TWO VERSIONS OF THE NIKODYM MAXIMAL FUNCTION ON THE HEISENBERG GROUP

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ABSTRACT. The classical Nikodym maximal function on the Euclidean plane \( \mathbb{R}^2 \) is defined as the supremum over averages over rectangles of eccentricity \( N \); its operator norm in \( L^2(\mathbb{R}^2) \) is known to be \( O(\log N) \). We consider two variants, one on the standard Heisenberg group \( \mathbb{H}_1 \) and the other on the polarized Heisenberg group \( \mathbb{H}^1_p \). The latter has logarithmic \( L^2 \) operator norm, while the former has the \( L^2 \) operator norm which grows essentially of order \( O(N^{1/4}) \). We shall imbed these two maximal operators in the family of operators associated to the hypersurfaces \( \{(x_1, x_2, \alpha x_1 x_2)\} \) in the Heisenberg group \( \mathbb{H}_1 \) where the exceptional blow up in \( N \) occurs when \( \alpha = 0 \).

1. INTRODUCTION

For each integer \( N \geq 2 \), let \( \mathcal{R}_N \) be the family of all rectangles centered at the origin whose eccentricity (the length of long side divided by the length of the short side) is \( N \). Then the classical Nikodym maximal function on the Euclidean plane is defined by

\[
M_{\mathcal{R}_N}f(x) = \sup_{R \in \mathcal{R}_N} \frac{1}{|R|} \int_R |f(x+y)| dy,
\]

where \( x \in \mathbb{R}^2 \) and \( f \) is a locally integrable function on \( \mathbb{R}^2 \). A. Córdoba [2] proved that

\[
\| M_{\mathcal{R}_N} \|_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)} \leq C(1 + \log N)^a
\]

with \( a = 2 \) to obtain results on the Bochner-Riesz means on \( \mathbb{R}^2 \). The sharp bound \( a = 1 \) in (1.1) was obtained by Strömberg [10].

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In this article we study the classical Nikodym maximal function on the Euclidean plane in the setting of the three dimensional Heisenberg group. We consider two realizations of the Heisenberg group. First let $H^1$ be the usual Heisenberg group identified with $\mathbb{R}^3$ endowed with the group multiplication
\[ x \cdot y = \left( x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1) \right) \]
where we use coordinates $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Next let $H^1_p$ be the polarized Heisenberg group endowed with the group law
\[ x \cdot_p y = \left( x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2 \right). \]
Associated with the two cases of Heisenberg groups, we consider two maximal averages over all rectangles of eccentricity $N$ supported on the hyperplane
\[ \Pi = \{ (x_1, x_2, 0) : x_1, x_2 \in \mathbb{R} \}. \]
We define the Nikodym maximal operator $\mathcal{M}_N$ associated with the standard Heisenberg group $H^1$ by
\[
(1.2) \quad \mathcal{M}_N f(x) = \sup_{R \in \mathcal{R}_N} \frac{1}{|R|} \int_R |f(x \cdot (y_1, y_2, 0))| dy_1 dy_2.
\]
On the polarized Heisenberg group $H^1_p$, the Nikodym maximal operator $\mathcal{M}_N^p$ is defined by
\[
(1.3) \quad \mathcal{M}_N^p f(x) = \sup_{R \in \mathcal{R}_N} \frac{1}{|R|} \int_R |f(x \cdot_p (y_1, y_2, 0))| dy_1 dy_2.
\]
The main purpose of this article is to prove the following results.

**Theorem 1.** There are positive constants $c$ and $C$ independent of $N$ such that
\[
c \log N \leq \| \mathcal{M}_N^p \|_{L^2(H^1) \to L^2(H^1)} \leq C (\log N)^{3/2}.
\]

**Theorem 2.** There are positive constants $c$ and $C$ independent of $N$ such that
\[
c N^{1/4} \leq \| \mathcal{M}_N \|_{L^2(H^1) \to L^2(H^1)} \leq C N^{1/4} (\log N)^2.
\]
Remark 1. We regard $M_N f(x)$ and $M^p_N f(x)$ as the maximal averages of $f$ over all tubes of eccentricity $N$ that are essentially embedded on the variable planes $\Pi(x)$ containing $x$ whose normal vectors are $a(x) = (x_2/2, -x_1/2, 1)$ and $a(x) = (0, -x_1, 1)$ respectively. We recently have found an interesting relation between the normal vector fields $a$ and the norms of the corresponding operator on $L^2(\mathbb{R}^3)$. We intend to take up these matters on the forthcoming paper [6].

