Regularity criteria of the magnetohydrodynamic equations in bounded domains or a half space

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\textbf{A B S T R A C T}

We study three-dimensional incompressible magnetohydrodynamic equations in bounded domains or a half space. We present new regularity criteria of weak solutions: a pair of weak solutions, \((u, b)\), become regular if \(u\) satisfies Serrin’s type conditions when we consider no-slip or slip boundary conditions for the velocity field, \(u\), and slip boundary conditions for the magnetic field, \(b\), in either bounded domains or a half space. In addition, in the case of a half-space with no-slip boundary conditions for \(u\) and slip boundary conditions for \(b\), we demonstrate that, if tangential components of \(u\) and normal component of \(b\) satisfy Serrin’s type conditions, then a pair of weak solutions, \((u, b)\), become regular.

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\textbf{1. Introduction}

We study the following three-dimensional magnetohydrodynamic equations (MHD):

\[
\begin{align*}
    u_t - \Delta u + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla \pi &= 0, \\
    b_t - \Delta b + (u \cdot \nabla)b - (b \cdot \nabla)u &= 0, \\
    \text{div} u &= 0, \\
    \text{div} b &= 0,
\end{align*}
\]

in \(Q_T := \Omega \times [0, T)\), \(\text{(1.1)}\)

where \(\Omega\) is a domain in \(\mathbb{R}^3\). Here \(u : Q_T \rightarrow \mathbb{R}^3\) is the flow velocity vector, \(b : Q_T \rightarrow \mathbb{R}^3\) is the magnetic vector and \(\pi = p + \frac{|b|^2}{2} : Q_T \rightarrow \mathbb{R}\) is the magnetic pressure. We consider the initial-boundary value problem of (1.1), which requires initial conditions.

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together with two types of boundary conditions defined as follows: Either

\[(B1) \quad u = 0 \quad \text{and} \quad b \cdot n = 0, \quad (\nabla \times b) \times n = 0,\]

or

\[(B2) \quad u \cdot n = 0, \quad (\nabla \times u) \times n = 0 \quad \text{and} \quad b \cdot n = 0, \quad (\nabla \times b) \times n = 0,\]

where \(n\) is the outward unit normal vector along boundary \(\partial \Omega\). In other words, we consider a slip boundary condition for the magnetic field and either no-slip or slip conditions for velocity field. The initial conditions satisfy the compatibility condition, i.e. \(\nabla \cdot u_0(x) = 0\) and \(\nabla \cdot b_0(x) = 0\). A weak solution pair \((u, b)\) of (1.1)-(1.2) with boundary conditions either (B1) or (B2) is regular in \(Q_T := (0, T) \times \Omega\) provided that \(\|u\|_{L^\infty(Q_T)} + \|b\|_{L^\infty(Q_T)} < \infty\). The notion of weak solutions will be introduced in Definition 3 of Section 2.

The MHD describe the dynamics of the interaction of electrically conducting fluids and electromagnetic forces such as conducting fluids, which are frequently generated in nature and industry, e.g., plasma and liquid metals (see e.g., [7]).

Many important contributions have been made on the existence, uniqueness and regularity of weak solutions to the MHD, and we list only some results relevant to our concerns. It has been shown that global weak solutions for MHD exist in finite energy space (see [8]) and classical solutions can exist locally in time (refer to [27] and [18] for the Navier–Stokes equations (NSE)). In particular, in the two-dimensional case, weak solutions become strong solutions, and therefore, strong solutions exist globally in time (see [8]). In the three-dimensional case, as shown in [31], if a weak solution pair \((u, b)\) are additionally in \(L^\infty(0, T; H^1(\mathbb{R}^3))\), they become regular. He and Xin proved in [16] that a weak solution pair \((u, b)\) become regular in the presence of a certain type of scaling invariant integral conditions, typically referred to as Serrin’s condition, for only the velocity field, \(u\), namely, \(u \in L^p(0, T; L^q(\mathbb{R}^3))\) with \(2/p + 3/q \leq 1\) and \(q > 3\). Here we emphasize that, for the case of whole space, additional conditions are imposed only on the velocity field, but not on the magnetic field. Such a result is, however, not known for domains with boundaries (see [16]). For a local interior case, various types of \(\epsilon\) regularity criteria and partial regularity results have been also established in terms of scaled norms in [17] and [21] (see [38] for boundary case). Compared to the result in [16], local regularity criteria require control of some scaled norms of magnetic fields as well as those of the velocity field. For the case of whole space, using the techniques of Besov spaces, the result of [16] has been improved (see e.g. [41] and [6]). Other types of regularity criteria can be referred to, for example, [5,40,42] and the related references therein.

The motivation of our study is that we do not know whether or not the result in [16] is also true for the case of bounded domains. One of the main difficulties for domains with boundaries is due to the fact that, unlike whole space \(\mathbb{R}^3\), controlling pressure is not obvious because of the absence of the boundary condition of pressure. To be more precise, in the case that \(\Omega = \mathbb{R}^3\), using the equation of pressure, we observe that the pressure \(\pi\) satisfies

\[\|\pi\|_{L^p(\mathbb{R}^3)} \leq C\left(\|u\|_{L^{2p}(\mathbb{R}^3)}^2 + \|b\|_{L^{2p}(\mathbb{R}^3)}^2\right), \quad 1 < p < \infty.\]

However, it is not known yet whether or not the estimate above holds for domains with boundaries. Therefore, methods of proof in \(\mathbb{R}^3\) do not seem to be applicable to the case of domains with boundaries. To overcome these difficulties, we use the maximal estimates of Stokes system for both cases of slip and no-slip boundary conditions, regarding the nonlinear term as an external force (see Lemma 4 in Section 2). Since such estimates of the Stokes system are also available for domain with boundaries, this approach allows for control of pressure and is useful for our analysis. In short, one of our main results is that the result for MHD in [16] can be extended to the cases of bounded domains and a
half-space with boundary data (B1) or (B2). To be more exact, the first of our main results reads as follows:

**Theorem 1.** Let \( \Omega \) be either a bounded domain with a smooth boundary in \( \mathbb{R}^3 \) or a half space \( \mathbb{R}^3_+ := \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0 \} \). Suppose that \((u, b)\) is a weak solution of (1.1) with initial conditions \( u_0, b_0 \in H^2(\Omega) \cap W^{1,q}(\Omega), q > 3 \) and boundary conditions (B1) or (B2). Assume further that \( u = (u_1, u_2, u_3) \) satisfies

\[
\sum_{k=1}^{3} \| u_k \|_{L^{p_k,q_k}(Q_T)} < \infty, \quad \frac{3}{p_k} + \frac{2}{q_k} \leq 1, \quad 3 < p_k, q_k \leq \infty. \tag{1.5}
\]

Then, \((u, b)\) become regular in \( \overline{Q_T} \).

**Remark 1.** We remark that, in the case that \((p_k, q_k) = (3, \infty)\) in (1.5), the result of Theorem 1 is also valid, provided that \( \| u_k \|_{L^{3,\infty}(Q_T)} \) is sufficiently small (see Remark 5 for more details).

In the absence of the magnetic field, \( b \), the equations (1.1) become Navier–Stokes equations; therefore, in the case of no-slip boundary conditions, the result in Theorem 1 immediately implies a well-known result, usually referred to as Serrin’s condition for the Navier–Stokes equations (see e.g., [29,32,24,11,30,4,34,12,36,28,26,35,20,9,14,15]). For the case of the Navier–Stokes equations with slip boundary data, it was shown in [3] that the Serrin’s conditions imply local boundary regularity of suitable weak solutions in a half space, and our result also holds for bounded domains. Although such a result may be known to experts, we were not able to find it in the literature.

**Corollary 1.** Let \( \Omega \) be either a bounded domain with a smooth boundary in \( \mathbb{R}^3 \) or a half space. Suppose that \( u \) is a weak solution of the Navier–Stokes equations, namely \( b = 0 \) in (1.1), with initial condition \( u_0 \in H^2(\Omega) \cap W^{1,q}(\Omega), q > 3 \) and with either no-slip or slip boundary conditions. Assume further that \( u = (u_1, u_2, u_3) \) satisfies

\[
\sum_{k=1}^{3} \| u_k \|_{L^{p_k,q_k}(Q_T)} < \infty, \quad \frac{3}{p_k} + \frac{2}{q_k} \leq 1, \quad 3 < p_k, q_k \leq \infty. \tag{1.6}
\]

Then \( u \) becomes regular in \( \overline{Q_T} \).

The second result is that, for the case of a half space, the control of tangential components of velocity and normal component of magnetic field imply regularity. A similar result was proved in [19] for whole space \( \mathbb{R}^3 \), and our extension is made, in particular, when no slip and slip boundary data are given to velocity and magnetic fields, respectively. To be more precise, we obtain the following.

**Theorem 2.** Suppose that \((u, b)\) is a weak solution of (1.1) in a half space with initial conditions \( u_0 \in H^2(\mathbb{R}^3_+) \cap W^{1,q}(\mathbb{R}^3_+), b_0 \in H^1(\mathbb{R}^3_+) \cap W^{1,q}(\mathbb{R}^3_+), q > 3 \) and boundary conditions (B1). Let \( Q_T := \mathbb{R}^3_+ \times [0, T) \). If the tangential components of the velocity \( \tilde{u} = (u_1, u_2) \) and the normal component of the magnetic field \( b_3 \) satisfy

\[
\| \tilde{u} \|_{L^{p,q}(Q_T)} < \infty, \quad \frac{3}{p} + \frac{2}{q} = 1, \quad 3 < p, q \leq \infty, \tag{1.6}
\]

\[
\| b_3 \|_{L^{3,\infty}(Q_T)} < \infty, \quad \frac{3}{r} + \frac{2}{s} = 1, \quad \frac{3p}{p-2} \leq r \leq \frac{3p}{p-3}, \tag{1.7}
\]

then a weak solution pair \((u, b)\) become regular in \( \overline{Q_T} \).
A direct consequence of Theorem 2 is the following.

**Corollary 2.** Suppose that $u$ is a weak solution to the Navier–Stokes equations, namely $b = 0$ in (1.1), with initial condition $u_0 \in H^2(\mathbb{R}^3_+) \cap W^{1,q}(\mathbb{R}^3_+)$, $q > 3$ and with no-slip boundary conditions. Let $Q_T := \mathbb{R}^3_+ \times [0, T)$. If the tangential components $\tilde{u} = (u_1, u_2)$ satisfy

$$
\|\tilde{u}\|_{L^p_{\text{loc}}(Q_T)} < \infty, \quad \frac{3}{p} + \frac{2}{q} = 1, \quad 3 < p \leq \infty,
$$

then $u$ becomes regular in $\overline{Q_T}$.

**Remark 2.** The result of Corollary 2 seems to be of independent interest. For regularity with the Navier–Stokes equations in a half space, it suffices to control only tangential components of the velocity field. There have been numerous results regarding component reduction for the case of whole spaces, but there is little data for domains with boundaries (compare to [22]); therefore, the result of Corollary 2 is new.

This paper is organized as follows. In Section 2, we recall the notion of weak solutions and review some known results. In Section 3 and Section 4, we present the proofs of Theorem 1 and Theorem 2, respectively. In Appendix A, we present the detailed proofs of the local existence of regular solutions.

### 2. Preliminaries

In this section, we introduce the notations and definitions used throughout this paper. We also recall some lemmas which are useful to our analysis. For $1 \leq q \leq \infty$ and a nonnegative integer $k$, $W^{k,q}(\Omega)$ indicates the standard Sobolev space with norm $\| \cdot \|_{k,q}$, i.e., $W^{k,q}(\Omega) = \{ u \in L^q(\Omega); \ D^\alpha u \in L^q(\Omega), \ 0 \leq |\alpha| \leq k \}$. As usual, $W^{k,q}_0(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ in $W^{k,q}(\Omega)$. When $q = 2$, we write $W^{k,q}(\Omega)$ (or $W^{k,q}_0(\Omega)$) as $H^k(\Omega)$ (or $H^k_0(\Omega)$). We also denote by $W^{k,q'}(\Omega)$ the dual space of $W^{k,q}_0(\Omega)$, where $q$ and $q'$ are Hölder conjugates and in case that $q = 2$ we write $W^{k,q'}(\Omega)$ as $H^{-k}(\Omega)$. Let $I$ be a finite time interval. For a function $f(x,t), \Omega \subset \mathbb{R}^3$, we denote $\| f \|_{L^p_{t,x}(\Omega)} = \| \int_I \| f |_{L^p(\Omega)} \|_{L^q(I)}$. For vector fields $u, v$ we write $(u \cdot v)_{j=1,2,3}$ as $u \otimes v$. All generic constants will be denoted by $C$, which may vary from line to line.

We recall first the definition of weak solutions.

**Definition 3 (Weak solutions).** Let $u_0, b_0 \in L^2_\sigma(\Omega)$. We say that $(u, b)$ is a weak solution of (1.1) if $u$ and $b$ satisfy the following:

(i) $u \in L^\infty(0,T); L^2(\Omega) \cap L^2([0, T); H^1(\Omega)), b \in L^\infty(0,T); L^2(\Omega) \cap L^2([0, T); H^1(\Omega))$.

(ii) $(u, b)$ satisfies (1.1) in the sense of distribution; that is

$$
\int_0^T \int_\Omega \left( \frac{\partial \phi}{\partial t} + \Delta \phi + (u \cdot \nabla) \phi \right) u \, dx \, dt + \int_0^T \int_\Omega u_0 \phi(x, 0) \, dx = \int_0^T \int_\Omega (b \cdot \nabla) b \, dx \, dt,
$$

$$
\int_0^T \int_\Omega \left( \frac{\partial \phi}{\partial t} + \Delta \phi + (u \cdot \nabla) \phi \right) b \, dx \, dt + \int_0^T \int_\Omega b_0 \phi(x, 0) \, dx = \int_0^T \int_\Omega (b \cdot \nabla) u \, dx \, dt,
$$

for all $\phi \in C_0^\infty(\Omega \times [0, T))$ with $\text{div} \phi = 0$, and
\begin{align*}
\int_{\Omega} u \cdot \nabla \psi \, dx &= 0, \\
\int_{\Omega} b \cdot \nabla \psi \, dx &= 0,
\end{align*}

for every \( \psi \in C_0^\infty(\Omega) \).

