PARTIAL REGULARITY OF MINIMUM ENERGY CONFIGURATIONS IN FERROELECTRIC LIQUID CRYSTALS

KYUNGKEUN KANG
Department of Mathematics
Yonsei University
50 Yonsei-ro, Seodaemun-gu
Seoul 120-749, South Korea

JINHAE PARK
Department of Mathematics
Chungnam National University
99 Daehak-ro, Yuseong-gu
Daejeon 305-764, South Korea

(Communicated by Fanghua Lin)

Abstract. Considered here is a system of smectic liquid crystals possessing polarizations described by the Oseen-Frank and Chen-Lubensky energies. We establish partial regularity of minimizers for the governing energy functional using the idea of \((c, \beta)\)-almost minimizer introduced in [9].

1. Introduction. We investigate regularity properties of equilibrium configurations of a ferroelectric liquid crystal occupying a bounded open domain \(\Omega\) in \(\mathbb{R}^3\) corresponding to minimizers of the governing energy functional for such a system.

As a state of matter between liquid and solid, liquid crystals have some of properties of solids and also exhibit a certain degree of fluidity. A uniaxial nematic liquid crystal consists of the rigid rod-like molecules which tend to align along their long axes. The average long axis defines the molecular director \(\mathbf{n}, |\mathbf{n}| = 1\). Upon lowering temperature from the nematic phase, a smectic C phase emerges and there is locally a one-dimensional formation of layers in such a way that the director \(\mathbf{n}\) is tilted away from the layer normal, but free to rotate around it. We describe the smectic phase by the director \(\mathbf{n}\) and the complex field \(\psi = \rho e^{i\omega}\): level sets of the phase function \(\omega\) correspond to the smectic layers and \(\rho\) denotes center of mass of the molecule. Among many others, there exists an interesting class of smectic C liquid crystals possessing a polarization vector field \(\mathbf{P}\). Typically, the polarization field tends to be parallel to \(\mathbf{n} \times \nabla \omega\). Such liquid crystals are called ferroelectric liquid crystals and potential application is in the design of large video displays and fast switches.

2010 Mathematics Subject Classification. Primary: 35J50, 35J47, 47J05.
Key words and phrases. Partial regularity, liquid crystal, minimizer, singularity.

K. Kang’s work was partially supported by NRF-2011-0028951 and J. Park’s work was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(grant number 2011-0014882).
In this paper, we study the energy functional
\[
\tilde{E}(\mathbf{n}, \mathbf{P}, \psi) = \int_\Omega \left\{ \tilde{F}_N(\mathbf{n}, \mathbf{P}) + \tilde{F}_{Sm}(\mathbf{n}, \psi) + \tilde{F}_p(\mathbf{P}, \psi) + \tilde{F}_{cl}(\mathbf{P}, E) + \tilde{g}(\mathbf{P}) \right\} \, dx
\]
subject to Maxwell's equations
\[
\begin{cases}
-\nabla \cdot ((\varepsilon \mathbf{I} + \varepsilon_a \mathbf{n} \otimes \mathbf{n}) \mathbf{E}) = \nabla \cdot \mathbf{P} \text{ in } \Omega, \\
-\nabla \cdot \mathbf{E} = 0 \text{ in } \mathbb{R}^3 - \Omega, \\
\nabla \times \mathbf{E} = 0 \text{ in } \mathbb{R}^3, \\
-((\varepsilon \mathbf{I} + \varepsilon_a \mathbf{n} \otimes \mathbf{n}) \mathbf{E} - \varepsilon_0 \mathbf{E}) \cdot \nu = \mathbf{P} \cdot \nu \text{ on } \partial \Omega,
\end{cases}
\]
where
\[
\tilde{F}_N = K_1(\nabla \cdot \mathbf{n})^2 + K_2(\mathbf{n} \cdot \nabla \times \mathbf{n} + \mathbf{r})^2 + K_3|\mathbf{n} \times (\nabla \times \mathbf{n}) - \gamma_0 \mathbf{P}|^2
\]
\[+ (K_2 + K_4)(\varepsilon) \nabla (\nabla \cdot \mathbf{n})^2 - (\nabla \cdot \mathbf{n})^2,
\]
\[
\tilde{F}_{Sm} = D(\nabla^2 \psi)^* (\nabla^2 \psi)^* + [C_{ij} n_i n_j + C_\perp (\delta_{ij} - n_i n_j)](\nabla^2 \psi)^* (\nabla^2 \psi)^* \\
+ r|\psi|^2 + \frac{g}{2} |\psi|^4,
\]
\[
\tilde{F}_p = B|\nabla \mathbf{P}|^2 + \frac{1}{2} K_c \left[ |(\mathbf{n} \times \nabla \psi \cdot \mathbf{P})(\mathbf{n} \cdot \nabla \psi)|^2 - \frac{1}{\mu^2} |\psi|^4 |\nabla \psi|^4 \right],
\]
\[
\tilde{F}_{cl} = -\frac{1}{2} (\varepsilon \mathbf{I}|\mathbf{E}|^2 + \varepsilon_a (\mathbf{n} \cdot \mathbf{E})^2) - \mathbf{P} \cdot \mathbf{E},
\]
\[
\tilde{g}(\mathbf{P}) = \frac{1}{4 \eta^2} |\mathbf{P}|^2 - P_0^2)^2 \text{ or } \frac{1}{6 \eta^2} |\mathbf{P}|^2 [(|\mathbf{P}|^2 - P_0^2)^2 - \gamma^2],
\]
with \(\varepsilon_0 > 0\) being the dielectric coefficient of the media, \(\varepsilon_\perp > |\varepsilon_a|\) and \(^*\) in \(\tilde{F}_{Sm}\) denoting the complex conjugate. The energy \(\tilde{F}_N\) is the Oseen-Frank energy and \(\tilde{F}_{Sm}\) is a modified Chen-Lubensky smectic energy due to Lukyanchuk [13] where the original Chen-Lubensky energy [4] is
\[
a_{i\perp} |\nabla \cdot \mathbf{D}_\perp |\psi|^2 - c_{i\perp} |\mathbf{D}_\perp |\psi|^2 + a_{i} |\mathbf{D}_i |\mathbf{D}_i |\psi|^2 + C_{i\perp} |\mathbf{D}_i |\psi|^2 + r |\psi|^2 + \frac{g}{2} |\psi|^4
\]
with \(\mathbf{D} = \nabla - i \mathbf{n}, \mathbf{D}_\perp = (\mathbf{n} \cdot \nabla - i \mathbf{n}), \mathbf{D}_i = \mathbf{D} - \mathbf{D}_\perp, a_{i\perp} > 0, a_i > 0\)

