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Qualitative Behavior of a Keller–Segel Model with Non-Diffusive Memory

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In this paper a one-dimensional Keller–Segel model with a logarithmic chemotactic-sensitivity and a non-diffusing chemical is classified with respect to its long time behavior. The strength of production of the non-diffusive chemical has a strong influence on the qualitative behavior of the system concerning existence of global solutions or Dirac-mass formation. Further, the initial data play a crucial role.

Keywords Chemotaxis; Keller–Segel model; Non-diffusive memory.

Mathematics Subject Classification Primary 35K57; Secondary 92C17.

1. Introduction

We consider a chemotaxis-system with a logarithmic chemotactic sensitivity and a non-diffusing chemical. The main question addressed is whether smooth solutions exist globally in time, or if a blowup happens. A crucial assumption is that the chemical is produced by the chemotactic species and decay terms do not occur. Thus a drift-diffusion equation is coupled to an ODE. In [6] Keller and Segel discussed traveling waves for a similar system, where for the chemical reaction kinetics just a decay term was considered. Thus existence of global solutions can always be expected. When varying the strength of the production an interesting long time behavior can be observed for the system, as introduced in [11] and formally explored in [9]. Existence of global solutions for linear production kinetics with respect to the chemotactic species was proved in [13]. For a fixed and strong production

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kinetics in [7] finite time blowup was shown for specific explicit initial data. The system considered in this paper arises from formal approximations of lattice models for slime-trail following of myxobacteria. The trails are reinforced by the bacteria themselves and serve as a non-diffusive information about the paths other bacteria have taken. In this paper though we are mostly interested in the peculiar qualitative behavior of solutions of this system, which is different from the classical Keller–Segel model and its variations, and requires other mathematical techniques. We will classify the system for a variety of production kinetics and types of initial data. The main aim is to find “critical conditions” for the switch between existence of global solutions and Dirac mass formation.

The system we study is

\[ u_t = u_{xx} - \left( \frac{u_{wx}}{w} \right)_x, \quad w_t = uw^\lambda \quad \text{for } t > 0, \quad x \in I = [a_1, a_2] \subset \mathbb{R}, \quad \text{and } \lambda \in [0, 1), \]

with periodic boundary conditions and positive initial data. Here \( u \) models the chemotactic species and \( w \) the non-diffusive memory. By setting \( \theta = \frac{1}{1-\lambda} \) and \( z = \frac{1}{(1-z)} u^\frac{\lambda}{3} \) we obtain

\[ u_t = u_{xx} - \theta \left( \frac{u z_x}{z} \right)_x, \quad z_t = u \quad \text{for } t > 0, \quad x \in I = [a_1, a_2]. \quad (1) \]

So \( \theta \in [1, \infty) \). In [13] a result for \( \lambda = 0 \), respectively \( \theta = 1 \) was obtained. In [7] a result for \( \lambda = 1 \) was given. We will have a closer look at the regime in between, where the interesting switch from the existence of global solutions toward Dirac mass formation is to be expected.

A remarkable feature of (1) for \( \theta = 1 \) is, that, at least formally, every space-dependent function \( u(x) \) is asymptotically a steady state for \( t \to \infty \). To see this heuristically, let \( u(x, t) \) be a solution of (1) and \( u(x, t) \to u(x) \) for \( t \to \infty \). Then \( z(x, t) \approx tu(x) \) for \( t \to \infty \). Thus formally the first equation in (1) yields

\[ u_t = u_{xx} - \theta \left( \frac{u z_x}{z} \right)_x \approx (u_{\infty})_{xx} - \left( u_{\infty} \frac{tu(x)}{tu_{\infty}} \right)_x = 0. \]

This suggests that for \( \theta = 1 \) and large times, \( u(x, t) \) can behave like any space-dependent function.

In Section 2 we prove this high sensitivity of the asymptotic behavior of the solutions on the initial data in case these solutions are close to being spatially homogeneous, since for this situation it is possible to use linearization arguments. Further Fourier analysis and a careful control of the nonlinear error terms will be relevant.

For \( \theta > 1 \) the chemotactic sensitivity in (1) becomes larger. So it would not be surprising to obtain mass aggregation for \( t \to \infty \) in certain situations, though not every solution can be expected to behave like this. Spatially homogeneous solutions remain homogeneous in time. For \( 1 < \theta < 3 \) in Section 3 first heuristic arguments are given to provide an intuitive explanation for the results shown thereafter. We will rigorously prove convergence of solutions to a Dirac mass for \( t \to \infty \) under suitable concentration conditions for the initial data. To do this, the mechanism of Dirac mass formation is first derived by formal asymptotics. Then it is rigorously proved that these asymptotics hold for a suitable class of initial data. The formal
asymptotics show that the solutions of (1) behave like stationary solutions of the first equation of our system near the aggregation region. This allows to approximate the second equation of (1) by an ODE coupled with an integral term, which can be solved explicitly, thus providing a description of the aggregation mechanism. Suitable a priori estimates allow to control the region, where the mass of $u$ does not concentrate for large values of $t$. Comparison arguments, using sub and supersolution, then provide estimates for the asymptotics of solutions in the region, where most of the mass concentrates.

In case $\theta > 3$ lengthy formal asymptotics, which are not presented here, suggest, that blowup of solutions happens in finite time. This situation would require different techniques and is not dealt with in this paper.

2. Qualitative Behavior of System (1) for $\theta = 1$

Throughout this section we will use the following notation for the functional spaces

$$H^k = H^k (I) = \left\{ f \in L^2 (I) : f^{(j)} (a_1) = f^{(j)} (a_2), \quad 0 \leq j \leq k - 1, \quad f^{(j)} \in L^2 (I), 0 \leq j \leq k \right\}$$

where $k = 1, 2, \ldots$. Notice that the definition of these spaces includes the periodicity of the functions. We are not mentioning this periodicity explicitly in the notation for the functional spaces for the sake of shortness.

We will show that the asymptotics for the long time dynamics for the solutions of (1) for $\theta = 1$ are strongly dependent on the choice of initial data. In this section $C$ will always denote a generic constant that can change from line to line. We will show that there exist global smooth solutions for system (1) with periodic boundary conditions. In [13] an $L^\infty$-estimate was proved for this case.

First note that with $z$ being a linear function in $t$, namely $at + b$, and $u$ being the constant $a$, we obtain homogeneous solutions of

$$u_t = u_{xx} - \left( \frac{z_x}{z} \right)_x, \quad z_t = u \quad \text{for} \ t > 0, \ x \in I := [0, \pi].$$

First we will study the stability of these solutions. Using the invariance under time translations of our equations, we can assume for $t \geq 1$ without loss of generality, that $z(x, t) = at$. Our main results of this section state, that for space-independent solutions $(a, at)$ of (1), and solutions $(u, z)$ with initial data $u_0 = u(x, 1), z_0 = z(x, 1)$ which are ‘sufficiently’ close to $(a, a)$, there exists $v_\infty \in H^2$ such that $u$ and $\zeta$ both converge in a suitable sense to $a + v_\infty$ for $t \to \infty$. The exact formulation of our theorems will be given later. We will consider solutions of type

$$u(x, t) = a + v(x, t), \quad z(x, t) = at + \zeta(x, t),$$

and assume that $v_0(x) = v(x, 1)$ and $\zeta_0(x) = \zeta(x, 1)$ are sufficiently small and regular. Since we will use asymptotic arguments, again further details will be specified in a natural manner later. Substituting (2) into (1), we have

$$v_t = v_{xx} - \left( \frac{(a + v) \zeta_x}{at + \zeta} \right)_x = v_{xx} - \left( \frac{\zeta_x}{at + \zeta} \right)_x - \left( \frac{\zeta_x}{at + \zeta} v - \frac{a \zeta_x}{at (at + \zeta)} \right)_x, \quad \zeta_t = v.$$

(3)
This equation will be analyzed by the aid of the Fourier-expansions

\[ v(x, t) = \sum_{n=-\infty}^{\infty} v_n(t) e^{inx}, \quad \zeta(x, t) = \sum_{n=-\infty}^{\infty} \zeta_n(t) e^{inx}. \]  

(4)

We will consider the linearized and the non-linear problem separately.

### 2.1. The Linearized Problem

First we linearize system (3) around the homogeneous solutions and obtain

\[ \frac{\partial v}{\partial t} = \frac{1}{l} \frac{\partial^2 v}{\partial x^2} - \frac{1}{t} \frac{\partial \zeta}{\partial x}, \quad \frac{\partial \zeta}{\partial t} = \frac{1}{l} \frac{\partial^2 \zeta}{\partial x^2}. \]  

(5)

We will show that the space-independent solution \((a, at)\) of (3) is stable on the linearized level, i.e., that perturbations with \(v, \zeta\) stay small in time.

**Theorem 2.1.** Suppose that \((v, \zeta) \in (C((0, \infty), H^2(I)))^2\) satisfies (5) a.e. \((0, \infty) \times I\). There exist \(\epsilon > 0\) and \(\delta = \delta(\epsilon)\) such that for \(\|v(1)\|_{H^2} + \|\zeta(1)\|_{H^2} < \delta\) one obtains \(\|v(t)\|_{H^2} \leq \epsilon\) and \(\|\zeta(t)\|_{H^2} \leq \epsilon t\) for all \(1 \leq t < \infty\). Moreover, there exists \(v_\infty \in H^2\) with

\[ \frac{1}{\pi} \int_0^\pi v_\infty e^{-inx} dx = \zeta_n(1) - (\zeta_n(1) - v_n(1)) \int_1^\infty \frac{e^{s(1-i)}}{s^2} ds \]

such that

\[ \|v(t) - v_\infty\|_{H^2} \to 0, \quad \text{and} \quad \left\| \frac{\zeta(t)}{t} - v_\infty \right\|_{H^2} \to 0 \text{ for } t \to \infty. \]

**Proof.** The Fourier coefficients of \(v\) and \(\zeta\) must satisfy

\[ v_n'(t) = -n^2 v_n(t) + \frac{1}{t} n^2 \zeta_n(t), \quad \zeta_n'(t) = v_n(t). \]

Therefore,

\[ \zeta_n'(t) = -n^2 v_n(t) + \frac{1}{t} n^2 \zeta_n(t). \]

Solving this ODE, we get

\[ \zeta_n(t) = A_n t + B_n t \int_t^\infty \frac{e^{-s} e^{s(1-i)}}{s^2} ds, \quad v_n(t) = A_n + B_n \int_t^\infty \frac{e^{-s}}{s^2} ds - \frac{B_n}{t} e^{-s}. \]  

(6)

where

\[ A_n = \zeta_n(1) - (\zeta_n(1) - v_n(1)) \int_1^\infty \frac{e^{s(1-i)}}{s^2} ds, \quad B_n = e^{\alpha} (\zeta_n(1) - v_n(1)). \]
Formula (6) is valid also for \( n = 0 \), thus \( \zeta_0(t) = A_0 t + B_0 \) and \( v_0(t) = A_0 \). Due to the assumptions on the initial data, we have \( \sum_{n=-\infty}^{\infty} (1 + n^4) \left( |v_n(1)|^2 + |\zeta_n(1)|^2 \right) \leq \delta^2 \), and \( v_n - \zeta_n/t = -B_n e^{-n^2 t}/t \) for all \( n \in \mathbb{Z} \). Thus

\[
\left\| v - \frac{\zeta}{t} \right\|_{H^2}^2 \leq \sum_{n=-\infty}^{\infty} \frac{(1 + n^4) B_n^2 e^{-2n^2 t}}{t^2} = \sum_{n=-\infty}^{\infty} (1 + n^4) |\zeta_n(1) - v_n(1)|^2 \frac{e^{2n^2 (1-t)}}{t^2}
\leq \frac{\| \zeta(1) - v(1) \|_{H^2}^2}{t^2} < \frac{\delta^2}{t^2}.
\]

Further

\[
\sum_{n=-\infty}^{\infty} (1 + n^4) A_n^2 \leq C \sum_{n=-\infty}^{\infty} (1 + n^4) \left( |v_n(1)|^2 + |\zeta_n(1)|^2 \right) \leq C \delta^2,
\]

and for any \( t \) with \( 1 \leq t < \infty \) we have

\[
\sum_{n=-\infty}^{\infty} (1 + n^4) B_n^2 \left( \int_t^\infty \frac{e^{-n^2 s}}{s^2} ds \right)^2 \leq \sum_{n=-\infty}^{\infty} (1 + n^4) (\zeta_n(1) - v_n(1))^2 \left( \int_t^\infty \frac{e^{-n^2 (1-s)}}{s^2} ds \right)^2 \leq \sum_{n=-\infty}^{\infty} (1 + n^4) (\zeta_n(1) - v_n(1))^2 \frac{1}{t^2} \leq \frac{\delta^2}{t^2}.
\]

Summing up all estimates, we obtain

\[
\| v \|_{H^2}^2 \leq \sum_{n=-\infty}^{\infty} (1 + n^4) \left[ A_n^2 + B_n^2 \left( \int_t^\infty \frac{e^{-n^2 s}}{s^2} ds \right)^2 + B_n^2 \frac{e^{-2n^2 t}}{t^2} \right] \leq C \delta^2, \quad 1 \leq t < \infty
\]

and

\[
\| v - v_n \|_{H^2}^2 \leq \sum_{n=-\infty}^{\infty} (1 + n^4) \left[ B_n^2 \left( \int_t^\infty \frac{e^{-n^2 s}}{s^2} ds \right)^2 + B_n^2 e^{-2n^2 t} \frac{1}{t^2} \right] \leq C \frac{\delta^2}{t^2}.
\]

By triangle inequality we also get the second estimate, which completes the proof. \( \square \)

### 2.2. Nonlinear Stability

At the end of this subsection we will state an analogous theorem to the linear case, because first we have to introduce the right notions for convergence.

