilon\}) \leq \frac{1}{\epsilon} \int |f_n - f| \, d\mu \to 0.$$

Hence, $f_n \to f$ in measure, and from the proof of Theorem 3.5.2 we can choose a subsequence n_k such that $g_k = f_{n_k}$,

$$\mu(E_k) \leq 2^{-k}, \qquad E_k = \{|g_k - g_{k+1}| \geq 2^{-k}\}.$$

From Theorem 3.5.2, $f_{n_k} = g_k \to f$ a.e..

Theorem. (3.5.4: Egorov's Theorem)

Let $\mu(E) < \infty$ and let $f_n \to f$ a.e.. Then, for any $\epsilon > 0$, there exist $F \subset E$ so that $\mu(E \setminus F) < \epsilon$ and $f_n \to f$ uniformly on F.

Proof. Since $f_n \to f$ *a.e.*, there exist $Z \subset E$ so that $\mu(Z) = 0$ and $f_n(x) \to f(x)$ for $x \in E \setminus Z$. It suffices to prove the theorem for the case when $Z = \emptyset$.

1. Let $E_{m,n} = \cap_{j=m}^{\infty} \{|f_j - f| < \frac{1}{n}\}$.
2. Then $\lim_{m\to\infty} \mu(E_{m,n}) = E$ for $n = 1, 2, \cdots$. (why?)
3. Hence, $\exists\ m_n$ s.t. $\mu(E \setminus E_{m_n,n}) \leq \epsilon 2^{-n}$.
4. Let $F = \cap_{n=1}^{\infty} E_{m_n,n}$. Then $\mu(E \setminus F) < \epsilon$. (why?)
5. Moreover, if $j > m_n$, then $\sup_{x\in F} |f_j(x) - f(x)| < \frac{1}{n}$. Hence, $f_n \to f$ uniformly on F.

Chapter 4. Product spaces

Throughout this chapter, we assume that $(X_j, \mathcal{M}_j, \mu_j), j = 1, 2$ is two $\sigma-$finite measure spaces. Recall that the measure μ_j is called -finite, if X is the countable union of measurable sets of finite measure. Let $X = X_1 \times X_2$ and let
$\mathcal{R} = \{E_1 \times E_2 \ : \ E_j \in \mathcal{M}_j\}$.
A product measure space $(X, \mathcal{M}, \mu)$ is constructed as follows:

- Define the pre-measure μ' on $\mathcal{R}$ by
  $\mu'(E_1 \times E_2) = \mu_1(E_1)\mu_2(E_2)$
- By Carathéodory's theorem, we obtain a complete measure μ on X whose σ-algebra of measurable sets contain the product algebra $\mathcal{M}_1 \otimes \mathcal{M}_2 := \sigma$-algebra generated by $\mathcal{R}$.
- Since μ_j is $\sigma-$finite, so is μ.

Theorem. (4.1.1: Fubini)

Assume $f \in L^1(d\mu)$. *Then*

$$\int_{X_2} f(x,y)d\mu_2(y) \in L^1(d\mu_1), \quad \int_{X_1} f(x,y)d\mu_1(x) \in L^1(d\mu_2)$$

and

$$\int_X fd\mu = \int_{X_1}\left[\int_{X_2} f(x,y)d\mu_2(y)\right]d\mu_1(x) = \int_{X_2}\left[\int_{X_1} f(x,y)d\mu_1(x)\right]d\mu_2(y)$$

The strategy of the proof.

1. Begin by proving the result for $f(x,y) = \chi_{E_1\times E_2}$. It is trivial!
2. Then, prove it for $f \in \mathcal{S}_{imple}$.
3. Finally, extend it for general $f \in L^1(d\mu)$.