Another version of the Chen-Lubensky energy introduced by Bauman and Phillips [2] takes the form
\[
\bar{F}_{Sm} = a_{i\perp} |\mathbf{D} \cdot \mathbf{D}_\perp |\psi|^2 - c_{i\perp} |\mathbf{D}_\perp |\psi|^2 + a_{i} |\mathbf{D} \cdot \mathbf{D}_i |\psi|^2 + C_{i\perp} |\mathbf{D}_i |\psi|^2 + r |\psi|^2 + \frac{g}{2} |\psi|^4.
\]
It turns out that this energy \(\bar{F}_{Sm}\) is comparable with the energy obtained by Leslie, Stewart, Carlsson and Nakagawa [12] which is available only for small deformations of smectic layers. The term \(F_P\) is the energy associated with the polarization \(\mathbf{P}\) with molecular chirality \(\mu\), dielectric susceptibility \(\chi_\perp\) [15], and \(\gamma\) depending on the temperature. The energy \(g(\mathbf{P})\) is a penalty term with \(\eta > 0\) because the polarization vector has the typical length \(P_0\) in the ferroelectric phases. Finally, the constitutive coefficients are assumed to satisfy
\[
\begin{cases}
D > 0, C_{i\perp} < 0, C_{i} > 0, \tau \geq 0, r < 0, g > 0, B > 0, \mu > 0, \chi_\perp > 0, \\
c_1 \geq K_2 + K_4 \geq c_0, \min\{K_1, K_3\} \geq K_2 + K_4, K_4 \leq 0, \quad K_c > 0.
\end{cases}
\]
for the Dirichlet problem of the classical Oseen-Frank energy (when \( \gamma_0 = 0 \)) [6, 14] were studied by Hardt, Kinderlehrer, and Lin [10]. It was shown in [10] that a minimizer \( \mathbf{n} \) of the classical Oseen-Frank energy is Hölder continuous on \( \Omega \setminus Z \) for some closed subset \( Z \) whose one dimensional Hausdorff measure is zero. For a special case of elastic constants, \( K_1 = K_2 = K_3 = K, K_4 = 0, \tau = 0 \), the classical Oseen-Frank energy reduces to

\[
\int_{\Omega} K |\nabla \mathbf{n}|^2 \, dx,
\]

which is the Dirichlet integral for harmonic maps. The partial regularity for harmonic maps was investigated by Schoen and Uhlenbeck [18, 19].

Regarding to the energy functional \( \tilde{\mathcal{E}} \), for \( |\psi| \) being constant, existence of minimizers with appropriate boundary conditions was studied in [16]. In the case of nonconstant \( |\psi| \), the \( c_1 \)-term in the smectic energy \( F_{Sm} \) gives rise to a difficulty since the \( c_1 \)-term is not bounded from below [1]. Bauman, Park, and Phillips [1] have recently found physically relevant boundary conditions for \( \psi \) and proven that the energy functional with these boundary conditions allows for minimizers in suitable classes of functions. But partial regularity for such minimizers of \( \tilde{\mathcal{E}} \) has not been studied in the literature. In this paper, we consider a simplified version \( \mathcal{E} \) of the energy functional \( \tilde{\mathcal{E}} \) obtained in [17] with a special bookshelf geometry that smectic layers are uniform and parallel to \( xy \)-plane. Then the governing energy functional \( \mathcal{E} \) can be reduced to

\[
\mathcal{E}(\mathbf{n}, \mathbf{p}, \varphi) = \int_{\Omega} \{ F_N + F_P + F_\varphi \} \, dx + \frac{1}{2}\varepsilon_0 \int_{\mathbb{R}^3 \setminus \Omega} |\nabla \varphi|^2 \, dx, \tag{5}
\]

subject to

\[
\begin{cases}
-\nabla \cdot ((\varepsilon_1 \mathbf{I} + \varepsilon_a \mathbf{n} \otimes \mathbf{n})\nabla \varphi) = \nabla \cdot \mathbf{p} \quad \text{in } \Omega, \\
-\Delta \varphi = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{\Omega}, \\
-[(\varepsilon_1 \mathbf{I} + \varepsilon_a \mathbf{n} \otimes \mathbf{n})\nabla \varphi - \varepsilon_0 \nabla \varphi] \cdot \nu = \mathbf{p} \cdot \nu \quad \text{on } \partial \Omega,
\end{cases} \tag{6}
\]

where \( a > 0, a^2 + c^2 = 1, |\mathbf{n}| = |\mathbf{p}| = 1 \), and

\[
F_N = K_1 (|\nabla \cdot \mathbf{n}|^2 + K_2 (\mathbf{n} \cdot \nabla \times \mathbf{n} + \tau)^2) + K_3 (|\nabla \times \mathbf{n}|^2 - (\mathbf{n} \cdot \nabla \times \mathbf{n} - \mathbf{p}|^2)
+ (K_2 + K_4) (tr(\nabla \mathbf{n})^2 - (\nabla \cdot \mathbf{n})^2) + \frac{1}{\varepsilon^2} \left( (n_3 - a)^2 + (n_1^2 + n_2^2 - c^2)^2 \right), \tag{7}
\]

\[
F_P = B |\nabla \mathbf{p}|^2 + \alpha [ (\mathbf{n} \times \mathbf{e}_3 \cdot \mathbf{p})^2 (\mathbf{n} \cdot \mathbf{e}_3)^2 - \chi_0^2] \tag{8},
\]

\[
F_\varphi = \frac{1}{2} (\varepsilon_\perp |\nabla \varphi|^2 + \varepsilon_a (\mathbf{n} \cdot \nabla \varphi)^2), \tag{9}
\]

with \( \alpha > 0, \varepsilon > 0 \), and a potential function \( \varphi \) for electric field \( \mathbf{E} \), i.e. \( \mathbf{E} = \nabla \varphi \). It is standard to show that the functional \( \mathcal{E} \) with a constraint (6) achieves a minimizer \( (\mathbf{n}, \mathbf{p}, \varphi) \) in \( W^{1,2}(\Omega, \mathbb{S}^2) \times W^{1,2}(\Omega, \mathbb{S}^2) \times W^{1,2}(\mathbb{R}^3) \) [1, 16]. A similar problem arises in the study of ferromagnetic materials [3, 9]. In this case, Hardt and Kinderlehrer [9] introduced a new terminology \((C, \beta)\)-almost energy minimizer so as to obtain partial regularity results. Based on the idea of \((C, \beta)\)-almost energy minimizer, we study partial regularity of minimizers for the problem (5)-(6) in this paper. One may consider our problem as a generalized version of the problem studied in [9] for ferromagnetics. As discussed in [9, 10], the set \( Z \) of singularities is defined as the set of points \( a \in \Omega \) for which the integral on the ball \( B_r(a) \) of radius \( r \) centered at
\begin{equation}
\frac{1}{r} \int_{B_r(a)} (|\nabla n|^2 + |\nabla p|^2) \, dx
\end{equation}
does not approach zero as \( r \to 0 \). In the proof of partial regularity, we use a hybrid inequality and energy decay estimate followed by Morrey’s lemma \([7]\). In the case that \( K_1 = K_2 = K_3 \), we establish a weaker version of interior monotonicity inequality which enables us to prove that there exist only a finite number of singularities inside \( \Omega \) for a minimizer. For general constants \( K_i (i = 1, 2, 3, 4) \), we prove that a minimizing pair \((n, p)\) is locally H{"o}lder continuous on \( \Omega \) except a subset \( Z \) with one dimensional Hausdorff measure zero.

