Communications in Partial Differential Equations

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Published online: 29 Aug 2007.

To cite this article: Dongho Chae, Kyungkeun Kang & Jihoon Lee (2007): On the Interior Regularity of Suitable Weak Solutions to the Navier-Stokes Equations, Communications in Partial Differential Equations, 32:8, 1189-1207

To link to this article: http://dx.doi.org/10.1080/03605300601088823

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On the Interior Regularity of Suitable Weak Solutions to the Navier–Stokes Equations

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We obtain the interior regularity criteria for the vorticity of “suitable” weak solutions to the Navier–Stokes equations. We prove that if two components of a vorticity belong to $L^q_t \cap L^r_x$ in a neighborhood of an interior point with $3/p + 2/q \leq 2$ and $3/2 < p < \infty$, then solution is regular near that point. We also show that if the direction field of the vorticity is in some Triebel–Lizorkin spaces and the vorticity magnitude satisfies an appropriate integrability condition in a neighborhood of a point, then solution is regular near that point.

Keywords Interior regularity criterion; Navier–Stokes equations.

Mathematics Subject Classification 35Q30; 76D05.

1. Introduction

In this paper, we study the interior regularity problem for suitable weak solutions to the Navier–Stokes equations in three dimension with zero external force:

\[
\begin{align*}
\frac{\partial v}{\partial t} + (v \cdot \nabla)v - \Delta v &= -\nabla p \\
\text{div} \, v &= 0
\end{align*}
\]

in $Q_T := \Omega \times (0, T)$, (1.1)

where $\Omega$ is a domain in $\mathbb{R}^3$, $v$ is the flow velocity, and $p$ is the scalar pressure. We are especially concerned with the initial boundary value problem on a bounded and smooth domain, and therefore, we require together with (1.1) initial and boundary
conditions:
\[
\begin{align*}
  v(x, 0) &= v_0(x), \quad x \in \Omega \\
  v(x, t) &= 0, \quad x \in \partial \Omega, \quad 0 < t < T.
\end{align*}
\tag{1.2}
\]

Here the initial data should satisfy the compatibility condition, i.e., \( v_0 = 0, \) \( x \in \partial \Omega \) and \( \text{div} v_0 = 0 \) in \( \Omega \). By suitable weak solutions we mean functions which solve the Navier–Stokes equations in the sense of distribution, satisfy some integrability conditions, and satisfy the local energy inequality (for details, see Definition 5 in Section 2). For a point \( z = (x, t) \) in \( \Omega \times (0, T] \), we denote
\[
B_x := \{ y \in \mathbb{R}^3 : |y - x| < r \}, \quad Q_{x,r} := B_x \times (t - r^2, t).
\]

A solution \( v \) is said to be regular at \( z \) if \( v \) is bounded in \( Q_{x,r} \) for some \( r > 0 \) and such point is called a regular point.

Since the fundamental results by Leray (1934) and Hopf (1951) on the existence for the weak solutions of the Navier–Stokes equations, the regularity question of weak solutions has been regarded as one of the most outstanding problems in the mathematical fluid mechanics. Imposing additional condition on the weak solutions,
\[
\|v\|_{L^p_t(L^q)} := \|v(\cdot, t)\|_{L^p(Q_T)} < \infty, \quad \frac{3}{p} + \frac{2}{q} \leq 1, \quad 3 \leq p \leq \infty,
\tag{1.3}
\]
it was proved that any weak solution becomes unique and regular in \( Q_T \). The first result in this direction was obtained in Ladyzhenskaya (1967), Ohayma (1960), Prodi (1959), and Serrin (1962), in the case that \( 3/p + 2/q < 1 \) and \( 3 < p \leq \infty \). Later, Fabes et al. (1972) extended Serrin’s result to the limiting case \( 3/p + 2/q = 1 \) and \( 3 < p \leq \infty \) for \( \Omega = \mathbb{R}^n \) (see also Sohr, 1983; Giga, 1986 for the results on a bounded domain). For the local problem, the interior case was proved by Struwe (1988) (see Takahashi, 1990 for the extension to the localization in Lorentz space), and this result was extended up to a flat boundary by the second author Kang (2004) and to curved boundary by Takahashi (1992) and Solonnikov (2002). The borderline case \( p = 3 \) and \( q = \infty \) was solved by Escauriaza et al. (2003). We also reference Kozono and Taniuchi (2000) for a refinement of the case \( p = \infty, q = 2 \), replacing \( \|v\|_{L^\infty_t} \) by \( \left( \int_0^T \|v\|_{\text{BMO}}^2 \right)^{1/2} \).

On the other hand, there have been works on regularity criteria concerning the gradient of velocity or vorticity rather than the velocity. Taking curl of the momentum equations of (1.1), we obtain the following vorticity equations:
\[
\frac{\partial \omega}{\partial t} + (v \cdot \nabla) \omega - \Delta \omega = (\omega \cdot \nabla)v,
\tag{1.4}
\]
where the vorticity \( \omega \) is defined by \( \omega = \text{curl} \, v \). By the Biot–Savart’s law, taking the incompressibility of \( v \) into account, we can represent \( v \) as the form of a singular integral of \( \omega \), namely,
\[
v(x, t) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{y \times \omega(x + y, t)}{|y|^3} dy.
\tag{1.5}
\]
for sufficiently rapidly decaying vorticity near infinity. Beirão da Veiga (1995) obtained a sufficient condition for regularity in terms of $\nabla v$ instead of the velocity, which is equivalent to the one in terms of the vorticity due to the Calderon–Zygmund inequality. More precisely, in Beirão da Veiga (1995), it was proved that if $\nabla v$ satisfies

$$\nabla v \in L^q(0, T; L^p(\mathbb{R}^3)), \quad \frac{3}{p} + \frac{2}{q} \leq 2, \quad \frac{3}{2} \leq p \leq \infty,$$

then $v$ is regular. This result was improved in Chae and Choe (1999), imposing the condition (1.7) below only for the two components of the vorticity. To be more precise, let $\tilde{\omega} = \omega_1 e_1 + \omega_2 e_2$, and $\omega_3 e_3 (e_i, i = 1, 2, 3$ are standard basis for $\mathbb{R}^3$). If the $\tilde{\omega}$ satisfies

$$\tilde{\omega} \in L^q(0, T; L^p(\mathbb{R}^3)), \quad \frac{3}{p} + \frac{2}{q} \leq 2, \quad \frac{3}{2} \leq p < \infty,$$

then $v$ becomes a regular solution (see Chae and Choe, 1999, Theorem 1).

One of our main results is to establish the local version of the regularity criteria (1.7) for the interior case.

**Theorem 1.** Let $z_0 = (x_0, t_0) \in Q_T, \ Q_{z_0}, \in Q_T, \ and \ e_i, \ i = 1, 2, 3$ be the standard basis for $\mathbb{R}^3$. If $v$ is a suitable weak solution of the Navier–Stokes equations (1.1)–(1.2) and $\tilde{\omega} := \omega_1 e_1 + \omega_2 e_2$, the planar parts of the vorticity $\omega = \text{curl} v$, where the derivatives are in the distribution sense, satisfy

$$\tilde{\omega} \in L^q(0, T; L^p(\mathbb{R}^3)), \quad \frac{3}{p} + \frac{2}{q} \leq 2, \quad \frac{3}{2} \leq p < \infty,$$

then $z_0$ is a regular point.

**Remark 1.** For the case $p = 3/2$ and $q = \infty$, if we add the smallness condition on $\tilde{\omega}$, i.e., $\|\tilde{\omega}\|_{L^q(0, T; L^p(\mathbb{R}^3))} < \varepsilon$ for a sufficiently small $\varepsilon > 0$, then $z_0$ is a regular point.

**Remark 2.** For the axially symmetric Navier–Stokes equations, the condition (1.8) is imposed on only the angle component of the vorticity in the interior (Chae and Lee, 2002) (compare to Neustupa and Pokorný, 2000 for the condition of velocity in the interior for axisymmetric solutions).