We consider the following Stokes system, which is the linearized Navier–Stokes equations:

\begin{align}
v_t - \Delta v + \nabla p &= f, \quad \text{div } v = 0 \quad \text{in } Q_T := \Omega \times (0, T) \tag{2.8}
\end{align}

with initial data \( v(x, 0) = v_0(x) \). As in (1.3) and (1.4), boundary data of \( v \) are again assumed to be either no-slip or slip conditions, namely

\begin{align}
v(x, t) &= 0, \quad x \in \partial \Omega \quad \text{or} \\
v \cdot n &= 0, \quad (\nabla \times v) \times n = 0, \quad x \in \partial \Omega. \tag{2.9} \tag{2.10}
\end{align}

Next, we recall maximal estimates of the Stokes system in terms of mixed norms (see [13, Theorem 5.1] and [33, Theorem 1.2] for no-slip and slip boundary cases, respectively).

**Lemma 4.** Let \( 1 < l, m < \infty \). Suppose that \( f \in L_{x,t}^{l,m}(Q_T) \) and \( v_0 \in D_1^{1-\frac{1}{m}, m} \), where \( D_1^{1-\frac{1}{m}, m} \) is a Banach space with the following norm (see e.g., [13]):

\[
D_1^{1-\frac{1}{m}, m}(\Omega) := \left\{ w \in L^1_{\sigma}(\Omega) ; \| w \|_{D_1^{1-\frac{1}{m}, m}} = \| w \|_{L^1_{\sigma}} + \left( \int_0^\infty \left\| \frac{1}{t} A_t^{-l} A_l w \right\|_{L^1_{\sigma}} \frac{dt}{t} \right)^{\frac{1}{m}} < \infty \right\},
\]

where \( A_l \) is the Stokes operator (see [13] and [33] for the details). If \((v, p)\) is the solution of the Stokes system (2.8) satisfying one of the boundary conditions (2.9) or (2.10), then the following estimate is satisfied:

\[
\| v_t \|_{L_{x,t}^{l,m}(Q_T)} + \| \nabla^2 v \|_{L_{x,t}^{l,m}(Q_T)} + \| \nabla p \|_{L_{x,t}^{l,m}(Q_T)} \leq C \| f \|_{L_{x,t}^{l,m}(Q_T)} + \| v_0 \|_{D_1^{1-\frac{1}{m}, m}(\Omega)}.
\]

Since \( D_1^{1-\frac{1}{m}, m}(\Omega) := [L_1(\Omega), W^{1,l}(\Omega))^{1-\frac{1}{m}, m} \), we note that \( \| v_0 \|_{D_1^{1-\frac{1}{m}, m}(\Omega)} \leq \| v_0 \|_{W^{1,l}(\Omega)} \) (see e.g., [1, Chapter 7]) and, therefore, \( \| v_0 \|_{D_1^{1-\frac{1}{m}, m}(\Omega)} \) in (2.11) can be replaced by \( \| v_0 \|_{W^{1,l}(\Omega)} \).

Next, we recall a Gagliardo–Nirenberg inequality (see e.g., [2, Lemma 3.1] and [25, Theorem 2.2]).

**Lemma 5.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( n \geq 1 \) and \( \partial \Omega \) is locally Lipschitz. Assume that \( u \in W^{1,p}(\Omega) \) and \( \int_\Omega u \, dx = 0 \). For every fixed number \( r \geq 1 \), there exists a constant \( C = C(n, p, r, \Omega) \) such that

\[
\| u \|_{L^q(\Omega)} \leq C \| \nabla u \|_{L^p(\Omega)}^{\frac{p}{q}} \| u \|_{L^r(\Omega)}^{1-\frac{p}{q}}, \tag{2.12}
\]

where \( p, q \geq 1 \), and \( \theta = (\frac{1}{r} - \frac{1}{q})(\frac{1}{r} - \frac{1}{p} + \frac{1}{2})^{-1} \).

We remark that \( \nabla \cdot u = 0 \) implies \( \int_\Omega u \, dx = 0 \) (see e.g., [2, p. 7]), and Lemma 5 is also true for the half space \( \mathbb{R}^3_+ \) as well as \( \mathbb{R}^3 \) (see e.g., [23, pp. 215–216]).
3. Proof of Theorem 1

Let $1 \leq q < \infty$ and we introduce a function space $X^q_t$ defined as follows

$$X^q_t = \left\{ f : \Omega \times [0,t) \rightarrow \mathbb{R}^3 \mid \| f \|_{X^q_t} := \limsup_{\tau \downarrow t} \| f (\tau) \|_{W^{1,q} (\Omega)} + \| f \|_{L^q ([0,t); W^{2,q} (\Omega))} < \infty \right\}.$$

**Proposition 1** (Local existence). Let $3 < q < \infty$ and $\Omega$ be either a bounded domain in $\mathbb{R}^3$ or a half-space $\mathbb{R}^3_+$. There exists $T_{\text{max}} \in (0, \infty)$, the maximal time of existence, such that, if $u_0, b_0 \in H^1 (\Omega) \cap W^{1,3} (\Omega)$, then there is a unique solution pair $(u, b)$ in (1.1) with boundary conditions (B1) or (B2) satisfying $u, b \in X^q_t$ for any $t < T_{\text{max}}$.

**Proof.** See Appendix A for the proof. \(\square\)

**Remark 3.** We remark that, if $T_{\text{max}} < \infty$ in Proposition 1,

$$\limsup_{t \nearrow T_{\text{max}}} (\| u \|_{X^q_t} + \| b \|_{X^q_t}) = \infty. \quad (3.13)$$

In Lemma 4 we review estimates of mixed norm for $v_t$, $\nabla^2 v$ and $\nabla p$ for the Stokes system. If the external force $f$ is slightly more regular, then we can have estimates of spatial derivatives of $v_t$, $\nabla^2 v$ and $\nabla p$. To be more precise, we have the following:

**Lemma 6.** Let $\Omega$ be either a bounded domain with a smooth boundary in $\mathbb{R}^3$ or a half space $\mathbb{R}^3_+$. Suppose that $(v, p)$ be a solution of (2.8), with initial condition $v_0 \in H^2 (\Omega)$ and boundary conditions (2.9) or (2.10). If $f \in H^1 \Omega$, $f (x, 0) \in L^2$ and $\partial_t f \in H^{-1} \Omega$, then $v_t$, $\nabla^2 v \in H^1 \Omega$ and $\nabla p \in H^1 \Omega$.

**Proof.** We use the method of difference of quotient with respect to $t$. Let

$$D^h_t (v) (x, t) := \frac{v (x, t + h) - v (x, t)}{h}.$$

Taking the difference of quotient $D^h_t$ to Eq. (2.8) and testing with $D^h_t (v)$, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |D^h_t (v)|^2 + \int_{\Omega} |\nabla D^h_t (v)|^2 = \int_{\Omega} D^h_t (f) D^h_t (v) \leq \| D^h_t (f) \|_{H^{-1} (\Omega)} \| D^h_t (v) \|_{H^1 (\Omega)} \leq C \| f_t \|_{H^{-1} \Omega} + \frac{1}{2} \| \nabla D^h_t (v) \|_{L^2}^2,$$

where Young’s inequality and Poincaré’s inequality are used. Integrating on $(0, T)$, we have

$$\| D^h_t (v) (\cdot, T) \|_{L^2}^2 + \| \nabla D^h_t (v) \|_{L^2 L^2} \leq C \| f_t \|_{H^{-1} \Omega} + \frac{1}{2} \| \nabla D^h_t (v) \|_{L^2}^2.$$

Hence, letting $h \rightarrow 0$, we get $\| \nabla v_t \|_{L^2 L^2} \leq C = C (\| f_t \|_{H^{-1} \Omega}, \| f_0 \|_{L^2}).$ Now, using $\nabla v_t \in L^2 L^2$, we consider the equation as an elliptic type, namely $-\Delta v + \nabla p = f - v_t$. Since $f, v_t$ are in $H^1 \Omega$, standard theory for steady-state Stokes system implies that $\Delta v$ and $\nabla p$ belong to $H^1 \Omega$. This completes the proof. \(\square\)
Lemma 7. Let $3 < q < \infty$ and $\Omega$ be either a bounded domain in $\mathbb{R}^3$ or a half-space $\mathbb{R}^3_+$. Suppose $(u, b)$ is a weak solution of (1.1) with initial conditions $u_0, b_0 \in W^{1,q}(\Omega)$ and boundary conditions (B1) or (B2). Assume further that

$$u, b \in L_c^{4,\infty}(\Omega \times [0, T]), \quad |u||\nabla u|, |b||\nabla b| \in L^{2,2}_c(\Omega \times [0, T]).$$

Then $u, b \in X_T^q$.

Proof. We first note that, due to Proposition 1, there exists $T^*$ such that $u, b \in X_t$ for all $t < T^*$, where $T^*$ is $T_{\text{max}}$ in Proposition 1. We claim that $T < T^*$ under the assumption (3.14). Suppose that this is not the case, i.e., $T \geq T^*$. Let $\tau$ be any number with $T^* - \epsilon < \tau < T^*$, where $\epsilon$ is sufficiently small, which will be specified later. For convenience, we denote $Q_\tau = \Omega \times (T^* - \epsilon, \tau)$. Here only the case of boundary condition (B2) is considered, since the case of boundary condition (B1) is similar. Reminding the identity $-\Delta u = \nabla \times \nabla \times u - \nabla(\nabla \cdot u)$, we note that

$$\int_\Omega u_t \cdot (-\Delta u) = \int_\Omega u_t \cdot (\nabla \times \nabla \times u) = \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla \times u|^2,$$

where we used the integration by parts and slip boundary conditions. We first show that $\nabla u, \nabla b \in L_c^{2,\infty}(Q_T^\tau)$, where $Q_{T^*} = \Omega \times (T^* - \epsilon, T^*)$. Indeed, we multiply the equations of (1.1) with $-\Delta u$ and $-\Delta b$, respectively, and sum them after integrating over $\Omega$ to obtain

$$\frac{1}{2} \frac{d}{dt} \int_\Omega (|\nabla \times u|^2 + |\nabla \times b|^2) + \int_\Omega (|\Delta u|^2 + |\Delta b|^2)$$

$$= \int_\Omega \nabla \pi \Delta u + \int_\Omega (u \cdot \nabla)u \Delta u - \int_\Omega (b \cdot \nabla)b \Delta u + \int_\Omega (u \cdot \nabla)b \Delta b - \int_\Omega (b \cdot \nabla)u \Delta b.$$

Integrating the above over $(T^* - \epsilon, \tau)$ and using $\|\nabla u\|_{L_2^2(\Omega)} \leq C\|\nabla \times u\|_{L_2^2(\Omega)}$ (see e.g., [39, Lemma 2.3]), we have

$$\frac{C}{2} \|\nabla u(\cdot, \tau)\|_{L_2^2}^2 + \|\nabla b(\cdot, \tau)\|_{L_2^2}^2 \leq \frac{C}{2} \|\nabla u(\cdot, T^* - \epsilon)\|_{L_2^2}^2 + \|\nabla b(\cdot, T^* - \epsilon)\|_{L_2^2}^2$$

$$+ \int_{T^* - \epsilon}^\tau \|\Delta u\|_{L_2^2}^2 + \|\Delta b\|_{L_2^2}^2 \, dx \, dt$$

$$\leq \int_{T^* - \epsilon}^\tau \|\nabla \pi\|_{L_2^2} \|\Delta u\|_{L_2^2} + \int_{T^* - \epsilon}^\tau \|\nabla \pi\|_{L_2^2} \|\Delta b\|_{L_2^2} + \int_{T^* - \epsilon}^\tau \|\nabla u\|_{L_2^2} \|\Delta u\|_{L_2^2}$$

$$+ \int_{T^* - \epsilon}^\tau \|\nabla b\|_{L_2^2} \|\Delta b\|_{L_2^2} + \int_{T^* - \epsilon}^\tau \|\nabla b\|_{L_2^2} \|\Delta b\|_{L_2^2}. \quad (3.15)$$
We observe that, via the hypothesis (3.14)

\[
\int_{T^* - \epsilon}^{\tau} \| (u \cdot \nabla) u \|_{L_x^2} \| \Delta u \|_{L_x^2} + \int_{T^* - \epsilon}^{\tau} \| (b \cdot \nabla) b \|_{L_x^2} \| \Delta u \|_{L_x^2} \\
\leq C \int_{T^* - \epsilon}^{\tau} \left( \| (u \cdot \nabla) u \|_{L_x^2}^2 + \| (b \cdot \nabla) b \|_{L_x^2}^2 \right) + \frac{1}{8} \int_{T^* - \epsilon}^{\tau} \| \Delta u \|_{L_x^2}^2.
\]

(3.16)

On the other hand, due to estimate (2.11), we have

\[
\int_{T^* - \epsilon}^{\tau} \| \nabla \pi \|_{L_x^2} \| \Delta u \|_{L_x^2} \leq C \int_{T^* - \epsilon}^{\tau} \| \nabla \pi \|_{L_x^2}^2 + \frac{1}{8} \int_{T^* - \epsilon}^{\tau} \| \Delta u \|_{L_x^2}^2 \\
\leq C \int_{T^* - \epsilon}^{\tau} \left( \| (u \cdot \nabla) u \|_{L_x^2}^2 + \| (b \cdot \nabla) b \|_{L_x^2}^2 \right) + C \| u(T^* - \epsilon) \|^2_{H^1(\Omega)} \\
\quad + \frac{1}{8} \int_{T^* - \epsilon}^{\tau} \| \Delta u \|_{L_x^2}^2.
\]