Remark 2. In the Euclidean space $\mathbb{R}^2$, if the collection $\mathcal{R}_N$ of all rectangles of eccentricity $N$ is replaced by that of all $1 \times 1/N$ rectangles, then the norm of the corresponding Nikodym maximal operator on $L^2(\mathbb{R}^2)$ is also known to be $O((\log N)^{1/2})$ in [2] with the lower bound $c_0(\log N)^{1/2}/(\log(\log N))^c$. Fairly recently, the bound $(\log N)^{1/2}$ is shown to be sharp (with respect to $N$) by U. Keich [4], where the sharp $L^p$ bounds with $p > 2$ are also obtained.

Remark 3. It would be interesting to relate the behavior of $M^p_N$ and $M_N$ to the results of curved Nikodym and Kakeya maximal operators considered by Bourgain [1], Minicozzi and Sogge [8] and Wisewell [12].

Remark 4. Consider the spherical maximal operator $S_{2n}$ on the $2n$ dimensional hyperplane of the $2n + 1$ dimensional Heisenberg group $\mathbb{H}^n$. A. Seeger and D. Müller [8] proved that $S_{2n}$ is bounded on $L^p(\mathbb{H}^n)$ when $p > \frac{2n}{2n-1}$ when $n \geq 2$. The unresolved case $n = 1$ leads us to consider the Nikodym maximal function on the plane of the Heisenberg group $\mathbb{H}^1$.

The proofs are based on the induction argument on the scale $N$ of eccentricity introduced in the Euclidean setting by S. Wainger [11], and the application of group Fourier transform on the Heisenberg group in combination with the Cotlar–Stein lemma used in [5].

We employ the induction argument [11] to reduce the $L^2(\mathbb{H}^1)$ estimation of the maximal operator to that of certain square sum operators. The application of the group Fourier transform for these square sums leads to the uniform $L^2(\mathbb{R}^1)$ estimations of oscillatory integral operators. It turns out that the phase functions of these integral operators are degenerate in the sense that their mixed
second derivatives vanish in the case of standard Heisenberg group $\mathbb{H}^1$, but do not vanish in the case of the polarized group $\mathbb{H}^1_p$. This non-vanishing curvature enables us to obtain the desired uniform estimation of the oscillatory integrals.

Organization. In Section 2, we employ the induction argument of [11] via the group Fourier transform to reduce to the uniform $L^2(\mathbb{R})$ estimations of a certain family of one dimensional oscillatory integral operators. In Section 3, we prove Theorem 1. For this purpose, we combine $TT^*$ methods and the Cotlar–Stein lemma to show the uniform $L^2(\mathbb{R})$ estimation of the oscillatory integrals. In Section 4, we obtain the lower bound in Theorem 2. In Section 5, we obtain the upper bound by using a similar argument to that of Section 3.

Notation. As usual, the notation $A \lesssim B$ for two scalar expressions $A, B$ will mean $A \leq CB$ for some positive constant $C$ independent of $A, B$ and $A \approx B$ will mean $A \lesssim B$ and $B \lesssim A$.

2. Induction and Group Fourier Transform

2.1. Hypersurfaces $\{(x_1, x_2, \alpha x_1 x_2)\}$ on $\mathbb{H}^1$. We interpret our two maximal operators as the members of the class of Nikodym type maximal operators associated with the hypersurfaces

$$\Pi_\alpha = \{(x_1, x_2, \alpha x_1 x_2) : x_1, x_2 \in \mathbb{R}\} \text{ where } \alpha \in \mathbb{R}$$

on the Heisenberg group $\mathbb{H}^1$. In proving Theorems 1 and 2, it suffices to assume that the angle between the long side of the rectangle $R \in \mathcal{R}_N$ and the $x_1$-axis is restricted to $[0, \pi/4]$. For such a rectangle $R$, we can find a parallelogram $R(k, r)$ with some $k \in \mathcal{D}_N = \{1, \ldots, N\}$ and $r > 0$ given by,

$$R(k, r) = \left\{ \left(y_1, y_2 + \frac{k}{N} y_1 \right) : -r < y_1 < r, -r/N < y_2 < r/N \right\} \tag{2.1}$$

satisfying the following engulfing property

$$R(k, r/10) \subset R \subset R(k, 10r). \tag{2.2}$$
Thus we now define $\mathcal{P}_N$ as the family of all parallelograms of the form (2.1),

$$
(2.3) \quad \mathcal{P}_N = \{ R(k, r) : k \in \mathcal{D}_N, r > 0 \}.
$$

Using the group isomorphism $I : \mathbb{H}^1 \to \mathbb{H}^1_p$ defined by $I(x_1, x_2, x_3) = (x_1, x_2, x_3 + x_1 x_2 / 2)$ where $I(x) \cdot_p I(y) = I(x \cdot y),$

