Multi-frequency topological derivative for imaging of perfectly conducting cracks
Development of non-iterative imaging algorithm

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Introduction
Survey on topological derivative
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Figure: Starting point of presentation.
Inverse scattering problem for imaging of arbitrary shaped curve-like perfectly conducting crack(s) completely hidden in a two-dimensional homogeneous conductor is considered.

Main purpose is **making a good initial guess** for Newton-type iterative algorithm.

Generally, non-iterative algorithm requires quite a number of incident fields.

**Topological derivative** based non-iterative imaging algorithm is developed.

Application of the topological derivative has been heuristic and lacks mathematical analysis.

We introduce **multi-frequency based topological derivative imaging function** and explore its structure.


\[ C: \text{perfectly conducting crack.} \]
\[ D: \text{simply connected domain with smooth boundary } B. \]
\[ k: \text{wavenumber.} \]
\[ u^\text{total}_m(\vec{x}; k): \text{time-harmonic electromagnetic total field} \]
\[
\begin{aligned}
\Delta u^\text{total}_m(\vec{x}; k) + k^2 u^\text{total}_m(\vec{x}; k) &= 0 \quad \text{in } D \setminus C \\
u^\text{total}_m(\vec{x}; k) &= 0 \quad \text{on } C.
\end{aligned}
\]
\[ \frac{\partial u^\text{total}_m(\vec{x}; k)}{\partial \vec{\nu}(\vec{x})} = \frac{\partial \exp(ik\vec{\theta}_m \cdot \vec{x})}{\partial \vec{\nu}(\vec{x})} \quad \text{on } B \]
\[ \vec{\nu}(\vec{x}): \text{unit outward normal to } \vec{x} \in B \]
\[ \vec{\theta}_m: \text{two-dimensional vector on the unit circle } S^1. \]
\[ k^2 \text{ is not an eigenvalue of (1).} \]
\[ u^\text{back}_m(\vec{x}; k) = \exp(ik\vec{\theta}_m \cdot \vec{x}): \text{background solution.} \]

**Residual functional:**

\[
\mathbb{R}(\mathcal{D}; k) := \frac{1}{2} \sum_{m=1}^{M} \left\| u^\text{total}_m(\vec{x}; k) - u^\text{back}_m(\vec{x}; k) \right\|_{L^2(\mathcal{B})}^2 \\
= \frac{1}{2} \sum_{m=1}^{M} \int_{\mathcal{B}} \left| u^\text{total}_m(\vec{x}; k) - u^\text{back}_m(\vec{x}; k) \right|^2 d\mathcal{B}(\vec{x}),
\]
Create a small crack \( \mathcal{L} \) of length \( 2\ell (\ll k) \) centered at 
\[ \bar{x}_s = (x^1_s, x^2_s) \in D \setminus \mathcal{B}. \]

Denote \( D \cup \mathcal{L} \) as that domain.

Due to the change of the topology of \( D \), we can consider the 
**topological derivative** \( \mathbb{R}_{TD}(\bar{x}_s; k) \):

\[
\mathbb{R}_{TD}(\bar{x}_s; k) = \lim_{\ell \to 0^+} \frac{\mathbb{R}(D \cup \mathcal{L}; k) - \mathbb{R}(D; k)}{\phi(\ell; k)},
\]

\( \phi(\ell; k) \to 0 \) as \( \ell \to 0^+ \).

Asymptotic expansion:

\[
\mathbb{R}(D \cup \mathcal{L}; k) = \mathbb{R}(D; k) + \phi(\ell; k)\mathbb{R}_{TD}(\bar{x}_s; k) + o(\phi(\ell; k)).
\]
We can introduce topological derivative\(^1\).

**Lemma (Topological derivative)**

Let \(\Re(f)\) denotes the real-part of \(f\). Then

\[
\Re_{\text{TD}}(\vec{x}_s; k) = \Re \left[ \sum_{m=1}^{M} v_{m}^{\text{adjnt}}(\vec{x}_s; k) u_{m}^{\text{back}}(\vec{x}_s; k) \right],
\]

where \(v_{m}^{\text{adjnt}}(\cdot; k)\) satisfies adjoint problem:

\[
\begin{align*}
\Delta v_{m}^{\text{adjnt}}(\vec{x}; k) + k^2 v_{m}^{\text{adjnt}}(\vec{x}; k) &= 0 \quad \text{in} \quad \mathcal{D} \\
\frac{\partial v_{m}^{\text{adjnt}}(\vec{x}; k)}{\partial \vec{n}(\vec{x})} &= u_{m}^{\text{total}}(\vec{x}; k) - u_{m}^{\text{back}}(\vec{x}; k) \quad \text{on} \quad \mathcal{B}
\end{align*}
\]

We introduce the **structure of topological derivative**\(^2\).

**Lemma**

For \( \tilde{x}_s \in \mathcal{D} \) and \( \tilde{x}_c \in \mathcal{C} \),

\[
\mathbb{R}_{TD}(\tilde{x}_s; k) \sim \Re \left[ \frac{i}{k} \sum_{m=1}^{M} \exp(ik\vec{\theta}_m \cdot (\tilde{x}_c - \tilde{x}_s)) \Phi(\tilde{x}_c, \tilde{x}_s; k) \right].
\]

Here \( \Phi(x, y; k) \) denotes the two-dimensional time-harmonic fundamental solution of Helmholtz equation

\[
\Phi(x, y; k) = -\frac{i}{4} H_0^1(k|x - y|),
\]

where \( H_0^1 \) denotes the Hankel function of the first kind.