This article is organized as follows. In section 2, we discuss basic properties of the energy functional and introduce some basic lemmas. The partial regularity results are presented in section 3.

2. Preliminaries. In this section, we discuss some basic properties and lemmas which are used in the later sections.

Throughout this paper, we assume that \( \Omega \) is an open and bounded domain in \( \mathbb{R}^n (n \geq 3) \) with a smooth boundary. We first note that the \((K_2 + K_4)\)-term in the energy with the Dirichlet boundary data is a null-Lagrangian \([10]\), meaning that \( \int_{\Omega} (tr(\nabla n)^2 - (\nabla \cdot n)^2) \) depends only on the value of \( n \) on \( \partial \Omega \). In our problem, this integral may not be bounded since no Dirichlet boundary condition for \( n \) on \( \partial \Omega \) is given in (5). Using the identity

\begin{equation}
tr(\nabla n)^2 = |\nabla n|^2 - |\nabla \times n|^2,
\end{equation}
we rewrite \( F_N \) as

\begin{align*}
F_N &= (K_1 - K_2 - K_4)(\nabla \cdot n)^2 + (K_2 + K_4)|\nabla n|^2 - (K_2 + K_4)|\nabla \times n|^2 \\
&\quad + K_2 n \cdot \nabla \times n + \tau)^2 + K_3 |n \times \nabla \times n - p|^2 \\
&\quad + \frac{1}{\epsilon^2} \left((n_3 - a)^2 + (n_1^2 + n_2^2 - c^2)^2\right).
\end{align*}

From (10), we estimate

\begin{align*}
K_2 (n \cdot \nabla \times n + \tau)^2 + K_3 |n \times \nabla \times n - p|^2 - (K_2 + K_4)|\nabla \times n|^2 \\
&\geq (\min\{K_2, K_3\} - (K_2 + K_4)) |\nabla \times n|^2 + 2K_2 \n \cdot \nabla \times n - 2p \cdot n \times \nabla \times n \\
&\quad + K_2 \tau^2 + K_3.
\end{align*}

From (4), it is easy to conclude the following lemma.

**Lemma 2.1.** There exist positive constants \( M_1, M_2, L_1, L_2 \) such that

\begin{equation}
M_1 \left( |\nabla n|^2 + |\nabla p|^2 \right) - L_1 \leq F_N + F_P \leq M_2 \left( |\nabla n|^2 + |\nabla p|^2 \right) + L_2.
\end{equation}

Now, let us introduce the admissible set

\[ \mathcal{A} = \left\{ (n, p, \varphi) : (n, p) \in W^{1,2}(\Omega, \mathbb{S}^2) \times W^{1,2}(\Omega, \mathbb{S}^2), \varphi \in W^{1,2}(\mathbb{R}^3) \text{ satisfy (6)} \right\}, \]

where \( \mathbb{S}^2 \) denotes the sphere with radius 1 centered at the origin in \( \mathbb{R}^3 \). By the standard theory of calculus of variations, there exists a minimizer \((n, p, \varphi) \in \mathcal{A}\) for \( E \) \([16]\). Let \( a \in \Omega \) be fixed. For convenience, we define

\begin{equation}
E_r(n, p) = \int_{B_r(a)} (F_n + F_P) \, dx,
\end{equation}

(12)
\[ \mathbb{E}_r(n, p) = \int_{B_r(a)} (|\nabla n|^2 + |\nabla p|^2) \, dx. \]  

(13)

In the sequel, we denote by \( C = C(\alpha, \beta, \ldots) \) a generic constant depending on the prescribed quantities \( \alpha, \beta, \ldots \), which may vary by line, unless any confusion is to be expected.

We next introduce lemmas which are important to obtain a hybrid inequality in the next section. The proofs of the first two lemmas can be found in [8, 9, 18].

**Lemma 2.2.** For any \( v \in W^{1,2}(\Omega, \mathbb{R}^3) \) with \( |v| = 1 \) on \( \partial \Omega \), there exists a function \( n \in W^{1,2}(\Omega, S^2) \) such that

\[ \int_{\Omega} |\nabla n|^2 \, dx \leq C \int_{\Omega} |\nabla v|^2 \, dx, \quad n = v \text{ on } \partial \Omega \]

(14)

for some constant \( C \), independent of \( v \).

**Lemma 2.3.** Let \( n \in W^{1,2}(\Omega, S^2) \) and \( a \in \Omega \). Then for a.e. \( r < \text{dist}(a, \partial \Omega) \) there exists a function \( m \in W^{1,2}(B_r(a), S^2) \) such that \( m = n \) on \( \partial B_r(a) \), and for any \( \mu \in \mathbb{R}^3 \),

\[ \int_{B_r(a)} |\nabla m|^2 \leq C \left( \int_{\partial B_r(a)} |\nabla_{\text{tan}} n|^2 \, dH^{n-1} \right) \frac{1}{r^{2n-2}} \]

for some \( C > 0 \), independent of \( a \) and \( r \). Here, \( \nabla_{\text{tan}} n \) denotes the tangential component of \( \nabla n \) and \( H^{n-1} \) is the \((n-1)\)-dimensional Hausdorff measure.

Next lemma is a slightly modified statement of the Lemma 1 in [20, p.32]. Since its verification is straightforward, its details are omitted here.