Differentiation Theory

5.1 Differentiation Theory of functions

Throughout this subsection, we consider a bounded function $f : [a, b] \to \mathbb{R}$. We will study **a necessary and sufficient condition that f' exist almost everywhere and**

$$f(y) - f(x) = \int_x^y f'(x) d\mu, \quad \mu((x, y)) = |y - x|.$$

- If f is Cantor function, then $f' = 0$ almost everywhere but
$$1 = f(1) - f(0) \neq 0 = \int_x^y f'(x) d\mu$$
- **Lebesgue's Theorem 5.1.1**: Every monotonic function $f : [a, b] \to \mathbb{R}$ is differentiable almost everywhere.
- Recall that the derivative of f at x exists if the following all four numbers are the same finite value:
$$\lim\ \inf_{h \to 0^{\pm}} \frac{f(x+h) - f(x)}{h}, \quad \lim\ \sup_{h \to 0^{\pm}} \frac{f(x+h) - f(x)}{h}$$

Definition. (Bounded Variations)

Let $f : [a, b] \to \mathbb{R}$ be a function. The total variation of f on $[a, x]$ is defined to be

$$T_f(a, x) = \sup_n \sup_{\mathcal{P}_n} \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|,$$

where $\mathcal{P}_n = \{a = x_0 < x_1 < \cdots < x_n = x\}$.
The class of functions of bounded variation on $[a, b]$ is denoted by $BV[a, b]$.

It is well known that the space $BV[a, b]$ is a Banach space with norm $\|f\|_{var} = T_f(a, b)$.

♣ If $f(x) = A \sin nx$, then $T_f(0, \pi) = An$.

Theorem. (5.1.3: Jordan Decomposition)

Every $f \in BV[a, b]$ can be written as two non-decreasing functions.

Proof. Let $T(x) = T_f(a, x)$. The theorem will be proved by showing $T - f$ and T are non-decreasing since $f = T - (T - f)$.

1. Let $x < y$. $T_f(x, y) = T(y) - T(x)$
2. From the definition,

   $$T(x) = T_f(a, x) \quad \text{is a montone non-decreasing function of } x$$

   since, for $x < y$, $T(y) = T(x) + T(x, y) \geq T(x)$.
3. Clearly, $|f(y) - f(x)| \leq T_f(x, y) = T(y) - T(x)$.
4. Hence, $f(y) - f(x) \leq T(y) - T(x)$.
5. Hence, $T(x) - f(x) \leq T(y) - f(y)$.

Definition. (5.1.5: Absolute continuous)

A function $f \in BV[a, b]$ *is absolute continuous iff* $\forall \epsilon > 0$, *there exist* δ *such that whenever a sequence of non-overlapping subintervals* $(x_j, y_j) \subset [a, b]$ *satisfies* $\sum_j (y_j - x_j) < \delta$, *then*

$$\sum_j |f(y_j) - f(x_j)| < \epsilon$$

Note that the Cantor function is not absolute continuous.

Theorem. (5.1.6: Absolute continuity)

If f' *exist almost everywhere,* $f' \in L^1(d\mu)$, *and*

$$f(x) = \int_a^x f'(x)\ d\mu, \qquad x \in (a, b]$$

then f *is absolute continuous.*

Proof.

We want to prove that for a given $\epsilon > 0$, there exist δ s.t. $\sum_j (y_j - x_j) < \delta \quad \Rightarrow \sum_j |f(y_j) - f(x_j)| < \epsilon$.

1. If $|f'|$ is bounded, then we choose $\delta = \frac{\epsilon}{\|f'\|_\infty}$ and

$$\sum_j |f(y_j) - f(x_j)| \leq \sum_j \int_{x_j}^{y_j} |f'| d\mu \leq C \sum_j (y_j - x_j) < \|f'\|_\infty \delta = \epsilon$$

2. If $f' \in L^1(d\mu)$ but not bounded, then we decompose

$$f' = g + h \text{ where } g \text{ is bounded and } \int |h| d\mu < \frac{\epsilon}{2}$$

This is possible because the bounded functions are dense in $L^1(d\mu)$. The results follows by choosing $\delta = \frac{\epsilon}{2\|g\|_\infty}$.

Theorem. (5.1.7: absolute + singular)

Let f be continuous and non-decreasing. Then f can be decomposed into **the sum of an absolute continuous function and a singular function,** *both monotone.*

Proof.