(3.17)

Now we estimate the fourth term in (3.15).

\[
\int_{T^* - \epsilon}^{\tau} \| (u \cdot \nabla b) \|_{L_x^2} \| \Delta b \|_{L_x^2} \leq \int_{T^* - \epsilon}^{\tau} \| u \|_{L_x^4} \| \nabla b \|_{L_x^4} \| \Delta b \|_{L_x^4} \leq \int_{T^* - \epsilon}^{\tau} \| u \|_{L_x^4} \| \nabla b \|_{L_x^4}^\frac{1}{2} \| \nabla b \|_{L_x^4}^\frac{1}{2} \| \Delta b \|_{L_x^4} \\
\leq C \int_{T^* - \epsilon}^{\tau} \| u \|_{L_x^8} \| \nabla b \|_{L_x^8} \| \Delta b \|_{L_x^8} \leq C \int_{T^* - \epsilon}^{\tau} \| u \|_{L_x^8}^8 \| \nabla b \|_{L_x^8}^2 + \frac{1}{4} \int_{T^* - \epsilon}^{\tau} \| \Delta b \|_{L_x^8}^2 \\
\leq C \| u \|_{L_x^8}^8 \| \nabla b \|_{L_x^8}^2 \| \Delta b \|_{L_x^8}^2 + \frac{1}{4} \int_{T^* - \epsilon}^{\tau} \| \Delta b \|_{L_x^8}^2.
\]

(3.18)

where the Hölder inequality, Sobolev embedding and the Young’s inequality are used. Similarly, we can estimate the last term in (3.15) as follows

\[
\int_{T^* - \epsilon}^{\tau} \| (b \cdot \nabla u) \|_{L_x^2} \| \Delta b \|_{L_x^2} \leq \int_{T^* - \epsilon}^{\tau} \| b \|_{L_x^8} \| \nabla u \|_{L_x^8} \| \Delta b \|_{L_x^8} \leq \int_{T^* - \epsilon}^{\tau} \| b \|_{L_x^8} \| \nabla u \|_{L_x^8} \| \nabla u \|_{L_x^8} \| \Delta b \|_{L_x^8} \\
\leq \int_{T^* - \epsilon}^{\tau} \| b \|_{L_x^8} \| \nabla u \|_{L_x^8} \| \Delta u \|_{L_x^8} \| \Delta b \|_{L_x^8}.
\]
Let \( p \) be any number with \( 1 < p < \infty \). We show next that \( u, b \in L^{4,p}_{x,t}(Q_T^+) \). Indeed, we first note that \( (u \cdot \nabla)u, (b \cdot \nabla)b, (u \cdot \nabla)b \) and \( (b \cdot \nabla)u \) are in \( L^{4,\infty}_{x,t}(Q_T^+) \) via the hypothesis (3.14), and the fact that \( \nabla u, \nabla b \in L^{2,\infty}_{x,t}(Q_T^+) \). Using the Stokes estimate (2.11) and Sobolev embedding, we have \( u \in L^{12,p}_{x,t}(Q_T^+) \). Indeed,

\[
\|u\|_{L^{12,p}_{x,t}} \leq C \|u\|_{W^{2,4}_{x,t}} \leq C \left( \|u\|_{W^{2,4}_{x,t}} + \|b\|_{W^{2,4}_{x,t}} \right) \leq C \left( \|u\|_{L^{12,p}_{x,t}} + \|b\|_{L^{12,p}_{x,t}} \right).
\]

It is now straightforward that \( (u \cdot \nabla)u \in L^{12,p}_{x,t}(Q_T^+) \) since \( u \in L^{12,p}_{x,t}(Q_T^+) \) and \( \nabla u \in L^{4,\infty}_{x,t}(Q_T^+) \). Similar arguments can be made for \( b \), and thus we can see that \( (b \cdot \nabla)b, (u \cdot \nabla)b \) and \( (b \cdot \nabla)u \) are in \( L^{12,p}_{x,t}(Q_T^+) \). Next, bootstrap arguments show that \( \nabla u \in L^{4,p}_{x,t}(Q_T^+) \). Indeed,

\[
\|\nabla u\|_{L^{4,p}_{x,t}} \leq C \|u\|_{L^{12,p}_{x,t}} \leq C \left( \|u\|_{L^{12,p}_{x,t}} + \|b\|_{L^{12,p}_{x,t}} \right) \leq C \left( \|u\|_{L^{4,p}_{x,t}} + \|b\|_{L^{4,p}_{x,t}} \right).
\]

We also determine that \( \nabla b \in L^{4,p}_{x,t}(Q_T^+) \). This can be shown similarly as in the case of \( \nabla u \), and thus we skip its details. Due to Sobolev embedding, it is straightforward that \( u, b \in L^{\infty,p}_{x,t}(Q_T^+) \).
Now we use estimates (A.52) and (A.53) to show that $\nabla u, \nabla b \in L^q_{x,t}(Q_{T^*})$. Let $p > 4$. For any $t$ with $t < T^*$, we compute via estimates (A.52)

$$
\|\nabla u(t)\|_{L^q_t} \leq C t^{-\frac{1}{2}} \|u_0\|_{L^q_x} + C \int_0^t (t-s)^{-\frac{1}{2}} \left( \|\nabla u\|_{L^q_t} + \|b\|_{L^q_t} \right) ds
$$

Similarly, we can obtain that

$$
\|\nabla b(t)\|_{L^q_t} \leq C t^{-\frac{1}{2}} \|b_0\|_{L^q_x} + C t^{-\frac{1}{2}} \left( \|\nabla u\|_{L^q_t} + \|\nabla b\|_{L^q_t} \right) ds
$$

$$(3.20)$$

Similarly, we can obtain that

$$
\|\nabla u(t)\|_{L^q_t} \leq C t^{-\frac{1}{2}} \|u_0\|_{L^q_x} + C t^{-\frac{1}{2}} \left( \|\nabla u\|_{L^q_t} + \|b\|_{L^q_t} \right) ds
$$

$$(3.21)$$

Since verification of (3.21) is similar to that of (3.20), we skip its details. Since $t$ is arbitrary, estimates (3.20) and (3.21) imply that $\nabla u, \nabla b \in L^q_{x,t}(Q_{T^*})$.

It remains to show that $\nabla^2 u, \nabla^2 b \in L^q_{x,t}(Q_{T^*})$. Indeed, we have shown that $u, b \in L^\infty_{x,t}(Q_{T^*})$ and $\nabla u, \nabla b \in L^q_{x,t}(Q_{T^*})$. Therefore, it is direct that $(u \cdot \nabla) u, (b \cdot \nabla) b, (u \cdot \nabla) b$ and $(b \cdot \nabla) u$ are in $L^q_{x,t}(Q_{T^*})$. Again due to linear estimates of Stokes and heat equations, we observe that $\nabla^2 u, \nabla^2 b \in L^q_{x,t}(Q_{T^*})$. Summing up, we have $u, b \in X^q_{\delta}$, which is contrary to the fact that $T^*$ is the maximal time of existence. Therefore, the hypothesis that $T \geq T^*$ is not true. This completes the proof of Lemma 7. □

We are ready to present the proof of Theorem 1.

**Proof of Theorem 1.** We argue by contradiction. Suppose that $T^*$ is the first time of singularity with $T^* \leq T$. Then $u$ and $b$ must satisfy for any $\delta > 0$,

$$
\limsup_{t \to T^*} \left( \|u(t, \cdot)\|^4_{L^2_x} + \|b(t, \cdot)\|^4_{L^2_x} \right)
$$

$$
+ \lim_{t \to T^*} \left( \int_{T^*-\delta}^t \|\nabla u(t, \cdot)\|^2_{L^2_x} + \|\nabla b(t, \cdot)\|^2_{L^2_x} \right) = \infty.
$$

$$(3.22)$$

As in Lemma 7, we consider only the case of boundary condition (B2), since the case of (B1) is much simpler. We observe first via integration by parts that

$$
- \int_\Omega \Delta u \cdot u |u|^2 = \sum_{i,j=1}^3 \int_\Omega u_{j,x_i} (u_j |u|^2)_{x_i} - \sum_{i,j=1}^3 \int_{\partial \Omega} u_{j,x_i} u_j |u|^2 n_i
$$

$$
= \frac{1}{2} \int_\Omega |\nabla |u|^2|^2 + \int_\Omega |u|^2 |\nabla u|^2 - \sum_{i,j=1}^3 \int_{\partial \Omega} u_{j,x_i} u_j |u|^2 n_i.
$$
Similarly, for $b$ we have
\[ -\int_\Omega \nabla b \cdot b |b|^2 = \frac{1}{2} \int_\Omega |\nabla |b|^2|^2 + \int_\Omega |b|^2 |\nabla b|^2 - \sum_{i,j=1}^3 \int_{\partial \Omega} b_j b_j |b|^2 n_i. \]

Multiplying the first equation of (1.1) with $|u|^2 u$ and integrating over $\Omega$, we have
\[
\frac{1}{4} \frac{d}{dt} \int_\Omega |u|^4 + \int_\Omega |\nabla u|^2 |u|^2 + \frac{1}{2} \int_\Omega |\nabla |u|^2|^2 = -\int_\Omega \nabla \pi |u|^2 u - \int_\Omega b \nabla (|u|^2 u) + \sum_{i,j=1}^3 \int_{\partial \Omega} u_j u_j |u|^2 n_i, \]

where we used integration by parts and divergence-free conditions of $u$ and $b$. Similarly, multiplying the equation of the magnetic field with $|b|^2 b$ and integrating over $\Omega$, we get
\[
\frac{1}{4} \frac{d}{dt} \int_\Omega |b|^4 + \int_\Omega |\nabla b|^2 |b|^2 + \frac{1}{2} \int_\Omega |\nabla |b|^2|^2 = -\int_\Omega b \nabla (|b|^2 b) u + \sum_{i,j=1}^3 \int_{\partial \Omega} b_j b_j |b|^2 n_i. \]

Summing the above estimates, we obtain
\[
\frac{1}{4} \frac{d}{dt} \int_\Omega (|u|^4 + |b|^4) + \int_\Omega (|\nabla u|^2 |u|^2 + |\nabla b|^2 |b|^2) + \frac{1}{2} \int_\Omega (|\nabla |u|^2|^2 + |\nabla |b|^2|^2) \]
\[ = -\int_\Omega \nabla \pi |u|^2 u - \int_\Omega b \nabla (|u|^2 u) - \int_\Omega b \nabla (|b|^2 b) u + \sum_{i,j=1}^3 \int_{\partial \Omega} u_j u_j |u|^2 n_i + \sum_{i,j=1}^3 \int_{\partial \Omega} b_j b_j |b|^2 n_i. \quad (3.23) \]

We note that the last two terms in (3.23) are estimated as follows
\[
\left| \sum_{i,j=1}^3 \int_{\partial \Omega} u_j u_j |u|^2 n_i \right| \leq C \int_\Omega |u|^3 |\nabla u|, \quad (3.24) \]
\[
\left| \sum_{i,j=1}^3 \int_{\partial \Omega} b_j b_j |b|^2 n_i \right| \leq C \int_\Omega |b|^3 |\nabla b|. \quad (3.25) \]

Here we will show the validity of (3.24) and the estimate (3.25) can be deduced in the same way. First, direct calculations show the identity
\[ \sum_{i=1}^3 u_{j,i} n_i = \left( (\nabla \times u) \times n \right)_j + u_{x_j} \cdot n, \quad \text{for } j = 1, 2, 3. \quad (3.26) \]

Using the (3.26) and slip boundary conditions, we get
\[ -3 \sum_{i,j=1}^{3} \int_{\partial \Omega} u_{j,i} u_{i} u_{j} \, dt = 3 \sum_{j=1}^{3} \int_{\partial \Omega} ((\nabla \times u) \times n)_{j} + u_{x} \cdot n) \, u_{j} u_{j} = -3 \sum_{j=1}^{3} \int_{\partial \Omega} u_{x} \cdot n u_{j} \, u_{j} \]

\[ = -3 \sum_{j=1}^{3} \int_{\partial \Omega} (u \cdot n)_{j} u_{j} + 3 \sum_{j=1}^{3} \int_{\partial \Omega} u \cdot n_{x} u_{j} u_{j} = -3 \sum_{j=1}^{3} \int_{\partial \Omega} u \cdot n_{x} u_{j} u_{j} \]

\[ \leq \| \nabla n \|_{L^{\infty} (\partial \Omega)} \int_{\partial \Omega} |u|^{4} \leq C \int_{\Omega} |\nabla |u|^{4} | \leq C \int_{\Omega} |u|^{3} |\nabla u|, \]

where we used the Trace Theorem (see e.g., [10, pp. 257–258]) and smoothness of boundary (in fact, \( C^{2} \) boundary is enough to control \( L^{\infty} \)-norm of \( \nabla n \)). Similarly, we can obtain the estimate (3.25). Let \( \epsilon \) be a sufficiently small positive number, which will be specified later. Integrating (3.23) in time over \((T^{*} - \epsilon, \tau)\) for any \( \tau \) with \( T^{*} - \epsilon < \tau < T^{*} \), we observe

\[ \int_{\Omega} (|u(\cdot, \tau)|^{4} \, dx + |b(\cdot, \tau)|^{4}) \, dx - \int_{\Omega} (|u(\cdot, T^{*} - \epsilon)|^{4} + |b(\cdot, T^{*} - \epsilon)|^{4}) \, dx \]

\[ + \int_{T^{*} - \epsilon}^{\tau} \int_{\Omega} |\nabla u|^{2} |u|^{2} \, dx \, dt + \int_{T^{*} - \epsilon}^{\tau} \int_{\Omega} |\nabla b|^{2} |b|^{2} \, dx \, dt \]

\[ + \frac{1}{2} \int_{T^{*} - \epsilon}^{\tau} \int_{\Omega} |\nabla u|^{2} \, dx \, dt + \frac{1}{2} \int_{T^{*} - \epsilon}^{\tau} \int_{\Omega} |\nabla b|^{2} \, dx \, dt \]

\[ \leq \int_{T^{*} - \epsilon}^{\tau} \int_{\Omega} |\nabla u|^{2} |u| \, dx \, dt + \int_{T^{*} - \epsilon}^{\tau} \int_{\Omega} |b|^{2} |u| |\nabla u| \, dx \, dt + \int_{T^{*} - \epsilon}^{\tau} \int_{\Omega} |b|^{2} |u| |\nabla b| \, dx \, dt \]

\[ + \int_{T^{*} - \epsilon}^{\tau} \int_{\Omega} |u|^{3} |\nabla u| \, dx \, dt + \int_{T^{*} - \epsilon}^{\tau} \int_{\Omega} |b|^{3} |\nabla b| \, dx \, dt \]

\[ := I + II + III + IV + V. \]