Let \( f(x, t) \) be the nonlinear part of (3) and let \( f_n \) denote the \( n \)th Fourier coefficient of \( f \), namely

\[
f(x, t) = -\left( \frac{\zeta_x}{at + \zeta} v - \frac{a \zeta_x}{at(at + \zeta)} \right)_x = -\left( \frac{\zeta_x}{at + \zeta} v - \frac{\zeta}{t} \right)_x = \sum_{n=-\infty}^{\infty} f_n(t) e^{inx},
\]
where \( v, \zeta \) are the solutions of (3). Recalling the Fourier expansion of \( v \) and \( \zeta \) in (4) and then comparing Fourier coefficients, we obtain

\[
\zeta''(t) + n^2 \zeta'(t) - \frac{1}{t} n^2 \zeta(t) = f_\nu(t). \tag{7}
\]

By setting \( \zeta_n(t) = t \Phi_n(t) \), we get

\[
\Phi_n'(t) = \frac{e^{-n^2 t}}{t^2} \int_1^t f_n(s) s e^{n^2 s} ds, \quad \text{so} \quad \Phi_n(t) = \int_1^t \frac{e^{-n^2 s}}{\zeta^2} \left( \int_1^s f_n(s) s e^{n^2 s} ds \right) d\zeta.
\]

The general solution of (7) is given by

\[
\zeta_n(t) = A_n t + B_n \int_1^\infty \frac{e^{-n^2 s}}{s^2} ds + t \int_1^t \frac{e^{-n^2 s}}{\zeta^2} \left( \int_1^s f_n(s) s e^{n^2 s} ds \right) d\zeta,
\]

\[
v_n(t) = A_n + B_n \int_1^\infty \frac{e^{-n^2 s}}{s^2} ds - \frac{B_n}{t} e^{-n^2 t} + t \int_1^t \frac{e^{-n^2 s}}{\zeta^2} \left( \int_1^s f_n(s) s e^{n^2 s} ds \right) d\zeta
\]

\[
+ \frac{e^{-n^2 t}}{t} \left( \int_1^t f_n(s) s e^{n^2 s} ds \right),
\]

where

\[
A_n = \zeta_n(1) - B_n \int_1^\infty \frac{e^{-n^2 s}}{s^2} ds, \quad \text{and} \quad B_n = e^{n^2} (\zeta_n(1) - v_n(1)).
\]

In the sequel we assume that the solutions \( \zeta(x, t) \) and \( v(x, t) \) belong to \( H^2 \), and that the initial data are small, i.e.,

\[
\sum_{n=\infty}^\infty (1 + n^4) \left( |v_n(1)|^2 + |\zeta_n(1)|^2 \right) < \epsilon.
\]

Next we introduce the norm

\[
\|\psi\|_{L^2}^2 = \int_{(L-1)^+}^L \|\psi(t)\|_{H^k(t)}^2 dt, \quad k \geq 0,
\]

for \( L > 1, (L-1)^+ = \max\{L-1, 1\} \). Then we obtain the following estimate

**Lemma 2.2.** Let \( \psi \) be sufficiently smooth in \( I \times [(L-1)^+, L] \). Then

\[
\sup_{(L-1)^+ < t < L} \|\psi(t)\|_{H^2} \leq C \left( \|\psi\|_{L^2} + \|\psi_{x}\|_{L^2} \right),
\]

which follows from elementary Sobolev estimates.

With this we can state a local existence result for small solutions.
Qualitative Behavior

251

Proposition 2.3. Suppose that \((v, \zeta) \in \left(C \left((0, \infty), H^2(I)\right) \right)^2\) satisfies (3) a.e. 
\((0, \infty) \times I\). There exist \(\epsilon > 0\) and \(\delta = \delta(\epsilon)\) such that for \(\|v(1)\|_{H^2} + \|\zeta(1)\|_{H^2} < \delta\) 
there exists \(T = T(\epsilon) > 0\) such that

\[
\sup_{L<T} ||v||_{L,2} < \epsilon, \quad \sup_{L<T} ||\zeta||_{L,2} < \epsilon T.
\]

Proof. This proof can be done by standard arguments for local existence, so again 
details are omitted. \(\square\)

Lemma 2.4. Let \(T \in (1, \infty]\) be the time in Proposition 2.3. Then for any \(t < T\)

\[
\|\zeta(t)\|_{L^\infty} + \|\zeta(t)\|_{H^2} \leq C \epsilon t.
\] (10)

Proof. Since \(\|\zeta(t)\|_{L^\infty} \leq C \|\zeta(t)\|_{H^2}\), it is sufficient to estimate \(\|\zeta(t)\|_{H^2}\). Let \(t \in ((L - 1)^+, L)\) where \(L < \min\{T, 2t\}\). Due to Lemma 2.2 and Proposition 2.3 we have

\[
\|\zeta(t)\|_{H^2} \leq C (\|\zeta\|_{L,2} + \|\zeta_t\|_{L,2}) \leq C (\|\zeta\|_{L,2} + \|v\|_{L,2}) \leq C \epsilon L \leq C \epsilon t.
\]

This completes the proof. \(\square\)

Lemma 2.5. Let \(T \in (1, \infty]\) be the time in Proposition 2.3. Then for any \(t < T\),

\[
\left\| v(t) - \frac{\zeta(t)}{t} \right\|_{H^2} \leq C \epsilon e^{-Ct}, \quad \|f(t)\|_{L^2} \leq C \epsilon^2 e^{-Ct}.
\]

Proof. With (10) we can estimate \(f(x, t)\) for \(t \in [1, T]\) as follows

\[
\|f(t)\|_{L^2(t)} \leq \left\| \left( \frac{\zeta}{t} + \frac{\zeta}{t} \right) v - \frac{\zeta}{t} \right\|_{L^2(t)} \leq C \left( \frac{1}{t} \|\zeta\|_{H^2} \left\| v - \frac{\zeta}{t} \right\|_{H^2} + \frac{1}{t^2} \|\zeta_t\|_{H^2} \left\| v - \frac{\zeta}{t} \right\|_{L^2} \right)
\]

\[
\leq C(\epsilon + \epsilon^2) \left\| v - \frac{\zeta}{t} \right\|_{H^2} \leq C \epsilon \left\| v - \frac{\zeta}{t} \right\|_{H^2}.
\] (11)

Since \(u_n' + n^2 v_n - n^2 t \zeta_n = f_n\), we have

\[
\frac{d}{dt} \left( v_n - \frac{\zeta_n}{t} \right) + n^2 \left( v_n - \frac{\zeta_n}{t} \right) + \frac{1}{t} \left( v_n - \frac{\zeta_n}{t} \right) = f_n.
\]

Multiplying with \(n^2(\nu - \frac{\zeta}{t})\), we get

\[
\frac{n^2}{2} \frac{d}{dt} \left( v_n - \frac{\zeta_n}{t} \right)^2 + n^4 \left( v_n - \frac{\zeta_n}{t} \right)^2 + n^2 \frac{d}{dt} \left( v_n - \frac{\zeta_n}{t} \right)^2 = f_n n^2 \left( v_n - \frac{\zeta_n}{t} \right).
\]

This implies that

\[
\frac{1}{2} \frac{d}{dt} \left\| v - \frac{\zeta}{t} \right\|_{H^1}^2 + \| v - \frac{\zeta}{t} \|_{H^2}^2 + \frac{1}{\sqrt{t}} \left\| v - \frac{\zeta}{t} \right\|_{H^1}^2 \leq \|f\|_{L^2} \left\| v - \frac{\zeta}{t} \right\|_{H^2}.
\]
Due to (11), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left\| v - \frac{\zeta}{t} \right\|^2_{H^1} + (1 - C\epsilon) \left\| v - \frac{\zeta}{t} \right\|^2_{H^2} \leq 0. \tag{12}
\]

Hence as long as \( \epsilon \) is sufficiently small, by integrating over \([1, t)\) we obtain
\[
\left\| v(t) - \frac{\zeta(t)}{t} \right\|_{H^1} \leq \left\| v_0 - \frac{\zeta_0}{t} \right\|_{H^1} e^{-Ct} \leq \epsilon e^{-Ct}. \tag{13}
\]
The second estimate of our lemma follows from (11) and (13). This completes the proof. □

Now we can state and prove the main theorem of this subsection.

**Theorem 2.6.** There exist \( \epsilon > 0 \) and \( \delta = \delta(\epsilon) \) such that for \( \|v(1)\|_{H^2} + \|\zeta(1)\|_{H^2} < \delta \) we have \( \|v\|_{L^2} \leq \epsilon \) and \( \|\zeta\|_{L^2} \leq \epsilon t \) for all \( t < \infty \). Moreover, there exists \( v_\infty \in H^2 \) such that
\[
\left\| \frac{\zeta(t)}{t} - v_\infty \right\|_{H^1} \rightarrow 0 \text{ for } t \rightarrow \infty,
\]
\[
\left\| v(t) - v_\infty \right\|_{H^1} \rightarrow 0 \text{ for } t \rightarrow \infty, \text{ and}
\]
\[
\left\| v - v_\infty \right\|_{L^2} \rightarrow 0 \text{ for } L \rightarrow \infty,
\]
where the norm \( \|\cdot\|_{L^2} \) is defined as in (8).

**Proof.** Let \( T \) be the time in Proposition 2.3. We claim that \( T = \infty \). Suppose that this is not the case, i.e., \( T < \infty \). Then either \( \|v\|_{L^2} > \epsilon \) or \( \|\zeta\|_{L^2} > \epsilon T \). Suppose that \( \|v\|_{L^2} > \epsilon \). For the case \( \|\zeta\|_{L^2} > \epsilon T \) we could argue similarly.

