Spatial distributions of islands in fractal surfaces and natural surfaces

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\textbf{ABSTRACT}

The size and spatial distributions of islands in random fractal surfaces and natural islands are investigated. Multi-scale island distributions from a fractal surface are analyzed to determine the size distribution of islands, such as the number of islands greater than a particular size, and this was found to exhibit a Korcak-type empirical relation. In the same multi-scale analysis, the spatial distribution of islands, which is the number of islands less than or equal to a particular distance between a pair of islands, is also investigated to determine the effects of differences in scale. The spatial distribution of distance between uniformly distributed islands is analytically calculated and compared with the island distributions of the fractal surface. We seek to determine the degree of resemblance between simulated spatial distributions and the natural island distribution of French Polynesia. We also estimate the size and spatial distributions of islands according to different sea levels.

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1. Introduction

Identification of island distributions, including the size and distance between islands, is important for contact surface research [1]. The contact between rough surfaces is generally restricted to a number of microscopic ‘actual contact areas’ located near the peaks or asperities of rough surfaces. The contact profile of rough surfaces is dependent on the intrinsic nature of the rough surface, showing a hierarchy of scales. In this multi-scale surface, the contact spots are affected by the occurrence of different contact spots at nearby points since the larger scale waviness will tend to cluster the next scale of asperity contacts into groups. The actual contact area between two rough surfaces can be modeled by the set of islands [2–4]. In such clusters of contacts, electrical contact resistance can be estimated by the number of distributions of the size and distance between islands [5–10].

Estimation of the spatial distributions of islands, with regard to size and distance, should benefit those who are particularly interested in analytical expression. The size distribution of islands is already characterized by a Korcak empirical number-area rule [11], which states that the relative number of islands whose area exceeds a threshold area can be expressed by a power law. Many works can identify complex systems by using this relationship [12–14]. Swingler [15] recently reported that the distance distribution of islands followed a Korcak-type distribution similar to the size distribution without considering the effects of differences in scale. We argue that the scale dependency of island distributions should be considered, because islands are formulated by rough surfaces, that may have different morphologies at different scales.

To estimate the general characteristics of spatial distributions, analytical formulations of spatial distributions should be confirmed by applying these formulations to simulated and natural island distributions. In this paper, we develop a probabilistic expression of spatial distributions and compare these simulated distributions with a natural example. In addition, we investigate the occurrence of fractal properties in these models. We chose the French Polynesian island chain as our natural world data. This cluster was used, because of its remoteness from
continents. Different heights were used in the calculations to determine if the fractal property was consistent at different sea levels. For the simulated island distribution, we generated a randomly rough surface using the random midpoint displacement algorithm [16]. Finally, we used this data to determine whether the Korcak-type law can accurately model the size and distance distributions of simulated and natural islands.

2. Size and spatial distribution of islands in fractal surfaces

We generate a fractal surface using the "random midpoint displacement" algorithm [16]. The fractal dimension of this surface is 2.5 and the length of the sides is 1. The island distribution is obtained when the surface is cut at any given height. Fig. 1 shows the distribution of islands for heights of 90%. Several island distributions are obtained from the original surface that is discretized at scales from $1/2^6$ to $1/2^{12}$, as shown in Fig. 2. For each surface, we use the same height as indicated in Fig. 1 to generate the island distribution. These results reveal that smaller islands tended to occur near larger islands and tended to cluster with the next scale of islands into groups.

From the several scaled island distributions, we next sought to determine the number of islands greater than a particular size "$a$," which can be formulated as $N(A \geq a) = m \int_a^\infty f(A)dA$. Fig. 3 shows a log-log plot of the number of islands greater than a particular size as opposed to the area of the islands for various sampling lengths (scale). These results indicate that above a certain value of island size in each sampling interval (scale), $N(A \geq a)$ tends to become a straight line and this line is common to all sampling intervals. However, below a certain island size, $N(A \geq a)$ is almost constant, indicating that the number of island sizes greater than the particular given size does not vary with respect to the island size range. This behavior occurs because in certain sampling intervals, $N(A \geq a)$ is affected by the critical size of islands, much like the cut-off frequency in signal processing analysis. This becomes increasingly evident as each additional $N(A \geq a)$ is plotted for different sampling intervals, which in turn conforms to the appropriate form of the power law. Hence, if $N(A \geq a)$ is obtained from the straight line of Fig. 3, the following equation can be obtained:

$$N(A \geq a) = m \int_a^\infty f(A)dA = C_1 A^{-K}$$

(1)

where $m$ is the number of islands per unit area and $C_1$ and $K$ are 0.0041 and 0.61, respectively.