1. f' exist almost everywhere by Lebesgue's theorem.
2. Since f is continuous, $\int_a^x \frac{f(t+h)-f(t)}{h} d\mu = \frac{1}{h}\int_x^{x+h} f(t)d\mu - \frac{1}{h}\int_a^{a+h} f(t)d\mu \to f(x) - f(a)$ as $h \to 0$.
3. Since $\frac{f(t+h)-f(t)}{h} \to f'(t)$ $a.e.$ as $h \to 0$, by Fatou's lemma,
$$\int_a^x f'(t)d\mu \leq \liminf \int_a^x \frac{f(t+h)-f(t)}{h} d\mu = f(x) - f(a)$$
4. Since $f' \geq 0$, $f' \in L^1(d\mu)$. Set $g = \int_a^x f'(t)dt$ and $h = f - g$. Then g is absolute continuous and $h' = 0 a.e.$.

Lebesgue-Radon-Nikodym Theorem

The goal is to understand the main structure of the following theorem which is a remarkable achievement.

Theorem. (5.3.6: Radon-Nikodym, Riesz representation)

Let μ and ν be $\sigma-$finite measures on a set X, and suppose that $\nu \ll \mu$ (ν is absolute continuous w.r.t.μ). Then

$$\exists \;\; f \in L^1(X, d\mu) \;\; s.t. \;\; \nu(E) = \int_E f \; d\mu$$

for all $E\mathcal{M}$ for which $\nu(E) < \infty$

- **Definition** : $\nu \ll \mu$ iff $\mu(E) = 0 \Rightarrow \nu(E) = 0$ for all $E \in \mathcal{M}$.
- **Definition** : μ and ν are mutually singular, writing $\mu \perp \nu$, iff

$$\exists \;\; E, F \in \mathcal{M} \;\; s.t. \;\; X = E \cup F, \; E \cap F = \emptyset, \;\; \mu(E) = 0 = \nu(F).$$

To prove Radon-Nikodym thm, we need to understand several concepts.

Throughout this section, X is a metric space.

Definition.

- *A real valued measure ν, defined on Borel subset of X, is called a signed measure if it can be written in the form $\nu = \mu_1 - \mu_2$ where μ_1 and μ_2 are positive Borel measures, at least one of which is finite.*
- *Define* **the total variation of** ν, *denoted by* $|\nu|$, *to be*

$$|\nu|(E) = \sup \left\{ \sum_j |\nu(E_j)| \right\}$$

where the sup is taken over all disjoint collection $\{E_j\}$ such that $E = \cup_j E_j$.

Theorem. (5.2.1: Jordan decomposition)

If ν is a signed measure, then its total variation $|\nu|$ is a positive, countably additive measure. Moreover, $\nu^+ = \frac{|\nu|+\nu}{2}$ and $\nu^- = \frac{|\nu|-\nu}{2}$ are positive measures. We thus have the decompositions:

$$\nu = \nu^+ - \nu^- \text{ (Jordan decomposition) } \& \quad |\nu| = \nu^+ + \nu^-$$

Proof. It is easy to prove the followings:

1. $|\nu|(\emptyset) = 0$ and $|\nu|$ is monotone and finitely additive.
2. If $\{E_j\}$ is a countable partition of E, then $|\nu|(\cup E_j) = \sum |\nu|(E_j)$.
3. ν^+ and ν^- are positive measures.

Theorem. (5.2.3: Hahn Decomposition)

Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of the signed measure ν. Then, $\nu^+ \perp \nu^-$, and the exist $P, N \in \mathcal{M}$ s.t.

$$P \cup N = X, \quad P \cap N = \emptyset, \quad \nu^+(N) = \nu^-(P) = 0$$

Moreover, $\nu(E) \geq 0, \ \forall \ E \subset P$, while $\nu(E) \leq 0, \ \forall \ E \subset N$.

Proof. Assume that $\nu^+(X) < \infty$ and let $\{E_j\}$ be a sequence s.t. $\nu(E_j) > \nu^+(X) - 2^{-j}$.