On the other hand, Constantin and Fefferman (1993) obtained sufficient conditions on $\xi(x, t) = \omega(x, t) / |\omega(x, t)|$, the direction of the vorticity, for the existence of a smooth solution of the Navier–Stokes equations in $\mathbb{R}^3$. They proved that if $\theta(x, y, t)$, the angle between $\xi(x, t)$ and $\xi(x + y, t)$, satisfies

$$|\sin \theta(x, y, t)| \leq C|y|,$$

then the solution becomes regular. In Beirão da Veiga and Berselli (2002), the condition (1.9) was relaxed in the form of the following regularity criterion:
Suppose there exist \( s \in [1/2, 1] \), a constant \( K > 0 \) and \( g \in L^q(0, T; L^p(\mathbb{R}^3)) \) where
\[
\frac{3}{p} + \frac{2}{q} = s - \frac{1}{2}, \quad q \in \left[ \frac{4}{2s-1}, \infty \right],
\]
such that \( |\sin \theta(x, y, t)| \leq g(x, t)|y|^s \) in a region where the vorticity at both \( x \) and \( x + y \) is larger than \( K \). Then the solution is regular. Moreover, for the case \( s \in [0, 1/2] \), additional condition on the integrability of vorticity together with its direction ensures the regularity of weak solutions (Beirão da Veiga, 2003): namely if
\[
|\sin \theta(x, y, t)| \leq |y|^s \quad \text{and} \quad \omega \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{3}{p} = s + 1
\]
for some \( s \in [0, 1/2] \) in the region where \( |\omega(x, t)| \) and \( |\omega(x + y, t)| \) are bigger than some large constant \( K > 0 \), then the solution becomes regular on \([0, T]\). The first author (Chae, 2007) recently unified and refined the above results regarding regularity conditions on the direction fields using the norms of the Triebel–Lizorkin type. The main result in Chae (2007) is as follows: Suppose there exist \( s \in (0, 1) \), \( q \in (3/(3 - s), \infty) \), \( p_1 \in (1, \infty) \), \( p_2 \in (1, 3/s) \), and \( r_1, r_2 \in [1, \infty] \) satisfying
\[
\frac{3}{p_1} + \frac{3}{p_2} + \frac{2}{r_1} + \frac{2}{r_2} \leq 2 + s, \quad \frac{s}{3} < \frac{1}{p_1} + \frac{1}{p_2} < \frac{2 + s}{3}, \quad \frac{1}{p_2} + \frac{1}{q} < 1 + \frac{s}{3}, \quad (1.10)
\]
such that
\[
\tilde{\xi}(x, t) \in L^{p_1}(0, T; \mathbb{T}_{p_1,q}(\mathbb{R}^3)), \quad \omega(x, t) \in L^{p_2}(0, T; L^{q_2}(\mathbb{R}^3)),
\]
then there is no singularity up to time \( T \). Here \( \mathbb{T}_{p_1,q}(\mathbb{R}^3) \) denotes Triebel–Lizorkin type of function space on \( \mathbb{R}^3 \), which will be briefly reviewed in section 2 (see Chae, 2007; Triebel, 1992 for the details). We also introduce, for our purpose, some local homogeneous Triebel–Lizorkin type function spaces \( \mathbb{T}_{p_1,q}(B_{xy}) \) (see Definition 4 in section 2 for the details). For the other applications of this type of spaces in related fluid equations we refer to Chae (2005, 2006a,b,c), while for the applications of the geometric structures of the ‘vortex stretching terms’ in the Euler equations and the quasi-geostrophic equations, we refer to Constantin (1994), Constantin et al. (1996), Deng et al. (2005), and references therein.

Our second main theorem is a localization of the main result in Chae (2007).

**Theorem 2.** Let \( v \) be a suitable weak solution of the Navier–Stokes equations (1.1)–(1.2) on \( Q_T \) and let \( z_0 = (x_0, t_0) \in Q_T \) and \( Q_{w,T} \subset Q_T \). We set \( \tilde{\xi}(x, t) = \omega(x, t)/|\omega(x, t)| \) (for \( \omega(x, t) \neq 0 \)), \( \omega = \text{curl} \, v \) as the directional field of the vorticity, where the derivatives in the curl operation is defined in the sense of distribution. Suppose there exist \( s \in (0, 1) \), \( q \in (1, \infty) \), \( p_1 \in (1, \infty) \), \( p_2 \in (1, 3/s) \), and \( r_1, r_2 \in [1, \infty] \) such that
\[
\frac{s}{3} < \frac{1}{p_1} + \frac{1}{p_2} < \frac{2 + s}{3}, \quad \frac{1}{p_2} + \frac{1}{q} < 1 + \frac{s}{3}, \quad \frac{3}{p_1} + \frac{3}{p_2} + \frac{2}{r_1} + \frac{2}{r_2} \leq 2 + s,
\]
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for which we have

\[ \bar{\zeta}(x, t) \in L^\infty(t_0 - r^2, t_0; \mathcal{F}_{p,q}(B_{x_0})), \quad \omega(x, t) \in L_{t,x}^{p_1}(Q_{x_0,r}). \]  

(1.12)

Then, \( z_0 \) is a regular point.

Remark 3. Since we know \( \omega \in L^2(Q_{x_0,r}) \) holds for the weak solutions, the second condition in (1.12) is unnecessary provided that \( s \geq 1/2 \), and therefore, imposing the above condition only on the directional field \( \zeta(x, t) \) for such case, namely, \( \zeta \in L^\infty(t_0 - r^3, t_0; \mathcal{F}_{p,q}(B_{x_0})) \) with \( 3/p_1 + 2/r_1 \leq s - 1/2 \), we obtain the regularity.

This paper is organized as follows: In section 2, we define the local version of Triebel–Lizorkin function spaces and recall the notion of suitable weak solutions that was introduced in Caffarelli et al. (1982) (this concept was earlier used by Scheffer, 1977) and slightly modified later in Lin (1998). We also remind a useful lemma to our analysis obtained previously in Neustupa and Penel (1999). In section 3, we obtain interior regularity criteria for the gradient of the velocity (Proposition 1) and for vorticity (Proposition 2), and then, using Biot–Savart’s law and above regularity criteria, we prove Theorem 1. Finally, using the geometric structures of the vortex stretching term, we present the proof of Theorem 2, adapting the methods used in Theorem 1.

2. Preliminaries

In this section we introduce some notations, review Triebel–Lizorkin function spaces, define suitable weak solutions, and recall a lemma proved in Neustupa and Penel (1999).

We start with notations. We denote by \( \Omega \) an open and smooth domain.

For \( 1 \leq q \leq \infty \), \( W^{k,q}(\Omega) \) denotes the usual Sobolev space, i.e., \( W^{k,q}(\Omega) = \{ u \in L^q(\Omega) : D^k u \in L^q(\Omega), 0 \leq |\alpha| \leq k \} \).

We denote by \( \bar{f}_E f \) the average of \( f \) on \( E \); i.e., \( \bar{f}_E f = \int_E f/|E| \).

Finally, by \( C = C(x, \beta, \ldots) \) we denote a constant depending on the prescribed quantities \( x, \beta, \ldots, \) which may change from line to line.