$$
\mathcal{M}_N^p f(x) \approx \sup_{R \in \mathcal{P}_N} \frac{1}{|R|} \int_R |f(I^{-1}(x_1, x_2, x_3)) \cdot_p I^{-1}(y_1, y_2, 0))| \, dy_1 dy_2
$$

$$
= \sup_{R \in \mathcal{P}_N} \frac{1}{|R|} \int_R |f_I(I^{-1}(x_1, x_2, x_3) \cdot (y_1, y_2, -y_1 y_2 / 2))| \, dy_1 dy_2
$$

where $f_I(x) = f(I(x))$. Thus, in order to prove Theorem 1, we work with the Nikodym maximal operator $\widetilde{\mathcal{M}}_N^p$ associated with the hypersurface $\Pi_{-1/2}$:

$$
\widetilde{\mathcal{M}}_N^p f(x) = \sup_{R \in \mathcal{P}_N} \frac{1}{|R|} \int_R |f (x \cdot (y_1, y_2, -y_1 y_2 / 2))| \, dy_1 dy_2.
$$

From (2.1), the support of the above integral $\{(y_1, y_2, -y_1 y_2 / 2) : (y_1, y_2) \in R\}$ with $R = R(k, r)$ is written as

$$
\left\{ \left( y_1, y_2 + \frac{k}{N} y_1, -\frac{1}{2} (y_2 + \frac{k}{N} y_1) y_1 \right) : -r < y_1 < r, -r/N < y_2 < r/N \right\}.
$$

Due to the group multiplication,

$$
\left( y_1, y_2 + \frac{k}{N} y_1, -\frac{1}{2} (y_2 + \frac{k}{N} y_1) y_1 \right) = (0, y_2, 0) \cdot \left( y_1, \frac{k}{N} y_1, -\frac{1}{2} \frac{k}{N} y_1^2 \right).
$$

Therefore we have,

$$
(2.4) \quad \widetilde{\mathcal{M}}_N^p f(x) \leq M_2 M_N^{-1/2} f(x)
$$

where

$$
M_N^p f(x) = \sup_{k \in \mathcal{D}_N, r > 0} \frac{1}{2r} \int_{-r}^{r} |f((x_1, x_2, x_3) \cdot (t, \frac{k}{N} t, \alpha \frac{k}{N} t^2))| \, dt,
$$

$$
M_2 f(x) = \sup_{r > 0} \frac{1}{2r} \int_{-r}^{r} |f((x_1, x_2, x_3) \cdot (0, t, 0))| \, dt.
$$

The operator $M_2$ is bounded on $L^p(\mathbb{H}^1)$ for all $1 < p \leq \infty$ since

$$
M_2 f(x) = \sup_{r > 0} \frac{1}{2r} \int_{-r}^{r} |f_I(x_1, x_2 + t, x_3 - x_1 x_2 / 2)| \, dt.
$$
which is the directional maximal function along the second axis. Hence, in proving Theorem 1, we have only to prove that for $\alpha = -1/2$,

$$
\|M^\alpha_N\|_{L^2(H^1) \to L^2(H^1)} \leq C(\log N)^{3/2}.
$$

By the change of variable $t' = -t$, we rewrite

$$
M^\alpha_N f(x) = \sup_{k \in D_N, r > 0} \frac{1}{2r} \int_{-r}^{r} f \left( x \cdot \left( t, \frac{k}{N} t, -\frac{k}{N} t^2 \right)^{-1} \right) dt
$$

where we note that $(y_1, y_2, y_3)^{-1} = (-y_1, -y_2, -y_3)$, which is the group inverse of $(y_1, y_2, y_3)$ in $H^1$. In proving (2.5), it suffices to consider the case that the support of integral is restricted to $[0, r]$ in (2.6) because of similarity. Let us choose a positive smooth function $\varphi$ supported $[1/2, 4]$ and $\varphi(t) \equiv 1$ on $[1, 2]$.