Recall the representation formula for \( v_n \)
\[
v_n(t) = A_n + B_n \int_1^t e^{-n^2 s^2} ds - \frac{B_n}{t} e^{-n^2 t}
+ \int_1^t \frac{e^{-n^2 t}}{\xi^2} \left( \int_1^\xi f_n(s)se^{\xi^2 s^2} ds \right) d\xi + \frac{e^{-n^2 t}}{t} \left( \int_1^t f_n(s)se^{\xi^2 s^2} ds \right), \tag{14}
\]
where
\[
A_n = \zeta_n(1) - B_n \int_1^\infty \frac{e^{-n^2 s^2}}{s^2} ds, \quad B_n = e^{n^2} (\zeta_n(1) - v_n(1)).
\]
It is sufficient to consider the nonlinear parts, i.e., the line marked by (14). For simplicity denote
\[
\Psi_n(t) := \int_1^t \frac{e^{-n^2 t}}{\xi^2} \left( \int_1^\xi f_n(s)se^{\xi^2 s^2} ds \right) d\xi + \frac{e^{-n^2 t}}{t} \left( \int_1^t f_n(s)se^{\xi^2 s^2} ds \right) = I_n(t) + II_n(t).
\]
Integrating by parts we obtain
\[ I_n(t) = \frac{e^{-n^2t}}{n^2} \int_1^t f_n(s)se^{n^2s}ds + \frac{1}{n^2} \int_1^t \frac{f_n(\xi)}{\xi} d\xi - \frac{2}{n^2} \int_1^t \frac{e^{-n^2\xi}}{\xi^2} \left( \int_1^\xi f_n(s)se^{n^2s}ds \right) d\xi. \]

Explicit computations show that
\[ \int_{(T-1)^+}^T \sum_{n=-\infty}^{\infty} n^4I_n^2(\tau) d\tau \leq C \int_{(T-1)^+}^T \sum_{n=-\infty}^{\infty} f_n^2(s)ds \]
\[ = C \|f\|^2_{L^2(Q_T)} \leq Ce^4 \int_1^T e^{-ct} dt \leq Ce^4. \]

Next consider
\[ |I_n(t)| \leq \frac{e^{-n^2t}}{t} \left( \int_1^t |f_n(s)|se^{n^2s}ds \right) \leq \int_1^t |f_n(s)|e^{n^2(t-s)}ds =: y(t). \]

Since the right-hand side is a solution of \( y'(t) + n^2y(t) = |f_n| \), one can estimate
\[ \int_{(T-1)^+}^T \sum_{n=-\infty}^{\infty} n^4I_n^2(\tau) d\tau \leq C \int_{(T-1)^+}^T \sum_{n=-\infty}^{\infty} f_n^2(s)ds = C \|f\|^2_{L^2(Q_T)} \leq Ce^4. \] (15)

This can be seen by defining \( y_n(t) = \int_1^t |f_n(s)|e^{n^2(t-s)}ds \). Thus \( y_n(t) + n^2y_n(t) = |f_n(t)| \). Let
\[ Y(x, t) = \sum_{n \neq 0} y_n(t)e^{inx}, \quad F(x, t) = \sum_{n \neq 0} |f_n(t)|e^{inx}. \]

Then for \( t \geq 1 \), \( y \) solves the following equation in \( I = [0, \pi] \)
\[ Y = Y_{xx} + F, \quad Y(x, 1) = 0. \]

Further, standard estimates, using the fact that we do not have a neutral eigenvalue, yield
\[ \|Y(\cdot, t)\|^2_{L^2} \leq C \int_1^T \|F(\cdot, t)\|^2_{L^2} dt. \]

With classical regularity theory for the heat equation we obtain
\[ \int_{(T-1)}^T \|Y(\cdot, t)\|^2_{H^2} \leq C \sup_{t-1 \leq s \leq t} \|Y(\cdot, s)\|^2_{H^2} + C \int_{(T-1)}^T \|F(\cdot, t)\|^2_{L^2} dt \]
\[ \leq C \int_1^T \|F(\cdot, t)\|^2_{L^2} dt. \]

Since
\[ \sum_n n^4I_n^2 = \sum_{n \neq 0} n^4|y_n(t)|^2 = \|Y(\cdot, t)\|^2_{H^2}, \]
estimate (15) follows. Summing up, we obtain
\[ ||v||_{T,2} \leq [\text{linear terms}] + \left( \int_{(T-1)^+}^{T} \sum_{n=-\infty}^{\infty} n^4 I_n^2(\tau) d\tau \right)^{\frac{1}{2}} + \left( \int_{(T-1)^+}^{T} \sum_{n=-\infty}^{\infty} n^4 \|I_n^2(\tau)\| d\tau \right)^{\frac{1}{2}} \]
\[ \leq C\delta + Ce^2. \]

This shows that \[ ||v||_{T,2} < \epsilon, \] which contradicts our hypothesis. Thus, \( T \) cannot be finite.

Next we show convergence. We will prove that \( I_n(t) = \zeta_n / t - A_n \in H^2 \) and \( I_n(t) \leq Ce^2 \) for all \( t \). By changing the order of integration and using Hölder’s inequality, we obtain
\[ |I_n(t)| = \left| \int_{1}^{t} f_n(s)e^{s^2} \int_{1}^{t} e^{-s^2} d\xi ds \right| \leq \left| \int_{1}^{t} |f_n(s)| s e^{s^2} ds \right| \left| \int_{1}^{t} e^{-s^2} d\xi ds \right| \]
\[ \leq C \int_{1}^{t} \frac{|f_n(s)| s ds}{s} \leq C \left( \int_{1}^{t} |f_n| e^{s^2} ds \right)^{\frac{1}{2}} \left( \int_{1}^{t} s^{-2} ds \right)^{\frac{1}{2}} \leq C \int_{1}^{t} |f_n| e^{s^2} ds. \]

Due to Lemma 2.5, we have that \( n^4 |I_n(t)| \leq Ce^4 \) for all \( t \). This implies that \( \zeta / t \) is in \( H^2 \) and converges to \( v_\infty \in H^2 \) for \( t \to \infty \), with
\[ v_\infty = \sum (v_\infty)_n e^{i\xi_n}, \] where \( (v_\infty)_n = A_n + I_n(\infty) \), and
\[ I_n(\infty) = \int_{1}^{\infty} e^{-s^2} \left( \int_{1}^{s} f_n(s)e^{s^2} ds \right) d\xi. \]

Now it is direct that
\[ \frac{\zeta}{t} \to v_\infty \text{ in } H^2 \text{ for } t \to \infty. \] (16)

From (12), we have
\[ \left\| v(t) - \frac{\zeta(t)}{t} \right\|_{H^1} \to 0 \text{ for } t \to \infty, \quad \left\| v(t) - \frac{\zeta(t)}{t} \right\|_{L,2} \to 0 \text{ for } L \to \infty. \] (17)

Combining (16) and (17), we obtain that \( v \) converges to \( v_\infty \) in the \( H^1 \) and the \( (L, 2) \)-norm, as defined in (8), since
\[ \| v - v_\infty \|_{H^1} + \left\| v - v_\infty \right\|_{L,2} \]
\[ \leq \left\| v - \frac{\zeta}{t} \right\|_{H^1} + \left\| \frac{\zeta}{t} - v_\infty \right\|_{H^1} + \left\| v - \frac{\zeta}{t} \right\|_{L,2} + \left\| \frac{\zeta}{t} - v_\infty \right\|_{L,2} \]
\[ \leq \left\| v - \frac{\zeta}{t} \right\|_{H^1} + \left\| v - \frac{\zeta}{t} \right\|_{L,2} + \left\| \frac{\zeta}{t} - v_\infty \right\|_{L,2}. \]

This completes the proof. \( \square \)
3. Qualitative Behavior of System (1) for $1 < \theta < 3$

In this section let $I = [-1, 1]$. The reason for this change of domain of integration is to fix the expected singularity at the origin and thus to avoid dealing with complicated shifts of its location. We consider

$$u_t = u_{xx} - \theta \left( \frac{z}{z} u \right)_x, \quad z_t = u \text{ in } I \times [0, \infty),$$

$$u(x, 0) = u_0(x), \quad z(x, 0) = z_0(x), \text{ with periodic boundary conditions.}$$

**Heuristics**

First, to get an intuitive insight, we give a heuristic argument regarding the blow-up asymptotics for this system for $t \to \infty$. After this we will go into the details of the rigorous analysis.

By a rescaling argument we can assume w.l.o.g. that $\int_I u \, dx = 1$. Then consider the simplified equation

$$\bar{z}_t = \frac{\bar{z}^\theta}{\int_I \bar{z}^\theta \, dx},$$

which results from setting $u_t = 0$, then solving the resulting equation for $u$, plugging the solution into the second equation and keeping the normalization condition in mind. Thus we expect this simplified equation to be a good approximation for the dynamics of the original problem for $t \to \infty$. Assuming that $z_0(0) > z_0(x)$ for any $x \in I \setminus \{0\}$ we can solve this equation and obtain

$$\bar{z}^{1-\theta}(x, t) = z_0^{1-\theta}(x) - (\theta - 1) \int_0^t \frac{ds}{\int_I \bar{z}^{\theta}(x, s) \, dx}.$$

We assume further that $z_0$ can be expanded near zero as follows

$$\bar{z}_0^{1-\theta}(x) = \bar{z}_0^{1-\theta}(0) + Bx^2 + h.o.t. \text{ for } x \to 0.$$

Here $B$ is a positive constant depending on the initial data. Continuing the heuristic argument, we thus obtain

$$\bar{z}^{1-\theta}(x, t) \approx \bar{z}_0^{1-\theta}(0) + Bx^2 - (\theta - 1) \int_0^t \frac{ds}{\int_I \bar{z}^{\theta}(x, s) \, dx}.$$

Define

$$\psi(t) := \bar{z}_0^{1-\theta}(0) - (\theta - 1) \int_0^t \frac{ds}{\int_I \bar{z}^{\theta}(x, s) \, dx}.$$

Thus $\bar{z}^{1-\theta}(x, t) \approx Bx^2 + \psi(t)$, and

$$\bar{z}(x, t) \approx \frac{1}{(Bx^2 + \psi(t))^{1/\theta}}.$$
Explicit computations show

\[-\frac{\theta - 1}{\psi'(t)} \approx \int \frac{dx}{t (Bx^2 + \psi(t))^{\frac{\theta}{\mu}}}.
\]

So we get \(\psi'(t) \approx -K\psi^\frac{\theta-1}{\theta} (t)\), where \(K\) is a positive constant. This yields \(\psi(t) \approx At^{-\frac{2(\theta-1)}{\theta}}\) with a constant \(A > 0\) for \(t \to \infty\). Since \(\psi(t) \to 0\) for \(t \to \infty\), we see that

\[t^\frac{\theta}{\mu} \approx -\int_{I} z^\frac{\theta}{\mu} (x,t) \, dx.
\]

Therefore, noting that \(\psi'(t) \approx -KA^{-\frac{\theta-1}{\theta}} t^{-\frac{\theta-1}{\theta}}\) for \(t \to \infty\), we obtain

\[\psi(t) \approx (\theta - 1) \int_{0}^{\infty} \frac{ds}{\int_{I} z^\theta(x,s) \, dx} \approx \frac{\theta - 1}{KA^{-\frac{\theta-1}{\theta}}},
\]

and

\[z(x,t) \approx \frac{1}{\left(Bx^2 + At^{-\frac{2(\theta-1)}{\theta}}\right)^{\frac{1}{\theta}}} = \frac{t^{\frac{2}{\theta}}}{\left(Bx^2 t^{\frac{2(\theta-1)}{\theta}} + A\right)^{\frac{1}{\theta}}},
\]

which ends our heuristic arguments.

Now we are ready to present rigorous arguments which justify the above given heuristics. A first idea for a quasi-steady state approximation of the system under consideration for \(1 < \theta < 3\) was given by Schwetlick [10]. In this section we will prove the following main theorem.

**Theorem 3.1.** There exist a family of initial data \(u_0, z_0 \in C^2,\tau\) such that the corresponding solutions \((u, z)\) of (1) satisfy \(u(x, t) \to \mu \delta(x)\) in the sense of measures, where \(\mu = \int_{I} u_0(x) \, dx\). Moreover, the following asymptotic formula hold for \(z(x, t)\)

(i) \(z(x, t) \to z_\infty(x)\) as \(t \to \infty\) uniformly in compact sets of \(I \setminus \{0\}\) for some function \(z_\infty \in C^2(I \setminus \{0\})\) which satisfies \(\lim_{r \to 0} x^\frac{\theta}{\mu} z_\infty(x) = B^{-\frac{1}{\mu^2}}\).

(ii) There exist constants \(A, B\) depending on the initial data such that

\[\lim_{t \to \infty} \frac{z\left(\frac{y}{t^{\frac{1}{1+\mu}}}, t\right)}{t^{\frac{2}{\mu}}} = \frac{1}{(By^2 + A)^{\frac{1}{\mu^2}}},
\]

uniformly in any compact sets \(\{y : |y| \leq C\}\).

**Remark.** As we will see in Assumption 3.5 later, the conditions on the initial data are, that \(u_0, z_0\) are symmetric, \(u_0\) is concentrated at the origin, and \(z_0\) behaves like a power law at the origin. For convenience we will also assume in the following w.l.o.g. that \(\mu = 1\).