**Lemma 2.4.** Let \( \{ (n_i, p_i) : i \in \mathbb{N} \} \in W^{1,2}(\Omega, S^2) \times W^{1,2}(\Omega, S^2) \) be a sequence satisfying \( \sup_i \int_{B_1(a)} (|\nabla n_i|^2 + |\nabla p_i|^2) < \infty \) where \( B_1 := B_1(x) \subset \Omega \). Suppose that for any pair of functions \( (m, q) \in W^{1,2}(B_1, S^2) \times W^{1,2}(B_1, S^2) \), the following conditions are satisfied:

(i) \( (n_i, p_i) = (m, q) \) on \( \partial B_1 \) for all \( i \in \mathbb{N} \),

(ii) there exists a sequence of positive numbers \( \{ C_i \}_{i \in \mathbb{N}} \) converging to zero as \( i \to \infty \) such that

\[ \int_{B_1} (|\nabla n_i|^2 + |\nabla p_i|^2) \leq \int_{B_1} (|\nabla m|^2 + |\nabla q|^2) + C_i. \]

Then there exists a subsequence, not relabeled, of \( \{ (n_i, p_i) : i \in \mathbb{N} \} \) such that \( (n_j, p_j) \) converges to \( (n, p) \) strongly in \( W^{1,2}(B_1) \). Moreover, \( n \) and \( p \) are harmonic maps in \( B_1 \).

3. **Partial regularity.** For an open and bounded domain \( \Omega \subset \mathbb{R}^3 \) with a smooth boundary, a pair \( (n, p) \in W^{1,2}(\Omega, S^2) \times W^{1,2}(\Omega, S^2) \) is said to be \((C, \beta)\)-almost minimizer in \( \Omega \) if for every \( a \in \Omega \) and for \( B_r(a) \subset \Omega \), there exist \( C > 0 \) and \( \beta \in (0,1) \) such that

\[ \mathcal{E}_r(n, p) \leq \mathcal{E}_r(m, q) + C r^{1+\beta}, \]

for any pair \( (m, q) \in W^{1,2}(\Omega, S^2) \times W^{1,2}(\Omega, S^2) \) with \( (m, q) = (n, p) \) in \( \mathbb{R}^3 \setminus B_r(a) \).

In the next lemma, we compare a minimizer for \( \mathcal{E} \) on the whole domain \( \Omega \) with a minimizer for \( \mathcal{E}_r \) on a local region \( B_r(a) \).
Lemma 3.1. Let \( a \in \Omega \) and \((n, p, \varphi) \in A\) be a minimizer for \( E \) in (5). Suppose further that \((n, p) \in W^{1,2}(B_r(a), \mathbb{S}^2) \times W^{1,2}(B_r(a), \mathbb{S}^2)\) be a minimizing pair for \( E_r(n, p) \) in (12) such that

\[
\hat{n} = n, \quad \hat{p} = p \quad \text{in } \Omega \setminus B_r(a),
\]

and \((\hat{n}, \hat{p}, \hat{\varphi})\) satisfies (6) by taking \( B_r(a) \) in replacement of \( \Omega \) and \( \hat{\varphi} = \varphi \) in \( \mathbb{R}^3 \setminus B_r(a) \). Then there exists a constant \( C > 0 \) independent of \( a, r, \) and \( \beta \) such that

\[
E_r(n, p) \leq E_r(\hat{n}, \hat{p}) + Cr^2. \tag{15}
\]

Proof. First, we notice that

\[
\int_\Omega (|\nabla \varphi|^2 + |\nabla \hat{\varphi}|^2) \, dx < C. \tag{16}
\]

For this, it suffices to show \( \int_\Omega |\nabla \varphi|^2 < C \) because the other case can be computed similarly. Indeed, using the fact that \( \varphi \) satisfies the equation (6), we have

\[
(\varepsilon_\perp - |\varepsilon_a|) \int_\Omega |\nabla \varphi|^2 \leq \int_\Omega \int |n \cdot \nabla \varphi|^2 \leq \int |n \cdot \nabla \varphi|^2 = \int_\Omega \int a \cdot \nabla \varphi \cdot \nu \varphi - \int_\Omega \nabla \cdot (a \cdot \nabla \varphi) \varphi = \int_\Omega (\varepsilon_0 \nabla \cdot p - p \cdot \nu) \varphi + \int_\Omega (\nabla \cdot \varphi) \varphi = -\varepsilon_\perp \int_\Omega |\nabla \varphi|^2 - \int_\Omega p \cdot \nabla \varphi.
\]

Therefore, it follows from the condition \( |p| = 1 \) that the inequality holds

\[
(\varepsilon_\perp - |\varepsilon_a|) \int_\Omega |\nabla \varphi|^2 + \epsilon_0 \int_{\mathbb{R}^3 \setminus \Omega} |\nabla \varphi|^2 \leq |\Omega|^{1/2} \|\nabla \varphi\|_{L^2(\Omega)}.
\]

With aid of the Young’s inequality, it is immediate that \( \int_\Omega |\nabla \varphi|^2 < C \) for some \( C > 0 \). In a similar fashion, we can obtain that \( \int_\Omega |\nabla \hat{\varphi}|^2 < C \). We skip its details. Now we are ready to show the estimate (15). Since \( E(n, p, \varphi) \leq E(\hat{n}, \hat{p}, \hat{\varphi}) \) and \( E_r(\hat{n}, \hat{p}) \leq E_r(n, p) \), we get

\[
0 \leq E_r(n, p) - E_r(\hat{n}, \hat{p}) \leq J,
\]

where

\[
J := \int_\Omega F_\varphi + \frac{1}{2} \int_{R^3 \setminus \Omega} \epsilon_0 |\nabla \hat{\varphi}|^2 - \int_\Omega F_\varphi - \frac{1}{2} \int_{R^3 \setminus \Omega} \epsilon_0 |\nabla \varphi|^2.
\]

As in computations for (16), we observe

\[
\int_\Omega F_\varphi = \frac{1}{2} \int_\Omega (\varepsilon_\perp I + \varepsilon_a n \otimes n) \nabla \varphi \cdot \nabla \varphi = -\frac{1}{2} \int_\Omega |\nabla \varphi|^2 - \frac{1}{2} \int_\Omega p \cdot \nabla \varphi.
\]

By the fact that \( p \cdot \nabla \varphi = \hat{p} \cdot \nabla \hat{\varphi} \) in \( \mathbb{R}^3 \setminus B_r(a) \) and using the above equality and (16), we estimate \( 2J \) as follows:

\[
2J = -\int_\Omega (\hat{p} \cdot \nabla \hat{\varphi} - p \cdot \nabla \varphi) = \int_{B_r(a)} (p - \hat{p}) \cdot \nabla \varphi + p \cdot (\nabla \varphi - \nabla \hat{\varphi}) \leq \|p - \hat{p}\|_{L^2(B_r(a))} \|\nabla \varphi\|_{L^2(B_r(a))} + \|p\|_{L^2(B_r(a))} \|\nabla \varphi - \nabla \hat{\varphi}\|_{L^2(B_r(a))} \leq C \|B_r(a)\|^{1/2} (\|\nabla \varphi\|_{L^2(\Omega)} + \|\nabla \hat{\varphi}\|_{L^2(\Omega)}) \leq Cr^2,
\]

where \( C \) is absolute constant depending on \( \Omega \) but not on \( a \in \Omega \) and \( r \). This completes the proof.
As a direct consequence of Lemma 3.1, we have the following.