Put

$$
\varphi_j(t) = \varphi(t/2^j)/2^j.
$$

For each $j \in \mathbb{Z}$, $k \in D_N$ and $\alpha \in \mathbb{R}$, we define a measure $\mu^\alpha_{j,k,N}$ by

$$
\mu^\alpha_{j,k,N}(f) = \int f \left( t, \frac{k}{N} t, \alpha \frac{k}{N} t^2 \right) \varphi_j(t) dt.
$$

Fix $k \in D$ and $r > 0$ where $2^{m-1} < r \leq 2^{m}$ for some $m \in \mathbb{Z}$. Then for $f \geq 0$,

$$
\frac{1}{r} \int_0^r f \left( x \cdot \left( t, \frac{k}{N} t, -\frac{k}{N} t^2 \right)^{-1} \right) dt \lesssim \sum_{j=\infty}^m \frac{2^j}{2^{m2^j}} \int_{2^{j-1}}^{2^j} f \left( x \cdot \left( t, \frac{k}{N} t, -\frac{k}{N} t^2 \right)^{-1} \right) dt \lesssim \sup_{j \in \mathbb{Z}, k \in D_N} f \ast \mu^\alpha_{j,k,N}(x_1, x_2, x_3) = \sup_{j \in \mathbb{Z}, k \in D_N} \mu^\alpha_{j,k,N} \ast [f]_3(x_1, x_2, -x_3).
$$

where $[f]_3(x_1, x_2, x_3) = f(x_1, x_2, -x_3)$ and $\ast$ is convolution on the Heisenberg group $H^1$. Here the last line follows from the identity

$$
[f]_3 \ast [g]_3(x_1, x_2, x_3) = [g \ast f]_3(x_1, x_2, x_3).
$$

Thus, we let

$$
\mathcal{M}^\alpha_N f(x_1, x_2, x_3) = \sup_{j \in \mathbb{Z}, k \in D_N} \mu^\alpha_{j,k,N} \ast |f|(x_1, x_2, x_3),
$$

(2.8)
and prove the following general result in Section 3.

**Proposition 1.** For any $\alpha \neq 0$ where $\alpha = -1/2$ corresponds to Theorem 1,

$$\|\mathcal{M}_N^\alpha f\|_{L^2(\mathbb{H}^1)} \leq C (\log N)^{3/2} \|f\|_{L^2(\mathbb{H}^1)}.$$

2.2. **Group Fourier transform.** Let $\mathcal{B}(L^2(\mathbb{R}))$ be the space of all bounded linear operators on $L^2(\mathbb{R})$. The group Fourier transform of $f \in L^1(\mathbb{H}^1) \cap L^2(\mathbb{H}^1)$ is defined as an operator-valued function from $\mathbb{R} \setminus \{0\}$ to $\mathcal{B}(L^2(\mathbb{R}))$ such that

$$\hat{f}(\lambda) = \int_{\mathbb{R}} \mathcal{F}^{2,3} f(x-y, \lambda(x+y)/2, \lambda) h(y) \, dy$$

where $\mathcal{F}^{2,3}$ is the Fourier transform with respect to the second and third variables. Note that $\hat{f}(\lambda) \in \mathcal{B}(L^2(\mathbb{R}))$ is well defined since the integral kernel is square integrable with respect to $x, y$ variables. We introduce the criteria of the $L^2(\mathbb{H}^1)$ boundedness of convolution type operators on the Heisenberg group.

**Proposition 2.** Let $G$ be a convolution operator defined by $Gf = K \ast f$ for $f \in \mathcal{S}(\mathbb{H}^1)$ and where the convolution kernel $K$ is a tempered distribution in $\mathcal{S}'(\mathbb{H}^1)$. Then $\|G\|_{L^2(\mathbb{H}^1) \to L^2(\mathbb{H}^1)} \leq \sup_{\lambda \in \mathbb{R} \setminus \{0\}} \|\hat{K}(\lambda)\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})}$.

**Proof.** For the proof of Proposition 2, we refer the reader to Chapter 1 of [3] and Chapter 11 of [9]. □

2.3. **Non vanishing versus vanishing mixed Hessian.** We write

$$\mu_{j,k,N}^\alpha (y_1, y_2, y_3) = \varphi_j(y_1) \delta \left( y_2 - \frac{k}{N} y_1 \right) \delta \left( y_3 - \alpha \frac{k}{N} y_1 \right),$$

where $\delta$ is the Dirac mass. By applying Proposition 2 and the formula (2.9), one is led to estimate the one dimensional oscillatory integral operator,

$$\left[ \tilde{\mu}_{j,k,N}^\alpha (\lambda) h \right](x) = \int_{\mathbb{R}} \mathcal{F}^{2,3} \mu_{j,k,N}^\alpha (x-y, \lambda(x+y)/2, \lambda) h(y) dy$$

$$= \int \exp \left( -i \lambda \frac{k}{N} P_{\alpha}(x,y) \right) \varphi_j(x-y) h(y) dy,$$
where the phase function \( P_\alpha(x, y) \) is given by
\[
P_\alpha(x, y) = \frac{1}{2}(x - y)(x + y) + \alpha(x - y)^2.
\]

Note that the mixed second derivative of the oscillatory function \( P_\alpha \) for \( \alpha \neq 0 \) does not vanish as
\[
[P_\alpha]_{xy}''(x, y) = -2\alpha \neq 0.
\]

This non-vanishing mixed second derivative condition enables us to apply integration by parts for the kernel of the operator
\[
\hat{\mu}_{j,k,N}^\alpha(\lambda) \left[ \hat{\mu}_{j,k,N}^\alpha(\lambda) \right]^* \to
\]
to prove Proposition 1. But the vanishing mixed second derivative condition \( \alpha = 0 \) in (2.12) yields the lower bound of Theorem 2, which shall be proved in Section 4.