To prove this theorem we need several steps. First we consider:
3.1. The Eigenvalue Problem

We define the differential operator

\[ \tilde{A}_\epsilon(f) := f_{xx} - \theta \left( \frac{z_x f}{z} \right) x = \left( f_x - \theta \left( \frac{z_x f}{z} \right) \right) x \quad \text{in } [-1, 1]. \]

Consider the eigenvalue problem \( \tilde{A}_\epsilon(f) = \lambda f \), i.e.,

\[ f_{xx} - \theta \left( \frac{z_x f}{z} \right) x = \lambda f. \]

Since we assumed periodic boundary conditions in \( \mathbb{R} \), we obtain \( f(-1) = f(1) \) and \( f'_x(-1) = f'_x(1) \). Now a set of functions \( \mathcal{U} \) is introduced, which is assumed to contain \( z \).

**Assumption 3.2.** Let \( 0 < \nu < 1 \) and let \( \mathcal{U} \) be a set of nonnegative functions such that for \( g \in \mathcal{U} \) the following conditions hold

1. \( g(\cdot, t) \in \mathbb{R}^{2\nu} \) is nonnegative and symmetric with respect to zero, i.e., \( g(-x, t) = g(x, t) \). Furthermore, there exists \( M > 0 \) such that

\[
T^{-\frac{1}{2}} \sup_{t} \left[ T^{-\frac{1}{2}} \sup_{|x_1|, |x_2| \leq T} \frac{|g_{xx}(x_1, t) - g_{xx}(x_2, t)|}{|x_1 - x_2|} \right] 
+ \sup_{t} \left[ \sup_{R \leq 1} R \frac{1}{2} \left( \sup_{R \leq 1} \frac{|g_{xx}(x_1, t) - g_{xx}(x_2, t)|}{|x_1 - x_2|} \right) \right] \leq M. \tag{18}
\]

2. There exist \( A, B, M > 0 \) such that

\[
\frac{t^{\frac{1}{2}}}{M \left( B x^{2(\nu-1)} + A \right)^{\frac{1}{2\nu}}} \leq g(x, t) \leq \frac{M t^{\frac{1}{2}}}{\left( B x^{2(\nu-1)} + A \right)^{\frac{1}{2\nu}}}. \tag{19}
\]

3. There exist \( A, B, M > 0 \) such that

\[
|g_x(x, t)| \leq M \frac{x^{\frac{2\nu}{2\nu-1}}}{\left( B x^{2(\nu-1)} + A \right)^{\frac{1}{2\nu}}}. \tag{20}
\]

4. There exists \( \epsilon_0 > 0 \) such that

\[
\left| \frac{g_x(x, t) - 1}{g(x, t)} \right| \leq \frac{\epsilon_0}{|x|}, \quad \left| \frac{g_x(x, t) - 1}{(\theta - 1) x} \right| \leq \frac{\epsilon_0}{|x|^2}.
\]

From now on, and in difference to the previous section, the appearing constants \( C = C(\theta, M) \) will depend on \( \theta \) and on \( M \). The same holds for the constants denoted by \( C_\delta, C_\gamma \).
Lemma 3.3. The operator $\tilde{A}_\cdot(\cdot)$ is self-adjoint with respect to a weighted integral with weight $\frac{dx}{x^\sigma}$. All eigenvalues are non-positive and the first eigenvalue $\lambda_0$ equals 0 with corresponding eigenfunction $z^0$.

Proof. We know that $h_x - \theta_x h = (\frac{1}{x^\sigma})_x x^\sigma$ for any $h$ and

$$\int f \tilde{A}_\cdot(f) \frac{dx}{x^\sigma} = -\int \left( f_x - \theta_x f \right) \frac{g}{x^\sigma} dx = -\int f \tilde{A}_\cdot(g) \frac{dx}{x^\sigma}.$$ 

It follows from standard arguments that all eigenvalues are non-positive (compare [1]) and that $z^0$ is an eigenvector corresponding to the eigenvalue 0. □

Proposition 3.4. Let $\lambda_1(t)$ be the second eigenvalue for the differential operator $\tilde{A}_\cdot(\cdot)$. Suppose that $z(\cdot, t) \in \mathcal{M}$ with $\mathcal{M}$ as in Assumption 3.2. Then there exists an absolute constant $C > 0$ independent of $z$ such that

$$\lambda_1(t) \leq -C \quad \text{for all } t.$$ 

Proof. Suppose that this is not the case. Then there exist a sequence of $t_m$, functions $z_m(\cdot, t_m) \in \mathcal{M}$, and eigenvalues $\lambda_{1,m}(t_m) \not\to 0$ for $t_m \to t_\infty$ (with $t_\infty$ being either finite or infinite), and corresponding eigenfunctions $\phi_{1,m}$ such that

$$\tilde{A}_{z_m}(\phi_{1,m}) = (\phi_{1,m})_{xx} - \theta \frac{(z_m)_x}{z_m} \phi_{1,m} = \lambda_{1,m} \phi_{1,m}.$$ 

Here we assume that the eigenfunction $\phi_{1,m}$ is normalized i.e., $\int \left| \phi_{1,m} \right|^2 \frac{dx}{z_m} = 1$.

- If $t_\infty < \infty$, then by Assumption 3.2 we have that $\|z_m\|_{L^2}$ is uniformly bounded, and $z_m$ converges to $z_\infty$ in $C^2$. Classical regularity theory, c.f. [2, Chapter 6], implies that $z_m \in C^{2,v}$ and $\|\phi_m\|_{L^2} \leq C$ for all $m$. Due to Sturm–Liouville theory, the eigenfunctions $\phi_{1,m}$ satisfy $\phi_{1,m}(0) = 0$ and $\phi_{1,m}(x) > 0$ for $x \in (0, 1)$. In addition, there exists $\phi_{1,\infty}$ such that $\phi_{1,m} \to \phi_{1,\infty}$ in $C^2$. Then the limiting equation becomes

$$\tilde{A}_{z_\infty}(\phi_{1,\infty}) = (\phi_{1,\infty})_{xx} - \theta \frac{(z_\infty)_x}{z_\infty} \phi_{1,\infty} = 0.$$ 

This equation can be solved explicitly and we obtain

$$\phi_{1,\infty}(x) = K z_\infty^0(x) + C \int_0^x \frac{z_{\infty}^0(\xi)}{z_\infty^0(\xi)} d\xi.$$ 

Since $\phi_{1,\infty}$ is periodic and $z_\infty \geq 0$, the integral term above must vanish, and thus $\phi_{1,\infty}(x) = K z_\infty^0(x)$. This yields $K = 0$, because $\phi_{1,\infty}(0) = 0$ and $z_\infty(0) > 0$.

Hence $\phi_{1,\infty} = 0$, which contradicts the fact that $\int \left| \phi_{1,\infty} \right|^2 \frac{dx}{z_\infty} = 1$.

- The case $t_\infty = \infty$. For any $0 < \delta < 1$ we note that $\|z_m\|_{L^2([1-\delta, 1])} \leq C_\delta$. Let $\delta_0 > 0$ be sufficiently small. Let $\psi(x) = C_\gamma x^\gamma$ with $\gamma > 1$, where $C_\gamma$ is a constant satisfying $C_\gamma \delta_0 = C_\delta$ in $[0, \delta_0]$. We show that $\psi \geq \phi_{1,m}$ for all $m$.
Indeed, for sufficiently small \( \epsilon = \epsilon(\gamma) \) we have
\[
\begin{align*}
\widetilde{A}_{z_m}(\psi) &= \gamma(\gamma - 1)x^{\gamma-2} - \frac{\theta}{\theta - 1}(\gamma - 1)x^{\gamma-2} - \theta \left( \frac{\left( \frac{z_m}{\psi} \right) (\gamma - 1)}{\theta - 1} \right) \psi \\
&= (\gamma - 1) \left( \gamma - \frac{\theta}{\theta - 1} \right) x^{\gamma-2} - \theta \left( \frac{z_m}{\psi} \right) x^{\gamma-2} - \theta \left( \frac{1}{\theta - 1} \right) \psi \\
&- \theta \left( \frac{z_m}{\psi} \right) x - \frac{1}{\theta - 1} \psi \\
&\leq (\gamma - 1) \left( \gamma - \frac{\theta}{\theta - 1} \right) x^{\gamma-2} + \frac{\epsilon}{\theta} x^2 + \theta \gamma \epsilon x^{\gamma-1} \\
&= (\gamma - 1) \left( \gamma - \frac{\theta}{\theta - 1} + \theta \epsilon \frac{1 + \gamma}{\gamma - 1} \right) x^{\gamma-2}.
\end{align*}
\]

So \( \widetilde{A}_{z_m}(\psi) \leq 0 \) in \([0, \delta_0]\) for \( 1 < \gamma < \frac{\theta}{\theta - 1} \) and \( \psi \) is a super-solution of \( \phi_{1,m} \) for all \( m \), due to the maximum principle,
\[
|\phi_{1,m}(x)| \leq C \gamma |x|, \quad 0 \leq x \leq \delta_0.
\]

There exists \( \phi_{1,\infty} \) such that \( \phi_{1,m} \to \phi_{1,\infty} \) uniformly in \( \mathbb{R}^2[\delta, 1] \) for any \( 0 < \delta < 1 \) and thus the limiting equation becomes
\[
\widetilde{A}_{z_\infty}(\phi_{1,\infty}) = (\phi_{1,\infty})_{xx} - \theta \left( \frac{z_{\infty}}{\psi} \right) \phi_{1,\infty} = 0.
\]

As in the previous case, this leads to a contradiction and completes the proof. \( \square \)

For convenience, we denote \( \langle g, h \rangle = \int_I g(x)h(x) \frac{dx}{z^0} \) for functions \( g \) and \( h \) which are integrable with respect to \( dx/z^0 \). We define \( v \) in terms of \( u := \frac{\partial v}{\partial z^0} + \bar{v} \). Note that \( \int_I v \, dx = 0 \) and \( \langle z^0, v \rangle = 0 \). Furthermore, \( v \) solves
\[
v_i = v_{xx} - \theta \left( \frac{z}{z^0} \right) x - \frac{z^0}{\int_{z} z^0 dx} \bar{v}.
\] (21)

For simplicity we denote
\[
\mathcal{R}(x, t) := -\left( \frac{z^0}{\int_I z^0 dx} \right) - \theta \frac{z^{\theta - 1} u}{\int_I z^0 dx} + \theta \frac{z^0 \int z^{\theta - 1} u dx}{(\int_I z^0 dx)^2}.
\] (22)

First we will make \textit{a priori} assumptions on \( v \), which will be shown to be fulfilled in Lemma 3.11.

\textbf{Assumption 3.5.} Let \( z(\cdot, t) \in \mathcal{D} \), where \( \mathcal{D} \) satisfies Assumption 3.2. Further suppose that
\[
|v(x, t)| \leq M \frac{z^0(x, t)}{\int_I z^0(y, t) dy},
\]
for a suitable constant \( M > 0 \).
Let us first give a technical lemma, which provides the main estimate to deduce a Sobolev inequality with weighted norm in Lemma 3.7. This technical lemma is an adaptation of a result given in [8] and provides one of the main estimates for the result stated thereafter. In this paper we use a formulation in [5] of the result in [8], which is more accessible. The proof of the following Lemma 3.6 will be given later.

**Lemma 3.6.** Let \( 1 < \theta < 3 \). Suppose that \( \zeta \in C^1([0,1]) \) with \( \zeta(0) = 0 \), and let \( z(,t) \in \mathcal{A} \), for all \( t \geq 1 \), with \( \mathcal{A} \) as in Assumption 3.2, for all \( t \geq 1 \). Then

\[
\left( \int_0^1 z^{(\theta-1)\theta} |\zeta|^p \, dx \right)^{\frac{1}{p}} \leq C \left( \int_0^1 z^\theta |\zeta|^2 \, dx \right)^{\frac{1}{2}}, \quad \text{for } p = \frac{6\theta - 2}{\theta + 1}.
\]

With this result we prove a Sobolev inequality with a weighted norm, where the weight is \( z^{-\theta} \).