**Corollary 3.2.** Suppose that \((n, p, \varphi) \in A\) is a minimizer for \(E\). Then \((n, p)\) is a \((C, \beta)\)-almost minimizer in \(\Omega\) for some \(C > 0\) and for any \(\beta \in (0, 1/2]\).

**Lemma 3.3.** (Hybrid inequality) Let \(a \in \Omega\). Suppose that \((n, p)\) be a \((C, \beta)\)-almost minimizer in \(\Omega\). Then for any \(B_r(a) \subset \Omega\) and for any \(0 < \lambda < 1\) there exists \(C_0 > 0\) such that the following inequality is satisfied:

\[
\left(\frac{\rho}{2}\right)^{-1} \int_{B_{\rho}(a)} (|\nabla n|^2 + |\nabla p|^2) \, dx \\
\leq \lambda \rho^{-1} \int_{B_{\rho}(a)} (|\nabla n|^2 + |\nabla p|^2) \, dx \\
+ C_0 \left(\rho^\beta + \lambda^{-1} \rho^{-3} \int_{B_{\rho}(a)} (|n - \mu_1|^2 + |p - \mu_2|^2) \, dx\right),
\]

(17)

where \(\mu_1, \mu_2 \in \mathbb{R}^3\) are arbitrary constant vectors.

**Proof.** By Lemma 2.2 and Lemma 2.3 as in [9], there exist \(m, q \in W^{1,2}(B_r(a), \mathbb{S}^2)\), \(m = n, q = p\) on \(\partial B_r(a)\), such that

\[
\int_{B_r(a)} (|\nabla m|^2 + |\nabla q|^2) \, dx \leq \delta r \int_{B_r(a)} (|\nabla n|^2 + |\nabla p|^2) \, dx \\
+ \delta^{-1} r^{-1} \int_{B_r(a)} (|n - \mu_1|^2 + |p - \mu_2|^2) \, dx.
\]

(18)

From (11), we obtain

\[
E_r(m, q) \leq C \int_{B_r(a)} (|\nabla m|^2 + |\nabla q|^2) \, dx + Cr^{1+\beta}.
\]

(19)

Since \((n, p)\) is a \((C, \beta)\)-almost minimizer, extending \(m\) and \(q\) to \(\Omega\) by \(m = n, q = p\) on \(\Omega \setminus B_r(a)\) we get

\[
E_r(n, p) - E_r(m, q) \leq Cr^{1+\beta}.
\]

(20)

Then for \(\rho/2 \leq r \leq \rho\), it follows from (11), (18), and (19) that

\[
\int_{B_{\rho}(a)} (|\nabla n|^2 + |\nabla p|^2) \, dx \leq \int_{B_{\rho}(a)} (|\nabla n|^2 + |\nabla p|^2) \, dx \\
\leq CE_r(n, p) + Cr^{1+\beta} \leq CE_r(m, q) + Cr^{1+\beta} \\
\leq C \left(\delta r \int_{B_r(a)} (|\nabla m|^2 + |\nabla q|^2) \, dx + Cr^{1+\beta}\right) \\
\leq C \left(\delta r \int_{B_r(a)} (|\nabla n|^2 + |\nabla p|^2) \, dx \\
+ \delta^{-1} r^{-1} \int_{B_r(a)} (|n - \mu_1|^2 + |p - \mu_2|^2) \, dx\right) + Cr^{1+\beta}.
\]

Hence we obtain the hybrid inequality (17) after a suitable choice of \(\delta\). 

We recall the functional \(E_r(n, p) := \int_{B_r(a)} (|\nabla n|^2 + |\nabla p|^2)\) defined in (13).
Lemma 3.4. (Energy decay estimate) Let \((n, p) \in W^{1,2}(\Omega; \mathbb{S}^2) \times W^{1,2}(\Omega; \mathbb{S}^2)\) be a \((C, \beta)\)-almost minimizer in \(\Omega\). There exist \(\epsilon_0, r_0, \eta\) and \(\theta < 1\) such that if \(\frac{1}{r_0} E_{\theta r_0}(n, p) < \epsilon_0^2\), then for any \(r\) with \(0 < r < r_0\)

\[
\frac{1}{\theta r} E_{\theta r}(n, p) \leq \theta \max\left\{ \eta r^\beta, \frac{1}{r} E_r(n, p) \right\}.
\]

Proof. We prove the lemma by contradiction. Suppose that for fixed \(\theta \in (0, 1/2)\), there exists \((C, \beta)\)-almost minimizer \((n_i, p_i)\) in \(\Omega\) and balls \(B_r(a_i) \subset \Omega\) such that

\[
c_i^2 := \frac{1}{r_i} \int_{B_r(a_i)} \left( |\nabla n_i|^2 + |\nabla p_i|^2 \right) dx \to 0, \quad (21)
\]

\[
r_i^{-\beta}(\theta r_i)^{-1} \int_{B_{\theta r_i}(a_i)} \left( |\nabla n_i|^2 + |\nabla p_i|^2 \right) dx \to \infty, \quad (22)
\]

\[
(\theta r_i)^{-1} \int_{B_{\theta r_i}(a_i)} \left( |\nabla n_i|^2 + |\nabla p_i|^2 \right) dx > \theta c_i^2. \quad (23)
\]

Clearly, we observe that \(c_i^2 r_i^{-\beta} \to 0\) as \(i \to \infty\). For simplicity, we set \(a_i = 0\) by translation. The blow-up sequence \(v_i : B_1 \to \mathbb{R}^3\) and \(q_i : B_1 \to \mathbb{R}^3\) are defined by

\[
v_i(x) = \epsilon_i^{-1}(n_i(r_i x) - \bar{n}_i), \quad q_i(x) = \epsilon_i^{-1}(p_i(r_i x) - \bar{p}_i),
\]

where \(\bar{n}_i, \bar{p}_i\) are the averages of \(n, p\) over \(B_r(0)\) respectively, and therefore \(v_i = 0, q_i = 0\). We denote \(B_r(0)\) by \(B_r\) for the rest part of the proof. Passing to a subsequence, if necessary, without changing notations we may assume that \(v_i\) and \(q_i\) converge weakly in \(W^{1,2}\) to \(v, q \in W^{1,2}(B_1; \mathbb{R}^3)\), respectively. In the following, we prove that \(q\) is harmonic and \(v\) satisfies an elliptic equation. Let \(\zeta \in C_0^\infty(B_1)\). In order to show that \(q\) is harmonic, we normalize \(q_i\) in the following manner:

\[
q_i^t(x) = \epsilon_i^{-1} \left( \frac{p_i(r_i x) + t \epsilon_i \zeta(x)}{|p_i(r_i x) + t \epsilon_i \zeta(x)|} - B_i^t \right) \quad \text{for } 0 < t < 1,
\]

where \(B_i^t\) is chosen so that \(q_i^t = 0\). For convenience, we set

\[
\hat{p}_i = \frac{p(r_i x) + t \epsilon_i \zeta(x)}{|p(r_i x) + t \epsilon_i \zeta(x)|}.
\]