1. For $A \subset E_j^c$,

$$\nu(A) \leq \nu^+(E_j^c) = \nu^+(X) - \nu^+(E_j) \leq \nu^+(X) - \nu(E_j) \leq 2^{-j}.$$

2. $\nu^-(E_j) = -\nu(E_j) + \nu^+(E_j) \leq -\nu(E_j) + \nu^+(X) < 2^{-j}$
3. Let $\limsup_j E_j = P$ & $\liminf_j E_j^c = N$
4. Then $\nu^+(N) = 0 = \nu^-(P)$.

Theorem. (5.3.2)

Let ν be finite and μ a positive measure on $(X, \mathcal{M})$. Then

$$\nu \ll \mu \quad \textit{iff} \quad \forall\ \epsilon > 0 \quad \exists\ \delta > 0 \ \ \textit{s.t.} \ \ \mu(E) < \delta \ \Rightarrow \ |\nu(E)| < \epsilon$$

Proof. By Jordan decomposition, it suffices to prove the result when $\nu = \nu^+$. The sufficiency $(\Leftarrow)$ is immediate; so we have only to establish its necessity.

1. To derive a contradiction, suppose $\nu \ll \mu$ but that $\epsilon - \delta$ condition fails.
2. Then $\exists\ \epsilon > 0$ and $\exists\ \{E_j\}$ s.t. $\nu(E_j) > \epsilon$ while $\mu(E_j) < 2^{-j}$.
3. Let $E = \limsup_j E_j = \cap_{k \geq 1} \cup_{j \geq k} E_j$.
4. By Fatou's lemma, $\nu(E) \geq \limsup_j \nu(E_j) \geq \epsilon$.
5. By monotone convergence theorem, $\mu(E) = \lim_k \mu(\cup_{j \geq k} E_j) \leq \lim_k \sum_{j=k}^{\infty} \mu(E_j) \leq \lim_k \sum_{j=k}^{\infty} 2^{-j} = \lim_k 2^{-k+1} = 0$.
6. From 4 and 5, ν is not absolutely continuous w.r.t. μ.

Theorem. (5.3.3)

Let $f \in L^1(X, d\mu)$, where μ is a positive measure. Define $\nu(E) = \int_E f \, d\mu$ for $E \in \mathcal{M}$. Then

$$\forall \, \epsilon > 0 \quad \exists \, \delta > 0 \;\; s.t. \;\; \mu(E) < \delta \;\Rightarrow\; \left| \int_E f \, d\mu \right| \; < \; \epsilon$$

Proof.

1. Define $\nu(E) = \int_E f \, d\mu$ for $E \in \mathcal{M}$.
2. Then $\nu \ll \mu$.
3. The result follows from 2 and 5.3.2

Lemma. (5.3.5)

Let ν be finite and positive, and put

$$\mathcal{F} = \left\{ f \in L^1(d\mu) \ : \ \nu(E) \geq \int_E f \ d\mu \ \text{ for all } \ E \in \mathcal{M} \right\}$$

Then $\exists$ unique $g \in \mathcal{F}$ s.t.

$$\int g d\mu \ = \ \sup_{f \in \mathcal{F}} \int \ f \ d\mu$$

Proof. The sup makes sense since $\mathcal{F} \neq \emptyset$ by Lemma 5.3.4.

1. Let $M = \sup_{f \in \mathcal{F}} \int \ f \ d\mu$ and let $\{f_j\}$ be a sequence in $\mathcal{F}$ s.t. $\int f_j d\mu \to M$.
2. Let $g_n = \max\{f_1, \ f_2, \cdots, f_n\}$. Then $g_n \nearrow$ and $M = \int \lim_j g_j \ d\mu$ by monotone convergence thm.