**Lemma 3.7.** Suppose that \( z(,t) \in \mathcal{A} \), where \( \mathcal{A} \) satisfies Assumption 3.2. Let \( h \in L^2(z^{-\theta} \, dx) \) with \( \int_I h = 0 \), where \( I = [-1,1] \). Then

\[
\left( \int_I |h|^p \, dx \right)^{\frac{1}{p}} \leq C \left( \int_I \left| h - \theta \frac{z}{z} h \right|^2 \, dx \right)^{\frac{1}{2}}, \quad \text{for } p = \frac{6\theta - 2}{\theta + 1}. \tag{23}
\]

Here \( C \) is an absolute constant independent of \( t, z, \) and \( \theta \), but it depends on \( M \).

**Proof.** We consider the following variational problem

\[
- \left( h - \theta \frac{z}{z} h \right)_x = \lambda \left( \int_I |h|^p - z^\theta \frac{z}{z} h \, dx \right)
\]

with \( \int_I |h|^p \, dx \mid_z = 1 \). Due to Assumption 3.2, \( z(,t) \) is non-singular for every finite \( t \), and thus classical theory for semi-linear elliptic boundary value problems with constraints, compare [12], implies that there exists \( \lambda(t) > 0 \) such that

\[
\left( \int_I |h|^p \, dx \right)^{\frac{1}{p}} \leq \frac{1}{\lambda(t)} \left( \int_I \left| h - \theta \frac{z}{z} h \right|^2 \, dx \right)^{\frac{1}{2}}.
\]

Our goal is to show that there exists \( k > 0 \) such that \( \lambda(t) \geq k \) for all \( t \geq t_0 \). Suppose that this not the case. Then there exist \( t_n, z_n, h_n \) with \( \int_I |h_n|^p \, dx \mid_z = 1 \) and \( \lambda_n(t_n) \downarrow 0 \) for \( t_n \to \infty \), possibly after choosing a suitable subsequence, such that

\[
\lambda_n = \int_I \left| (h_n) - \theta \frac{z_n}{z_n} h_n \right|^2 \, dx \mid_{z_n^p}
\]

We introduce a new function \( \varphi_n := \frac{h_n}{z_n^{\theta}} \). Then, since we want to minimize the constant \( C \) in (23), the given problem can be rewritten as an eigenvalue problem

\[
- \left( z^{(\theta-1)\theta} \varphi_n \right)_x = \lambda_n \left( \int_I z^{(\theta-1)\theta} |\varphi_n|^{p-2} \varphi_n - \frac{\int_I z^{(\theta-1)\theta} |\varphi_n|^{p-2} \varphi_n \, dx}{\int_I z^{(\theta-1)\theta} \, dx} \right),
\]
because

\[ \int |h_n|^p \frac{dx}{z_n} = \int |\varphi_n|^p z_n^{(p-1)\theta} dx. \]  

(24)

Additionally, the normalization and orthogonality condition have to be fulfilled, namely

\[ \int \frac{z_n}{|\varphi_n|} \varphi_n dx = 1, \quad \int \varphi_n^2 dx = 1 \quad \text{and} \quad \lambda_n = \int |(\varphi_n)|^2 dx. \]

Now expressing \( \varphi_n = \varphi_n(0) + \psi_n \) we can estimate (24) by

\[ \int |\varphi_n|^p z_n^{(p-1)\theta} dx \leq \int |\varphi_n(0, t)|^p z_n^{(p-1)\theta} dx + \int |\psi_n(x, t)|^p z_n^{(p-1)\theta} dx. \]  

(25)

To control the first term on the right-hand side of (25), we estimate \( |\varphi_n(0, t)| \). First, by Hölder’s inequality we obtain

\[ |\varphi_n(x, t)| - |\varphi_n(0, t)| \leq \left( \int_0^x |(\varphi_n)|^2 d\xi \right)^{\frac{1}{2}} \left( \int_0^x \frac{d\xi}{z_n^0} \right)^{\frac{1}{2}} \leq \sqrt{\lambda_n} \left( \int_0^x \frac{d\xi}{z_n^0(\xi, t)} \right)^{\frac{1}{2}}. \] 

(26)

Therefore, we have

\[ |\varphi_n(x, t)| \leq |\varphi_n(0, t)| + \sqrt{\lambda_n} \left( \int_0^x \frac{d\xi}{z_n^0(\xi, t)} \right)^{\frac{1}{2}}. \] 

(27)

Due to (26), we get \( |\psi_n| \leq \sqrt{\lambda_n} \left( \int_0^x \frac{d\xi}{z_n^0(\xi, t)} \right)^{\frac{1}{2}} \). Since \( \int z_n^0 \varphi_n dx = \varphi_n(0, t) \int z_n^0 dx + \int z_n^0 \psi_n dx \), we obtain due to (27)

\[ |\varphi_n(0, t)| = \left| \int z_n^0 \psi_n dx \right| \leq \sqrt{\lambda_n} \int z_n^0 \left( \int_0^x \frac{d\xi}{z_n^0(\xi, t)} \right)^{\frac{1}{2}} dx \leq C \sqrt{\lambda_n} t^{\frac{1}{2(n+1)}}. \]

Here we used that \( \int z_n^0 \left( \int_0^x \frac{d\xi}{z_n^0(\xi, t)} \right)^{\frac{1}{2}} dx \leq C. \) Due to Assumption 3.2, 2., and because \( \int (y^2 + a)^{-\frac{(p-1)(n+1)}{2(n-1)}} dy < \infty \) we compute

\[ \int z_n^{(p-1)\theta} |\varphi_n(0)|^p dx \leq C \lambda_n^{\frac{p}{2}} t^{-\frac{n+1}{4}} \int z_n^{(p-1)\theta} dx \leq C \lambda_n^{\frac{p}{2}} t^{-\frac{n+1}{4}}. \]
Now we estimate the second term in (25). This is done by Lemma 3.6. For $p > 2$ we obtain
\[
\left( \int_I z_n^{(p-1)\theta} |\varphi_n|^p \, dx \right)^{\frac{1}{p}} \leq \left( \int_I z_n^{(p-1)\theta} |\varphi_n(0, t)|^p \, dx \right)^{\frac{1}{p}} + \left( \int_I z_n^{(p-1)\theta} |\psi_n|^p \, dx \right)^{\frac{1}{p}}
\]
\[
\leq C \hat{\beta}_a t^{-\frac{\theta}{2p-1}} + C \left( \int_I z_n^\theta |(\psi_n)_x|^2 \, dx \right)^{\frac{1}{2}} \leq C \hat{\beta}_n \to 0 \quad \text{for} \ n \to \infty.
\]
This is a contradiction to our hypothesis and thus completes the proof of Lemma 3.7. \qed

Now we give the proof of our technical Lemma 3.6. If $z$ would behave like a power law, we could have mainly used the estimate given in [5] to obtain our result. But unfortunately this is not the case everywhere, so that we have to introduce boundary layer estimates.

**Proof of Lemma 3.6.** For convenience, we denote $\alpha = \frac{\theta - 1}{2p-1}$. First the contributions where $z$ is large are analyzed. For this, as can be seen from Assumption 3.2, we have to look at a specific domain of integration. So we show for a smooth function $\zeta$ with $\zeta(0) = 0$ that
\[
\left( \int_0^{2r^{-\alpha}} z^{(p-1)\theta} |\zeta|^p \, dx \right)^{\frac{1}{p}} \leq C \left( \int_0^{2r^{-\alpha}} z^\theta |\zeta_x|^2 \, dx \right)^{\frac{1}{2}}.
\]
Indeed, due to Assumption 3.2 and with the change of variables $y = r^x$ and $\zeta,y = \zeta(r^{-x}y)$, we have
\[
\left( \int_0^{2r^{-\alpha}} z^{(p-1)\theta} |\zeta|^p \, dx \right)^{\frac{1}{p}} \leq \left( \int_0^{2r^{-\alpha}} \frac{t^{2p-1p\theta}}{(x^2 + a)^{\frac{p+1p\theta}{2p}}} |\zeta|^p \, dx \right)^{\frac{1}{p}}
\]
\[
= r^{\frac{2p-1-p\theta}{p}} \left( \int_0^{2r^{-\alpha}} \frac{1}{2y^2 + a} \frac{|\zeta|^p}{y^2} \, dy \right)^{\frac{1}{p}}
\]
\[
\leq C r^{\frac{2p-1-p\theta}{p}} \left( \int_0^{2r^{-\alpha}} \frac{1}{2y^2 + a} \frac{|\zeta_x|^2}{y^2} \, dy \right)^{\frac{1}{2}}
\]
\[
\leq C r^{\frac{2p-1-p\theta}{p}} \left( \int_0^{2r^{-\alpha}} z^\theta |\zeta_x|^2 \, dx \right)^{\frac{1}{2}}
\]
since $\frac{2p-1-p\theta}{p} - \frac{\theta}{2p-1} - \frac{1}{2} = 0$. Next we do a further splitting
\[
\zeta(x, t) = \eta(xr^x)\zeta(x, t) + (1 - \eta(xr^x))\zeta(x, t) := \tilde{\zeta}(x, t) + \hat{\zeta}(x, t),
\]
where $\eta$ is a standard cut-off function such that $\eta(y) = 1$ for $y \leq 1$ and $\eta = 0$ if $y \geq 2$. Then, since $\tilde{\zeta}$ is supported in $[0, 2r^{-2}]$ and $\tilde{\zeta}(0) = 0$, using (28), we get

$$\left( \int_0^1 \varepsilon^{(p-1)\theta} \left| \tilde{\zeta}_s \right|^p dx \right)^{\frac{1}{p}} \leq C \left( \int_0^1 \varepsilon^{\theta} \left| \tilde{\zeta}_s \right|^2 dx \right)^{\frac{1}{2}}.$$  

Because $\tilde{\zeta}_x = \eta(x^2)\tilde{\zeta}_x + t^2\eta'(x^2)\tilde{\zeta}$, we obtain

$$\left( \int_0^1 \varepsilon^{(p-1)\theta} \left| \tilde{\zeta} \right|^p dx \right)^{\frac{1}{p}} \leq C \left( \int_0^1 \varepsilon^{\theta} \left| \tilde{\zeta}_x \right|^2 dx \right)^{\frac{1}{2}} + C \left( \int_0^1 \varepsilon^{\theta} t^{2s} |\eta| \tilde{\zeta}^2 dx \right)^{\frac{1}{2}}. \quad (29)$$

We now give an estimate for the second term on the right-hand side of (29). Noting that $\eta'$ is supported in $(r^{-2}, 2r^{-2})$, we compute

$$\left( \int_0^1 \varepsilon^{\theta} t^{2s} |\eta| \tilde{\zeta}^2 dx \right)^{\frac{1}{2}} \leq \left( \int_{1-r^{-2}}^{2r^{-2}} \frac{t^{2n}}{(x^{2s} + a)^{\frac{1}{2}}} t^{2s} |\zeta| \tilde{\zeta}^2 dx \right)^{\frac{1}{2}}$$

$$= t^{\frac{2n}{a} + \frac{1}{2}} \left( \int_0^2 \frac{1}{(y^2 + a)^{\frac{1}{2}}} |\zeta| \tilde{\zeta}^2 dy \right)^{\frac{1}{2}}$$

$$\leq t^{\frac{2n}{a} + \frac{1}{2}} \left( \int_0^2 \frac{1}{(y^2 + a)^{\frac{1}{2}}} |\zeta| \tilde{\zeta}^2 dy \right)^{\frac{1}{2}}$$

$$\leq C t^{\frac{2n}{a} + \frac{1}{2}} \left( \int_0^2 \frac{1}{(y^2 + a)^{\frac{1}{2}}} |\zeta| \tilde{\zeta}^2 dy \right)^{\frac{1}{2}}$$

$$\leq C \left( \int_0^1 \varepsilon^{\theta} \left| \tilde{\zeta}_x \right|^2 dx \right)^{\frac{1}{2}}, \quad (30)$$

where we used $\psi_n(0) = 0$. Combining (29) and (30), we obtain

$$\left( \int_j \varepsilon^{(p-1)\theta} \left| \tilde{\zeta} \right|^p dx \right)^{\frac{1}{p}} \leq C \left( \int_j \varepsilon^{\theta} \left| \tilde{\zeta}_x \right|^2 dx \right)^{\frac{1}{2}}.$$  

It remains to show that

$$\left( \int_j \varepsilon^{(p-1)\theta} \left| \tilde{\zeta} \right|^p dx \right)^{\frac{1}{p}} \leq C \left( \int_j \varepsilon^{\theta} \left| \tilde{\zeta}_x \right|^2 dx \right)^{\frac{1}{2}}. \quad (31)$$

Keeping in mind that $\tilde{\zeta}$ vanishes in $[0, r^{-2}]$, we note that $\varepsilon$ is comparable to $|x|^{\frac{-2n}{a}}$. We know that $|z| \leq C|x|^{\frac{1}{2n}}$. On the other hand, since $x > t^{-2}$, we have $|x|^{\frac{2n}{a}} \leq C(x^2 + t^{-2})^{-\frac{1}{2n}} \leq Cx$. Therefore,

$$\left( \int_j \varepsilon^{(p-1)\theta} \left| \tilde{\zeta} \right|^p dx \right)^{\frac{1}{p}} \leq C \left( \int_j x^{-\frac{2(p-1)\theta}{a}} \left| \tilde{\zeta} \right|^p dx \right)^{\frac{1}{p}}.$$
where we used a Sobolev inequality with weight (see, e.g., [8, Theorem 1 and corollaries in 2]).