For convenience, we denote \( Q_{\tau} := \Omega \times (T^{*} - \epsilon, \tau) \). Using Hölder’s inequality, the first term \( I \) can be estimated as follows

\[ I \leq \sum_{k=1}^{3} \int_{T^{*} - \epsilon}^{\tau} \left\| \nabla \pi \right\|_{L^{2}} \left\| u \right\|_{L^{2}}^{2} \left\| u \right\|_{L^{2}} \left\| u \right\|_{L^{2}} \sup_{T^{*} - \epsilon < t < \tau} \left\| u(\cdot, t) \right\|_{L^{2}}^{4} \]

\[ \leq \sum_{k=1}^{3} \int_{T^{*} - \epsilon}^{\tau} \left\| \nabla \pi \right\|_{L^{2}} \left\| u \right\|_{L^{2}}^{2} \left\| u \right\|_{L^{2}}^{2} \left\| u \right\|_{L^{2}}^{2} \sup_{T^{*} - \epsilon < t < \tau} \left\| u(\cdot, t) \right\|_{L^{2}}^{4} \]
where \( \theta_k = 2/q_k \). For convenience of computation, we denote \( C_\varepsilon := \| u(\cdot, T^* - \varepsilon) \|_{W^{1,2}(\Omega)} \). Using the estimate (2.11), we continue to estimate \( I \) as

\[
I \leq C \sum_{k=1}^{3} \left( \| (u \cdot \nabla) u \|_{L^2(Q,T)} + \| (b \cdot \nabla) b \|_{L^2(Q_T)} + C_\varepsilon \right)
\times \| \nabla|u|^{q_k-2} \|_{L^2(Q_T)} \| u_k \|_{L^{p_k,q_k}(Q_T)} \sup_{T^*-\varepsilon < t < T} \| u(\cdot, t) \|_{L^4_x}^{4 q_k}
\leq C \sum_{k=1}^{3} \| u \|_{L^2(Q_T)} \| \nabla|u|^{q_k-2} \|_{L^2(Q_T)} \| u_k \|_{L^{p_k,q_k}(Q_T)} \sup_{T^*-\varepsilon < t < T} \| u(\cdot, t) \|_{L^4_x}^{4 q_k}
+ C \sum_{k=1}^{3} \| b \|_{L^2(Q_T)} \| \nabla|u|^{q_k-2} \|_{L^2(Q_T)} \| u_k \|_{L^{p_k,q_k}(Q_T)} \sup_{T^*-\varepsilon < t < T} \| b(\cdot, t) \|_{L^4_x}^{4 q_k}
+ C \sum_{k=1}^{3} \| u \|_{L^2(Q_T)} \| \nabla|u|^{q_k-2} \|_{L^2(Q_T)} \| u_k \|_{L^{p_k,q_k}(Q_T)} \sup_{T^*-\varepsilon < t < T} \| u(\cdot, t) \|_{L^4_x}^{4 q_k}
+ C \sum_{k=1}^{3} \| b \|_{L^2(Q_T)} \| \nabla|u|^{q_k-2} \|_{L^2(Q_T)} \| u_k \|_{L^{p_k,q_k}(Q_T)} \sup_{T^*-\varepsilon < t < T} \| b(\cdot, t) \|_{L^4_x}^{4 q_k}.
\]

(3.27)

Next we estimate \( II \). Following similar computations as in \( I \), we get

\[
II \leq \int_{T^* - \varepsilon}^{T} \int_{\Omega} |b|^2 |u| \| \nabla u | \ dx dt
\leq C \sum_{k=1}^{3} \| u \|_{L^2(Q_T)} \| \nabla|b|^{q_k} \|_{L^2(Q_T)} \| u_k \|_{L^{p_k,q_k}(Q_T)} \sup_{T^*-\varepsilon < t < T} \| b(\cdot, t) \|_{L^4_x}^{4 q_k}.
\]

(3.28)

In the same manner, we estimate \( III \)

\[
III \leq C \sum_{k=1}^{3} \| b \|_{L^2(Q_T)} \| \nabla|b|^{q_k} \|_{L^2(Q_T)} \| u_k \|_{L^{p_k,q_k}(Q_T)} \sup_{T^*-\varepsilon < t < T} \| b(\cdot, t) \|_{L^4_x}^{4 q_k}.
\]

(3.29)

For \( IV \) and \( V \), using Hölder’s inequality, we have

\[
IV + V \leq C \varepsilon^2 \left( \| u \|_{L^2(Q_T)} \sup_{T^*-\varepsilon < t < T} \| u(\cdot, t) \|_{L^4_x}^{2} + \| b \|_{L^2(Q_T)} \sup_{T^*-\varepsilon < t < T} \| b(\cdot, t) \|_{L^4_x}^{2} \right).
\]

Summing up (3.27)–(3.29) and using Young’s inequality, we obtain

\[
\frac{1}{4} \int_{\Omega} \left| u(\cdot, \tau) \right|^4 dx + \frac{1}{4} \int_{\Omega} \left| b(\cdot, \tau) \right|^4 dx \leq \frac{1}{4} \int_{\Omega} \left( \left| u(\cdot, T^* - \varepsilon) \right|^4 + \left| b(\cdot, T^* - \varepsilon) \right|^4 \right) dx
\]
\[
+ \int_{T^* - \varepsilon}^{T} \int_{\Omega} |\nabla u|^2 |u|^2 dx dt + \int_{T^* - \varepsilon}^{T} \int_{\Omega} |\nabla b|^2 |b|^2 dx dt
\]
\[
+ \frac{1}{2} \int_{T^* - \varepsilon}^{T} \int_{\Omega} |\nabla u|^2 dx dt + \frac{1}{2} \int_{T^* - \varepsilon}^{T} \int_{\Omega} |\nabla b|^2 dx dt.
\]
\[
\begin{align*}
&\leq C \sum_{k=1}^{3} \|u\| \|\nabla u\|_{L^2(Q_T)}^{2q_k} \|u_k\|_{L^q_{x,t}}^{q_k} \sup_{T^*-\epsilon < t < \tau} \|u(\cdot, t)\|_{L^4_{x,t}}^4 + C \sum_{k=1}^{3} \|b\| \|\nabla b\|_{L^2(Q_T)}^{q_k-2} \|u_k\|_{L^q_{x,t}}^{q_k} \sup_{T^*-\epsilon < t < \tau} \|u(\cdot, t)\|_{L^4_{x,t}}^4 \\
&\quad + C C_{\epsilon} \sum_{k=1}^{3} \|u\| \|\nabla u\|_{L^2(Q_T)}^{q_k-2} \|u_k\|_{L^q_{x,t}}^{q_k} \sup_{T^*-\epsilon < t < \tau} \|u(\cdot, t)\|_{L^4_{x,t}}^4 + C C_{\epsilon}^2 \\
&\quad + C \left(\sum_{k=1}^{3} \|u_k\|_{L^q_{x,t}}^{q_k} + \epsilon\right) \left(\sup_{T^*-\epsilon < t < \tau} \|u(\cdot, t)\|_{L^4_{x,t}}^4 + \sup_{T^*-\epsilon < t < \tau} \|b(\cdot, t)\|_{L^4_{x,t}}^4\right).
\end{align*}
\]

Since the above estimate holds for all \(t\) with \(T^*-\epsilon < t < \tau\), we obtain

\[
\sup_{T^*-\epsilon < t < \tau} \left(\|u(\cdot, t)\|_{L^4_{x,t}}^4 + \|b(\cdot, t)\|_{L^4_{x,t}}^4\right) + \int_{T^*-\epsilon}^{\tau} \int_{\Omega} (|\nabla u|^2 + |\nabla b|^2) \, dx \, dt
\]

\[
+ \frac{1}{2} \int_{T^*-\epsilon}^{\tau} \int_{\Omega} (|\nabla u|^2 + |\nabla b|^2) \, dx \, dt
\]

\[
\leq \int_{\Omega} |u(\cdot, T^*-\epsilon)|^4 \, dx + \int_{\Omega} |b(\cdot, T^*-\epsilon)|^4 \, dx + C C_{\epsilon}^2 + C \left(\sum_{k=1}^{3} \|u_k\|_{L^q_{x,t}}^{q_k} + \epsilon\right) \times \left(\sup_{T^*-\epsilon < t < \tau} \|u(\cdot, t)\|_{L^4_{x,t}}^4 + \sup_{T^*-\epsilon < t < \tau} \|b(\cdot, t)\|_{L^4_{x,t}}^4\right).
\]

With sufficiently small \(\epsilon\) so that \(\sum_{k=1}^{3} \|u_k\|_{L^q_{x,t}}^{q_k} + \epsilon\leq \frac{1}{2t}\) with a constant \(C\) in the above estimate, we have

\[
\|u(\cdot, t)\|_{L^4_{x,t}}^4 + \|b(\cdot, t)\|_{L^4_{x,t}}^4 + \frac{1}{2} \|\nabla u\|^2_{L^2(Q_T)}
\]

\[
+ \frac{1}{2} \|\nabla b\|^2_{L^2(Q_T)} + \frac{1}{2} \|\nabla u\|^2_{L^2(Q_T)} + \frac{1}{2} \|\nabla b\|^2_{L^2(Q_T)}
\]

\[
\leq 2 \|\nabla u(\cdot, T-\epsilon)\|_{L^2(\Omega)}^4 + \|b(\cdot, T-\epsilon)\|_{L^2(\Omega)}^4 + C C_{\epsilon}^2.
\]
We denote, for simplicity, \( Q_\epsilon = \Omega \times (T^* - \epsilon, T^*) \). Since \( \tau \) is arbitrary with \( \tau < T^* \), we obtain

\[
\| u(\cdot, t) \|_{L^4_{t,x}}^{Q_\epsilon} + \| b(\cdot, t) \|_{L^4_{t,x}}^{Q_\epsilon} + \frac{1}{2} \| \nabla u \|_{L^2(\Omega)}^2 + \frac{1}{2} \| \nabla b \|_{L^2(\Omega)}^2 \leq C,
\]

where \( C \) is a constant depending on \( \| u(\cdot, T^* - \epsilon) \|_{W^{1/2}(\Omega)} \). This is contrary to the hypothesis of (3.22). Therefore, \( T^* \) cannot be a maximal time of existence less than or equal to \( T \). This completes the proof. \( \square \)

**Remark 5.** We suppose that \( \| u \|_{L^3_{t,x}((Q_T^*)} < \delta \). Following a similar procedure as in Theorem 1, we can obtain

\[
(\| u(\cdot, \tau) \|_{L^4_{t,x}}^4 + \| b(\cdot, \tau) \|_{L^4_{t,x}}^4) - (\| u(\cdot, T^* - \epsilon) \|_{L^4_{t,x}}^4 + \| b(\cdot, T^* - \epsilon) \|_{L^4_{t,x}}^4)
+ \left( \| \nabla u \|_{L^2(\Omega)}^2 + \| \nabla b \|_{L^2(\Omega)}^2 \right)
\leq C \| u \|_{L^3_{t,x}(Q_T^*)} \| u(\cdot, \tau) \|_{L^4_{t,x}((Q_T^*)} + \| b(\cdot, \tau) \|_{L^4_{t,x}((Q_T^*)} + CC_\epsilon
+ \frac{1}{2} \left( \| \nabla u \|_{L^2(\Omega)}^2 + \| \nabla b \|_{L^2(\Omega)}^2 \right).
\]

Therefore, if \( \delta \) is sufficiently small, the result of Theorem 1 is also true.

### 4. Proof of Theorem 2

In this section, the proof of Theorem 2 will be given. We note first that in case \( \mathbb{R}^3_+ \), the slip boundary conditions (B2) are rewritten in terms of components of vectors as

\[
u_{1,3} = u_{2,3} = u_3 = 0, \quad b_{1,3} = b_{2,3} = b_3 = 0 \quad \text{on} \quad \{x_3 = 0\}.
\]

(4.30)

Before presenting the proof of Theorem 2, we start with an observation, which is useful to our purpose (compare to [19] for the case of whole space).