In fact, \(\hat{p}\) is defined a.e. since \(|p| = 1\) a.e. in \(B_r(0)\). Using the almost minimality, we obtain

\[
-C \epsilon_i^{-2} r_i^{-\beta} \leq \epsilon_i^{-2} r_i^{-1} (\mathcal{E}_r(n_i, \hat{p}_i) - \mathcal{E}_r(n_i, p_i)).
\]

Note that for any \(\delta > 0\),

\[
\left| \int_{B_{r_i}} n_i \times \nabla \times n_i \cdot (p_i - \hat{p}_i) dx \right| \leq C_0 \delta \int_{B_{r_i}} |\nabla n_i|^2 dx + 2 \delta^{-1} r_i^3, \quad (24)
\]
for some fixed $C_0 > 0$. Using that $\epsilon_i^{-2} r_i^{-1} \int_{B_{r_i}} |\nabla n_i|^2 \, dx \leq 1$, we have
\[
-C \epsilon_i^{-2} r_i^3 - C \epsilon_i^{-2} r_i^3 \leq C \epsilon_i^{-2} r_i^{-1} (\mathcal{E}_{r_i}(n_i, \tilde{p}_i) - \mathcal{E}_{r_i}(n_i, p_i))
\]
\[
= C \epsilon_i^{-2} r_i^{-1} \left( K_3 \int_{B_{r_i}} n_i \times \nabla n_i \cdot (p_i - \tilde{p}_i) \, dx \right)
\]
\[
+ C \epsilon_i^{-2} r_i^{-1} \left( B \int_{B_{r_i}} (|\nabla \tilde{p}_i|^2 - |\nabla p_i|^2) \, dx + D r_i^3 \right)
\]
\[
\leq C_0 C K_3 \delta + C (2 K_3 \delta^{-1} + D) \epsilon_i^{-2} r_i^2
\]
\[
+ B C \epsilon_i^{-2} r_i^{-1} \int_{B_{r_i}} (|\nabla \tilde{p}_i|^2 - |\nabla p_i|^2) \, dx,
\]
where $K_3$ and $B$ are constants in (4). We calculate
\[
\epsilon_i^{-2} r_i^{-1} \int_{B_{r_i}} (|\nabla \tilde{p}_i|^2 - |\nabla p_i|^2) \, dx = \int_{B_{r_i}} (|\nabla q_i|^2 - |\nabla q_i|^2) \, dx.
\]
\[
= \int_{B_{r_i}} |\nabla q_i|^2 \left( \frac{1}{|p_i|^2} - 1 \right) + \frac{2 t|\nabla q_i \cdot \nabla \zeta|^2}{|p_i|^2} + \frac{i^2 |\nabla \zeta|^2}{|p_i|^2} \, dx + Dr_i^2,
\]
where $p_i(x) = p_i(x, r, \tau) + t \epsilon_i(x)$ and for some constant $D > 0$.

Let us choose $\delta = t^2$. Since $||p_i|| - 1 \leq t \epsilon_i ||\zeta||_L^2$ and $\epsilon_i^{-2} r_i^2 \leq \epsilon_i^{-2} r_i^2 \to 0$ as $i \to \infty$, we let $i \to 0$, divide by $t$ and take $t$ to zero to deduce that
\[
0 \leq 0 + C \int_{B_{r_i}} (\nabla q \cdot \nabla \zeta) + 0.
\]

Replacing $\zeta$ by $-\zeta$, we have opposite inequality so that $q$ is harmonic. Hence mean-value property of harmonic functions yields
\[
\frac{1}{r^3} \int_{B_{r_i}} |q - q_{r_i}|^2 \, dx \leq C r^2,
\]
where $q_{r_i}$ is the average of $q$ over $B_{r_i}$.

Next, we claim that $v$ satisfies an elliptic equation. Without loss of generality, we may assume that the average $n_i$ of $n_i$ over $B_{r_i}$ is proportional to $(0, 0, 1)$ by choosing appropriate rotations $R_i$ for each $i$. We split $F_{\chi} + F_p + F_\varphi$ into $F + G$ as
\[
F = (K_2 + K_3) |\nabla n_i|^2 + V(n, \nabla n),
\]
\[
V = (K_1 - K_2 - K_4) (\nabla \cdot n_i)^2 - (K_2 + K_4) |\nabla \nabla n_i|^2 + K_2 (n_i \cdot \nabla \nabla n_i)^2
+ K_3 |n_i \times \nabla \times n_i|^2,
\]
\[
G = F_\varphi + F_p + 2 K_2 \tau n_i \cdot \nabla \nabla n_i - 2 K_3 p_i \cdot n_i \times \nabla \nabla n_i + (K_3 + K_2 \tau^2)
+ \frac{1}{\epsilon_i^2} \left( (n_3 - a)^2 + (n_1 + n_2 - c^2)^2 \right).
\]

Using $s$ as a variable for $\nabla n$, the Euler-Lagrange equation for $n$ is given by
\[
- \text{div} \left[ F_{n} - (n \otimes n) V_n \right] + X(n, \nabla n) + Y(n, \nabla n, p) + G_n = 0,
\]
\[
- \text{div} \left[ G_{n} - (n \otimes n) G_{n} \right] + (I - n \otimes n) A_n (\nabla \varphi, n) = 0,
\]
where $A(\nabla \varphi, n) = F_\varphi$ and
\[
X(n, \nabla n) = (I - n \otimes n) V_n - \nabla n (n V_s) - (V_s \cdot \nabla n)n,
\]
\[
Y(n, \nabla n, p) = (I - n \otimes n) G_n - \nabla n (n G_s) - (G_s \cdot \nabla n)n.
\]
Note that $|X(n, \nabla n)| \leq C|\nabla n|^2$ for some $C$.