Continue the proof

4. To prove $g_n \in \mathcal{F}$, $\int_E g_n \, d\mu \le \nu(E)$.
   Proof.
   - Let $E_1 = E \cap \{g_n = f_1\}$ and $E_j = E \cap \{g_n = f_j\} \setminus \cup_{k=1}^{j-1} E_k,\ j = 2, \cdots, n$.
   - Then $E = \cup_{j=1}^{n} E_j$ and E_j are disjoint. Hence,

$$\int_E g_n \, d\mu = \sum_{j=1}^{n} \int_{E_j} f_j \, d\mu \le \sum_{j=1}^{n} \nu(E_j) = \nu(E)$$

4. From 4 and monotone convergence theorem, $g = \lim_j g_j \in \mathcal{F}$.

Theorem. (5.3.6: Radon-Nikodym)

Let μ and σ be $\sigma-$finite measures on a set X, and suppose that $\nu \ll \mu$. Then $\exists\ f \in L^1(X, d\mu)$ s.t.

$$\nu(E) = \int_E f \, d\mu \quad \textit{for all } \nu(E) < \infty \quad (E \in \mathcal{M})$$

Proof.

1. From Jordan decomposition, the proof can be reduced to the case in which μ, ν are positive and finite.
2. Let f be the maximal function generated by Lemma 5.3.5.
3. Then $\nu_1(E) = \nu(E) - \int_E f d\mu$ satisfies $\nu_1 \geq 0$ and $\nu_1 \ll \mu$.
4. If $\nu_1 \neq 0$, $\exists\ \epsilon > 0$ & E' s.t. $\nu_1(E') > 0$ and

$$\nu_1(E'') \geq \epsilon\ \mu(E'') \quad \text{for } E'' \subset E' \qquad \text{(Why?)}$$

Hence, $\nu(E) = \int_E f + \epsilon \chi_{E'}\, d\mu$ for $E \in \mathcal{M}$, contradicting the maximality of f.

6.2 Duality and Radon-Nikodym Theorem

Definition.

Let $\mathcal{B} = L^p(X, d\mu)$, $1 \leq p < \infty$. We knew that $\mathcal{B}$ is a Banach space with norm $\| \cdot \| = \| \cdot \|_p$.

- *A bounded linear functional on $\mathcal{B}$ is a mapping $\ell : \mathcal{B} \longrightarrow \mathbb{R}$ such that $|\ell(f)| \leq C\|f\|$ for all $f \in \mathcal{B}$.*
- *We denote by $\mathcal{B}^*$ the space of linear functionals on $\mathcal{B}$.*
- *Define $\|\ell\|_{\mathcal{B}^*} = \|\ell\|_* = \left\{ \frac{|\ell(f)|}{\|f\|} \ : \ f \in \mathcal{B} \ \& \ \|f\| \neq 0 \right\}$*

It is easy to prove that $\mathcal{B}^*$ is a normed vector space, that is,

- $\|\ell\|_* \geq 0$ and $\|\ell\| = 0 \Leftrightarrow \ell = 0$. Here, $\ell = 0$ means that $\ell(f) = 0$ for all $f \in \mathcal{B}$.
- $\|\lambda \ell\|_* = |\lambda| \ \|\ell\|_*$ & $\|\ell_1 + \ell_2\|_* = \|\ell_1\|_* + \|\ell_2\|_*$.

Dual & Isometry: $L^{\frac{p}{p-1}} = (L^p)^*$

Theorem. (6.2.2: Isometric injection : $L^{\frac{p}{p-1}} \hookrightarrow (L^p)^*$)

Let $1 < p < \infty$ and $q = \frac{p}{p-1}$. For $g \in L^q$, define $\ell_g : \mathcal{B} \to \mathbb{R}$ by

$$\ell_g(f) = \int g\ f\ d\mu, \quad \text{for } f \in L^p = \mathcal{B}.$$

Then $\ell_g \in \mathcal{B}^*$ and $\|\ell_g\|_* = \|g\|_q$. This means that the injection $L^q \hookrightarrow (L^p)^*$ defined by $g \mapsto \ell_g$ is an ***isometric injection*** *of L^q into $(L^p)^*$.*

Proof.