For \( \tilde{z} \) we have that \( \tilde{z}_x = (1 - \eta(xt^2))\zeta_x - t\eta'(xt^2)\zeta \). Following a similar procedure as for the estimate (30), we can show that

\[
\left( \int_I z^0 \left| \tilde{z}_x \right|^2 dx \right)^{\frac{1}{2}} \leq C \left( \int_I z^0 \left| \zeta_x \right|^2 dx \right)^{\frac{1}{2}}.
\]

Summarizing the above estimates, we obtain

\[
\left( \int_I z^{(p-1)\eta} \left| \tilde{z} \right|^p dx \right)^{\frac{1}{p}} \leq C \left( \int_I z^0 \left| \zeta_x \right|^2 dx \right)^{\frac{1}{2}}.
\]

Then (31) and (32) lead to

\[
\left( \int_0^1 z^{(p-1)\eta} \left| \tilde{z} \right|^p dx \right)^{\frac{1}{p}} \leq \left( \int_0^1 z^{(p-1)\eta} \left| \tilde{z} \right|^p dx \right)^{\frac{1}{p}} + \left( \int_0^1 z^{(p-1)\eta} \left| \tilde{z} \right|^p dx \right)^{\frac{1}{p}} \leq C \left( \int_0^1 z^0 \left| \zeta_x \right|^2 dx \right)^{\frac{1}{2}}.
\]

This completes the proof of our lemma and extends the result given in [5]. \( \square \)

**Lemma 3.8.** Suppose that \( z(x,t) \) and \( v(x,t) \) satisfy Assumptions 3.2 and 3.5, respectively. Then for any \( \delta > 0 \) there exist constants \( C \), independent of \( \delta \), and \( C_\delta \), dependent upon \( \delta \), such that the following estimates are satisfied:

\[
\langle v, v \rangle \leq \frac{C}{t^{1/3}} \quad \text{and} \quad |v(x,t)| \leq \frac{C_\delta}{t^{1/3}} \quad \text{for} \quad |x| > \delta.
\]

**Proof.** First we see, that

\[
\int_I v v_l \frac{dx}{z^0} = \int_I \left( v_x - \theta \frac{z_x}{z} v \right) \frac{dx}{z^0} + \int_I \mathcal{R}(x,t) v \frac{dx}{z^0} = \langle v, \tilde{A}_x(v) \rangle + \langle \mathcal{R}, v \rangle.
\]

Now we split the first term on the right-hand side into half and since \( \langle z^0, v \rangle = 0 \), due to Proposition 3.4, we have \( \frac{1}{2} \langle v, \tilde{A}_x(v) \rangle \leq -C \langle v, v \rangle \). Thus we obtain

\[
\frac{d}{dt} \langle v, v \rangle + \frac{C}{2} \langle v, v \rangle + \frac{1}{2} \left( v_x - \theta \frac{z_x}{z} v, v_x - \theta \frac{z_x}{z} v \right) \leq |\langle \mathcal{R}, v \rangle| + \frac{C}{t} \langle v, v \rangle.
\]

Using Hölder's inequality and (23), we have

\[
\frac{d}{dt} \|v\|_{L^2(z^{-\eta}dx)}^2 + C \|v\|_{L^2(z^{-\eta}dx)}^2 + C \left\| v_x - \theta \frac{z_x}{z} v \right\|^2_{L^2(z^{-\eta}dx)} \leq \|\mathcal{R}\|_{L^p(z^{-\eta}dx)} \|v\|_{L^p(z^{-\eta}dx)}
\]

264 Kang et al.
Summing up, we obtain
\[ v_t + \|v\|_{L^p(\mathbb{R}^d)}^2 + \epsilon \|v\|_{L^p(\mathbb{R}^d)}^2 \leq C \|R\|_{L^p(\mathbb{R}^d)}^2 \leq C \|R\|_{L^p(\mathbb{R}^d)}^2 + C \epsilon \left\| v - \theta \frac{\partial}{\partial z} v \right\|_{L^2(\mathbb{R}^d)}^2, \]
where \( p \) is given as in (23) and \( p' = (6\theta - 2)/(5\theta - 3) \) is its H"older conjugate.

Summing up, we obtain
\[ \frac{d}{dt} \|v\|_{L^2(\mathbb{R}^d)}^2 + C \|v\|_{L^2(\mathbb{R}^d)}^2 + C \left\| v - \theta \frac{\partial}{\partial z} v \right\|_{L^2(\mathbb{R}^d)}^2 \leq C \|R\|_{L^p(\mathbb{R}^d)}^2. \]

Due to Assumption 3.2, we compute
\[
\|R\|_{L^p(\mathbb{R}^d)}^2 = \left( \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x} G(\xi, t) \right|^p dx \right)^{\frac{2}{p}} \leq \frac{C}{\left( \int_{\mathbb{R}^d} g(\xi) dx \right)^{\frac{1}{2}}} \left( \int_{\mathbb{R}^d} \left( t^{2(\theta-1)} - \theta \right) dx \right)^{\frac{1}{p'}} = C t^\frac{\delta}{\alpha - \gamma}.
\]
Therefore, \( \langle v, v \rangle \leq C t^\frac{\delta}{\alpha - \gamma} \) and so \( |v(x, t)| \leq C t^\frac{\delta}{\alpha - \gamma} \) for \( |x| > \delta \). This completes the proof.

### 3.2. Estimates for the Solution near \( x = 0 \)

To show the asymptotic behavior of \( u \), we especially need estimates for \( x \) near 0. For this we introduce a so-called internal variable
\[
\xi = \frac{\theta}{\alpha - \gamma} x, \quad v(x, t) = t^{\frac{\delta}{\alpha - \gamma}} G(\xi, t), \quad z(x, t) = t^{\frac{\gamma}{\alpha - \gamma}} Z(\xi, t),
\tag{33}
\]
and define \( x = (\theta - 1)/(3 - \theta) \) and \( \gamma = 2/(3 - \theta) \). Due to Lemma 3.8, we have
\[
|G(\xi, t)| \leq C t^{\frac{\delta}{\alpha - \gamma} + \frac{\delta}{\alpha - \gamma} + \frac{2}{\alpha - \gamma}} = C t^{\frac{\delta}{\alpha - \gamma} - \frac{2(\alpha - \gamma)}{\alpha - \gamma}} = C t^{\frac{\delta}{\alpha - \gamma}} \quad \text{for} \quad |\xi| \geq t^\delta \quad \text{for any} \quad \delta > 0.
\]

If \( z \) satisfies Assumption 3.2, then \( Z(\xi, t) \approx (\xi^2 + a)^{-\frac{\delta}{\alpha - \gamma}} \). Furthermore, under Assumption 3.5, one can easily see that
\[
|G(\xi, t)| \leq M \frac{Z^0}{\Gamma(t)}, \quad \text{with} \quad \Gamma(t) = \int_{-\tau}^{\tau} Z^0(\xi, t) d\xi.
\]

Since \( \Gamma(t) \leq C \) for all \( t > 0 \), we obtain \( |G(\xi, t)| \leq CMZ^0 \). Recall (22), in terms of the new variables, explicit computations show that
\[
\mathcal{R}(\xi, t) = \theta t^{\delta - 1} \left( \frac{Z^0}{\Gamma^2(t)} + \frac{Z^0 \Lambda(t)}{\Gamma^2(t)} + \frac{Z^0 \Upsilon(t)}{\Gamma^2(t)} \right),
\]
where
\[
\Lambda(t) = \int_{-\tau}^{\tau} Z^0(\xi, t) d\xi, \quad \Upsilon(t) = \int_{-\tau}^{\tau} Z^0(\xi, t) G(\xi, t) d\xi.
\]
Now $\Gamma(t), \Lambda(t)$, and $\Upsilon(t)$ are uniformly bounded for any $t$ as long as $z$ and $v$ satisfy Assumptions 3.2 and 3.5, respectively. For convenience, denote 

$$\mathcal{R}_1(\xi, t) = \theta \left( \frac{Z^{2\theta-1} \Lambda(t)}{\Gamma^3(t)} + \frac{Z^{2\theta} \Lambda(t)}{\Gamma(t)} + \frac{Z^{2\theta} \Upsilon(t)}{\Gamma(t)} \right).$$

By change of variable, due to (21), $G$ solves

$$\frac{\theta - 1}{3 - \theta} \frac{d}{d\xi} \left( G + \xi \frac{\partial G}{\partial \xi} \right) + t^{\frac{\theta - 1}{\theta}} \frac{\partial G}{\partial t} = t^{\frac{\theta - 1}{\theta}} \left[ G_{\xi\xi} - \theta \left( \frac{Z}{\xi} \right) \right] + \mathcal{R}(\xi, t). \quad (34)$$

Simplifying (34), we have

$$G_{\xi\xi} - \theta \left( \frac{Z}{\xi} \right) = t^{\frac{\theta - 1}{\theta}} \left( G + \xi \frac{\partial G}{\partial \xi} \right) + t^{\frac{\theta - 1}{\theta}} \frac{\partial G}{\partial t} - \theta t^{\frac{\theta - 1}{\theta}} \frac{Z^{2\theta-1}}{\Gamma(t)} G + t^{\frac{\theta - 1}{\theta}} \mathcal{R}_1.$$

In the next lemma we formulate the asymptotic behavior of $u$ under Assumption 3.2.

**Lemma 3.9.** If $z(x, t)$ satisfies Assumption 3.2, then

$$\left| u(x, t) - \frac{z^0}{\int z^0 \, dx} \right| \leq \varepsilon(t) \frac{z^0}{\int z^0 \, dx},$$

where $\varepsilon(t) \leq C t^{-\beta}$ for some $\beta > 0$.

The proof of this lemma relies on

**Lemma 3.10.** Suppose that $z(x, t)$ and $v(x, t)$ satisfy Assumptions 3.2 and 3.5, respectively. Let $G$ be defined as in (33) so that $G$ solves equation (34). Then there exists a super-solution for $G$ in the set $\mathcal{R}_1 \leq \delta t^{(0-1)/(3-\theta)}$ for sufficiently small $\delta > 0$.