To identify the spatial distribution, we consider the distance "$s$" between a pair of islands. We define this distance as the distance between two area centroids of a pair of islands. The number of islands less than or equal to a particular distance "$s$," whose definition is $N(s \leq s') = S(s) \int_{s'}^\infty p(s)ds$, is obtained. To explore the characteristics of this spatial distribution, we calculate the spatial distributions of island distributions with different scales from $1/2^6$ to $1/2^{12}$. We also confined the islands to the square of $0.25 < x, y < 0.75$, as shown in Fig. 1, to avoid the effect of the boundaries, including the distances from islands inside the square to those outside the square. Thus, we need to use an algorithm that, for this range, would yield results representative of an infinite region. Fig. 4 shows a log-log plot of the number of islands less than or equal to a particular distance against distance per $2\pi m^2$ for various sampling lengths (scale). As scales become finer, the function $N(s \leq s')/2\pi m^2$ becomes linear except for the region of lower and higher distances. This is increasingly evident as additional $N(s \leq s')/2\pi m^2$ points are plotted for different sampling intervals, which in turn allowed the appropriate form of the power law to be extracted, leading to the derivation of a power law relationship of the spatial cumulative distribution of inter-distance as

$$N(s \leq s')/2\pi m^2 = S_c \int_{s'}^\infty p(s)ds = C_2 s^H$$

(2)

where $C_2$ and $H$ are 2.675 and 1.3, respectively. It is also conjectured that the behavior of the lower distance region is affected by the scale and that the behavior of the higher distance is limited by the total region of island distribution.

The fact that the function $N(s \leq s')$ has a power law relation with a specific slope in the fractal limits suggests that a different slope may have been obtained for a uniform distribution of islands. To test this hypothesis, we calculate the distribution of distances for a completely random distribution of islands. The analysis is given in the Appendix and the results are summarized as

$$p(s) = 2\pi m^2 s \left( \pi - 2 \int_{1-s}^1 \arccos \left( \frac{1-r^2 - s^2}{2rs} \right) dr \right);$$

for $0 < s < 1$,

$$= 4\pi m^2 s \left( \pi - \arccos \left( \frac{1-r^2 - s^2}{2rs} \right) \right) dr;$$

for $1 < s < 2$.

(3)

Within the limits of the uniform distribution over all space, but still restricting the distances to those terminating at one of the islands inside the unit circle, the distribution is

$$p^*(s) = 2m^2 \pi^2 s.$$

(5)
The cumulative distributions of distance per unit area $N(s \leq s')/2\pi m^2$ are plotted and indicated by the dashed line in Fig. 4. The slope of the cumulative distribution increases quadratically with $s$, because of the area of a circle of radius $s$ centered on the given island. Thus, the cumulative distribution for a uniform distribution of islands has a slope of 2:1 which is steeper than the slope of the spatial distribution of islands obtained from the fractal surface.

### 3. Size and spatial distribution of natural islands

The size distribution of islands has already been modeled using Korcak's empirical number-area rule [11]. An examination of data from the whole world yields a Korcak's exponent of patchiness, $K$, of 0.65, indicating that the fractal dimension of its patch (coast line) is 1.3. More
local estimates using restricted regions range from $K = 0.5$ for Africa up to $K = 0.75$ for Indonesia and North America. Distributions with high Korcak's exponents of patchiness have more small patches, or simply patchier than distributions characterized by smaller values of $K$ [17]. Thus, it is interesting to find a degree of resemblance between spatial distributions of simulations and actual island distribution.

Large natural island distributions can be found near continents. The island distributions near continent, or large chunks of land are limited by the continent or the chunk of land. However, the distribution of islands in ocean is not affected by the edge of the region. For this reason, we select the islands of French Polynesia, located at a longitude of $-135^\circ$ to $-155^\circ$ and a latitude of $-6^\circ$ to $-26^\circ$, to study, as shown in Fig. 5(a) [18].

To specify the island distribution, data is collected from the shuttle radar topography mission (SRTM) [19], because of the accuracy and real-time nature of its data. Pictures obtained from the SRTM have a resolution of 90 m at the equator, meaning that 1 pixel equals $8.333 \times 10^{-4}$ degrees. The contour of islands is extracted from the height data of the region.

The centroid of the islands’ area is calculated from the contour line of each island. Most of the centroids are located within the islands, but some are also located outside. The locations of the centroids in the cluster of islands are shown in Fig. 5(b). The islands are highly clustered at the longitude of $-140^\circ$ to $-150^\circ$ and the latitude of $-14^\circ$ to $-20^\circ$. Outside this region, the islands are sparsely distributed. We also trace the island distribution according to different sea levels from 0 m to 6 m.

Fig. 6(a) shows that the number of islands is greater than a particular size, $N(A \geq a)$ according to different sea levels. There is a power law relation for sea level of 0 m and 6 m, as shown in Fig. 6(b). It is evident that the number of islands greater than a particular size have a linear relation in the log–log plot, indicating that this distribution is a Korcak-type distribution. For two extremities of sea levels, the power law expressions is $9000A^{-0.79}$ for a sea level of 0 m and $3500A^{-0.822}$ for a sea level of 6 m. The Korcak’s exponents for other sea levels are between $-0.79$ and $-0.822$. As the sea level increases, so do the number of islands.