Next, we review some function spaces, so called Triebel–Lizorkin spaces. The functions in the Triebel–Lizorkin space on \( \Omega \) are characterized by differences (e.g., see Triebel, 1992, 5.2.2, p. 245). Let

\[ \Delta^M_h f(x) = \sum_{j=0}^{M} (-1)^{M-j} \binom{M}{j} f(x + jh), \quad M \in \mathbb{N}, \quad x, h \in \mathbb{R}^n, \]

where \( \binom{M}{j} \) is the binomial coefficient. We set

\[ V^M(x, t) = \{ h \in \mathbb{R}^n : |h| < t \text{ and } x + \tau h \in \Omega, \text{ for all } 0 \leq \tau \leq M \}. \]

We define the operator \( d^M_{t,a} \) as follows:

\[ d^M_{t,a} f(x) = \left( \int_{V^M(x,t)} |\Delta^M_h f(x)|^\alpha dh \right)^{\frac{1}{\alpha}}, \quad x \in \Omega, \quad t > 0. \]

Now we recall the definition of Triebel–Lizorkin space (see Triebel, 1992, p. 245).
Definition 3. Let \(0 < p < \infty, 0 < q \leq \infty, 1 \leq r \leq \infty, s > n \max \left\{ \frac{1}{p} - \frac{1}{r}, \frac{1}{q} - \frac{1}{r}, 0 \right\} \), \(0 < u \leq r\), and \(M > s\). Then the Triebel–Lizorkin function space, \(F_{p,q}^s\), is given as follows:

\[
F_{p,q}^s = \left\{ f \in L^{\max(p,r)}(\Omega) : \|f\|_p + \left\| \left( \int_0^1 t^{-uq}d_{r,q}f(t)^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_p < \infty \right\}
\]

in the sense of quasi-norms.

For the whole domain \(\mathbb{R}^3\), the first author (Chae, 2007) considered the homogeneous Triebel–Lizorkin norm for obtaining the regularity criterion on the direction of vorticity. For \(0 < s < 1\), \(1 \leq p \leq \infty\), and \(1 \leq q \leq \infty\), the seminorm of \(\hat{\mathcal{F}}_{p,q}^s\) is defined as follows:

\[
\|f\|_{\hat{\mathcal{F}}_{p,q}^s} = \begin{cases} \left\| \int_{\mathbb{R}^3} \frac{|f(x+y) - f(x)|^q}{|y|^{s+q}} dy \right\|_{L^p(\mathbb{R}^3, dx)} & \text{if } 1 \leq p \leq \infty, 1 \leq q < \infty \\
\text{ess sup} \frac{|f(x+y) - f(x)|}{|y|^s} \left\| \right\|_{L^p(\mathbb{R}^3, dx)} & \text{if } 1 \leq p \leq \infty, q = \infty.
\end{cases}
\]

We observe that \(\hat{\mathcal{F}}_{\infty,\infty}^s \equiv C^s\) (here \(\hat{\mathcal{F}}_{\infty,\infty}^s\) is understood as quotient space by the equivalence relation that two functions are equivalent if difference of them is a constant) and \(\mathcal{F}_{p,2} := \hat{\mathcal{F}}_{p,2} \cap L^p = L^p(\mathbb{R}^3) = (1 - \Delta)^{-s} L^p(\mathbb{R}^3)\). In Definition 3, set \(u = r = q\) and \(M = 1\). We define the localized Triebel–Lizorkin type quasi-norms as follows.

Definition 4. For \(0 < s < 1\) and \(\Omega \Subset \Omega\),

\[
\|f\|_{\mathcal{F}_{p,q}^s(\Omega)} = \begin{cases} \left\| \int_0^1 t^{-uq}d_{r,q}f(t)^q \frac{dt}{t} \right\|_{L^p(\Omega, dx)} & \text{if } 1 \leq p \leq \infty, 1 \leq q < \infty \\
\text{ess sup} \frac{|f(x+y) - f(x)|}{|y|^s} \left\| \right\|_{L^p(\Omega, dx)} & \text{if } 1 \leq p \leq \infty, q = \infty.
\end{cases}
\]

Observe that \(\hat{\mathcal{F}}_{\infty,\infty}^s(\Omega) = C^s(\Omega)\).

We note that there are works about the solution of the Navier–Stokes equations in Triebel–Lizorkin space (see Cannone and Planchon, 1999).

Next we recall the definition of suitable weak solutions for the Navier–Stokes equations. The existence of suitable weak solutions was proved in Caffarelli et al. (1982) and a slightly modified definition, which we follow in this paper, was used in Lin (1998).

Definition 5. Let \(Q_T = \Omega \times I\) where \(\Omega \subset \mathbb{R}^3\) and \(I = [0, T]\). A pair \((v, p)\) is a suitable weak solution (1.1)–(1.2) of the Navier–Stokes equations if the following conditions are satisfied:

(a) The functions \(v : Q_T \to \mathbb{R}^3\) and \(p : Q_T \to \mathbb{R}\) satisfy

\[
v \in L^\infty(I; L^2(\Omega)) \cap L^2(I; W^{1,2}(\Omega)), \quad p \in L^2(Q_T).
\]
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(b) The functions $v$ and $p$ solve the Navier–Stokes equations (1.1) in $Q_T$ in the sense of distributions and $v$ satisfies the boundary and initial conditions (1.2).

(c) The functions $v$ and $p$ satisfy the local energy inequality

$$
\int_\Omega |v(x, t)|^2 \phi(x, t) dx + 2 \int_0^t \int_\Omega |\nabla v(x, t')|^2 \phi(x, t') dx dt' 
\leq \int_0^t \int_\Omega (|v|^2 (\partial_t \phi + \Delta \phi) + (|v|^2 + 2p)v \cdot \nabla \phi) dx dt' 
$$

for almost all $t \in (0, T)$ and for all nonnegative functions $\phi \in C_0^\infty(Q_T)$.

It is known that the onedimensional parabolic Hausdorff measure of the set $\mathcal{F}$ of possible interior singular points of suitable weak solutions is zero, denoted $\mathcal{H}^1(\mathcal{F}) = 0$ (see Caffarelli et al., 1982). On the other hand, it is also known that for weak solutions there exists a set $E \subset I = [0, T]$ such that $E$ is closed, of 1/2-dimensional Hausdorff measure zero, and solutions are regular in $I \setminus E$ (see e.g., Foias and Temam, 1979; Galdi, 2000; Heywood, 1980). $E$ can be, in fact, written as $\bigcup_{i \in \mathcal{J}} I_i$, where set $\mathcal{J}$ is at most countable, and $I_i = (x_i, \beta_i)$ are disjoint open intervals in $[0, T]$. In accordance with the terminology used in Galdi (2000), in what follows, we call the instant time $\beta_i$ an epoch of possible irregularity. Here we note that the left end point, $x_i$, of $I_i$ is a regular point from the construction of such open interval (see e.g., Leray, 1934, Structure Theorem or Galdi, 2000, Theorem 6.3).

We conclude this section by recalling a fact proved in Neustupa and Penel (1999, Lemma 1) for suitable weak solutions. We slightly modify the statement for our purpose. Since its verification can be done by following the proof of Lemma 1 in Neustupa and Penel (1999) line by line, we omit the proof.

**Lemma 6.** Let $z_0 = (x_0, t_0) \in Q_T$. Suppose $v$ is a suitable weak solution of the Navier–Stokes equations (1.1)–(1.2) in $Q_T$ and $t_0$ be an epoch of possible irregularity. Then there exist positive numbers $\tau$, $r_1$, and $r_2$ with $r_1 < r_2$ such that the followings are satisfied

(a) $\tau$ is sufficiently small so that $t_0$ is only one epoch of possible irregularity in time interval $[t_0 - \tau, t_0]$.

(b) The closure of $B_{r_2}(x_0) \times (t_0 - \tau, t_0)$ is contained in $Q_T$, i.e., $\overline{B_{r_2}(x_0)} \times [t_0 - \tau, t_0] \subset Q_T$.

(c) $((\overline{B_{r_2}(x_0)} - B_{r_1}(x_0)) \times [t_0 - \tau, t_0]) \cap \mathcal{F} = \emptyset$, where $\mathcal{F}$ is the set of possible singular points of $v$.

(d) $v$, $v_t$, and $p$ are, together with all their space derivatives, continuous on $\overline{(B_{r_2}(x_0) - B_{r_1}(x_0)) \times [t_0 - \tau, t_0]}$.