**Proof.** First, we look for a super-solution of the form

$$G_1(\xi, t) = \tilde{S}(\xi, t) + \tilde{U}(\xi, t) = t^{(\xi^2+1)} S(\xi, t) + t^{-2(\xi^2+1)} U(\xi, t),$$

where

$$S(\xi, t) = Z^0(\xi, t) \int_0^\xi Z^{-\theta}(y, t) \int_y^\infty \mathcal{R}_1(\eta, t) d\eta dy,$$

and where $\tilde{U}$ solves

$$U_{\xi\xi} - \theta \left( \frac{Z}{\xi} \right) = \frac{K_1}{(\xi^2 + a)^{1/\alpha}}.$$
Here $K_1$ is a constant, which will be specified later. Since $\mathcal{R} \approx Z^\theta$, up to multiplicative constants depending on $M$, one can check that $S_{\xi} - \theta \left( \frac{Z_\xi}{Z} \right) \approx \mathcal{R}_1$ and
\[ |S| + |\xi S| \approx \frac{1}{(\xi^2 + a)^{\frac{r}{2}}} \quad |S_t| \approx \frac{1}{r(\xi^2 + a)^{\frac{r}{2}}}. \tag{35} \]
again, all up to multiplicative constants depending on $M$. Similarly, we can show
\[ |U| + |\xi U| \approx \frac{1}{(\xi^2 + a)^{\frac{r}{2}}} \quad |U_t| \approx \frac{1}{r(\xi^2 + a)^{\frac{r}{2}}}. \tag{36} \]
Now we define a differential operator $\mathcal{H}_z$ as follows
\[ \mathcal{H}_z(f) = -t^{-2s} \frac{\partial f}{\partial t} + f_{\xi} - \theta \left( \frac{Z_{\xi}}{Z} f \right) - \alpha t^{-2s-1} \left( f + \xi \frac{\partial f}{\partial \xi} \right) + \theta t^{-2s-1} \frac{Z^{\theta-1}}{\Gamma^2(t)} f, \]
and compute
\[ \mathcal{H}_z(\varphi(t)) = -t^{-2s} \frac{\partial \varphi}{\partial t} + t^{-2s-1}\mathcal{R}_1 + t^{-2(2s+1)} \frac{K_1}{(\xi^2 + a)^{\frac{r}{2}}} \]
\[ - \alpha t^{-2s-1} \left( \varphi + \xi \frac{\partial \varphi}{\partial \xi} \right) + \theta t^{-2s-1} \frac{Z^{\theta-1}}{\Gamma^2(t)} \varphi \]
\[ \approx t^{-2s-1} \left( \varphi + \xi \frac{\partial \varphi}{\partial \xi} \right) + \theta t^{-2s-1} \frac{Z^{\theta-1}}{\Gamma^2(t)} \varphi, \tag{37} \]
by using (35) and (36) and where $C_{x, \theta}$ is a constant depending on $x$ and $\theta$.

Next we are looking for a super-solution of (34), which is of the form
\[ \bar{G}(\xi, t) = \gamma(t) Z^\theta(\xi, t) + t^{-2s-1} \gamma(t) \psi(\xi, t) + e^{-\mu(t-\bar{t})} Q(\xi, t), \tag{38} \]
where $\bar{t}$ in the last term is fixed and
\[ \gamma(t) = C_d t^{-1}, \quad \psi(\xi, t) = K_2 Z^\theta(\xi, t) \int_0^\xi Z^{-\theta}(y, t) \int_{\gamma}^\infty Z^\theta(v, t) \, dv \, dy, \]
with constants $K_2$ and $\mu$, which will be specified later, and with $C_d > 2C$, where $C$ is the absolute constant appearing in Lemma 3.8. For convenience, denote
\[ \varphi_2 := \gamma(t) Z^\theta(\xi, t) + t^{-2s-1} \gamma(t) \psi(\xi, t), \quad \varphi_3 := e^{-\mu(t-\bar{t})} Q(\xi, t). \]
We have
\[ \psi_{\xi \xi} - \theta \left( \frac{Z_{\xi}}{Z} \psi \right) = \frac{K_2}{(\xi^2 + a)^{\frac{r}{2}}}, \tag{39} \]
We compute the following quantity

\[ |\psi| + |\xi\psi| \approx \frac{1}{(\xi^2 + a)^{\frac{1}{2}}} \text{,} \quad |\psi| \approx \frac{1}{t(\xi^2 + a)^{\frac{1}{2}}} . \tag{40} \]

With (39) and (40), we can show that

\[
\mathcal{H}_Z(\theta_2) = -t^{-2s} \frac{\partial \theta_2}{\partial t} + t^{-2s-1} \chi(t) \frac{K_z}{(\xi^2 + a)^{\frac{1}{2}}} - \alpha t^{-2s-1} \left( \theta_2 + \xi \frac{\partial \theta_2}{\partial \xi} \right) + t^{-2s-1} \theta Z^{\theta s-1} \theta_2
\]

\[ \approx t^{-2s-1} \chi(t) \left( \frac{1 + K_z}{(\xi^2 + a)^{\frac{1}{2}}} + t^{-4s-2} \chi(t) \left( \frac{1}{(\xi^2 + a)^{\frac{1}{2}}} + \frac{1}{(\xi^2 + a)^{\frac{1}{2}}} \right) . \tag{41} \]

Finally, choose \( \theta_3 = e^{-\mu(t-\xi)} Q(\xi, t) \) where \( Q \) satisfies

\[ -t^{-2s} Q, + Q_{\xi\xi} - \theta \left( \frac{Z\xi}{Z} \right) \leq \alpha t^{-2s-1} \left( Q + \xi \frac{\partial Q}{\partial \xi} \right) - \theta t^{-2s-1} \frac{Z^{\theta s-1}}{\Gamma^2(t)} Q - \mu t^{-2s} Q \]

and \( Q(\xi, t) > 0 \) on the boundary \( |\xi| = \delta t^{\frac{1}{4}} \). We obtain a solution satisfying (42) in a perturbative manner. To do this we take \( Q_0(\xi, t) = a(Z(\xi, t))^\theta \), where \( a \) is a constant of order one to be determined. Then \( Q_0 \) solves

\[ Q_{0,\xi\xi} - \theta \left( \frac{Z_0\xi}{Z} \right) = 0 . \tag{43} \]

We look for solutions of (42) of the form

\[ Q(\xi, t) = Q_0(\xi, t) + Q_1(\xi, t) , \]

where \( Q_1 \) satisfies

\[
Q_{1,\xi\xi} - \theta \left( \frac{Z_1\xi}{Z} \right) \leq -2 \left( \chi t^{-2s-1} Q_0 + \alpha t^{-2s-1} \xi \left| \frac{\partial Q_0}{\partial \xi} \right| \right.
\]

\[ + \theta t^{-2s-1} \frac{Z^{\theta s-1}}{\Gamma^2(t)} Q_0 + \mu t^{-2s} Q_0 + t^{-2s} \left| \frac{\partial Q_0}{\partial t} \right| . \tag{44} \]

with \( Q_1(0, t) = 0 \). Now it remains show that \( Q = Q_0 + Q_1 \) satisfies (42). Suppose that \( \delta \) is sufficiently small. Assume that \( Q_1 \) satisfies

\[ |Q_1| \leq Q_0, \quad \left| \frac{\partial Q_1}{\partial \xi} \right| \leq \left| \frac{\partial Q_0}{\partial \xi} \right| , \quad \left| \frac{\partial Q_1}{\partial t} \right| \leq \left| \frac{\partial Q_0}{\partial t} \right| \tag{45} \]

in the set \( |\xi| \leq \delta \),. We will check this condition a posteriori. First we prove (42). We compute the following quantity

\[ J \equiv -t^{-2s} Q, + Q_{\xi\xi} - \theta \left( \frac{Z\xi}{Z} \right) - \alpha t^{-2s-1} \left( Q + \chi \frac{\partial Q}{\partial \xi} \right) + \theta t^{-2s-1} \frac{Z^{\theta s-1}}{\Gamma^2(t)} Q + \mu t^{-2s} Q . \]
Using $Q = Q_0 + Q_1$ as well as (43), we obtain

$$J = Q_{1,\xi} - \theta \left( Q_1 \frac{Z_{\xi}}{Z} \right)_{\xi} - r^{-2s} Q_{0,t} - r^{-2s} Q_{1,t} - x t^{-2s-1} \left( Q_0 + \frac{\partial Q_0}{\partial \xi} \right)$$

$$- x t^{-2s-1} \left( Q_1 + \frac{\partial Q_1}{\partial \xi} \right) + \theta t^{-2s-1} \frac{Z_{\theta}^{1/2}}{\Gamma^2(t)} Q_0$$

$$+ \theta t^{-2s-1} \frac{Z_{\alpha}^{1/2}}{\Gamma^2(t)} Q_1 + \mu t^{-2s} Q_0 + \mu t^{-2s} Q_1.$$

Thus it follows that

$$J \leq Q_{1,\xi} - \theta \left( Q_1 \frac{Z_{\xi}}{Z} \right)_{\xi}$$

$$+ \left[ x t^{-2s-1} Q_0 + x t^{-2s-1} \xi \frac{\partial Q_0}{\partial \xi} \right] + \theta t^{-2s-1} \frac{Z_{\theta}^{1/2}}{\Gamma^2(t)} Q_0 + \mu t^{-2s} Q_0 + r^{-2s} |Q_{0,t}|$$

$$+ \left[ x t^{-2s-1} Q_1 + x t^{-2s-1} \xi \frac{\partial Q_1}{\partial \xi} \right] + \theta t^{-2s-1} \frac{Z_{\alpha}^{1/2}}{\Gamma^2(t)} Q_1 + \mu t^{-2s} Q_1 + r^{-2s} |Q_{1,t}|.$$
Using $Q_{1,2}(0, t) = 0$, we have

$$Q_{1,2}(\xi, t) = 2\mu t^{-2s} (Z(\xi, t))^\theta \int_0^t d\zeta \left[ (Z(\xi, t))^{-\theta} \int_\zeta^{\infty} \mathcal{Q}_0(\eta, t) \, d\eta \right].$$

We can now estimate the behavior of $Q_{1,2}(\xi, t)$ for $\xi \gg 1$ and see how to choose $\mu$. Using Assumption 3.2, it follows that $\mathcal{Q}_0$ behaves like $\xi^{-\frac{2\theta}{\alpha}}$ for large $\xi$. It is then easy to see that $Q_{1,2}$, up to multiplicative constants, behaves like

$$Q_{1,2} \approx \mu t^{-2s} \frac{1}{\xi^{2\theta}} = \mu t^{-2s} \xi^{-\frac{2\theta}{\alpha}},$$

for large $\xi$, where $\alpha = \frac{\theta - 3}{\theta}$.

Now we can compare $Q_0$ with $Q_{1,2}$. Note that $Q_0 > 0$ for $|\xi| \approx \delta t^\frac{2}{\alpha}$. Moreover

$$Q_0 \approx \xi^{-\frac{2\theta}{\alpha}}, \quad Q_{1,2} \approx \mu t^{-2s} \xi^{-\frac{2\theta}{\alpha}}.$$

Therefore, for $\mu$ of order one and $\delta$ small it follows that $|Q_{1,2}| \ll Q_0$. In a similar manner, we can show that $|Q_{1,2}| \ll |Q_0|$ and $|Q_{1,2}| \ll |Q_0|$. Details are omitted. This completes the estimate (45).

Now, we conclude that $\mathcal{H}_x(\mathring{g}_1) \leq 0$, which yields $\mathcal{H}_x(\mathring{g}) \leq 0$. This is the case, since with (37) and (41), we obtain

$$\mathcal{H}_x(\mathring{g}_1 + \mathring{g}_2 + \mathring{g}_3 - G) \leq \mathcal{H}_x(\mathring{g}_1) + \mathcal{H}_x(\mathring{g}_2) - \mathcal{H}_x(G)$$

$$\approx t^{-2s-1}(K_1 + 1)(\xi^2 + a)^{-\frac{1}{2\theta}} + t^{-3s-1}(\xi^2 + a)^{-\frac{1}{2\theta}}$$

$$+ t^{-2s-1}(t) \left( \frac{1}{(\xi^2 + a)^{2\theta \alpha}} + \frac{1}{(\xi^2 + a)^{2\theta \alpha}} \right),$$

where we used that $\mathcal{H}_x(\mathring{g}) = t^{-2s-1}\mathring{g}_{1i}$. Suppose that $t$ is as large as needed, which can be obtained by setting $t \geq t_0$ for an arbitrarily large number $t_0$. By choosing constants $K_1$ and $K_2$ such that $K_1 + 1 < 0$ and $K_2 + 1 < 0$, we obtain that $\mathcal{H}_x(\mathring{g}_1 + \mathring{g}_2 + \mathring{g}_3 - G) < 0$.