2.4. Induction argument on the scale of \( N \). The proof of Proposition 1 is based on the following induction argument. Assume that
\[
\| M_\alpha^N \|_{L^2(\mathbb{R}^1) \to L^2(\mathbb{R}^1)} \leq C (\log N)^{3/2}, \tag{2.13}
\]
where \( C \) does not depend on \( N \). Under this assumption we show that
\[
\| M^N_{2N} \|_{L^2(\mathbb{R}^1) \to L^2(\mathbb{R}^1)} \leq C (\log 2N)^{3/2}, \tag{2.14}
\]
where \( C \) in (2.13) and (2.14) is to be the same one. This combined with the obvious case \( N = 2^1 \) implies that (2.13) holds for every positive integer \( N \) of the form \( 2^\ell \) by induction on \( \ell \), which yields the desired result for every positive integer \( N \). Now we prove (2.14) under the assumption of (2.13). For any \( j \in \mathbb{Z} \),
\[
\sup_{k \in \mathcal{D}_{2N}} \left| \mu_{j,k,2N}^\alpha * f(x_1, x_2, x_3) \right| \leq \sup_{k \in \mathcal{D}_N} \left| \left( \mu_{j,2k-1,2N}^\alpha - \mu_{j,2k,2N}^\alpha \right) * f(x_1, x_2, x_3) \right| + \sup_{k \in \mathcal{D}_N} \left| \mu_{j,2k,2N}^\alpha * f(x_1, x_2, x_3) \right|. \tag{2.15}
\]

Note that on the last term of (2.15),
\[
\mu_{j,2k,2N}^\alpha = \mu_{j,k,N}^\alpha \quad \text{for all } j \in \mathbb{Z}, \; k \in \mathcal{D}_N.
\]
Hence we take the suprema over \( j \in \mathbb{Z} \) on both sides of (2.15) to obtain that

\[
\| M_2^\alpha f \|_{L^2(\mathbb{H}^3)} \leq \| G^\alpha f \|_{L^2(\mathbb{H}^1)} + \| M_N^\alpha f \|_{L^2(\mathbb{H}^1)},
\]

where

\[
G^\alpha f(x_1, x_2, x_3) = \left( \sum_{j \in \mathbb{Z}, k \in D_N} \left| (\mu_{j,2k-1,2N}^\alpha - \mu_{j,2k,2N}^\alpha) * f(x_1, x_2, x_3) \right|^2 \right)^{1/2}.
\]

In proving (2.14) it suffices to show that for \( \alpha \neq 0 \),

\[
\| G^\alpha f \|_{L^2(\mathbb{H}^1)} \leq C^* (\log N)^{1/2} \| f \|_{L^2(\mathbb{H}^1)},
\]

since (2.17), (2.16) and (2.13) with \( C > 10C^* \) lead us to obtain that

\[
\| M_2^\alpha f \|_{L^2(\mathbb{H}^1) \to L^2(\mathbb{H}^1)} \leq C^* (\log N)^{1/2} + C (\log N)^{3/2} \leq C (\log(2N))^{3/2}.
\]

For the estimate in (2.17), we now use the group Fourier transform. For each \( k \in D_N, j \in \mathbb{Z} \) and fixed \( \lambda, \alpha \neq 0 \), put

\[
T_{j,k,N}^{\alpha,\lambda} = \mu_{j,2k-1,2N}^\alpha(\lambda) - \mu_{j,2k,2N}^\alpha(\lambda)
\]

where \( \mu_{j,k,N}^\alpha(\lambda) \) is the group Fourier transform which is expressed in (2.10). In proving (2.17), by Proposition 2, it suffices to prove that there exists a constant \( C > 0 \) independent of \( \lambda \) such that for \( \alpha \neq 0 \),

\[
\| G^{\lambda} h \|_{L^2(\mathbb{R}^3)} \leq C (\log N)^{1/2} \| h \|_{L^2(\mathbb{R}^3)},
\]

where

\[
|G^{\lambda} h(x)| = \left( \sum_{j \in \mathbb{Z}, k \in D_N} \left| T_{j,k,N}^{\alpha,\lambda} h(x) \right|^2 \right)^{1/2}.
\]