**Remark 4.** We remark that all assertions except for the property (a) in Lemma 6 are valid for any point $z = (x, t) \in Q_T$ for suitable weak solutions. Main point in this argument is to use the fact that one-dimensional Hausdorff measure of possible singular set is zero for suitable weak solutions (see the proof of Lemma 1 in Neustupa and Penel, 1999 for the details). For the property (a), it cannot be, however, omitted that $t_0$ is assumed to be an epoch of possible irregularity.
3. Proofs of Main Theorems

In this section we present the proofs of our main theorems. We start with the following proposition, which is a local version of the regularity criteria for \( \nabla u \) proved in Beirão da Veiga (1995).

**Proposition 1.** Let \( z_0 = (x_0, t_0) \in Q_T \) and \( Q_{0,t_0} \subset Q_T \). If \( v \) is a suitable weak solution of the Navier–Stokes equations (1.1)–(1.2), and \( \nabla u \) satisfies

\[
\nabla v \in L^\gamma_{\text{loc}}(Q_{0,t_0}), \quad \frac{3}{\gamma} + \frac{2}{q} \leq 2, \quad \frac{3}{2} \leq \gamma \leq \infty, \tag{3.1}
\]

then \( z_0 \) is a regular point.

**Proof.** Suppose \( t_0 \) is an epoch of possible irregularity. Suppose further that \( r_1, r_2 \), and \( \tau \) are the positive numbers in Lemma 6. We denote, for convenience, \( B_1 := B_{x_0,r_1} \) and \( B_2 := B_{x_0,r_2} \). Now we localize the equations (1.1) near \( x_0 \) in the spacial variables. More precisely, we choose a cut-off function \( \varphi \in C_0^\infty(B_2) \) such that \( \varphi = 1 \) on \( B_1 \). Following the method in Neustupa and Pokorný (2000), we set \( u = \varphi v - V \), where \( V \) satisfies \( \text{div} V = v \cdot \nabla \varphi \). We note that \( V \) can be chosen to be compactly supported in \( B_2 \setminus B_1 \) and all the space derivatives of \( V \) and \( \partial_t V \) are regular (see e.g., Galdi, 1994).

It is straightforward that \( u \) satisfies the following equations:

\[
\partial_t u + (u \cdot \nabla) u = h - \nabla(p \varphi) + \Delta u, \quad \text{div} u = 0, \tag{3.2}
\]

where

\[
h = -\partial_t V - (\varphi v \cdot \nabla)V - (V \cdot \nabla)(\varphi v) + (V \cdot \nabla)V + (\varphi v \cdot \nabla)v \\
+ \varphi(\varphi - 1)(v \cdot \nabla) v - 2(\nabla \varphi \cdot \nabla)v - v \Delta \varphi + \Delta V + p \nabla \varphi.
\]

One can see that \( h(t, t) \) is supported only in \( \overline{B_2} \setminus B_1 \) for each \( t \in [t_0 - \tau, t_0] \) and, moreover, sufficiently regular in the region. Multiplying \( |u|u \) on the both sides of (3.2) and integrating over \( B_2 \), we have the following inequality

\[
\frac{d}{dt} \int_{B_2} |u|^3 \, dx + \int_{B_2} |\nabla |u|^2| \, dx \leq C \int_{B_2} |p \varphi| |\nabla u| |u| \, dx + C \int_{B_2} |h||u|^2 \, dx := I_1 + I_2.
\]

Taking the divergence on the both sides of (3.2), we obtain

\[
-\Delta(p \varphi) = -\text{div} h + \sum \partial_i \partial_j (u, u).
\]

Noting that the above equation can be extended to the whole domain, we can have, due to the standard elliptic estimates for \( \Delta \),

\[
\|p \varphi\|_k \leq C(\|(-\Delta)^{-1} \text{div} h\|_k + \|u\|_{H^k}), \quad 1 < k < \infty.
\]

We consider first the case that \( \gamma \neq \frac{3}{2} \). Using the above inequality, the Hölder inequality, the Young inequality and the Gagliardo–Nirenberg inequality, \( I_1 \) and \( I_2 \) can be estimated as follows:

\[
I_1 \leq C\|p \varphi\|_{\frac{3}{3-\gamma}} \|u\|_{\frac{3}{2-\gamma}} \|\nabla u\|_{\gamma} \leq C\|u\|_{H_{\gamma}} \|\nabla u\|_{\gamma} + C\|u\|_{H_{\gamma}} \|\nabla u\|_{\gamma},
\]

\[
I_2 \leq C\|p \varphi\|_{\frac{3}{3-\gamma}} \|u\|_{\frac{3}{2-\gamma}} \|\nabla u\|_{\gamma} \leq C\|u\|_{H_{\gamma}} \|\nabla u\|_{\gamma} + C\|u\|_{H_{\gamma}} \|\nabla u\|_{\gamma}.
\]
Using the Gronwall's inequality, we obtain
\[ \leq C \|u\|_{3}^{\frac{2+\gamma}{\gamma}} \|\nabla |u|^{\frac{1}{2}}\|_{2}^{\frac{1}{2}} \|\nabla u\|_{y} + C \|u\|_{3}^{\frac{2-\gamma}{\gamma}} \|\nabla |u|^{\frac{1}{2}}\|_{2}^{\frac{1}{2}} \|\nabla u\|_{y} \]
\[ \leq C \|u\|_{3}^{\frac{2+\gamma}{\gamma}} \|\nabla u\|_{2}^{\frac{2+\gamma}{\gamma}} + \frac{1}{2} \|\nabla |u|^{\frac{1}{2}}\|_{2}^{\frac{1}{2}} + C \|u\|_{3}^{\frac{2-\gamma}{\gamma}} \|\nabla u\|_{\gamma}^{\frac{2+\gamma}{\gamma}} \]
\[ \leq C \|u\|_{3}^{\frac{2+\gamma}{\gamma}} \|\nabla u\|_{2}^{\frac{2+\gamma}{\gamma}} + \frac{1}{2} \|\nabla |u|^{\frac{1}{2}}\|_{2}^{\frac{1}{2}} + C \]

and
\[ I_{2} \leq C \|u\|_{3}^{2} \leq C \|u\|_{3}^{2} + C. \]

Using the Gronwall’s inequality, we obtain
\[ \sup_{t_{0}-\tau \leq t \leq t_{0}} \|u(\cdot, t)\|_{3}^{2} + \int_{t_{0}-\tau}^{t_{0}} \|\nabla |u|^{\frac{1}{2}}(\cdot, t)\|_{2}^{\frac{1}{2}} \|\nabla u(\cdot, t)\|_{\gamma}^{\frac{2+\gamma}{\gamma}} dt \leq C \int_{t_{0}-\tau}^{t_{0}} \|\nabla u(\cdot, t)\|_{\gamma}^{\frac{2+\gamma}{\gamma}} dt. \]

From the assumption (3.1), we have the boundedness of \( \|u(\cdot, t)\|_{3} \) in time variable. Therefore, due to Escauriaza et al. (2003), \( t_{0} \) should be a regular point of \( u \). For the case \( \gamma = 3/2 \), it is immediate that \( u \in L^{3}_{t}L^{3}_{x}(Q_{t_{0}, \frac{t_{0}}{2}}) \) by the Sobolev embedding theorem, which leads to the same conclusion as previous case.