1. Clearly, ℓ_g is linear on $\mathcal{B}$.
2. According to Hölder inequality,

$$|\ell_g(f)| = \|fg\|_1 \leq \|f\|_p \|g\|_{\frac{p}{p-1}} \quad \text{for all } \ f \in L^p = \mathcal{B}$$

3. From the definition of $\|\cdot\|_*$ and 2, $\quad \|\ell_g\|_* \leq \|g\|_q$.

Proof of isometric injection : $L^{\frac{p}{p-1}} \hookrightarrow (L^p)^*$

4. If $g \neq 0$ and

$$f = \frac{|g|^{p-1} sgn(g)}{\|g\|_q^{q-1}} \quad \text{where} \quad sgn(g) = \frac{g}{|g|}$$

   then

$$\ell_g(f) = \|g\|_q$$

5. From 4, $\|\ell_g\|_* \geq \|g\|_q$
6. From 3 and 5, $\|\ell_g\|_* = \|g\|_q$
7. Hence, $g \mapsto \ell_g$ is an **isometric injection** of L^q into $(L^p)^*$.

♣ ♣ ♣ ♣

Question : What about $(L^p)^* \hookrightarrow L^q$? YES! See the next Theorem 6.2.3.

Theorem. (6.2.3: Representation combining Radon-Nikodym)

Let X be a metric space and let μ be a Borel measure with $\mu(X) < \infty$. For $1 < p < \infty$ and $\ell \in (L^p(X, d\mu))^*$, there exist $g \in L^{\frac{p}{p-1}}$ such that

$$\ell(f) = \int fg \, d\mu \quad \text{for all } f \in L^p$$

- **SUPER important!** This idea with $p = 2$ provides the key concept of Lax-Milgram (uniqueness and existence of PDE, Finite Element Method, and so on).
- To understand this theorem completely, we need to study Hahn-decomposition, signed measure, absolute continuity, Lebesgue differentiation, etc. However, it is a disaster that many recent mathematicians do not know its usefulness and basic structure even after they mastered Real analysis. This is the main reason why I introduce the theorem without knowledge of them here to provide a rough insight.

Theorem. (6.2.3: Continue...)

♣♢♡ *For* $1 < p < \infty$ *and* $\ell \in \mathcal{B}^* = (L^p(X, d\mu))^*$*, there exist* $g \in L^{\frac{p}{p-1}}$ *such that* $\ell(f) = \int fg \; d\mu$ *for all* $f \in L^p = \mathcal{B}$

Sketch of the proof. Let us understand its structure roughly.

- Denote $B_\epsilon(x) = \{y \in X \;:\; |y - x| < \epsilon\}$. Due to absolute continuity of ♠ , we can define Radon-Nikodym derivative in some sense

$$g(x) := \lim_{\epsilon \to 0} \frac{\ell(\chi_{B_\epsilon(x)})}{\mu(B_\epsilon(x))}, \quad x \in X$$

- Hence, $\ell(\chi_E) = \int_E g \; d\mu$ for any measurable set E.
- Hence, $\ell(\phi) = \int g \; \phi \; d\mu$ for any $\phi \in \mathcal{S}_{imple}$.
- The result follows from the facts that $\mathcal{S}_{imple}$ is dense in L^p and $\ell : L^p \to \mathbb{R}$ is bounded and linear.

6.3 Distribution functions

Given a real valued function f on a measure space $(X, \mathcal{M}, \mu)$ its distribution function $\lambda_f : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}$ by

$$\lambda_f(\alpha) \;=\; \mu(\{x : \; |f(x)| > \alpha\})$$

Theorem. (6.3.2: Prove it for $\mathcal{S}_{imple}$. EASY!)

Suppose that $\lambda_f(\alpha)$ is finite for all $\alpha \in \mathbb{R}^+$. For any continuously increasing function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$,

$$\int \eta(|f|) d\mu = - \int \eta(\alpha) \; d\lambda_f(\alpha).$$

In particular, for $1 \leq p < \infty$,

$$\int |f|^p d\mu = - \int_0^\infty \alpha^p \; d\lambda_f(\alpha) = p \int_0^\infty \alpha^{p-1} \; \lambda_f(\alpha) \; d\alpha$$