In order to apply the maximum principle, we need $g_1 + g_2 + g_3 > G$ for $t = \bar{t}$ and $G \geq C t^{-\frac{\theta}{\alpha}}$ for $|\xi| = \delta t^\alpha$. Positivity of $g_1 + g_2 + g_3 - G$ for $t = \bar{t}$ follows since $g_3$ is the largest term among $\{g_i : i = 1, 2, 3\}$ and $g_3 > G$ for $t = \bar{t}$. On the other hand, at the boundary $|\xi| = \delta t^\alpha$ the inequality $G \leq C t^{-\frac{\theta}{\alpha}}$ results from $g_3 > 0$ and $g_2 > C t^{-\frac{\theta}{\alpha}}$ for $|\xi| = \delta t^\alpha$. Note that $g_3$ is added to control the “small nonlinear terms”, which are very small when compared to $g_2$. Summing up all above given arguments, we conclude that $g_1 + g_2 + g_3 \geq G$ is a super-solution for $G$. □

After the construction of this super-solution, we can now prove Lemma 3.9.

Proof of Lemma 3.9. Since the super-solution given above is bounded by $C t^{-\frac{\theta}{\alpha}} Z(\xi, t)$ for large $t$ with $t \geq \bar{t} + \log(\bar{t})$, it follows that there exists $\beta > 0$ such
that the super-solution is bounded by $C t^{-\beta} Z^0(\xi, t)$ for all $t \geq t_0$, and thus, back in the original variable, we obtain

$$
\left| u(x, t) - \frac{z^0}{\int_t^\infty \frac{dx}{z^0}} \right| \leq C t^{-\beta} t^{\frac{\alpha_0}{\beta}} Z^0(\xi, t) \frac{dz}{\int_t^\infty Z^0(\xi, t) d\xi} \leq C t^{-\beta} \frac{z^0(x, t)}{\int_t^\infty z^0(x, t) dx},
$$

where we used $\int_t^\infty Z^0(\xi, t) d\xi \leq C$, with $C = C(\theta, M)$. This completes the proof of our lemma.

Finally, we conclude the proof of the main theorem in this section with

\textbf{Lemma 3.11.} There exist solutions $u, z$ which satisfy all conditions of Assumption 3.2 for all $t \geq t_0$.

\textbf{Proof.} Without loss of generality, the initial time of our problem is $t_0$, since the system under consideration is invariant under time translations $t \to t - t_0$. Our choice of initial data $u(\cdot, t_0)$ and $z(\cdot, t_0)$ is sufficiently smooth and moreover, $u(\cdot, t_0)$ is assumed to be very close to the expected asymptotic behavior $t^{\frac{\alpha_0}{\beta}} \left( R^{x_2 t^{\frac{\alpha_0}{\beta} - 1}} + A \right)^{-\frac{\alpha}{\beta}}$. More precisely, it will be assumed that $u(\cdot, t_0)$ satisfies 1–4 in Assumption 3.2, with $M > 1, M - 1$ small and $t = t_0$.

First we derive the estimates for the derivatives and the Hölder norms in 1–4 of Lemma 3.11.

For all $|y| \leq 1$, we obtain following estimates for all $|y| \leq 1$

$$
|v^y_r(y, \tau)| \leq \frac{2 \epsilon (\bar{t})}{(\bar{t})^{\frac{\alpha}{\beta}}} |z^y_r(y, \tau)| \leq 2.
$$

On the other hand $v^y_r$ and $z^y_r$ satisfy

\begin{align}
\label{eq:62}
v_{r, \tau} &= v_{r, yy} - \theta \left( \frac{z^y_r}{z^r} v_r \right)_y - \frac{1}{R^2} \left( \frac{z^y_r}{x(\tau)} \right)_\tau, \\
\label{eq:63}
z_{r, \tau} &= \frac{z^y_r}{x(\tau)} + v_r,
\end{align}

with

$$
x(\tau) = \int_0^\tau z^0 dx \approx (\bar{t})^{\frac{\alpha}{\beta}(\frac{\alpha_0}{\beta})}.
$$

The term $\left( \frac{z^y_r}{x(\tau)} \right)_\tau$ can be shown to be sufficiently smooth and small by using (47), since

$$
\left( \frac{z^y_r}{x(\tau)} \right)_\tau = \theta \frac{z^y_r}{x(\tau)} + v_r - \frac{\chi(\tau)}{x^2(\tau)} z^y_r.
$$
Note that $\phi(t) \approx (t) \frac{2^{\alpha - 1}}{A(t)}$. One can easily see that the above terms are small contributions compared to other terms on the right hand side of (46). In order to obtain $e^{2x}$ estimates, we first take two spatial derivatives in (47) and obtain

$$\left( \frac{\partial^2 z_R}{\partial y^2} \right)_t = \frac{\theta (t-1) z_{R}^{\theta-2}}{\alpha(t)} \left( \frac{\partial z_R}{\partial y} \right)^2 + \frac{\theta z_{R}^{\theta-1}}{\alpha(t)} \frac{\partial^2 z_R}{\partial y^2} + \frac{\partial^2 v_R}{\partial y^2}.$$ 

The above equation indicates that we have to control $\frac{\partial^2 z_R}{\partial y^2}$. In fact, an interior regularity result for $v_R$ in the region $\frac{3}{4} \leq |y| \leq \frac{3}{4}$ is needed. We introduce a cutoff $\xi(y)$ which equals 1 for $\frac{3}{4} \leq |y| \leq \frac{3}{4}$ and vanishes for $|y| - 1 > \frac{1}{2}$. Then for $\xi(y) v_R =: \tilde{v}_R$, it follows

$$\tilde{v}_{R,t} = \tilde{v}_{R,yy} - \theta \left( \frac{z_{R,y}}{z_R} \right) y - 2v_{R,y} \xi - v_R \xi_y + \theta \left( \frac{z_{R,y} v_R \xi_y}{z_R} \right) - \frac{1}{R^2} \left( \frac{z_R \xi}{\alpha(t)} \right),$$

The equation for $\tilde{v}_R$ is similar to the one for $v_R$ except for some source terms that are of order $\frac{\alpha(t)}{t^{2\alpha}}$. Since $\tilde{v}_R$ vanishes, it follows that as long as $z_R$ satisfies Assumption 3.1, the fundamental solution of the equation satisfied by $\tilde{v}_R$ decreases exponentially in $t$, and the $C^{2+\cdot}$ derivatives in space also decay exponentially by standard regularizing effects. More precisely, we obtain two types of contributions for the derivatives of $\tilde{v}_R$, one of which is the part associated to the initial data starting at $t = i$ that decreases exponentially, and a second part associated to the source term which is of order $\frac{\alpha(t)}{t^{2\alpha}}$. Due to the decay of the function $\frac{1}{\alpha(t)}$, we can obtain a similar decay for the derivatives of $z_R$ and the Hölder estimates, by using derivatives of $v_R$ as source terms in the equation (47). This gives the desired estimate for any $t \geq 2$. If $t \leq 2$ we obtain similar results for $\|v_R(t)\|_{C^2} \leq \|z_R\|_{C^2}$ using the regularity of the initial data $v_0(x)$, $z_0(x)$. In particular for $t \in [i, i + 1]$ we can derive

$$\|v_R(i + 1)\|_{C^{2+\cdot}} \leq \sigma \|v_R(i)\|_{C^{2+\cdot}} + \frac{C \phi(i)}{(i) \frac{2^{\alpha}-1}{2^{\alpha}}},$$

$$\|z_R(i + 1)\|_{C^{2+\cdot}} \leq \sigma \|z_R(i)\|_{C^{2+\cdot}} + \frac{C \phi(i)}{(i) \frac{2^{\alpha}-1}{2^{\alpha}}},$$

where $0 < \sigma < 1$ due to the exponential decay of the solutions for the initial data mentioned above. The main contribution is due to the sources. Usual iterative methods yield the global smallness estimates as desired. Taking the supremum for all the admissible values of $R$ and returning to the original variables $(x, t)$, we obtain all estimates defined in Assumption 3.2.

The regularity estimates (18) and (20) imply that $u(t, x) = \frac{\alpha(t)}{i^\alpha} (1 + \eta(x, t))$, where

$$|\eta(x, t)| + (|x| + t^{-\frac{\alpha-1}{2}}) |\eta_x(x, t)| + (x^2 + t^{-\frac{2(\alpha-1)}{\alpha}}) |\eta_{xx}(x, t)| \leq \epsilon(t).$$

(48)

We then have $z_x(x, t) = \frac{\alpha(t)}{i^\alpha} (1 + \eta(x, t))$. Therefore, integrating in time,

$$z^{1-\theta}(x, t) = z_0^{1-\theta}(x) - \theta(t - 1) \int_0^t ds \frac{\eta(x, s) ds}{i^\alpha} \left( \frac{2^{\alpha-1}}{2^{\alpha}} \right) dy - (\theta - 1) \int_0^t \frac{\eta(x, s) ds}{i^\alpha} \left( \frac{2^{\alpha-1}}{2^{\alpha}} \right) dy.$$ 

(49)
Due to (48) and using the fact that $C_1 t^{-\frac{1}{\beta}} \leq \int_0^t \zeta^\theta(x, t) dx \leq C_2 t^{-\frac{1}{\beta}}$, we obtain that the function defined by means of $\Psi(x) = \int_0^\infty \frac{\eta(x, t) ds}{\int_t \zeta^\theta(y, s) dy}$ is bounded for $x \in I$. Moreover, it turns out that $\Psi \in \mathcal{C}^2(I)$. Indeed, we can write

$$\Psi_{xx}(x) = \int_0^{[x]^{-\frac{1}{\beta}}} \frac{\eta_{xx}(x, s) ds}{\int_t \zeta^\theta(y, s) dy} + \int_{[x]^{-\frac{1}{\beta}}}^\infty \frac{\eta_{xx}(x, s) ds}{\int_t \zeta^\theta(y, s) dy}.$$  

Using (48), we obtain

$$|\Psi_{xx}(x)| \leq \int_{[x]^{-\frac{1}{\beta}}}^\infty \frac{1}{s^{1+\beta}} e(s) ds + \int_{[x]^{-\frac{1}{\beta}}}^\infty \frac{e(s)}{s^{1+\beta} |x|^2} ds$$

$$\leq \int_{[x]^{-\frac{1}{\beta}}}^\infty \frac{1}{s^{1+\beta}} ds + C |x|^{\beta - \beta}/|x|^{2} \leq C.$$ 

Then (49) can be rewritten as

$$z^{1-\theta}(x, t) = z_0^{1-\theta}(x) - (\theta - 1) \int_0^\infty \frac{ds}{\int_t \zeta^\theta(y, s) dy} - \Psi(x)$$

$$\quad + (\theta - 1) \int_t^\infty \frac{\eta(x, s) ds}{\int_t \zeta^\theta(y, s) dy} + (\theta - 1) \int_t^\infty \frac{\eta(x, s) ds}{\int_t \zeta^\theta(y, s) dy}.$$ 

Since $\lim_{t \to \infty} z^{1-\theta}(0, t) = 0$ due to (19), we get

$$z^{1-\theta}(0) - (\theta - 1) \int_0^\infty \frac{ds}{\int_t \zeta^\theta(y, s) dy} - \Psi(0) = 0.$$ 

We then have

$$z^{1-\theta}(x, t) = z_0^{1-\theta}(x) + (\theta - 1) \int_t^\infty \frac{\eta(x, s) ds}{\int_t \zeta^\theta(y, s) dy} + (\theta - 1) \int_t^\infty \frac{\eta(x, s) ds}{\int_t \zeta^\theta(y, s) dy},$$  

(50)

where $(z_\infty(x))^{1-\theta} = [z_0^{1-\theta}(x) - z_0^{1-\theta}(0)] - [\Psi(x) - \Psi(0)]$. Using the fact that $z_\infty \in \mathcal{C}^2$, and $z_\infty^{1-\theta}(x) = Bx^2 + o(x^2)$ as $x \to 0$, where we used that $z_\infty^{1-\theta}(0) = (z_0^{1-\theta})_x(0) = 0$, we then note that the formal arguments at the beginning of Section 3 become rigorous and the asymptotics (ii) in Theorem 3.1 follows. Notice that (i) in Theorem 3.1 is an automatic consequence of (50). This completes the proof of our main theorem. □

**Remark 3.12.** In our model we intentionally did not include a degradation term for the chemical signal. This was done in order to keep the number of parameters minimal which drive the system towards blowup or global existence of solutions, and allows us to classify the system with respect to this behavior more easily.

The limit function $z_\infty(x)$ in general depends on the initial data. Due to the above mentioned reduction of the model, we have the following effect. For $t \to \infty$ the chemotactic signal $z$ is non-zero also in regions where no cells are present, since it is not degraded.
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