Now we suppose that \( t_{0} \) is a singular time which is not an epoch of irregularity. We observe first that there exists a time \( t^{*} \) with \( t_{0} - t^{2} < t^{*} < t_{0} \) such that \( t^{*} \) is a regular time. We also note that there exists \( \tilde{r}_{1}, \tilde{r}_{2} \) with \( 0 < \tilde{r}_{1} < \tilde{r}_{2} < r \) such that \( v \) is regular on \( (B_{\tilde{r}_{1}} \backslash B_{\tilde{r}_{2}}) \times [t^{*}, t_{0}] \). As mentioned earlier in Remark 4, this can be done by using the fact that one-dimensional Hausdorff measure of possible singular set is zero for suitable weak solutions. We now claim that \( v \) is regular in \( B_{t_{0}^{\frac{1}{3}}} \times [t^{*}, t_{0}] \). Suppose that this is not the case. Then there is a point \( (y, s) \in B_{t_{0}^{\frac{1}{3}}} \times [t^{*}, t_{0}] \) such that \( v \) is singular at \( (y, s) \) and \( v \) is regular for all \( (x, t) \in B_{t_{0}^{\frac{1}{3}}} \times [t^{*}, t_{0}] \) and \( t < s \), i.e., \( (y, s) \) is the point where irregularity of \( v \) occurs in the first place in \( B_{t_{0}^{\frac{1}{3}}} \) later in time \( t^{*} \). Now we take a local neighborhood of \( (y, s) \) which is contained in \( B_{t_{0}^{\frac{1}{3}}} \times [t^{*}, t_{0}] \). Following the same proof as the previous case of an epoch of irregularity, we can show that \( (y, s) \) is, in fact, a regular point, which leads to a contradiction to the assumption that \( (y, s) \) is a singular point. Therefore, \( v \) should be regular in \( B_{t_{0}^{\frac{1}{3}}} \times [t^{*}, t_{0}] \), which immediately implies that \( (x_{0}, t_{0}) \) is a regular point. This completes the proof. 

Next, we give local regularity criteria of vorticity, instead of \( \nabla v \) by controlling \( \nabla v \) in terms of vorticity.

**Proposition 2.** Let \( z_{0} = (x_{0}, t_{0}) \in Q_{r} \) and \( Q_{w,r} \subset Q_{r} \). Suppose that \( \omega \) be the vorticity of a suitable weak solution \( v \) of the Navier–Stokes equations (1.1)–(1.2). If \( \omega \) satisfies
\[ \omega \in L^{p,q}_{t,x}(Q_{z_{0},t}), \quad \frac{3}{p} + \frac{2}{q} \leq 2, \quad \frac{3}{2} \leq p < \infty, \tag{3.3} \]
then \( z_{0} \) is a regular point.

**Proof.** We choose \( \psi \in C_{0}^{\infty}(\mathbb{R}^{3}) \) cut-off functions \( \varphi \) satisfying \( \text{supp} \varphi \subset B_{t_{0},r} \) and \( \varphi(x) = 1 \) on \( B_{t_{0},r} \). From the divergence free condition, we obtain the system
\[ \nabla \cdot (v\psi) = v \cdot \nabla \varphi, \quad \nabla \times (v\psi) = \omega \varphi + v \times \nabla \varphi. \tag{3.4} \]
We have, from the above system (3.4), the following integral formula for \( \psi(x) \):

\[
\psi(x) = K(v \cdot \nabla)(x) + \tilde{K}(\omega \psi)(x) + \tilde{K}(v \times \psi)(x),
\]

where

\[
 K(f)(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y)f(y)}{|x-y|^3} \, dy \quad \text{and} \quad \tilde{K}(g)(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times g(y)}{|x-y|^3} \, dy
\]

defined for scalar field \( f \) and vector field \( g \), respectively. Therefore, from (3.5), we obtain

\[
\|v\|_{L^p(B_{2r_1} \frac{1}{2})} \leq C(\|v\|_{L^p(B_{r_0} \frac{3}{2})} + \|\omega\|_{L^p(B_{r_0} \frac{3}{2})}), \quad 1 < p < \infty.
\]

It is known that \( v \in L^1_t L^2_x(Q_{t_0 \infty}) \) (see e.g., Foias et al., 1981). On the other hand, it is immediate that \( v \in L^\infty_t L^{q,p}_x(Q_{t_0 \infty}) \) from the fact that \( v \in L^\infty_t L^{q,p}_x(Q_{t_0 \infty}) \). Interpolating the spatial and temporal exponents of \( v \), we can see that \( v \in L^p_t L^q_x(Q_{t_0 \infty}) \), with \( 2/q + 3/p < 2 \) and \( 3/2 \leq p \leq \infty \). Combining the assumption (3.3) and the estimation (3.7), it is direct that \( \nabla v \in L^p_t L^q_x(Q_{t_0 \infty}) \), with \( 2/q + 3/p < 2 \) and \( \frac{3}{2} \leq p < \infty \). Therefore, due to Proposition 1, \( z_0 \) is a regular point for \( v \). This completes the proof. \( \square \)

Now we are ready to prove Theorem 1.

**Proof of Theorem 1.** Suppose that \( t_0 \) is an epoch of irregularity. Let \( r_1 \) and \( r_2 \) with \( r_1 < r_2 \) be the positive numbers in Lemma 6. We take smooth cut-off functions \( \varphi \) and \( \tilde{\varphi} \) satisfying

\[
\varphi(x) = \begin{cases} 
1 & \text{in } B_{r_1}, \\
0 & \text{in } \mathbb{R}^3 \setminus B_{r_1 + r_1^{\alpha/2}},
\end{cases} \quad \tilde{\varphi}(x) = \begin{cases} 
1 & \text{in } B_{r_2 + r_2^{1/2}}, \\
0 & \text{in } \mathbb{R}^3 \setminus B_{r_2}.
\end{cases}
\]

Multiplying \( \varphi^3(x) \) on the both sides of (1.4) and defining \( \sigma := \omega \varphi^3 \), we have

\[
\partial_t \sigma + (v \cdot \nabla)\sigma = v \cdot \nabla \varphi^3 \omega + (\sigma \cdot \nabla)\sigma + \Delta \sigma - 2 \nabla \omega \cdot \nabla \varphi^3 - \omega \Delta \varphi^3.
\]

Multiplying \( \sigma \) to the equations (3.8), and integrating by parts, we obtain

\[
\frac{1}{2} \frac{d}{dt} \| \sigma \|^2_2 + \| \nabla \sigma \|^2_2 = \int_{\Omega} (\sigma \cdot \nabla)v \cdot \sigma \, dx + \int_{\Omega} v \cdot \nabla \varphi^3 \sigma \omega \, dx - 2 \int_{\Omega} \nabla \omega \cdot \nabla \varphi^3 \sigma \, dx - \int_{\Omega} \omega \Delta \varphi^3 \sigma \, dx := I + II + III + IV.
\]

We decompose the vorticity \( \omega \) as follows: \( \omega = \tilde{\omega} + \omega', \tilde{\omega} = \omega_1 e_1 + \omega_2 e_2, \omega' = \omega_3 e_3 \). The divergence and curl of \( \varphi \tilde{\varphi} \) are given as in (3.4). Solving the above system, we obtain a similar integral formula for \( v \tilde{\varphi} \) as the formula (3.5). Taking the derivative, we have the expression of the \( \nabla(v \tilde{\varphi})(x) \)

\[
\nabla(v \tilde{\varphi})(x) = \mathcal{P}(v \cdot \nabla \tilde{\varphi})(x) + \mathcal{F}(\omega \tilde{\varphi})(x) + \mathcal{F}(v \times \tilde{\varphi})(x)
\]

\[
= \mathcal{P}(v \cdot \nabla \tilde{\varphi})(x) + \mathcal{F}(\omega \tilde{\varphi})(x) + \mathcal{F}(\partial_t \tilde{\varphi})(x) + \mathcal{F}(v \times \tilde{\varphi})(x).
\]

(3.10)
where

\[
\mathcal{P}(f)(x) = -\text{P.V.} \int_{\mathbb{R}^3} \left( \frac{1}{4\pi |x-y|^3} I_3 - \frac{3}{4\pi} \frac{(x-y) \otimes (x-y)}{|x-y|^5} f(y) \right) dy + \frac{1}{3} I_3 f(x),
\]

for scalar function \( f \) and \( 3 \times 3 \) identity matrix \( I_3 \) and

\[
\mathcal{P}(g)(x) = -\text{P.V.} \int_{\mathbb{R}^3} \left( \frac{1}{4\pi |x-y|^3} \frac{1}{|x-y|^3} \frac{3}{4\pi} \frac{[(x-y) \times g(y)] \otimes (x-y) h}{|x-y|^5} \right) dy
\]

\[+ \frac{1}{3} g(x) \times h,\]

for vector field \( g \) and constant vector \( h \). We first estimate \( I \) in (3.9).