**Lemma 8.** Suppose that \((u, b)\) is a weak solution of (1.1) in a half space with initial conditions \(u_0 \in H^1_0(\mathbb{R}^3_+) \cap W^{1,q}(\mathbb{R}^3_+), \ b_0 \in H^1(\mathbb{R}^3_+) \cap W^{1,q}(\mathbb{R}^3_+), q > 3 \) and boundary conditions (B1). Assume further that the tangential components of the velocity \( \tilde{u} = (u_1, u_2) \) and the normal component of the magnetic field \( b_3 \) satisfy the following integrability conditions:

\[
\| \tilde{u} \|_{L^{p,q}_{t,x}(\mathbb{R}^3_+ \times (0, T)))} < \infty, \quad \frac{3}{p} + \frac{2}{q} = 1, \quad 3 < p \leq \infty,
\]

\[
\| b_3 \|_{L^{r,s}_{t,x}(\mathbb{R}^3_+ \times (0, T))} < \infty, \quad \frac{3}{r} + \frac{2}{s} = 1, \quad \frac{3p}{p-2} \leq r \leq \frac{3p}{p-3}.
\]

Then, \( \tilde{b} = (b_1, b_2) \in L^3_{t,x}((\mathbb{R}^3_+ \times (0, T))) \) and satisfies

\[
\sup_{0 \leq t \leq T} \| \tilde{b}(t) \|_{L^3}^3 + \| \nabla \tilde{b} \|_{L^2}^2 \leq C \| \tilde{b}(0) \|_{L^3}^3 \exp(\| \tilde{u} \|_{L^{p,q}_{t,x}}^q + \| b_3 \|_{L^{r,s}_{t,x}}^q) \| \tilde{u} \|_{L^{p,q}_{t,x}}^q)
+ C \| b_3 \|_{L^{r,s}_{t,x}}^q \| \tilde{u} \|_{L^{p,q}_{t,x}}^q \exp(\| \tilde{u} \|_{L^{p,q}_{t,x}}^q + \| b_3 \|_{L^{r,s}_{t,x}}^q) \| \tilde{u} \|_{L^{p,q}_{t,x}}^q).
\]
Proof. We observe first, via integration by parts, that

\[- \int_{\mathbb{R}^3_+} \Delta \tilde{b} \cdot \tilde{b} |\tilde{b}| = \sum_{i=1}^{3} \sum_{j=1}^{2} \int_{\mathbb{R}^3_+} b_{j,x_i} (b_j |b_j|) x_i - \sum_{i=1}^{3} \sum_{j=1}^{2} \int_{\partial \mathbb{R}^3_+} b_{j,x_i} (b_j |b_j|) n_i \]

\[= \sum_{i=1}^{3} \sum_{j=1}^{2} \int_{\mathbb{R}^3_+} b_{j,x_i} b_{j,x_i} |b_j| + \frac{3}{2} \sum_{i=1}^{3} \sum_{j=1}^{2} \int_{\mathbb{R}^3_+} |b_j| x_i |b_j| = \frac{8}{9} \int_{\mathbb{R}^3_+} |\nabla |\tilde{b}|^3|^2. \]

The boundary term, \( \sum_{i=1}^{3} \sum_{j=1}^{2} \int_{\partial \mathbb{R}^3_+} b_{j,x_i} (b_j |b_j|) n_i \), vanishes in the above identity, because of the boundary condition (4.30) and \( n = (0, 0, -1) \) on \( \{x_3 = 0\} \). Multiplying both sides of (1.1) with \( |\tilde{b}| b \) and integrating by parts, we obtain via the above observation

\[\frac{1}{3} \frac{d}{dt} \int_{\mathbb{R}^3_+} |\tilde{b}|^3 + \frac{8}{9} \int_{\mathbb{R}^3_+} |\nabla |\tilde{b}|^3|^2 = \int_{\mathbb{R}^3_+} (b \cdot \nabla) \tilde{b} |\tilde{b}| dx = - \int_{\mathbb{R}^3_+} (b \cdot \nabla) (\tilde{b} |\tilde{b}|) \tilde{u} dx = J. \quad (4.31)\]

We compute \( J \) as follows

\[|J| \leq \sum_{j=1}^{2} \int_{\mathbb{R}^3_+} b_j (\tilde{b} |\tilde{b}|) x_j \tilde{u} dx + \int_{\mathbb{R}^3_+} b_3 (\tilde{b} |\tilde{b}|) x_3 \tilde{u} dx \]

\[\leq \int_{\mathbb{R}^3_+} |\nabla |\tilde{b}|^3|^2 |\tilde{b}|^3 |\tilde{u}| dx + \int_{\mathbb{R}^3_+} |b_3| |\nabla |\tilde{b}|^3|^2 |\tilde{b}|^3 |\tilde{u}| dx = J_1 + J_2. \]

Using Hölder’s inequality, Young’s inequality and the interpolation inequality, we continue to estimate \( J_1 \) as

\[|J_1| \leq \| |\nabla |\tilde{b}|^3|^2 \|_{L^2} \| |\tilde{b}|^3 |\tilde{u}| \|_{L^2} \leq C \| |\tilde{b}|^3 \|_{L^{2p}}^2 \| \tilde{u} \|_{L^2}^2 + \epsilon \| |\nabla |\tilde{b}|^3|^2 \|_{L^2}^2 \]

\[\leq C |\tilde{u}|_{L^p}^2 \| |\tilde{b}|^3 \|_{L^p}^{2p-6} \| \nabla |\tilde{b}|^3 \|_{L^2}^6 \| |\tilde{b}|^3 \|_{L^2}^2 + 2 \epsilon \| |\nabla |\tilde{b}|^3|^2 \|_{L^2}^2 \]

\[\leq C |\tilde{u}|_{L^p}^{2p-3} \| |\tilde{b}|^3 \|_{L^2}^2 + 2 \epsilon \| |\nabla |\tilde{b}|^3|^2 \|_{L^2}^2. \quad (4.32)\]

Let \( \theta := p - \frac{2p}{r} - 2 \). We then observe that

\[\frac{1}{r} = \frac{p - 3}{3p} \theta + \frac{p - 2}{3p} (1 - \theta), \quad \frac{1}{s} = \frac{q - 2}{2q} \theta + \frac{q - 2}{3q} (1 - \theta). \quad (4.33)\]

We set \( \kappa \) and \( \alpha \) as

\[\frac{1}{\kappa} = \frac{1}{6} \theta + \frac{p - 2}{6p} (1 - \theta), \quad \alpha = \theta \frac{s(q - 2)}{2q}. \]

Noting that \( \frac{1}{r} + \frac{1}{p} + \frac{1}{s} = \frac{1}{2} \), we estimate \( J_2 \) as follows
Using Gronwall's inequality and Hölder inequality and (4.33), we obtain
\[
|J_2| \leq \left\| \nabla |\tilde{b}|^{\frac{2}{3}} \right\|_{L^2} \left\| b_3 \right\|_{L^2} \left\| b_3 \right\|_{L^2} \left\| \tilde{u} \right\|_{L^2} + \epsilon \left\| \nabla |\tilde{b}|^{\frac{2}{3}} \right\|_{L^2}^2
\leq C \left\| b_3 \right\|_{L^2}^2 \left\| \tilde{b} \right\|_{L^2} \left\| \tilde{u} \right\|_{L^2}^2 + \epsilon \left\| \nabla |\tilde{b}|^{\frac{2}{3}} \right\|_{L^2}^2
\leq C \left\| b_3 \right\|_{L^2}^2 \left\| \tilde{b} \right\|_{L^2} \left\| \tilde{u} \right\|_{L^2}^2 + \epsilon \left\| \nabla |\tilde{b}|^{\frac{2}{3}} \right\|_{L^2}^2
\leq C \left\| b_3 \right\|_{L^2}^2 \left\| \tilde{b} \right\|_{L^2} \left\| \tilde{u} \right\|_{L^2}^2 + \epsilon \left\| \nabla |\tilde{b}|^{\frac{2}{3}} \right\|_{L^2}^2.
\] (4.34)

Using Young's inequality, we continue to estimate \( J_2 \) as
\[
|J_2| \leq C \left\| b_3 \right\|_{L^2}^2 \left\| \tilde{b} \right\|_{L^2} \left\| \tilde{u} \right\|_{L^2}^2 + C \left\| b_3 \right\|_{L^2} \left\| \tilde{b} \right\|_{L^2} \left\| \tilde{u} \right\|_{L^2}^2 + \epsilon \left\| \nabla |\tilde{b}|^{\frac{2}{3}} \right\|_{L^2}^2
= C \left\| b_3 \right\|_{L^2}^2 \left\| \tilde{b} \right\|_{L^2} \left\| \tilde{u} \right\|_{L^2}^2 + C \left\| b_3 \right\|_{L^2} \left\| \tilde{b} \right\|_{L^2} \left\| \tilde{u} \right\|_{L^2}^2 + \epsilon \left\| \nabla |\tilde{b}|^{\frac{2}{3}} \right\|_{L^2}^2.
\] (4.35)

The second term in (4.35) can be estimated as follows
\[
\left\| b_3 \right\|_{L^2}^{\frac{2(1-\alpha)}{\alpha}} \left\| \tilde{b} \right\|_{L^2}^{\frac{2}{\alpha}} \left\| \tilde{u} \right\|_{L^2}^{2 \frac{1-\alpha}{\alpha}} \leq C \left\| b_3 \right\|_{L^2}^{\frac{2(1-\alpha)}{\alpha}} \left\| \tilde{u} \right\|_{L^2}^{2 \frac{1-\alpha}{\alpha}} + C \left\| \tilde{b} \right\|_{L^2}^{\frac{2}{\alpha}} \left\| \tilde{u} \right\|_{L^2}^{2 \frac{1-\alpha}{\alpha}}
\leq C \left\| b_3 \right\|_{L^2}^{\frac{2(1-\alpha)}{\alpha}} \left\| \tilde{u} \right\|_{L^2}^{2 \frac{1-\alpha}{\alpha}} + C \left\| \tilde{u} \right\|_{L^2}^{2 \frac{1-\alpha}{\alpha}} \left\| \tilde{b} \right\|_{L^2}^{\frac{2}{\alpha}}
+ 2\epsilon \left\| \nabla \tilde{b} \right\|_{L^2}^{\frac{2}{\alpha}}.
\] (4.36)

where we used Young's inequality and estimations as we did for \( J_1 \). Summing (4.32) and (4.35)–(4.36), we have
\[
\frac{1}{3} d \frac{d}{dt} \left\| \tilde{b} \right\|_{L^2}^{3} \leq C \left\| \tilde{u} \right\|_{L^2}^{2 \frac{3}{\alpha}} + C \left\| b_3 \right\|_{L^2}^{2 \frac{3}{\alpha}} \left\| \tilde{u} \right\|_{L^2}^{2 \frac{3}{\alpha}} + C \left\| \tilde{b} \right\|_{L^2}^{3 \frac{1-\alpha}{\alpha}} \left\| \tilde{u} \right\|_{L^2}^{2 \frac{3}{\alpha}}.
\] (4.37)

Using Gronwall's inequality and Hölder inequality and (4.33), we obtain
\[
\left\| \tilde{b}(t) \right\|_{L^1}^{3} \leq C \left\| \tilde{b}(0) \right\|_{L^1}^{3} \exp \left( \int_0^T \left( \left\| \tilde{u} \right\|_{L^2}^{\frac{3}{\alpha}} + \left\| b_3 \right\|_{L^2}^{\frac{3}{\alpha}} \left\| \tilde{u} \right\|_{L^2}^{2 \frac{3}{\alpha}} \right) dt \right)
+ \int_0^T \left( \left\| b_3 \right\|_{L^2}^{\frac{3}{\alpha}} \left\| \tilde{u} \right\|_{L^2}^{2 \frac{3}{\alpha}} \left\| \tilde{b} \right\|_{L^2}^{3 \frac{1-\alpha}{\alpha}} \left\| \tilde{u} \right\|_{L^2}^{2 \frac{3}{\alpha}} \right) dt \exp \left( \int_0^T \left( \left\| \tilde{u} \right\|_{L^2}^{\frac{3}{\alpha}} + \left\| b_3 \right\|_{L^2}^{\frac{3}{\alpha}} \left\| \tilde{u} \right\|_{L^2}^{2 \frac{3}{\alpha}} \right) dt \right).
\]
We note that, due to \(2\alpha/\theta = 3(1-\alpha)/(1-\theta) = s(q-2)/q\)

\[
\int_0^T \|b_3\|^{2}_{L_x^p} \|\tilde{u}\|^2_{L^p} \leq \|b_3\|^{2}_{L_x^{p,q,t}} \int_0^T \|b_3\|^{\frac{3(1-\alpha)}{1-\theta}} \|\tilde{u}\|^{\frac{3}{1-\theta}}_{L_x^{p,q,t}} \|\tilde{u}\|^2_{L^p}.
\]

Therefore, \(\tilde{b}(t) \in L^3(\mathbb{R}^3)\) for all \(t \leq T\). This completes the proof. \(\Box\)

**Remark 6.** It is worth mentioning that Lemma 8 is also true for bounded domains. In addition, we can see that Lemma 8 is valid even for the case of slip boundary data for \(u\). Since its verification is rather straightforward, we skip its details.

Now we are ready to present the proof of Theorem 2.