For each $i$, $n_i$ satisfies

\[ -\text{div } [F_s - (n_i \otimes n_i) V_s] + X(n_i, \nabla n_i) + Y(n_i, \nabla n_i, p_i) \]
\[ -\text{div } [G_s - (n_i \otimes n_i) G_s] + (I - n_i \otimes n_i) A_{n_i}(\nabla \varphi_i, n_i) = 0 \quad \text{in } \mathbb{B}_r. \tag{28} \]

For any $\xi \in W^{1,2}_0(\mathbb{B}_1, \mathbb{R}^3) \cap L^\infty(\mathbb{B}_1, \mathbb{R}^3)$, multiplying $(28)$ by $\xi(x)$, integrating over $\mathbb{B}_r$, and then plugging $\nabla n_i = \epsilon_i \nabla v_i$ into the integration followed by dividing by $r_i \epsilon_i$ and taking $i \to \infty$ we obtain

\[ \int_{\mathbb{B}_1} [F_s(e, \nabla v) - (e \otimes e) V_s(e, \nabla v)] \cdot \nabla \xi \, dx = 0. \tag{29} \]

In fact, we can prove $(29)$, using Lemma 2.4 and the fact that $||\nabla v_i||_{L^2} \leq C < \infty$, $\epsilon_i^{-2} r_i^2 \to 0$ as $i \to \infty$, and

\[ n_i \to e \quad \text{in } L^2, \quad \nabla v_i \to \nabla v \quad \text{in } L^2, \]

together with the following estimates

\[ \left| \int_{\mathbb{B}_r} (I - n_i \otimes n_i) A_{n_i}(\nabla \varphi_i, n_i) \cdot \xi \left( \frac{x}{r_i} \right) \right| \leq C r_i^{1+\beta}, \]
\[ \left| \int_{\mathbb{B}_r} [G_s - (n_i \otimes n_i) G_s] \cdot \nabla \xi \left( \frac{x}{r_i} \right) \, dx \right| \leq C r_i^2, \]
\[ \left| \int_{\mathbb{B}_r} Y(n_i, \nabla n_i, p_i) \cdot \xi \left( \frac{x}{r_i} \right) \, dx \right| \leq C (r_i^3 + \epsilon_i r_i^2), \]
\[ \left| \int_{\mathbb{B}_r} X(n_i, \nabla n_i) \cdot \xi \left( \frac{x}{r_i} \right) \, dx \right| \leq C \epsilon_i^2 r_i, \]
\[ \int_{\mathbb{B}_r} [F_s - (n_i \otimes n_i) V_s] \cdot \nabla \xi \left( \frac{x}{r_i} \right) \, dx \]
\[ = \epsilon_i r_i \int_{\mathbb{B}_1} [F_s(n_i, \nabla v) - (n_i \otimes n_i) V_s(n_i, \nabla v_i)] \cdot \nabla \xi(x) \, dx. \]

With $\xi = (\xi_1, \xi_2, 0)$, the equation $(29)$ becomes

\[ \int_{\mathbb{B}_1} F_s(e, \nabla v) \cdot \nabla \xi \, dx = 0. \]

We see that $v' = (v_1, v_2)$ satisfies

\[ -\text{div } F'_s(e, \nabla v') = 0, \tag{30} \]

where $F'_s$ is the first two rows of the matrix $F_s$. Since $F'_s(e, \eta) \cdot \eta = F'_s(e, \eta) \cdot \eta \geq \alpha |\eta|^2$ for all $\eta = (\eta_j)$ with $\eta_j = 0$ and $\bar{v} = 0$, $(30)$ is an elliptic PDE and by the standard linear elliptic theory [7, p.78] we obtain

\[ \frac{1}{r^3} \int_{\mathbb{B}_r} |v - \bar{v}_r|^2 \, dx \leq C r^2 \int_{\mathbb{B}_1} |v|^2 \, dx \quad \text{for } 0 \leq r \leq 1, \tag{31} \]

where $\bar{v}_r$ is the average of $v$ over $\mathbb{B}_r$. We finally have obtained

\[ \frac{1}{r^3} \int_{\mathbb{B}_r} \left( |v - \bar{v}_r|^2 + |q - q_r|^2 \right) \, dx \leq C r^2. \tag{32} \]
Using the hybrid inequality, iteration gives as in [9, p.1244]
\[
\frac{1}{\theta r_1} \int_{B_{r_1}} (|\nabla n|^2 + |\nabla p|^2) < \theta \epsilon^2,
\]
which leads a contradiction to (23). This completes the proof. \(\square\)

**Lemma 3.5.** Let \( a \in \Omega \). Suppose that \((n, p, \varphi) \in A\) is a minimizer for \(E\) in (5). Assume further that there exist \( \epsilon > 0 \) and \( r_0 > 0 \) with \( B_{r_0}(a) \subset \Omega \) such that
\[
\frac{1}{r_0} \int_{B_{r_0}(a)} (|\nabla n|^2 + |\nabla p|^2) \, dx < \epsilon^2.
\]
Then there exists \( \beta \in (0, 1) \) such that for any \( r \) with \( r < r_0 \),
\[
\frac{1}{r} \int_{B_{r}(a)} (|\nabla n|^2 + |\nabla p|^2) \, dx \leq C r^\beta.
\]
Moreover, \( n \) and \( p \) are Hölder continuous at \( a \).

**Proof.** The decay estimate (34) can be obtained from the previous Lemma 3.4 by the standard method of iteration. It then follows that \( n \) and \( p \) are Hölder continuous in a neighborhood of \( a \). Here, we just give the sketch of proof since its verification is rather standard. Using the estimate in Lemma 3.4, iteration leads to the inequality
\[
E_{\theta r_1}(n, p) \leq \theta^{2k} E_r(n, p) + \eta \frac{\theta^{1 - \beta}}{1 - \theta^{1 - \beta}} (r \theta^k)^{1 + \beta},
\]
where \( 0 < \theta < 1 \) and for some \( \eta > 0 \). Let \( r_1 > 0 \) be a fixed positive number. For any \( 0 < \rho < r_1 \) we can choose a positive integer \( m \) such that \( \theta^m r_1 < \rho \leq \theta^{m-1} r_1 \).

Due to the above estimate, we have
\[
E_\rho(n, p) \leq E_{\theta^{k-1} r_1}(n, p) \leq \frac{\eta}{1 - \theta^{1 - \beta}} (r_1 \theta^m)^{1 + \beta} + \theta^{2(m-1)} E_{r_1}(n, p)
\]
\[
\leq \frac{\eta}{1 - \theta^{1 - \beta}} \theta^{1 + \beta} + \frac{1}{\theta^2} \rho^2 E_{r_1}(n, p) \leq C (\rho^{1 + \beta} + \rho^2) \leq C \rho^{1 + \beta},
\]
where \( C = C(r_1, \theta, \beta, \eta) > 0 \). Therefore, we obtain \( \frac{1}{\rho} E_\rho(n, p) \leq C \rho^\beta \). This completes the proof. \(\square\)

**Theorem 3.6.** Let \((n, p, \varphi) \in A\) be a minimizer for \(E\) in (5). Then \( n \) and \( p \) are locally Hölder continuous in \( \Omega \setminus Z \), where \( Z \) is a closed subset of \( \Omega \) of one dimensional Hausdorff measure zero. In addition, if \( K_1 = K_2 = K_3 = K > 0 \) in (4), then the possible singular set \( Z \) is a set of finite points in \( \Omega \).