Fig. 7 shows the number of islands less than or equal to a particular distance $N(s \leq s')$ according to different sea levels. The function $N(s \leq s')$ is relatively linear except for the regions of the lower and higher distances. For sea level of 0 m and 6 m, the linear region is fitted by a line represented by the following equation: $N(s \leq s')/2\pi m^2 = 45.0s^{-0.9282}$ and $N(s \leq s')/2\pi m^2 = 150.0s^{-0.993}$, respectively.
Despite the changes in sea level, a power law relationship is maintained. More interestingly, if we plot the number of islands less than or equal to a particular distance per unit area $N(s/s')/(2\pi m^2)$ for different sea levels, as shown in Fig. 8, the curves are merged into one, indicating a power law relationship. The linear line is represented by the equation: $N(s/s')/(2\pi m^2) = 1.66 \times 10^{-5}s^{1.0}$.

4. Discussion

We investigate the size and spatial distribution of islands in fractal surfaces and natural islands to determine if these island distributions conform to a Korcak-type empirical relationship. Our results show that the size and spatial distributions of islands do follow a Korcak-type relationship. There is also good agreement between the spatial distributions of simulated and actual island distributions. The size and spatial distribution of the island distribution are obtained from the random fractal surfaces having a fractal dimension of 2.5. The Korcak exponent is related to the Korcak dimension $D_k$ such that $K = D_k/D_E$ [17] where $D_E$ is the dimension of the embedding Euclidian space. The Korcak exponent and the Korcak dimension of the random fractal surface are $K = 0.61$, $D_k = 2$, respectively, giving $D_k = 1.22$. The Korcak dimension is related to the fractal dimension of surface $D$ as $D_k = D - 1$. Thus, the fractal dimension of surface is 2.22. The fractal dimension of surface is still larger than the Korcak dimension. The hypothesis that the Korcak dimension and the fractal dimension of surface are not identical has already been reported [20]. Thus, the analysis process and results can be validated. The Korcak dimension of the island distribution of French Polynesia ranges from 1.58 to 1.64 according to different sea levels. The corresponding fractal dimension of the surface is around 2.6 or greater.

Our current results can be compared with the size and spatial distribution results reported by Swingler [15], who investigated the distribution of contact spots according to various contact forces. The Korcak exponent $K$ and the slope of spatial distribution $H$ for larger contact forces approaches 1.26 and 1.10, respectively. The Korcak exponent of this size distribution is much larger than the Korcak exponent of our natural and simulated island distributions. It has already been reported that contact spot formation shows different trends according to the sampling interval of the contact surface. Thus, the results of Swingler’s study may be due to his use of one specific sampling interval, indicating that the sampling interval is limited by the resolution of the measuring device. The slope of the spatial distribution $H$ decreased from 1.48 to 1.10 as the contact forces increased. This trend shows that when the contact area increases, spots are not uniformly distributed and could possibly be clustered, because the slope is far away from 2.

The spatial distribution of islands in a random fractal surface, $N(s/s')$ has a power law relation. We obtained this result from investigating island distributions on a multi-scale surface. The linear behavior of the log–log plot of $N(s/s')$ becomes more evident as the scale becomes finer. We believe that the current linear line is almost saturated, but there is the possibility that small changes in the slope may have occurred. However, there is a distinct difference between the slope of the spatial cumulative distribution of inter-island distance for uniformly distributed islands and fractal surfaces, indicating that the slope of uniformly distributed islands is steeper than that of non-uniformly distributed islands. If the distribution of islands is statistically uncorrelated, the probability of an island at a specific location is unaffected by the actual occurrence of an island at a nearby point. However, since the long waviness of a surface will tend to cluster the islands of small scale roughness into groups, the distributions are correlated, meaning that the size of islands is related to the distance between islands. Thus, it is possible that the difference in slope is a property of the resultant clustering. We did not examine this hypothesis thoroughly in the current analysis, but we did show that the size distribution and the spatial distribution of islands have different Korcak exponents.
This work is focused on the investigation of the fractal properties according to scales for the size and spatial distribution of islands. The outcome of this fractality can be represented as a fractal dimension, which is a typical example of fractality. However, this fractal dimension of single exponent is not enough to describe its dynamics; instead, a continuous spectrum of exponents may be needed [21,22]. Thus, a multifractal analysis that can generate multifractal spectra, which illustrates how scaling varies over the original datasets, can fulfill the understanding of the fractal characteristics of the size and spatial distribution of islands. One possible extension to the multifractal analysis in future work is to calculate all Renyi (generalized) dimensions of the French Polynesian islands. These extensions are the subject of ongoing research.