**Proof of Theorem 2.** We argue by contradiction. Suppose that \(T^* \) be the first time of singularity and \(T^* \leq T\). Multiplying the velocity equations of (1.1) with \(-\Delta' u\) and integrating, we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla' u|^2 + \int_{\mathbb{R}^3} |\nabla' \nabla u|^2 = \int_{\mathbb{R}^3} (u \cdot \nabla) u \Delta' u - \int_{\mathbb{R}^3} (b \cdot \nabla) b \Delta' u
\]

\[
= \int_{\mathbb{R}^3} \sum_{j=1}^2 \left( u_j \frac{\partial u}{\partial x_j} \right) \Delta' u + \int_{\mathbb{R}^3} \left( u_3 \frac{\partial u}{\partial x_3} \right) \Delta' u - \int_{\mathbb{R}^3} (b \cdot \nabla) b \Delta' u
\]

\[
:= I + II + III,
\]

where we used the divergence-free condition so that the nonlinear term vanishes. Firstly, we estimate \(II\) as

\[
II = \sum_{k=1}^2 \sum_{i=1}^3 \int_{\mathbb{R}^3} (u_3 u_{i,x_3}^3) u_{i,x_i,x_k} = -\sum_{k=1}^2 \sum_{i=1}^3 \int_{\mathbb{R}^3} (u_3 u_{i,x_3}) u_{i,x_k}
\]

\[
= -\sum_{k=1}^2 \sum_{i=1}^3 \int_{\mathbb{R}^3} u_{3,x_i} u_{i,x_3} u_{i,x_k} - \sum_{k=1}^2 \sum_{i=1}^3 \int_{\mathbb{R}^3} u_3 u_{i,x_k} u_{i,x_k} := II_1 + II_2.
\]

We continue to estimate \(II_2\) as

\[
II_2 = -\frac{1}{2} \sum_{k=1}^2 \sum_{i=1}^3 \int_{\mathbb{R}^3} u_3 |u_{i,x_i}|^2
\]

\[
= \frac{1}{2} \sum_{k=1}^2 \sum_{\beta=1}^2 \int_{\mathbb{R}^3} u_{\beta,x_\beta} |u_{i,x_k}|^2 = \sum_{\beta=1}^2 \sum_{i=1}^3 \int_{\mathbb{R}^3} u_{\beta,x_\beta} u_{i,x_k} u_{i,x_k}.
\]

On the other hand, using the divergence-free condition,
Young's inequality. Applying Young's inequality to the second term in (4.38), we observe that

\[ II_1 = -\sum_{k=1}^{2} \sum_{i=1}^{2} \int_{\mathbb{R}^3_+} u_{3,x_i} u_{i,x_k} u_{i,x_k} = \sum_{k=1}^{2} \sum_{i=1}^{2} \int_{\mathbb{R}^3_+} u_{3,x_i} u_{3,x_k} \]

\[ = \sum_{k=1}^{2} \sum_{i=1}^{2} \int_{\mathbb{R}^3_+} (u_{3,x_i} u_{i,x_k}) x_i \]

\[ = \sum_{k=1}^{2} \sum_{i=1}^{2} \int_{\mathbb{R}^3_+} (u_{3,x_i} x_k) x_i - \sum_{k=1}^{2} \sum_{i=1}^{2} \int_{\mathbb{R}^3_+} (u_{3,x_i} x_k) x_i. \]

Using Hölder's inequality with \( \frac{1}{2} + \frac{p-2}{2p} = 1 \), we note that

\[ |II_1| \leq \|\tilde{u}\|_{L^p} \|\nabla' u\|_{L^2} \|\nabla' u\|_{L^2}. \]

One can also see that

\[ |I| \leq \|\tilde{u}\|_{L^p} \|\nabla' u\|_{L^2} \|\Delta' u\|_{L^2}, \quad |III| \leq \|b|\|\nabla b\|_{L^2} \|\Delta' u\|_{L^2}. \]

Summarizing all estimates above, we obtain

\[ \frac{1}{2} \frac{d}{dt} \|\nabla' u\|_{L^2}^2 + \|\nabla' u\|_{L^2}^2 \leq \|\tilde{u}\|_{L^p} \|\nabla' u\|_{L^2} \|\nabla' u\|_{L^2} + \|b|\|\nabla b\|_{L^2} \|\Delta' u\|_{L^2}. \quad (4.38) \]

Let \( \theta = 1 - \frac{3}{p} \). The first term of the right-hand side in (4.38) can be estimated as follows

\[ \|\tilde{u}\|_{L^p} \|\nabla' u\|_{L^2} \|\nabla' u\|_{L^2} \leq \|\tilde{u}\|_{L^p} \|\nabla' u\|_{L^2}^{\theta} \|\nabla' u\|_{L^2}^{1-\theta} \|\nabla' u\|_{L^2} \]

\[ \leq C \|\tilde{u}\|_{L^p} \|\nabla' u\|_{L^2}^{\theta} \|\nabla' u\|_{L^2}^{1-\theta} \|\nabla' u\|_{L^2} \]

\[ \leq C \|\tilde{u}\|_{L^p} \|\nabla' u\|_{L^2}^{\theta} \|\nabla' u\|_{L^2}^{2-2\theta} + \frac{1}{16} \|\nabla' u\|_{L^2}^2 \]

\[ \leq C \|\tilde{u}\|_{L^p} \|\nabla' u\|_{L^2}^{\theta} \|\nabla' u\|_{L^2}^2 + \frac{1}{16} \|\nabla' u\|_{L^2}^2 \]

\[ \leq C \|\tilde{u}\|_{L^p} \|\nabla' u\|_{L^2}^2 + \frac{1}{8} \|\nabla' u\|_{L^2}^2, \]

where we used Hölder's inequality with \( \frac{p-2}{2p} = \frac{\theta}{2} + \frac{1-\theta}{6} \), interpolations, Sobolev embedding, and Young's inequality. Applying Young's inequality to the second term in (4.38), we observe that

\[ \frac{d}{dt} \|\nabla' u\|_{L^2}^2 + \frac{13}{16} \|\nabla' u\|_{L^2}^2 \leq C \|\tilde{u}\|_{L^p} \|\nabla' u\|_{L^2}^2 + C \|b|\|\nabla b\|_{L^2}^2. \quad (4.39) \]
Next, multiplying the third equation of the velocity field in (1.1) with $-\Delta u_3$ and integrating, we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}^+} |\nabla u_3|^2 + \int_{\mathbb{R}^+} |\Delta u_3|^2 = \int_{\mathbb{R}^+} (u \cdot \nabla)u_3 \Delta u_3 - \int_{\mathbb{R}^+} (b \cdot \nabla)b_3 \Delta u_3 + \int_{\mathbb{R}^+} \pi \Delta u_3. \tag{4.40}
\]

We first estimate the second term in the right-hand side of (4.40) as
\[
\left| \int_{\mathbb{R}^+} (b \cdot \nabla)b_3 \Delta u_3 \right| \leq \|b||\nabla|b\| \|\nabla^2 u_3\|_{L^2}. \tag{4.41}
\]

Next, we estimate the first term in the right-hand side of (4.40) as
\[
\int_{\mathbb{R}^+} (u \cdot \nabla)u_3 \Delta u_3 = \sum_{j=1}^{3} \int_{\mathbb{R}^+} u_j u_3, x_j \Delta u_3 = \sum_{j=1}^{2} \int_{\mathbb{R}^+} u_j u_3, x_j \Delta u_3 + \int_{\mathbb{R}^+} u_3 u_3, x_3 \Delta u_3 := A_1 + A_2.
\]

The term $A_1$ is directly estimated as follows
\[
|A_1| \leq \|\tilde{u}\|_{L^p} \|\nabla' u_3\|_{L^{2p}} \|\nabla^2 u_3\|_{L^2}. \tag{4.42}
\]

Continuing computations for $A_2$, we have
\[
A_2 = - \sum_{k=1}^{3} \int_{\mathbb{R}^+} \left(u_3 u_3, x_k \right) u_3, x_k = - \sum_{k=1}^{3} \int_{\mathbb{R}^+} (u_3, x_k u_3, x_3 + u_3 u_3, x_3, x_k) u_3, x_k
\]
\[
= - \sum_{k=1}^{3} \int_{\mathbb{R}^+} u_3, x_k |u_3, x_k|^2 - \frac{1}{2} \sum_{k=1}^{3} \int_{\mathbb{R}^+} u_3 |u_3, x_k|^2
\]
\[
= - \sum_{k=1}^{3} \int_{\mathbb{R}^+} u_3, x_k |u_3, x_k|^2 + \frac{1}{2} \sum_{k=1}^{3} \int_{\mathbb{R}^+} u_3, x_j |u_3, x_k|^2 = - \frac{1}{2} \sum_{k=1}^{3} \int_{\mathbb{R}^+} u_3, x_j |u_3, x_k|^2
\]
\[
= \frac{1}{2} \sum_{k=1}^{3} \sum_{\beta=1}^{2} \int_{\mathbb{R}^+} u_\beta, x_k |u_3, x_k|^2 = - \sum_{k=1}^{3} \sum_{\beta=1}^{2} \int_{\mathbb{R}^+} u_\beta u_3, x_k u_3, x_3, x_k.
\]

Therefore, we obtain
\[
|A_2| \leq \|\tilde{u}\|_{L^p} \|\nabla u_3\|_{L^{2p}} \|\nabla' u_3\|_{L^2}. \tag{4.43}
\]

Summing (4.42) and (4.43), we obtain
\[
|A| \leq C \|\tilde{u}\|_{L^p} \|\nabla u_3\|_{L^{2p}} \|\nabla^2 u_3\|_{L^2}. \tag{4.44}
\]
Next, we estimate the third term in the right-hand side of (4.40). Using the equation of pressure and integration by parts, we observe that

\[
\int_{\mathbb{R}^3} \pi_{x_3} \Delta u_3 = - \int_{\mathbb{R}^3} \Delta \pi u_{3,x_3} = \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} (u_i u_j)_{x_j x_i} + (b_i b_j)_{x_j x_i} u_{3,x_3} \]

\[
= \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} u_{i,x_j} u_{j,x_i} u_{3,x_3} + \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} b_{i,x_j} b_{j,x_i} u_{3,x_3} \]

\[
= \sum_{i=1}^{3} \sum_{j=1}^{2} \int_{\mathbb{R}^3} u_{i,x_j} u_{j,x_i} u_{3,x_3} + \sum_{i=1}^{3} \int_{\mathbb{R}^3} u_{i,x_3} u_{3,x_i} u_{3,x_3} + \int_{\mathbb{R}^3} u_{3,x_3} u_{3,x_3} u_{3,x_3} \]

\[
+ \sum_{i=1}^{3} \sum_{j=1}^{2} \int_{\mathbb{R}^3} b_{i,x_j} b_{j,x_i} u_{3,x_3} + \sum_{i=1}^{3} \int_{\mathbb{R}^3} b_{i,x_3} b_{3,x_i} u_{3,x_3} + \int_{\mathbb{R}^3} b_{3,x_3} b_{3,x_3} u_{3,x_3} \]

\[:= J_1 + J_2 + J_3 + J_4 + J_5 + J_6.\]

We consider the first term \( J_1 \). Via the divergence-free condition, we have

\[
J_1 = \sum_{i=1}^{3} \sum_{j=1}^{2} \int_{\mathbb{R}^3} u_{i,x_j} u_{j,x_i} u_{3,x_3} = - \sum_{i=1}^{3} \sum_{j=1}^{2} \int_{\mathbb{R}^3} (u_i u_j)_{x_j x_i} u_j \]

\[
= - \sum_{i=1}^{3} \sum_{j=1}^{2} \int_{\mathbb{R}^3} (u_{i,x_j} u_{3,x_3} + u_{i,x_3} u_{3,x_i}) u_j = - \sum_{i=1}^{3} \sum_{j=1}^{2} \int_{\mathbb{R}^3} u_{i,x_j} u_{3,x_3} u_j.\]

Using Hölder's inequality, we obtain

\[
|J_1| \leq \|\tilde{u}\|_{L^p} \|\nabla' u\|_{L^{2p}}^{\frac{2p}{2p-2}} \|\nabla^2 u_3\|_{L^2}.\]

Next consider \( J_4 \).

\[
J_4 = \sum_{i=1}^{3} \sum_{j=1}^{2} \int_{\mathbb{R}^3} b_{i,x_j} b_{j,x_i} u_{3,x_3} = - \sum_{i=1}^{3} \sum_{j=1}^{2} \int_{\mathbb{R}^3} (b_{i,x_j} u_{3,x_3})_{x_i} b_j = - \sum_{i=1}^{3} \sum_{j=1}^{2} \int_{\mathbb{R}^3} b_j b_{i,x_j} u_{3,x_3}.\]

therefore,

\[
|J_4| \leq \|b\|_{L^2} \|\nabla b\|_{L^2} \|\nabla^2 u_3\|_{L^2}.\]

For \( J_2 \), we observe that

\[
J_2 = \sum_{i=1}^{2} \int_{\mathbb{R}^3} u_{i,x_3} u_{3,x_i} u_{3,x_3} = - \sum_{i=1}^{2} \int_{\mathbb{R}^3} (u_{3,x_i} u_{3,x_3})_{x_i} u_i.\]
Thus, it is clear that
\[ |J_2| \leq \sum_{i=1}^{2} \| \tilde{u} \|_{L^p_x} \| \nabla u_3 \|_{L^2} \frac{2}{L^p_x} \| \nabla^2 u_3 \|_{L^2}. \]

Next, we consider \( J_5 \) and \( J_6 \) together. We note that some cancellation occurs between the two terms. More precisely, we observe that
\[
J_5 = 2 \sum_{i=1}^{2} \int_{\mathbb{R}^3_+} b_{1, x_i} b_{3, x_i} u_{3, x_3} = - \sum_{i=1}^{2} \int_{\mathbb{R}^3_+} (b_{3, x_i} u_{3, x_3}) x_i b_i
\]
\[
= - \sum_{i=1}^{2} \int_{\mathbb{R}^3_+} b_i b_{3, x_i} x_{3, x_3} - \sum_{i=1}^{2} \int_{\mathbb{R}^3_+} b_i b_{3, x_i} x_{3, x_3}
\]
\[
= \sum_{i=1}^{2} \int_{\mathbb{R}^3_+} b_i b_{3, x_i} x_{3, x_3} + \sum_{i=1}^{2} \int_{\mathbb{R}^3_+} b_i b_{3, x_i} x_{3, x_3} - \sum_{i=1}^{2} \int_{\mathbb{R}^3_+} b_i b_{3, x_i} x_{3, x_3}
\]
\[
= - J_6 + \sum_{i=1}^{2} \int_{\mathbb{R}^3_+} b_i b_{3, x_i} x_{3, x_3} - \sum_{i=1}^{2} \int_{\mathbb{R}^3_+} b_i b_{3, x_i} x_{3, x_3},
\]
where the divergence-free condition for \( b \) is used. Hence, it follows that
\[ |J_5 + J_6| \leq \| b \|_{L^p_x} \| \nabla b \|_{L^2} \| \nabla^2 u_3 \|_{L^2}. \]