\[
|I| \leq C \left| \int_{\Omega} \int_{\Omega} (\sigma(x, t) \cdot \hat{y}) ((\hat{\omega} \varphi)(x+y, t) \times (\hat{\omega} \varphi^k(x, t)) \cdot \hat{y}) \frac{dy}{|y|^3} dx \right|
\]

\[+ C \left| \int_{\Omega} \int_{\Omega} (\sigma(x, t) \cdot \hat{y}) ((\hat{\omega} \varphi)(x+y, t) \times (\hat{\omega} \varphi^k(x, t)) \cdot \hat{y}) \frac{dy}{|y|^3} dx \right|
\]

\[+ C \left| \int_{\Omega} \int_{\Omega} (\sigma(x, t) \cdot \hat{y}) ((\hat{\omega} \varphi)(x+y, t) \times (\hat{\omega} \varphi^k(x, t)) \cdot \hat{y}) \frac{dy}{|y|^3} dx \right|
\]

\[+ \int_{\Omega} |\sigma(x, t)|^2 |\mathcal{P}(v \cdot \nabla \varphi)| dx + \int_{\Omega} |\sigma(x, t)|^2 |\tilde{\mathcal{P}}(v \times \nabla \varphi)| dx
\]

\[:= I_1 + I_2 + I_3 + I_4 + I_5,\]  

(3.13)

with \( \hat{y} = \frac{y}{|y|^3} \). We estimate \( I_1, \ldots, I_5 \), separately.

\[
I_1 \leq C \|\tilde{\omega} \varphi\|_2 \|\sigma\|_2 \leq C \|\tilde{\omega} \varphi\|_2 \|\sigma\|_2 \frac{\|\tilde{\omega} \varphi\|_2}{16} \|\nabla \sigma\|_2^2, \quad \frac{3}{2} < \gamma < \infty,
\]

where we used the Calderon–Zygmund type estimate. Similarly, \( I_2 \) can be estimated.

\[
I_2 \leq C \|\tilde{\omega} \varphi\|_2 \|\sigma\|_2 \frac{\|\tilde{\omega} \varphi\|_2}{16} \|\nabla \sigma\|_2^2.
\]

(3.14)

To estimate \( I_3 \), we note that

\[
\|\hat{\omega} \varphi^k\|_2 \frac{\|\tilde{\omega} \varphi\|_2}{16} \leq C \left( \int_{\mathbb{R}^3 \setminus \mathbb{B}_{\eta_0}} |\hat{\omega} \varphi^k|^2 dx + \int_{\mathbb{R}^3 \setminus \mathbb{B}_{\eta_0 \gamma} \setminus \mathbb{B}_{\eta_0}} |\hat{\omega} \varphi^k|^2 dx \right).
\]

Due to Lemma 6, we have

\[
\int_{\mathbb{R}^3 \setminus \mathbb{B}_{\eta_0 \gamma}} |\hat{\omega} \varphi^k|^2 dx < \infty.
\]
Using the interpolation argument and the Gagliardo–Nirenberg inequality, we obtain

\[ I_3 \leq C \| \tilde{\phi} \|_{L^2} \| \sigma \|_{L^2}^{\frac{2}{3}} \left( \| \sigma \|_{L^2}^{\frac{2}{3}} + C \right) \]
\[ \leq C \| \tilde{\phi} \|_{L^2} \| \sigma \|_{L^2}^{\frac{2}{3}} + C \| \tilde{\phi} \|_{L^2} \| \sigma \|_{L^2}^{\frac{2}{3}} + \frac{1}{16} \| \nabla \sigma \|_{L^2}^2 \]
\[ \leq C \| \tilde{\phi} \|_{L^2} \| \sigma \|_{L^2}^{\frac{2}{3}} + C \| \tilde{\phi} \|_{L^2} + \frac{1}{16} \| \nabla \sigma \|_{L^2}^2. \]

We have

\[ I_4 \leq C \| \sigma \|_{L^2} \| \theta (v \cdot \nabla \phi) \|_{L^2} \leq C \| \sigma \|_{L^2}^2 + \frac{1}{8} \| \nabla \sigma \|_{L^2}^2, \]

and

\[ I_5 \leq C \| \sigma \|_{L^2} \| \tilde{\theta} (v \times \nabla \phi) \|_{L^2} \leq C \| \sigma \|_{L^2}^2 + \frac{1}{8} \| \nabla \sigma \|_{L^2}^2. \]

On the other hand, \( II \) can be estimated by virtue of the Young inequality, the Hölder inequality and the Gagliardo–Nirenberg inequality as follows:

\[ |II| \leq C \int_{\Omega} |v||\sigma|^\frac{2}{3-1} |\omega|^{\frac{2}{7}} dx \]
\[ \leq C \left( \int_{\Omega} |v|^2 dx \right)^\frac{2}{7} \left( \int_{\Omega} |\omega|^2 dx \right)^\frac{2}{7} \left( \int_{\Omega} |\sigma|^\frac{2(2-1)}{3-1} dx \right)^\frac{4}{7} \]
\[ \leq \int_{\Omega} |\omega|^2 dx + C \left( \int_{\Omega} |\sigma|^2 dx \right)^\frac{3}{5} \left( \int_{\Omega} |\nabla \sigma|^2 dx \right)^\frac{2}{5} \]
\[ \leq \int_{\Omega} |\omega|^2 dx + \frac{1}{16} \int_{\Omega} |\nabla \sigma|^2 dx + C \int_{\Omega} |\sigma|^2 dx. \]

\( III \) and \( IV \) can be estimated rather easily as follows:

\[ |III| \leq C \int_{\Omega} |\omega||\nabla \sigma| dx + C \int_{\Omega} |\omega||\sigma| dx \]
\[ \leq \frac{1}{16} \int_{\Omega} |\nabla \sigma|^2 dx + C \int_{\Omega} |\sigma|^2 dx + C \int_{\Omega} |\omega|^2 dx, \]

and

\[ |IV| \leq C \int_{\Omega} |\sigma|^2 dx + C \int_{\Omega} |\omega|^2 dx. \]

Collecting all the estimates above, we obtain

\[ \frac{d}{dt} \| \sigma \|_{L^2}^2 + \| \nabla \sigma \|_{L^2}^2 \leq C \left( 1 + \| \tilde{\phi} \|_{L^2}^{\frac{2}{3}} \right) \| \sigma \|_{L^2}^2 + C \| \omega \|_{L^2}^2 + \| \tilde{\phi} \|. \]
Due to the Gronwall inequality, we have
\[ \sup_{t_0 - \tau \leq t \leq t_0} \| \varpi(t) \|_2^2 + \int_{t_0 - \tau}^{t_0} \| \nabla \varpi(t) \|_2^2 \, dt \]
\[ \leq \left( \| \varpi(t_0 - \tau) \|_2^2 + C \int_{t_0 - \tau}^{t_0} \| \varpi(t) \|_2^2 \, dt + C \tau^2 \left( \int_{t_0 - \tau}^{t_0} \| \tilde{\omega} \|_2^2 \, dt \right)^{\frac{2}{3}} \right) \times \exp \left[ C \int_{t_0 - \tau}^{t_0} \left( 1 + \| \tilde{\omega} \|_2^2 \right) \, dt \right]. \]

Since it is well-known that \( \int_{t_0 - \tau}^{t_0} \| \varpi(t) \|_2^2 \, dt < \infty \) from the energy estimates. We have \( \varpi \in L_{x,t}^{\infty,2}(\Omega) \cap L^2(t_0 - \tau, t_0; W^{1,2}(\Omega)) \) and, therefore, we obtain \( \varpi \in L_{x,t}^{\infty,2}(Q_{t_0}) \cap L_{x,t}^{2,6}(Q_{t_0}). \) From Proposition 2, we conclude that \( z_0 \) is a regular point.

Next we consider the case that \( t_0 \) is a singular time that is not an epoch of possible irregularity. This case can be treated in the same way as we did in the proof of Proposition 1 and, therefore, we omit the details.