3. Proof of Theorem 1

In this section we shall prove (2.18) in order to prove Proposition 1.
3.1. Decomposition into local and global parts. Take a function \( \psi \in C_c^\infty([-2, 2]) \) such that \( 0 \leq \psi \leq 1 \) and \( \psi(u) = 1 \) for \( |u| \leq 1/2 \). Put \( \eta(u) = \psi(u) - \psi(2u) \). We choose the integer \( a \) satisfying

\[
2^{a-1} < 2^{10}(|\alpha| + 1) \leq 2^a.
\]

Note that \( a > 10 \). For fixed \( \lambda \) and \( \alpha \), let

\[
S_{j,k,N}^{\text{loc}} h(x) = \int \exp \left( i\lambda \frac{k}{N} P_{\alpha} (x, y) \right) \varphi_j (x - y) \psi \left( \frac{y}{2^{j+a}} \right) h(y) dy
\]

(3.2)

\[
S_{j,k,N}^{m} h(x) = \int \exp \left( i\lambda \frac{k}{N} P_{\alpha} (x, y) \right) \varphi_j (x - y) \eta \left( \frac{y}{2^{m}} \right) h(y) dy,
\]

where \( m > j + a \). Then,

\[
\hat{\mu}_{j,k,N}^2 (\lambda) = S_{j,k,N}^{\text{loc}} + \sum_{m=j+a+1}^{\infty} S_{j,k,N}^{m}.
\]

We omit \( \lambda \) and \( \alpha \) from the notation \( S_{j,k,N}^{\text{loc}} \) and \( S_{j,k,N}^{m} \) for simplicity. We set

\[
T_{j,k,N}^{\text{loc}} = S_{j,2k-1,2N}^{\text{loc}} - S_{j,2k,2N}^{\text{loc}}
\]

\[
T_{j,k,N}^{\text{glo}} = \sum_{m=j+a+1}^{\infty} (S_{j,2k-1,2N}^{m} - S_{j,2k,2N}^{m}).
\]

In proving (2.18), we show that there is a constant \( C > 0 \) independent of \( \lambda \) such that

\[
\left\| \left( \sum_{j \in \mathbb{Z}, k \in \mathcal{D}_N} \left| T_{j,k,N}^{\text{loc}} h \right|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^1)} \leq C (\log N)^{1/2} \| h \|_{L^2(\mathbb{R}^1)},
\]

(3.3)

and

\[
\left\| \left( \sum_{j \in \mathbb{Z}, k \in \mathcal{D}_N} \left| T_{j,k,N}^{\text{glo}} h \right|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^1)} \leq C (\log N)^{1/2} \| h \|_{L^2(\mathbb{R}^1)}.
\]

(3.4)
3.2. **Local part estimate (3.3).** It suffices to show that for each fixed nonzero $\alpha$ and $\lambda$,

\begin{equation}
\|T_{j,k,N}^{\text{loc}}\|_{op} \lesssim \left( \frac{k}{N} 2^{2j}\lambda \right)^{-1/2} \min \left\{ 1, \left| \frac{1}{N} 2^{2j}\lambda \right| \right\},
\end{equation}

since from (3.5)

\begin{equation*}
\left\| \left( \sum_{j \in \mathbb{Z}, k \in D_N} |T_{j,k,N}^{\text{loc}} h|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R})}^2 \\
\lesssim \sum_{j \in \mathbb{Z}, k \in D_N} \left( \frac{k}{N} 2^{2j}\lambda \right)^{-1} \min \left\{ 1, \left| \frac{1}{N} 2^{2j}\lambda \right| \right\} \|h\|_{L^2(\mathbb{R})}^2 \\
\lesssim (\log N) \|h\|_{L^2(\mathbb{R})}^2.
\end{equation*}

**Proof of (3.5).** Our proof is based on the $TT^*$ method. In order to estimate the $L^2$ norm of

\[ T_{j,k,N}^{\text{loc}} = S_{j,2k-1,2N}^{\text{loc}} - S_{j,2k,2N}^{\text{loc}}, \]

we compute the integral kernel $S(x, z)$ of the operator $S_{j,k,N}$ $[S_{j,k,N}^{\text{loc}}]^*$:

\[ S(x, z) = \int \exp \left( i\lambda \frac{k}{N} \left( P_\alpha(x, y) - P_\alpha(z, y) \right) \right) \varphi_j(x-y) \varphi_j(z-y) \psi \left( \frac{y}{2^{j+a}} \right)^2 \, dy. \]

where we recall

\[ P_\alpha(x, y) = \frac{1}{2} (x-y)(x+y) + \alpha(x-y)^2. \]