The usage of the size and spatial distribution can shed light on the identification of a probability function for contact spots between rough surfaces. Several research areas such as tribology and contact physics have highlighted the importance of size and spatial distribution of contact spots in developing theories. Furthermore, knowledge of the size and spatial distributions of electrical contacts is also important for determining their contact resistance. Specifically, expressions of the size and spatial distributions of electrical contacts is an example of fractality. However, this fractal dimension of islands also showed some resemblance. For the French Polynesian islands, the Korcak exponent of the size distribution was 2, steeper than the slope of the spatial distribution of islands from a fractal surface. The spatial distributions of simulated and natural islands also showed some resemblance. For the French Polynesia islands, the Korcak exponent of the size distribution changed minimally according to different sea levels. The spatial distribution followed a power law, and this held true even for different sea levels.

5. Conclusion

The size and spatial distributions of islands in random surfaces as well as natural islands were investigated. We analyzed the size distribution of islands from a random fractal surface according to the multi-scale distribution of islands. It is confirmed that the size distribution of islands, such as the number of islands greater than a particular size, exhibits a Korcak type empirical relationship with fractal behavior. In the same multi-scale analysis, we observed that the spatial distribution of islands, which is the number of islands less than or equal to a particular distance, followed a power law relationship as the scale of the surface became finer. The spatial distribution of the inter-island distance for uniformly distributed islands were also analytically calculated, and showed that the slope of the log–log plot for the spatial distribution was 2, steeper than the slope of the spatial distribution of islands from a fractal surface. The spatial distributions of simulated and natural islands also showed some resemblance. For the French Polynesian islands, the Korcak exponent of the size distribution changed minimally according to different sea levels. The spatial distribution followed a power law, and this held true even for different sea levels.

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Appendix A. Distribution of inter-island distances in the uniform distribution of islands

A square is harder to integrate than the circle, but exhibit similar truncation behavior, therefore we chose to consider a circle. Suppose we have a uniform distribution of islands \( m \) per unit area over the unit circle \( r < 1 \), so that the number of islands in an area \( \Delta A \) is \( m\Delta A \). If there is an island at the point \( r, 0 \), the probability of there being another island at a distance in the range \( (s, s + \Delta s) \) is \( mL\Delta s \), where \( L \) is the length of the circular arc of radius \( s \) centered on \( r, 0 \) that lies within the unit circle.

From Fig. A.1, this arc length is

\[
L = 2s(\pi - \theta),
\]

for \( 1 - r < s < 1 + r \). For smaller values of \( s \) \( (s < 1 - r) \), \( \frac{L}{2\pi} \), and \( L = 0 \) for \( s > 1 + r \). Now in the circular annulus of thickness \( \Delta r \) and radius \( r \), there are \( 2\pi m r\Delta r \) islands, each of which contributes \( mL\Delta s \) distances, so the number of inter-island distances in \( (s, s + \Delta s) \) is

\[
p(s)\Delta s = 2\pi m^2\Delta s \int_0^1 Lrdr.
\]

We need to split the range of integration because of the discontinuous nature of \( L \). For \( s < 1 \), we obtain

\[
\int_0^1 Lrdr = \int_0^{1-s} (2\pi s)rdr + \int_1^{1-s} 2s(\pi - \theta)rdr,
\]

which is more conveniently grouped as

\[
\int_0^1 Lrdr = 2\pi s \int_0^1 rdr - 2s \int_1^{1-s} \theta dr = s(\pi - 2 \int_1^{1-s} \theta dr).
\]

For \( 1 < s < 2 \),

\[
\int_0^1 Lrdr = \int_0^{s-1} 2s(\pi - \theta)rdr.
\]

From Fig. A.1, we have

\[
\theta = \arccos \left(\frac{1 - r^2 - s^2}{2rs}\right).
\]

Fig. A.1. Uniform distribution of islands over a unit circle.
Thus, for $0 < s < 1$,

$$p(s) = \frac{2\pi m^2 s}{s} \int_{-\pi}^{\pi} \frac{1}{2s} \arccos \left( \frac{1 - r^2 - s^2}{2rs} \right) rdr,$$

while for $1 < s < 2$,

$$p(s) = 4\pi m^2 s \int_{-1}^{1} (\pi - \theta) rdr.$$

In contrast, if we extended the uniform distribution over all space, but still restricted the distances to those terminating at one of the islands inside the unit circle, we get the simpler expression

$$p'(s) = 2m^2\pi^2 s.$$

The exact expression and the limit are plotted in Fig. A.2. The truncation error begins to be significant at $s = 0.2$. If we plotted the cumulative distribution, the truncation error would probably start occurring at a larger value of $s$ because derivatives are more sensitive than integrals.

References