Next, we present the proof of Theorem 2.

Proof of Theorem 2. Suppose that \( t_0 \) is an epoch of irregularity. Without loss of generality, we assume \( x_0 = 0. \) Let \( \tau, r_1 \) and \( r_2 \) with \( r_1 < r_2 \) be the positive numbers in Lemma 6. Furthermore, \( r_1 \) and \( r_2 \), in particular, can be taken to satisfy \( 5r_1 < r_2 \) and \( r_1 < 2/a. \) We choose \( \zeta_0 \) cut-off functions \( \varphi \) and \( \tilde{\varphi} \) satisfying
\[ \varphi(x) = \begin{cases} 1 & \text{in } B_{r_1}, \\ 0 & \text{in } \mathbb{R}^3 \setminus B_{2r_1} \end{cases}, \quad \tilde{\varphi}(x) = \begin{cases} 1 & \text{in } B_{\frac{1}{2}r_1}, \\ 0 & \text{in } \mathbb{R}^3 \setminus B_{2r_1}. \end{cases} \]

Since we have the same equation for divergence and curl of \( v(x) \) as in (3.4), we have the same local expression of \( v \) as in (3.5). On \( x \in B_{1/n}, \) we have the local expression of \( \nabla v(x) \) same as (3.10). As in the proof of Theorem 1, we define \( \varpi := \omega \varphi^4. \) Then the equation (1.1) of the Navier–Stokes equations can be written in the local equation
\[ \partial_t \varpi + (v \cdot \nabla)\varpi = v \cdot \nabla \varphi^4 \omega + (\varpi \cdot \nabla)v + \Delta \varpi - 2\nabla \omega \cdot \nabla \varphi^4 - \omega \Delta \varphi^4. \quad (3.15) \]

Multiplying \( \varpi \) on the both sides of (3.15) and integrating over the domain, we have
\[ \frac{1}{2} \frac{d}{dt} \| \varpi \|_2^2 + \| \nabla \varpi \|_2^2 \]
\[ = \int_{\Omega} (\varpi \cdot \nabla)v \cdot \varpi \, dx + \int_{\Omega} v \cdot \nabla \varphi^4 \omega \varpi \, dx \]
\[ - 2 \int_{\Omega} \nabla \omega \cdot \nabla \varphi^4 \varpi \, dx - \int \omega \Delta \varphi^4 \varpi \, dx \]
\[ := I + II + III + IV. \]
Using the fact $\zeta(x, t) \times \xi(x, t) = 0$ and (3.10), we rewrite $I$ as follows:

$$I = \frac{3}{4\pi} \int_\Omega \int_{B_{\frac{1}{2}}} (\xi(x, t) \cdot \hat{y}) [((\xi(x + y, t) - \xi(x, t)) \times \zeta(x, t) \cdot \hat{y}]$$

$$\times |(\omega \hat{\phi})(x + y, t)| \frac{dy}{|y|} |\sigma(x, t)|^2 dx$$

$$+ \int_\Omega (\sigma \cdot \mathcal{P}(v \cdot \nabla \phi)) \sigma dx + \int_\Omega (\sigma \tilde{\mathcal{P}}(v \times \nabla \phi)) \sigma dx$$

$$:= I_1 + I_2 + I_3,$$

with $\hat{y} = \frac{y}{|y|}$. We estimate

$$|I_1| \leq C \int_\Omega \left( \int_{B_{\frac{1}{2}}} \frac{\xi(x + y, t) - \xi(x, t)}{|y|^{3+\sigma}} dy \right)^\frac{p}{2} \left( \int_{B_{\frac{1}{2}}} \frac{|\omega \hat{\phi}(x + y, t)|^{\frac{p}{q}}}{|y|^{3+\sigma}} dy \right)^\frac{q}{2} |\sigma(x, t)|^2 dx$$

$$\leq C \left( \int_{B_{\frac{1}{2}}} \frac{\xi(x + y, t) - \xi(x, t)}{|y|^{3+\sigma}} dy \right)^\frac{p}{2} \left( \int (\frac{\xi(x + y, t) - \xi(x, t)}{|y|^{3+\sigma}} dy \right)^\frac{q}{2} \left( \int \mathcal{J}_{\alpha}(|\omega \hat{\phi}|^q) \right)^\frac{1}{q} \|\sigma\|_{p_1}^2, \quad (3.16)$$

where

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{2}{p_3} = 1, \quad \frac{1}{q} + \frac{1}{q'} = 1,$$

and $\mathcal{J}_\alpha(\cdot), 0 < \alpha < 3$, is the Riesz potential defined as

$$\mathcal{J}_\alpha(f)(x) = \frac{\gamma(\alpha)}{\pi} \int_{B_{\frac{1}{2}}} \frac{f(x + y)}{|y|^{3+\alpha}} dy, \quad \gamma(\alpha) = 2\pi \Gamma(\frac{3}{2}) \Gamma(\frac{\alpha}{2}) \Gamma(1 - \frac{\alpha}{2}).$$

We restrict $q \in \left(\frac{3}{2}, \infty\right]$ from $0 < sq < 3$. By the Hardy–Littlewood–Sobolev inequality, we estimate

$$\left\| \mathcal{J}_{sq}(|\omega \hat{\phi}|^q) \right\|^\frac{1}{q} \leq C \|\omega \hat{\phi}\|_{sq} = C \|\omega \hat{\phi}\|_{sq} = C \|\omega \hat{\phi}\|_{sq}, \quad (3.17)$$

where $r$ satisfies $rq = 3\bar{p}_2/(3 + sq)$, i.e., $1/r = q/\bar{p}_2 + sq/3$ (we require that $\bar{p}_2 > q$). We note that, for $y \in B_{\bar{p}}$, $y$ can be expressed in spherical coordinates i.e., $y = r\sigma$ with $\sigma \in S^2$ and $0 \leq r \leq R$, where $S^2$ is a unit sphere. Using Fubini's Theorem and the expression of the spherical coordinates, we estimate

$$\int_{B_{\frac{1}{2}}} \frac{\xi(x + y, t) - \xi(x, t)}{|y|^{3+sq}} dy \leq C \int_{S^2} \int_0^{\tilde{r}_1} \int_0^{\tilde{r}_2} r^{-3+sq} |\xi(x + r\sigma) - \xi(x)|^q r^2 \sigma^3 dr d\sigma$$

$$\leq C \int_{S^2} \int_0^{\tilde{r}_1} \int_0^{\tilde{r}_2} r^{-3+sq} |\xi(x + r\sigma) - \xi(x)|^q r^2 dr d\sigma$$

$$- C \int_{S^2} \int_0^{\tilde{r}_1} |\xi(x + r\sigma) - \xi(x)|^q r^2 dr d\sigma + C$$

\[\text{\hspace{10em}}\]
\[ \leq C \int_{\mathbb{R}^d} \int_0^{\hat{r}} |\tilde{\zeta}(x + r\sigma) - \tilde{\zeta}(x)|^q r^2 \int_r^{\hat{r}} t^{-\frac{1}{4} - \sigma} dt \, dr \, d\sigma + C \]
\[ \leq C \int_0^{1} t^{-\frac{1}{4} - \sigma} \int_{V(t;\varepsilon)} |\tilde{\zeta}(x + y) - \tilde{\zeta}(x)|^q dy \, dt + C \]
\[ \leq \int_0^{1} t^{-\sigma} d_{\eta}(x) \tilde{\zeta}(x) \frac{dt}{T} + C. \]  
(3.18)

For the second inequality above, we used the fact that \(|\tilde{\zeta}| = 1\). Using the interpolation inequality, we have
\[ \|w\|_{\hat{p}_1}^2 \leq C \|w\|_2^{2 - \frac{3}{2} - \frac{2}{q}} \|w\|_6^{\frac{3}{2} + \frac{1}{q}}. \]  
(3.19)