Again, with the aid of the divergence-free condition, we can estimate \( J_3 \) as follows
\[
J_3 = - \sum_{\beta=1}^{2} \int_{\mathbb{R}^3_+} u_{3, x_3} u_{3, x_3} u_{\beta, x_3} = \sum_{\beta=1}^{2} \int_{\mathbb{R}^3_+} (u_{3, x_3} u_{3, x_3}) x_3 u_{\beta, x_3,
\]
which implies that
\[ |J_3| \leq \| \tilde{u} \|_{L^p_x} \| \nabla u_3 \|_{L^2} \frac{2}{L^p_x} \| \nabla \nabla^2 u_3 \|_{L^2}. \]

Summing all estimates for \( J_1, \ldots, J_6 \), we obtain
\[
|J| \leq C \| \tilde{u} \|_{L^p_x} \left( \| \nabla' u \|_{L^2} \frac{2}{L^p_x} + \| \nabla u_3 \|_{L^2} \frac{2}{L^p_x} \right) \| \nabla^2 u_3 \|_{L^2} + \| b \|_{L^p_x} \| \nabla b \|_{L^2} \| \nabla^2 u_3 \|_{L^2}
\]
\[
\leq C \| \tilde{u} \|_{L^p_x} \left( \| \nabla' u \|_{L^2}^2 + \| \nabla u_3 \|_{L^2}^2 \right) + C \| b \|_{L^p_x} \| \nabla b \|_{L^2} + \frac{3}{16} \| \nabla^2 u_3 \|_{L^2}. \quad (4.45)
\]

Summing (4.41), (4.44) and (4.45), we have
\[
\frac{d}{dt} \| \nabla u_3 \|^2_{L^2_t} + \| \nabla^2 u_3 \|^2_{L^2_t} \leq C \| \tilde{u} \|_{L^p_t} \left( \| \nabla' u \|_{L^\infty_t}^{2p} + \| \nabla u_3 \|_{L^2_t}^{2p} \right) \| \nabla^2 u_3 \|_{L^2_t} + C \| |b| \nabla b| \|_{L^2_t} \| \nabla^2 u_3 \|_{L^2_t} \\
\leq C \| \tilde{u} \|_{L^p_t}^{2p} \| \nabla^2 u \|_{L^2_t}^2 + \| \nabla u_3 \|_{L^2_t}^2 \} + C \| |b| \nabla b| \|_{L^2_t}^2 \\
+ \frac{3}{16} \left( \| \nabla^2 u_3 \|_{L^2_t}^2 + \| \nabla \nabla u \|_{L^2_t}^2 \right),
\] (4.46)

where Young's inequality is used.

Next, multiplying the magnetic equation of (1.1) with $|b|^2 b$ and integrating, we have

\[
\frac{1}{4} \frac{d}{dt} \int_\Omega |b|^4 + \int_\Omega |\nabla b|^2 |b|^2 + \frac{1}{2} \int_\Omega |\nabla |b| |^2 \]

\[
= \int_\Omega (b \cdot \nabla) |b|^2 b = \sum_{k=1}^{3} \sum_{j=1}^{3} \int_\Omega b_k u_{j,x_k} |b|^2 b_j = \sum_{k=1}^{3} \sum_{j=1}^{2} \int_\Omega b_k u_{j,x_k} |b|^2 b_j + \sum_{k=1}^{3} \int_\Omega b_k u_{3,x_k} |b|^2 b_3 \\
= -\sum_{k=1}^{3} \sum_{j=1}^{2} \int_\Omega b_k u_j \langle b \rangle^2 b_{j,x_k} + \sum_{k=1}^{3} \int_\Omega b_k u_{3,x_k} |b|^2 b_3 + \int_\Omega b_k u_{3,x_k} |b|^2 b_3 := K_1 + K_2 + K_3.
\]

Via Hölder's inequality, Sobolev embedding and Young's inequality,

\[
|K_1| \leq \| \tilde{u} \|_{L^p_t} \| |b| \|_{L^2_t} \| \nabla b \|_{L^2_t} \leq \| \tilde{u} \|_{L^p_t} \| |b| \|_{L^2_t}^{2p} \| \nabla b \|_{L^2_t}^{1-\theta} \| |b| \nabla b| \|_{L^2_t} \\
\leq C \| \tilde{u} \|_{L^p_t} \| |b| \|_{L^2_t}^{2p} \| \nabla b \|_{L^2_t}^{2-\theta} \leq C \| \tilde{u} \|_{L^p_t}^{2p} \| |b| \|_{L^2_t}^4 + \frac{1}{16} \| \nabla |b| |^2 \|_{L^2_t}^2, \] (4.47)

where $\theta = 1 - \frac{3}{p}$. Using the result of Lemma 8, we control $K_2$ as follows

\[
|K_2| \leq \int_\Omega \langle b \rangle^2 |b_3| |\nabla u_3| \leq \| b_3 \|_{L^6_t} \| b \|_{L^2_t} \| \nabla u_3 \|_{L^6_t} \\
\leq C \| b_3 \|_{L^6_t} \| b \|_{L^2_t}^{2p} \| |b| \|_{L^2_t} \| \nabla u_3 \|_{L^6_t} \\
\leq C \| b_3 \|_{L^6_t}^{2p} \| b \|_{L^4_t}^4 + \frac{1}{16} \left( \| \nabla^2 u_3 \|_{L^2_t}^2 + \| \nabla |b| |^2 \|_{L^2_t}^2 \right), \] (4.48)

where $\theta = 1 - \frac{3}{2}$. Due to the divergence-free condition and integration by parts, we estimate the third term $K_3$ as

\[
K_3 = \int_\Omega b_3^2 |b|^2 u_{3,x_3} = \sum_{j=1}^{2} \int_\Omega u_j (b_3^2 |b|^2)_{x_j} ;
\]

therefore, it follows that
solutions become regular. Hence, by introduce a space of smooth functions, which are compactly supported and divergence free, denoted Appendix A

Then, from the estimate (4.51), we can see that

Therefore, we obtain

where $\theta = 1 - \frac{3}{p}$. Summing (4.47), (4.48) and (4.49), we have

Summing (4.39), (4.46) and (4.50), we get

Gronwall's inequality implies

Therefore, we obtain

Then, from the estimate (4.51), we can see that $\|u_3\|_{L^{\infty}_{x,t}} < C$, and it follows, due to Theorem 1, that solutions become regular. Hence, $T^*$ cannot be a maximal time of existence less than or equal to $T$. The proof is completed.

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Appendix A

In this section, we will present the proof of Proposition 1 using the semigroup method. First, we introduce a space of smooth functions, which are compactly supported and divergence free, denoted by

$$C_{0,\sigma}^\infty(\Omega) = \left\{ w \in C_0^\infty(\Omega) \mid \text{div } w = 0 \right\}.$$
Let $1 < q < \infty$. We recall the Helmholtz decomposition for $L^q(\Omega)$, which is $L^q(\Omega) = L^q_\sigma(\Omega) \oplus G^q(\Omega)$, where

$$L^q_\sigma(\Omega) = C^\infty_0(\Omega) \cap L^q(\Omega), \quad G^q(\Omega) = \{ \nabla p \in L^q(\Omega) \mid p \in L^q_{\text{loc}}(\Omega) \}.$$

For a sufficiently smooth domain $\Omega$, we note that the solenoidal space $L^q_\sigma(\Omega)$ is characterized as

$$L^q_\sigma(\Omega) = \{ w \in L^q(\Omega) \mid \text{div } w = 0 \text{ in } \Omega, \ w \cdot n = 0 \text{ on } \partial \Omega \}.$$

Let $P$ be a linear continuous projection from $L^q(\Omega)$ onto $L^q_\sigma(\Omega)$. We then have

$$\|Pw\|_{L^q(\Omega)} \leq C_q \|w\|_{L^q(\Omega)}$$

for any $w \in L^q(\Omega)$, $1 < q < \infty$.

Next, we define the Stokes operators $A^{(n)}_q$ and $A^{(s)}_q$ as follows

$$A^{(n)}_q u = -P \Delta u, \quad u \in D(A^{(n)}_q) := L^q_\sigma(\Omega) \cap W^{2, q}(\Omega) \cap W^{1, q}_0(\Omega),$$

$$A^{(s)}_q u = -P \Delta u, \quad u \in D(A^{(s)}_q) := \{ u \in L^q_\sigma(\Omega) \cap W^{2, q}(\Omega) \mid (\nabla \times u) \times n = 0 \text{ on } \partial \Omega \}.$$

In addition, we define the linear operator $B_q$ as

$$B_q b = \nabla \times (\nabla \times b), \quad b \in D(B_q) = \{ b \in L^q_\sigma(\Omega) \cap W^{2, q}(\Omega) \mid (\nabla \times b) \times n = 0 \text{ on } \partial \Omega \}.$$

For $A^{(n)}_q$ and $A^{(s)}_q$, we have

$$\|\nabla e^{-A^{(n)}_q t} u\|_{L^q(\Omega)} \leq C t^{-\frac{1}{2}} \|u\|_{L^q(\Omega)},$$

$$\|\nabla e^{-A^{(s)}_q t} u\|_{L^q(\Omega)} \leq C t^{-\frac{1}{2}} \|u\|_{L^q(\Omega)} \quad \text{(A.52)}$$

with the robin boundary for all $q \in (1, \infty)$ (see [33, p. 262]). Due to above relation, we write $A_q$ for $A^{(n)}_q$ and $A^{(s)}_q$, respectively. We also recall the following estimate:

$$\|\nabla e^{-B_q t} b\|_{L^q(\Omega)} \leq C t^{-\frac{1}{2}} \|b\|_{L^q(\Omega)} \quad \text{(A.53)}$$

with the slip boundary (see [37, Theorem 2]).

**Proof of Proposition 1.** We first show the existence of solutions by the Banach Fixed Point Theorem. We suppose that $\|u_0\|_{X_0} + \|b_0\|_{X_0} < \frac{R}{2}$ for some $R > 0$. We set

$$S = \{ (u, b) \in X_T \mid \|u\|_{X_T} + \|b\|_{X_T} \leq R + 1, \ u|_{\partial \Omega} = 0, \ (\nabla \times b) \times n|_{\partial \Omega} = 0 \},$$

$$S^{(s)} = \{ (u, b) \in X_T \mid \|u\|_{X_T} + \|b\|_{X_T} \leq R + 1, \ (\nabla \times u) \times n|_{\partial \Omega} = 0, \ (\nabla \times b) \times n|_{\partial \Omega} = 0 \}.$$

We then note that $S^{(n)}$ and $S^{(s)}$ are a closed subset of $X_T$. For $(u, b) \in S$ and $t \in [0, T]$, we define

$$\Phi(u, b)(t) = \begin{pmatrix} \Phi_1(u, b)(t) \\ \Phi_2(u, b)(t) \end{pmatrix} := \begin{pmatrix} e^{-tA_q} u_0 + \int_0^t e^{-(t-s)A_q} P ((u \cdot \nabla) u + (b \cdot \nabla) b) ds \\ e^{-tB_q} b_0 + \int_0^t e^{-(t-s)B_q} ((u \cdot \nabla b) - (b \cdot \nabla u)) ds \end{pmatrix}.$$
First, we claim that $\Phi(u, b) \in S$ for sufficiently small $T > 0$. Using the estimate (A.52), we estimate $\|\Phi_1(u, b)(t)\|_{L^q_t(L^q_x)}$ as follows

$$
\|\nabla \Phi_1(u, b)(t)\|_{L^q_t(L^q_x)} \leq \|\nabla e^{-tA_u}u_0\|_{L^q_t(L^q_x)} + \int_0^t \|\nabla e^{-(t-s)A_u}P(u \cdot \nabla)u\|_{L^q_t(L^q_x)} ds
$$

$$
+ \int_0^t \|e^{-(t-s)A_u}P(b \cdot \nabla)b\|_{L^q_t(L^q_x)} ds. \tag{A.54}
$$

We first note that

$$
\|\nabla e^{-tA_u}u_0\|_{L^q_t(L^q_x)} \leq \|\nabla u_0\|_{L^q_t(L^q_x)}. \tag{A.55}
$$

The second term of (A.54) can be estimated as follows

$$
\int_0^t \|\nabla e^{-(t-s)A_u}P(u \cdot \nabla)u\|_{L^q_t(L^q_x)} ds \leq C \int_0^t (t-s)^{-\frac{1}{2}} \|\nabla u\|_{L^q_t} ds \leq C \int_0^t (t-s)^{-\frac{1}{2}} \|\nabla u\|_{L^q_t} ds
$$

$$
\leq C \int_0^t (t-s)^{-\frac{1}{2}} \|\nabla u\|_{L^q_t}^2 ds \leq Ct^\frac{1}{2} \|\nabla u\|_{L^q_t}^2. \tag{A.56}
$$

Similarly,

$$
\int_0^t \|\nabla e^{-(t-s)A_u}P(b \cdot \nabla)b\|_{L^q_t(L^q_x)} ds \leq C \|\nabla b\|_{L^q_t(L^q_x)}^2 t^\frac{1}{2}. \tag{A.57}
$$

Summing (A.56) and (A.57), we obtain

$$
\|\nabla \Phi_1(u, b)(t)\|_{L^q_t(L^q_x)} \leq \|\nabla u_0\|_{L^q_t(L^q_x)} + Ct^\frac{1}{2} \left(\|\nabla u\|_{L^q_t(L^q_x)}^2 + \|\nabla b\|_{L^q_t(L^q_x)}^2\right). \tag{A.58}
$$

For $\Phi_2(u, b)(t)$, via similar arguments as $\Phi_1(u, b)(t)$,

$$
\|\nabla \Phi_2(u, b)(t)\|_{L^q_t(L^q_x)} \leq \|\nabla b_0\|_{L^q_t(L^q_x)} + Ct^\frac{1}{2} \left(\|\nabla u\|_{L^q_t(L^q_x)}^2 + \|\nabla b\|_{L^q_t(L^q_x)}^2\right). \tag{A.59}
$$

Since its verification of (A.59) is almost the same as above, we skip its details.