The derivative of the phase function with respect to the $y$ variable is

\begin{equation}
[P_\alpha]_y'(x, y) - [P_\alpha]_y'(z, y) = -2\alpha (x-z),
\end{equation}

which enables us to apply integration by parts to obtain that

\[ |S(x, z)| \leq \frac{C_\alpha 2^{-j}}{\left( \frac{k}{N} 2^{2j}\lambda \left| x-z \right| \right)^2 + 1}. \]
By Schur’s test,

\[
\|S_{j,k,N}^{\text{loc}}\|_{op} = \sqrt{\|S_{j,k,N}^{\text{loc}} [S_{j,k,N}^{\text{loc}}]^*\|_{op}} \\
\leq \sqrt{\sup_z \int |S(x,z)| \, dx} \lesssim \left| \frac{k}{N} 2^{2j} \lambda \right|^{-1/2}.
\]

Hence

\[
\|T_{j,k,N}^{\text{loc}}\|_{op} \lesssim \|S_{j,2k,N}^{\text{loc}}\|_{op} + \|S_{j,2k,N}^{\text{loc}}\|_{op} \lesssim \left| \frac{k}{N} 2^{2j} \lambda \right|^{-1/2}.
\]

In order to gain \(|\frac{1}{N} 2^{2j} \lambda|\) in (3.5), we need to use the mean value theorem as well as integration by parts. Since \(T_{j,k,N}^{\text{loc}} = S_{j,2k-1,N}^{\text{loc}} - S_{j,2k,N}^{\text{loc}}\), we write

\[
T_{j,k,N}^{\text{loc}} f(x) = \int e^{i\lambda \frac{k}{N} P_\alpha(x,y)} U_{j,k,N}(x,y) f(y) \, dy,
\]

where

\[
U_{j,k,N}(x,y) = \left( e^{-i\frac{2\lambda}{N} P_\alpha(x,y)} - 1 \right) \varphi_j(x - y) \psi(y/2^j + a).
\]

Thus the integral kernel \(T(x,z)\) of \(T_{j,k,N}^{\text{loc}}[T_{j,k,N}^{\text{loc}}]^*\) is

\[
T(x,z) = \int \exp\left(i \lambda \frac{k}{N} (P_\alpha(x,y) - P_\alpha(z,y))\right) U_{j,k,N}(x,y) \overline{U_{j,k,N}(z,y)} \, dy.
\]

By using the mean value theorem and the support condition of the above integral such that \(|y| \lesssim 2^j\), \(|x| \lesssim 2^j\) and \(|x-y| \approx 2^j\),

\[
|U_{j,k,N}(x,y)| \lesssim \min \left\{ 1, \left| \frac{1}{N} 2^{2j} \lambda \right| \right\} 2^{-j}, \quad (3.9)
\]

\[
\left| \frac{\partial U_{j,k,N}}{\partial y} (x,y) \right| \lesssim \left| \frac{1}{N} 2^j \lambda \right| 2^{-j},
\]

\[
\left| \frac{\partial^2 U_{j,k,N}}{\partial y^2} (x,y) \right| \lesssim \max \left\{ \left| \frac{\lambda}{N} \right|, \left| \frac{1}{N} 2^j \lambda \right|^2 \right\} 2^{-j}.
\]

The above inequalities also hold when \(U_{j,k,N}(x,y)\) is replaced by \(U_{j,k,N}(z,y)\).

Using the first inequality of (3.9),

\[
|T(x,z)| \lesssim \left| \frac{\lambda}{N} 2^{2j} \right|^2 2^{-j}. \quad (3.10)
\]
Applying integration by parts twice, we get

\[ T(x, z) = \int \exp \left( i\lambda \frac{k}{N} \left( P_\alpha(x, y) - P_\alpha(z, y) \right) \right) \left( \frac{\partial}{\partial y} \right)^2 \left[ \frac{U_{j,k,N}(x, y) U_{j,k,N}(z, y)}{\left( \lambda \frac{k}{N} 2^{\alpha(x - z)} \right)^2} \right] \, dy. \]

By using (3.9) and the support condition, we obtain that

\[ |T(x, z)| \lesssim \left| \frac{\lambda 2^{2j}}{N} \right|^{2-2j} \left| \frac{k}{N} \lambda (x - z) \right|^{2}. \]  