(we require here that \(2 < \hat{p}_1 < 6\), i.e., \(0 < 1/p_1 + 1/\hat{p}_2 < 2/3\)). Combining (3.16)--(3.19), we derive
\[ |I_1| \leq C \left( \left\| \tilde{\zeta} \right\|_{\hat{p}_1} \left( \zeta(x) \right) + 1 \right) \|\omega\tilde{\varphi}\|_{\hat{p}_2}^{2 - \frac{3}{2} - \frac{2}{q}} \|\sigma\|_6^{\frac{3}{2} + \frac{1}{q}} \]
\[ \leq C \left( \left\| \tilde{\zeta} \right\|_{\hat{p}_1} \left( \zeta(x) \right) + 1 \right) \|\omega\tilde{\varphi}\|_{\hat{p}_2}^{2 - \frac{3}{2} - \frac{2}{q}} \|\sigma\|_6^{\frac{3}{2} + \frac{1}{q}} + \epsilon \|\sigma\|_6^2, \]  
(3.20)

where we used the Young’s inequality \(ab \leq C \frac{a^p}{u} + \epsilon \frac{b^p}{u}\) with
\[ a = \left\| \tilde{\zeta} \right\|_{\hat{p}_1} \left( \zeta(x) \right), \quad b = \|\sigma\|_6^{\frac{1}{2} + \frac{1}{q}}, \]
and
\[ u = \frac{2\hat{p}_1\hat{p}_2}{2\hat{p}_1\hat{p}_2 - 3(p_1 + \hat{p}_2)}, \quad u' = \frac{2p_1\hat{p}_2}{3(p_1 + \hat{p}_2)}. \]

We note that \(\epsilon\) can be chosen to be arbitrarily small. Setting \(3\hat{p}_2/(3 + s\hat{p}_2) = p_2\), we have \(\hat{p}_2 = 3p_2/(3 - sp_2)\). Combining this equality with the previous condition, \(\hat{p}_2 > q'\), we get \(1/p_2 + 1/q < 1 + s/3\). Substituting \(\hat{p}_2\) into (3.20), we obtain
\[ |I_1| \leq C \left( \left\| \tilde{\zeta} \right\|_{\hat{p}_1}^Q \left( \zeta(x) \right) + 1 \right) \|\omega\tilde{\varphi}\|_{\hat{p}_2}^Q \|\sigma\|_6^2 + \frac{1}{8} \|\sigma\|_6^2, \]  
(3.21)

where we set \(Q = 2p_1p_2/((2 + s)p_1p_2 - 3p_1 - 3p_2)\). Therefore, we have the restrictions
\[ \frac{s}{3} \leq \frac{1}{p_1} + \frac{1}{p_2} \leq \frac{2 + s}{3}. \]

By the Calderón–Zygmund inequality, we also estimate
\[ |I_2| \leq C \|\sigma\|_3^2 \|v\|_3 \leq C \|\sigma\|_3^2 \|v\|_3^2 + \frac{1}{8} \|\nabla\sigma\|_2^2 \leq C \|\sigma\|_3^2 (\|v\|_3^2 + \|\nabla v\|_2^2) + \frac{1}{8} \|\nabla\sigma\|_2^2. \]
and

\[ |I_3| \leq C \|\sigma\|_2^3 \|v\|_2^3 + \frac{1}{8} \|\nabla \sigma\|_2^2 \leq C \|\sigma\|_2^3 (\|v\|_2^3 + \|\nabla v\|_2^3) + \frac{1}{8} \|\nabla \sigma\|_2^2. \]

On the other hand, \( II \) can be estimated by virtue of the Young inequality, the Hölder inequality and the Gagliardo–Nirenberg inequality as follows:

\[
|II| \leq C \int_\Omega |\omega|^{\frac{3k+1}{k}} |\omega|^{\frac{1}{2}} dx \leq C \left( \int_\Omega |\omega|^{2} dx \right)^{\frac{k}{2}} \left( \int_\Omega |\omega|^{\frac{2k+1}{k}} dx \right)^{\frac{1}{2}} \left( \int_\Omega |\sigma|^{\frac{2k}{k-1}} dx \right) \frac{k+1}{k+2},
\]

\[
\leq C \int_\Omega |\omega|^{2} dx + C \left( \int_\Omega |\omega|^{2} dx \right)^{\frac{k}{k+2}} \left( \int_\Omega |\nabla \omega|^{2} dx \right)^{\frac{1}{k+2}} \quad (3.22)
\]

\( III \) and \( IV \) can be estimated rather easily as follows:

\[
|III| \leq C \int_\Omega |\omega| \nabla |\omega| d\Omega + C \int_\Omega |\omega| |\sigma| d\Omega
\]

\[
\leq C \int_\Omega |\nabla |\omega|^{2} dx + C \int_\Omega |\omega|^{2} dx + C \int_\Omega |\omega|^{2} dx, \quad (3.23)
\]

and

\[
|IV| \leq C \int_\Omega |\omega|^{2} dx + C \int_\Omega |\omega|^{2} dx. \quad (3.24)
\]

Collecting all the estimates on \( I_1, I_2, I_3, II, III, \) and \( IV, \) we have the following inequality

\[
\frac{d}{dt} \|\sigma\|_{L^2_t}^2 + \|\nabla \sigma\|_{L^2_t}^2 \leq C \left( \|\xi\|_{W^{0,\infty}_{2,2}}^0 + 1 \right) \|\omega \tilde{\varphi}\|_{L^0_t}^0 \|\sigma\|_{L^2_t}^2 + C(\|\nabla v\|_{L^2_t}^2 + 1) \|\sigma\|_{L^2_t}^2 + C \|\omega\|_{L^2_t}^2.
\]

The Gronwall Lemma provides us with

\[
\|\sigma\|_{L^2_t}^2 \leq \left( \|\sigma_0\|_{L^2_t}^2 + C \|\omega\|_{L^2_t}^2 \right) \exp \left[ C \int_{t_0}^{t} \left( \|\xi(t)\|_{W^{0,\infty}_{2,2}}^0 + 1 \right) \|\omega \tilde{\varphi}(t)\|_{L^2_t}^2 dt \right]
\]

\[
\quad + C \int_{t_0}^{t} (\|\nabla v(t)\|_{L^2_t}^2 + 1) dt
\]

\[
\leq \left( \|\sigma_0\|_{L^2_t}^2 + C \|\omega\|_{L^2_t}^2 \right) \exp \left[ C \left( \int_{t_0}^{t} \left( \|\xi(t)\|_{W^{0,\infty}_{2,2}}^0 + 1 \right) dt \right)^{\frac{q}{2}} \right]
\]

\[
\quad \times \left( \int_{t_0}^{t} \|\omega \tilde{\varphi}(t)\|_{L^2_t}^2 dt \right)^{\frac{q}{2}} 2^{-\frac{q}{2} - \frac{q}{2} - \frac{q}{2}}
\]

\[
\quad + C \int_{t_0}^{t} (\|\nabla v(t)\|_{L^2_t}^2 + 1) dt, \quad (3.25)
\]
where $\sigma_0 = \omega(\cdot, t_0 - \tau) \varphi^\delta$. Therefore, it is immediate that we have $\omega \in L^\infty(t_0 - \tau, t_0; L^2(B_{2r_1}))$. By Proposition 2, we conclude that $z_0$ is a regular point.

Next we consider the case that $t_0$ is a singular time that is not an epoch of possible irregularity. This case can be treated in the same way as the case of Proposition 1 and Theorem 2, and, therefore, the details are omitted. □

**Remark 5.** In this paper, for simplicity, zero external force is assumed. A standard modification of the proofs, however, gives the same conclusion for non-zero external force provided that such force is sufficiently regular.

**Acknowledgments**

This research is supported partially by KOSEF grant no. R01-2005-000-10077-0. D. Chae is supported by KRF Grant (MOEHRD, Basic Research Promotion Fund). K. Kang is supported by KRF-2006-331-C00020. J. Lee is supported by KRF-2006-311-C00007.

**References**