Next, we estimate $\|\Phi(u, b)(t)\|_{W^{2,q}_{t,x}L^q_x(\Omega_t)}$. We consider

$$
\|\nabla^2 \Phi_1(u, b)(t)\|_{L^q_t(L^q_x)} \leq \|\nabla^2 e^{-tA_u}u_0\|_{L^q_t(L^q_x)} + \int_0^t \|\nabla e^{-(t-s)A_u}Q(u \cdot \nabla)u\|_{L^q_t(L^q_x)} ds
$$

$$
+ \int_0^t \|\nabla e^{-(t-s)A_u}Q(b \cdot \nabla)b\|_{L^q_t(L^q_x)} ds. \tag{A.60}
$$
where we used integration by parts. The first term is estimated as follows
\[
\| \nabla^2 e^{-tA}u_0 \|_{L^q_t(\Omega)} \leq \| \nabla e^{-tA} \nabla u_0 \|_{L^q_t(\Omega)} \leq C t^{-\frac{1}{2}} \| \nabla u_0 \|_{L^q_t(\Omega)}.
\]
The second term of right-hand side in (A.60) can be estimated as follows
\[
\int_0^t \| \nabla e^{-t-s}A \nabla (u \cdot \nabla u) \|_{L^q_t(\Omega)} \, ds \\
\leq \int_0^t \| \nabla e^{-t-s}A (|\nabla u|^2 + |u \nabla^2 u|) \|_{L^q_t(\Omega)} \, ds \\
= \int_0^t \| \nabla e^{-t-s}A |\nabla u|^2 \|_{L^q_t(\Omega)} \, ds + \int_0^t \| \nabla e^{-t-s}A |u \nabla^2 u| \|_{L^q_t(\Omega)} \, ds := I_1 + I_2.
\]
We first estimate $I_1$ as
\[
|I_1| = \int_0^t \| \nabla e^{-t-s}A |\nabla u|^2 \|_{L^q_t(\Omega)} \, ds \leq C \int_0^t (t-s)^{-\frac{1}{2}} \| \nabla u \|_{L^q_t(\Omega)} \| \nabla u \|_{L^q_{x,t}(\Omega)} \, ds \\
\leq C \| \nabla u \|_{L^q_{x,t}(Q_t)} \int_0^t (t-s)^{-\frac{1}{2}} \| \nabla^2 u \|_{L^q_t(\Omega)} \, ds \leq C t^{\frac{q-2}{2}} \| \nabla u \|_{L^q_{x,t}(Q_t)} \| \nabla^2 u \|_{L^q_{x,t}(Q_t)}.
\]
Next, we estimate $I_2$ as
\[
|I_2| = \int_0^t \| \nabla e^{-t-s}A |u \nabla^2 u| \|_{L^q_t(\Omega)} \, ds \leq C \int_0^t (t-s)^{-\frac{1}{2}} \| u \|_{L^q_t(\Omega)} \| \nabla^2 u \|_{L^q_t(\Omega)} \, ds \\
\leq C \int_0^t (t-s)^{-\frac{1}{2}} \| u \|_{L^q_{x,t}(\Omega)} \| \nabla^2 u \|_{L^q_t(\Omega)} \, ds \leq C \int_0^t (t-s)^{-\frac{1}{2}} \| \nabla u \|_{L^q_t(\Omega)} \| \nabla^2 u \|_{L^q_{x,t}(\Omega)} \, ds \\
\leq C \| \nabla u \|_{L^q_{x,t}(Q_t)} \int_0^t (t-s)^{-\frac{1}{2}} \| \nabla^2 u \|_{L^q_t(\Omega)} \, ds \leq C t^{\frac{q-2}{2}} \| \nabla u \|_{L^q_{x,t}(Q_t)} \| \nabla^2 u \|_{L^q_{x,t}(Q_t)}.
\]
Summing $I_1$ and $I_2$, we get
\[
\| \nabla^2 e^{-t-s}A \mathbb{P}(u \cdot \nabla) u \|_{L^q_t(\Omega)} \leq C t^{\frac{q-2}{2}} \| \nabla u \|_{L^q_{x,t}(Q_t)} \| \nabla^2 u \|_{L^q_{x,t}(Q_t)}.
\]
In the same manner, we have
\[
\| \nabla^2 e^{-t-s}A \mathbb{P}(b \cdot \nabla) b \|_{L^q_t(\Omega)} \leq C t^{\frac{q-2}{2}} \| \nabla b \|_{L^q_{x,t}(Q_t)} \| \nabla^2 b \|_{L^q_{x,t}(Q_t)}.
\]
We estimate $\Phi$. Therefore, combining estimates (A.58), (A.59), (A.61) and (A.62), for sufficiently small $T$, we then obtain

$$
\| \nabla^2 \Phi(u, b)(t) \|_{L^q_t(Q_t)} 
\leq C t^{-\frac{\frac{\gamma}{2}}{2}} \| u_0 \|_{W^{1, q}_X(Q_t)} + C t^{\frac{\gamma}{2}} \left( \| u \|_{L^q_t(Q_t)} \| \nabla^2 u \|_{L^q_t(Q_t)} + \| b \|_{L^q_t(Q_t)} \| \nabla^2 b \|_{L^q_t(Q_t)} \right).
$$

Integrating over time, we obtain

$$
\int_0^t \| \nabla^2 \Phi(u, b)(\tau) \|_{L^q_t(Q_t)} d\tau 
\leq C t^{-\frac{\frac{\gamma}{2}}{2}} \| u_0 \|_{W^{1, q}_X(Q_t)} + C t^{\frac{\gamma}{2}} \left( \| u \|_{L^q_t(Q_t)} \| \nabla^2 u \|_{L^q_t(Q_t)} + \| b \|_{L^q_t(Q_t)} \| \nabla^2 b \|_{L^q_t(Q_t)} \right).
$$

A.61

Similar computations lead to the following estimate:

$$
\int_0^t \| \nabla^2 \Phi_2(u, b)(\tau) \|_{L^q_t(Q_t)} d\tau 
\leq C t^{-\frac{\frac{\gamma}{2}}{2}} \| b_0 \|_{W^{1, q}_X(Q_t)} + C t^{\frac{\gamma}{2}} \left( \| u \|_{L^q_t(Q_t)} \| \nabla^2 u \|_{L^q_t(Q_t)} + \| b \|_{L^q_t(Q_t)} \| \nabla^2 b \|_{L^q_t(Q_t)} \right).
$$

A.62

Combining estimates (A.58), (A.59), (A.61) and (A.62), for sufficiently small $T$, we then obtain

$$
\| \Phi(u, b) \|_{X_T} \leq \| \Phi_1(u, v) \|_{X_T} + \| \Phi_2(u, b) \|_{X_T} \leq \frac{R}{2} + C(R + 1) T^{\frac{1}{2}} \leq \frac{R + 1}{2}.
$$

Therefore, $\Phi$ maps $S$ onto itself for sufficiently small $T$.

Next, we show that $\Phi$ is indeed a contraction mapping for a small $T > 0$.

$$
\| \Phi(u_1, b_1) - \Phi(u_2, b_2) \|_{X_T} 
\leq \| \Phi(u_1, b_1) - \Phi(u_2, b_2) \|_{W^{1, q}_X} + \| \Phi(u_1, b_1) - \Phi(u_2, b_2) \|_{W^{2, q}_X} 
\leq \| \Phi_1(u_1, b_1) - \Phi_1(u_2, b_2) \|_{W^{1, q}_X} + \| \Phi_1(u_1, b_1) - \Phi_1(u_2, b_2) \|_{W^{2, q}_X} 
+ \| \Phi_2(u_1, b_1) - \Phi_2(u_2, b_2) \|_{W^{1, q}_X} + \| \Phi_2(u_1, b_1) - \Phi_2(u_2, b_2) \|_{W^{2, q}_X}.
$$

We estimate $\| \Phi_1(u_1, b_1) - \Phi_1(u_2, b_2) \|_{W^{1, q}_X}$ and compute
We now estimate
\[ \| \nabla (\Phi_1(u_1, b_1) - \Phi_1(u_2, b_2)) \|_{L^q_t} \leq \int_0^t \| \nabla (e^{-(t-s)A_\theta} \nabla [(u_1 \cdot \nabla) u_1 - (u_2 \cdot \nabla) u_2]) \|_{L^q_x} ds \]
\[ + \int_0^t \| \nabla (e^{-(t-s)A_\theta} \nabla [(b_1 \cdot \nabla) b_1 - (b_2 \cdot \nabla) b_2]) \|_{L^q_x} ds \]
\[ := P_1 + P_2. \]

We continue to compute \( P_1 \) as
\[ P_1 = C \int_0^t \| \nabla (e^{-(t-s)A_\theta} (P[(u_1 - u_2) \cdot \nabla] u_1 + (u_2 \cdot \nabla) (u_1 - u_2))) \|_{L^q_x} ds \]
\[ \leq Ct^\frac{1}{2} \| \nabla (u_1 - u_2) \|_{L^q_{x,t}} \| \nabla u_1 \|_{L^q_{x,t}} + \| \nabla u_2 \|_{L^q_{x,t}} \| \nabla (u_1 - u_2) \|_{L^{q,\infty}_{x,t}} \]
\[ = Ct^\frac{1}{2} \| \nabla (u_1 - u_2) \|_{L^{q,\infty}_{x,t}} (\| \nabla u_1 \|_{L^q_{x,t}} + \| \nabla u_2 \|_{L^q_{x,t}}). \] (A.63)

Following similar procedures, we get
\[ P_2 \leq Ct^\frac{1}{2} \| \nabla (b_1 - b_2) \|_{L^q_{x,t}} (\| \nabla b_1 \|_{L^{q,\infty}_{x,t}} + \| \nabla b_2 \|_{L^{q,\infty}_{x,t}}). \] (A.64)

Summing (A.63) and (A.64) and integrating over time, we obtain
\[ \| \Phi_1(u_1, b_1) - \Phi_1(u_2, b_2) \|_{W^{1,q}_{x,t}} \leq Ct^\frac{1}{2} \| \nabla (u_1 - u_2) \|_{L^q_{x,t}} (\| \nabla u_1 \|_{L^{q,\infty}_{x,t}} + \| \nabla u_2 \|_{L^{q,\infty}_{x,t}}) \]
\[ + Ct^\frac{1}{2} \| \nabla (b_1 - b_2) \|_{L^q_{x,t}} (\| \nabla b_1 \|_{L^{q,\infty}_{x,t}} + \| \nabla b_2 \|_{L^{q,\infty}_{x,t}}) \]
\[ \leq Ct^\frac{1}{2} (R + 1) \| (u_1, b_1) - (u_2, b_2) \|_{W^{1,q}_{x,t}}. \] (A.65)

We now estimate \( \| \Phi_1(u_1, b_1) - \Phi_1(u_2, b_2) \|_{W^{2,q}_{x,t}} \). Via integration by parts, we have
\[ \| \nabla^2 (\Phi_1(u_1, b_1) - \Phi_1(u_2, b_2)) \|_{L^q_t} \leq \int_0^t \| \nabla (e^{-(t-s)A_\theta} \nabla (P[(u_1 \cdot \nabla) u_1 - (u_2 \cdot \nabla) u_2])) \|_{L^q_x} ds \]
\[ + \int_0^t \| \nabla (e^{-(t-s)A_\theta} \nabla (P[(b_1 \cdot \nabla) b_1 - (b_2 \cdot \nabla) b_2])) \|_{L^q_x} ds \]
\[ := P_3 + P_4. \]

We continue to compute \( P_3 \) as
\[ P_3 = C \int_0^t \| \nabla (e^{-(t-s)A_\theta} \nabla (P[(u_1 - u_2) \cdot \nabla] u_1 + (u_2 \cdot \nabla) (u_1 - u_2))] \|_{L^q_x} ds \]
\[ \leq Ct^\frac{1}{2q} \| \nabla (u_1 - u_2) \|_{L^{q,\infty}_{x,t}} \| \nabla^2 u_1 \|_{L^{q,\infty}_{x,t}} + \| \nabla u_2 \|_{L^{q,\infty}_{x,t}} \| \nabla^2 (u_1 - u_2) \|_{L^{q,\infty}_{x,t}). \] (A.66)
In the same manner, we have
\[
P_4 \leq C t^{\frac{q-2}{2q}} \left( \| \nabla (b_1 - b_2) \|_{L_q^q, t}^{q, q} + \| \nabla^2 b_1 \|_{L_q^q, t}^{q, q} + \| \nabla b_2 \|_{L_q^q, t}^{q, q} \right).
\] (A.67)

Summing (A.66) and (A.67), we obtain
\[
\| \Phi_1(u_1, b_1) - \Phi_1(u_2, b_2) \|_{W_{x,t}^{2, q}} \leq C t^{\frac{1}{2}} (R + 1)^{\frac{1}{2}} \left( \| u_1 - u_2 \|_{L_q^q, t} + \| b_1 - b_2 \|_{L_q^q, t} \right). \] (A.68)

Thus, due to (A.65) and (A.68),
\[
\| \Phi_1(u_1, b_1) - \Phi_1(u_2, b_2) \|_{X_T} \leq C (R + 1)^{\frac{1}{2}} T^{\frac{1}{2}} \left( \| u_1 - u_2 \|_{L_q^q, t} + \| b_1 - b_2 \|_{L_q^q, t} \right). \]

Similarly,
\[
\| \Phi_2(u_1, b_1) - \Phi_2(u_2, b_2) \|_{X_T} \leq C (R + 1)^{\frac{1}{2}} T^{\frac{1}{2}} \left( \| u_1 - u_2 \|_{L_q^q, t} + \| b_1 - b_2 \|_{L_q^q, t} \right). \]

This shows that, if \( T \) is sufficiently small, \( \Phi \) is a contraction mapping on \( S \). Standard arguments of contraction mapping imply the existence of solutions in class \( S \) before the maximal time, \( T_{\text{max}} \), of existence. This completes the proof. \( \Box \)

References