**Proof.** We set
\[
Z = \left\{ a \in \Omega : \limsup_{r \to 0} r^{-1} \int_{B_r(a)} (|\nabla n|^2 + |\nabla p|^2) \, dx > 0 \right\}.
\]
Then \( Z \) is a closed set, which is of 1-dimensional Hausdorff measure zero. Due to Lemma 3.5, it is immediate that \( n \) and \( p \) are locally Hölder continuous in \( \Omega \setminus Z \). Hence we obtain the first statement.

It remains to verify the second statement. From now on, we suppose that \( K_1 = K_2 = K_3 = K > 0 \) and \( a \in Z \). For convenience, we write \( B_r(a) \) as \( B_r \) without any confusion. In order to show that \( Z \) is at most discrete set for such case, we need the following monotonicity type inequality as in [9, p.1245]
\[
\frac{1}{s} \int_{B_s} (K|\nabla n|^2 + B|\nabla p|^2) - \frac{1}{r} \int_{B_r} (K|\nabla n|^2 + B|\nabla p|^2) \geq -C(s^\beta - r^\beta)
\]
for some $C > 0$ and $0 < r < s < \text{dist}(a, \partial \Omega)$. Here $B$ is the constant in (4). Next we define $\mathbf{n}, \mathbf{p}$ by
\[
(\hat{n}(x), \hat{p}(x)) = \begin{cases} 
(n(a + t \frac{x-a}{|x-a|}), p(a + t \frac{x-a}{|x-a|})) & \text{if } x \in B_t(a), \\
(n(x), p(x)) & \text{if } x \in \Omega \setminus B_t(a),
\end{cases}
\]
for $r < t < s$. Then the almost minimality property of $(\mathbf{n}, \mathbf{p})$ yields
\[
\mathcal{E}_t(\mathbf{n}, \mathbf{p}) \leq \mathcal{E}_t(\mathbf{n}, \mathbf{p}) + ct^{1+\beta}.
\]
(36)

Since $K_1 = K_2 = K_3 = K$, using (10) we have
\[
K_1(\nabla \cdot \mathbf{n})^2 + K_2(\mathbf{n} \cdot \nabla \times \mathbf{n})^2 + K_3|\mathbf{n} \times \nabla \times \mathbf{n}|^2 + (K_2 + K_4)(\text{tr} (\nabla \mathbf{n})^2 - (\nabla \cdot \mathbf{n})^2)
\]
\[
= K(|\nabla \mathbf{n}|^2 + K_4(\text{tr} (\nabla \mathbf{n})^2 - (\nabla \cdot \mathbf{n})^2)).
\]
(37)

It follows from (36) and (37) that
\[
\int_{B_t(a)} (K|\nabla \mathbf{n}|^2 + B|\nabla \mathbf{p}|^2) \, dx + K_4 \int_{\partial B_t(a)} ((\nabla \mathbf{n}) \mathbf{p} - (\nabla \cdot \mathbf{n}) \mathbf{p}) \cdot \nu \, d\mathcal{H}
\]
\[
\leq \int_{B_t(a)} (K|\nabla \mathbf{n}|^2 + B|\nabla \mathbf{p}|^2) \, dx + K_4 \int_{\partial B_t(a)} ((\nabla \mathbf{n}) \mathbf{p} - (\nabla \cdot \mathbf{n}) \mathbf{p}) \cdot \nu \, d\mathcal{H} + Ct^{1+\beta}.
\]

Since $\mathbf{n} = \hat{n}$ on $\partial B_t(a)$, we get
\[
\int_{B_t(a)} (K|\nabla \mathbf{n}|^2 + B|\nabla \mathbf{p}|^2) \, dx \leq \int_{\partial B_t(a)} (K|\nabla \mathbf{n}|^2 + B|\nabla \mathbf{p}|^2) \, d\mathcal{H} + Ct^{1+\beta}
\]
\[
= t \int_{\partial B_t(a)} (K|\nabla \mathbf{n}|^2 + B|\nabla \mathbf{p}|^2) \, d\mathcal{H} + ct^{1+\beta}
\]
\[
= t \frac{d}{dt} |_{r=t} \int_{B_t(a)} (K|\nabla \mathbf{n}|^2 + B|\nabla \mathbf{p}|^2) - t \int_{\partial B_t(a)} \left( K \left| \frac{\partial \mathbf{n}}{\partial r} \right|^2 + B \left| \frac{\partial \mathbf{p}}{\partial r} \right|^2 \right) + Ct^{1+\beta}
\]
for some $D > 0$. Dividing by $t^2$ followed by integrating with respect to $t$ from $r$ to $s$ yields
\[
\frac{1}{s} \int_{B_t(a)} (K|\nabla \mathbf{n}|^2 + B|\nabla \mathbf{p}|^2) - \frac{1}{r} \int_{B_t(a)} (K|\nabla \mathbf{n}|^2 + B|\nabla \mathbf{p}|^2)
\]
\[
\geq \int_r^s \frac{1}{t} \int_{\partial B_t(a)} \left( K \left| \frac{\partial \mathbf{n}}{\partial r} \right|^2 + B \left| \frac{\partial \mathbf{p}}{\partial r} \right|^2 \right) \, d\mathcal{H} - C(s^3 - r^3).
\]
(38)

Hence we obtain
\[
\frac{1}{s} \int_{B_t(a)} (K|\nabla \mathbf{n}|^2 + B|\nabla \mathbf{p}|^2) - \frac{1}{r} \int_{B_t(a)} (K|\nabla \mathbf{n}|^2 + B|\nabla \mathbf{p}|^2) \geq -C(s^3 - r^3).
\]

The rest of the proof remains almost the same as in [9] with minor changes, and therefore, we skip its details. \hfill \Box

Remark 1. 1. It is not clear whether or not the condition $K_1 = K_2 = K_3$ is necessary in Theorem 3.6 to ensure the size of $Z$ is finite, and thus we leave it open.

2. For the boundary regularity, i.e. $a \in \partial \Omega$, the previous arguments up to Lemma 3.3 replacing $B_{r}(a)$ by $\Omega \cap B_{r}(a)$ still work with some changes, but it is not obvious how to get a decay rate corresponding to Lemma 3.4 for this case.

Acknowledgments. The authors would like to think the referees for their careful reading the paper with valuable comments and suggestions.
REFERENCES


Received July 2011; revised September 2012.

E-mail address: kkang@yonsei.ac.kr
E-mail address: jhpark2003@cnu.ac.kr, jhpark2003@gmail.com