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For several wavenumbers \( \{k_f : f = 1, 2, \cdots, F \} \) define a normalized topological derivative function

\[
R_{MTD}(\vec{x}_s; F) = \frac{1}{F} \sum_{f=1}^{F} \frac{R_{TD}(\vec{x}_s; k_f)}{\max_{\vec{x}_s \in \mathcal{D}} |R_{TD}(\vec{x}_s; k_f)|}.
\]  

Based on the structure of topological derivative, analysis of (2) is carried out for the following three cases of interest\(^3\).

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\(^3\)W.-K. Park, Fast shape reconstruction of perfectly conducting cracks by using multi-frequency topological derivative function, preprint
Theorem (First case: symmetric incident directions)

Under the **small number of symmetric incident directions** configuration, the structure of (2) is

\[
\mathbb{R}_{MTD}(\vec{x}_s; F) \sim \sum_{m=1}^{M} \frac{1}{|\vec{x}_c - \vec{x}_s| \sqrt{1 - \left( \theta_m \cdot \frac{\vec{x}_c - \vec{x}_s}{|\vec{x}_c - \vec{x}_s|} \right)^2}},
\]

for large enough number of \( F \).
Theorem (Second case: non-symmetric incident directions)

Under the **small number of non-symmetric incident directions configuration**, the structure of (2) becomes

\[
\mathbb{R}_{\text{MTD}}(\vec{x}_s; F) \sim \sum_{m=1}^{M} \frac{1 - \frac{2}{\pi} \sin^{-1} \left( \vec{\theta}_m \cdot \frac{\vec{x}_c - \vec{x}_s}{|\vec{x}_c - \vec{x}_s|} \right)}{|\vec{x}_c - \vec{x}_s| \sqrt{1 - \left( \vec{\theta}_m \cdot \frac{\vec{x}_c - \vec{x}_s}{|\vec{x}_c - \vec{x}_s|} \right)^2}},
\]

for large enough number of \( F \).
Theorem (Third case: enough number of incident directions)

Under the **large enough number of incident directions** configuration, the following relationship holds in the sense of distribution

$$\mathcal{R}_{MTD}(\vec{x}_s; F) \sim \frac{1}{|\vec{x}_c - \vec{x}_s|} \delta(\vec{x}_c, \vec{x}_s)$$

for large enough number of $F$. Here, $\delta(\vec{x}_c, \vec{x}_s)$ denotes the Dirac delta function.
\( \mathbb{R}_{\text{MTD}}(\vec{x}_s; F) \) reaches its maximum value at \( \vec{x}_s \in \mathcal{D} \) that satisfies

\[ |\vec{x}_c - \vec{x}_s| = 0. \]

However at the points \( \vec{x}_s \in \mathcal{D} \) such that

\[ \vec{\theta}_m \cdot \frac{\vec{x}_c - \vec{x}_s}{|\vec{x}_c - \vec{x}_s|} = \pm 1 \]

map of \( \mathbb{R}_{\text{MTD}}(\vec{x}_s; F) \) also shows large magnitude so that unexpected replicas \( C' \) will appear.

Symmetric incident direction configuration will gives a better imaging result than non-symmetric one.
Figure: At the location $\vec{x}_s$ such that $\vec{x}_s = \vec{x}_c$ (black colored vector) and $\vec{\theta}_m \cdot \frac{\vec{x}_c - \vec{x}_s}{|\vec{x}_c - \vec{x}_s|} = \pm 1$ (green colored vector), true shape of $C$ and ghost replicas $C'$ will appear, respectively.
• Adopted wavenumber is taken of the form

\[ k_f = \frac{2\pi}{\lambda_f} \]

for \( f = 1, 2, \ldots, F \).

• \( \lambda_f \): given wavelength which will be equi-distributed between \( \lambda_1 = 0.7 \) and \( \lambda_F = 0.4 \).

• Incident direction:

\[ \hat{\theta}_m := \left( \cos \frac{2(m-1)\pi}{M}, \sin \frac{2(m-1)\pi}{M} \right), \]

for \( m = 1, 2, \ldots, M \).

• A white Gaussian noise with 15dB signal-to-noise ratio (SNR) is added to the unperturbed data.
(a) $M = 4$ case

(b) $M = 16$ case

**Figure:** Map of $\mathbb{R}_{MTD}(\vec{x}_s; 16)$.
Figure: Map of $\mathbb{R}_{MTD}(\vec{x}_s; 16)$.
Figure: Map of $\mathcal{R}_{\text{MTD}}(\bar{x}_s; 16)$. 

(a) $M = 4$ case  

(b) $M = 16$ case
Image of single crack
Imaging of two separated cracks
Influence of multi-frequency
Symmetric vs non-symmetric configuration

Figure: Map of $\mathbb{R}_{TD}(\vec{x}_s)$ with $\lambda = 0.4$. 

(a) $M = 32$ case
(b) $M = 64$ case
Figure: Map of $\mathbb{R}_{\text{MTD}}(\vec{x}_s; 16)$. 

(a) $M = 5$ case

(b) $M = 5$ case
(a) $M = 64$ case

(b) true crack

**Figure:** Map of $R_{MTD}(\mathbf{x}_s; 64)$. 
Figure: Map of $\mathcal{R}_{\text{MTD}}(\vec{x}_s; 64)$. 

(a) $M = 64$ case

(b) true crack
- Proposed multi-frequency topological derivative is **useful for imaging of perfectly conducting crack(s)**.
- An improvement to overcome some limitations is required.
- Extension to the imaging of crack with **zero Neumann condition** is a forthcoming work.
- Application of **three dimensional problem** will be an interesting subject.
Thank you very much.

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Some limitations: large curvature
Some limitations: oscillating
Conclusion and perspective
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