From (3.10) and (3.12),

\[ |T(x, z)| \lesssim \left| \frac{\lambda 2^{2j}}{N} \right|^{2-2j} \left| \frac{k}{N} \lambda (x - z) \right|^{2+1}. \]

Thus by Schur’s test,

\[ \|T_{j,k,N}[T_{j,k,N}]^*\|_\text{op} \lesssim \frac{\lambda 2^{2j}|2-j|}{\left| \frac{k}{N} 2^{j} \lambda \right|}. \]

Hence,

\[ \|T_{j,k,N}\|_\text{op} \lesssim \left| \frac{k}{N} 2^{2j} \lambda \right|^{-1/2} \left| \frac{1}{N} 2^{2j} \lambda \right|, \]

which combined with (3.8) yields (3.5).

\[ \square \]

3.3. Global part estimate (3.4). In view of \( T_{j,k,N}^m = \mathcal{S}_{j,2k-1,2N}^m - \mathcal{S}_{j,2k,2N}^m \),

\[ T_{j,k,N}^m h(x) = \int \exp \left( i\lambda \left( \frac{2k}{2N} \right) P_\alpha(x, y) \right) \left( \exp \left( i\lambda \left( \frac{-1}{2N} \right) P_\alpha(x, y) \right) - 1 \right) \times \varphi_j(x - y) \eta \left( \frac{y}{2^m} \right) h(y) \, dy. \]

Thus the integral kernel for \( T_{j,k,N}^m = \sum_{m=j+a+1}^\infty \mathcal{T}_{j,k,N}^m \) is supported in the set given by

\[ \{(x, y) : |x - y| \approx 2^j \text{ and } |y| > 2^{j+a} \} \subset \{(x, y) : |x| \approx |y| > 2^{j+a} \}, \]
where $a > 10$. By using this support condition and (3.13),
\[
\sum_{j \in \mathbb{Z}, k \in \mathcal{D}_N} \int |T_{j,k,N}^{glo} h(x)|^2 \, dx = \sum_{j \in \mathbb{Z}, k \in \mathcal{D}_N} \sum_{m \in \mathbb{Z}} \int_{2^{m-1} < |x| < 2^m} |T_{j,k,N}^{glo} h(x)|^2 \, dx \\
\leq \sum_{j \in \mathbb{Z}, k \in \mathcal{D}_N} \sum_{m \in \mathbb{Z}} \int |(T_{j,k,N}^{m-1} + T_{j,k,N}^m + T_{j,k,N}^{m+1}) h(x)|^2 \, dx \\
\leq 3 \sum_{m \in \mathbb{Z}} \sum_{j \in \mathbb{Z}, k \in \mathcal{D}_N} \int |T_{j,k,N}^m h(x)|^2 \, dx
\]
where we note that the support of $h$ on the last line is contained $\{y : |y| > 2^m\}$.

Therefore, in proving (3.4) it suffices to show that for each fixed $m \in \mathbb{Z}$,
\[
(3.14) \quad \left\| \left( \sum_{j \in \mathbb{Z}, k \in \mathcal{D}_N} |T_{j,k,N}^m h|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^1)} \lesssim (\log N)^{1/2} \| h \|_{L^2(\mathbb{R}^1)}.
\]

The size of derivative of phase function on the composition kernel of $S_{j,k,N}^m [S_{j,k,N}^m]^*$ is the same as (3.6). This, in the same way as (3.7), yields
\[
(3.15) \quad \| S_{j,k,N}^m \|_{op} \lesssim \left| \frac{k}{N} 2^{2j/\lambda} \right|^{-1/2}.
\]

Thus, we get $\| T_{j,k,N}^m \|_{op} \lesssim \left| \frac{k}{N} 2^{2j/\lambda} \right|^{-1/2}$ as in (3.8). But when we use the mean value theorem on the region $|y| \approx 2^m \gg 2^{j+a}$,
\[
(3.16) \quad \left| \exp \left( -i \frac{\lambda}{2N} P_a(x,y) \right) - 1 \right| \lesssim \left| \frac{1}{2N} 2^{2j+m} \lambda \right| \quad \left| \frac{\partial}{\partial y} \left( \exp \left( -i \frac{\lambda}{2N} P_a(x,y) \right) - 1 \right) \right| \lesssim \left| \frac{1}{2N} 2^{2m} \lambda \right|.
\]

This leads (3.9) where $2^j$ is replaced by $2^m$. From this,
\[
(3.17) \quad \| T_{j,k,N}^m \|_{op} \lesssim \left| \frac{k}{N} 2^{2j/\lambda} \right|^{-1/2} \left| \frac{1}{N} 2^{2j+m+\lambda} \right|
\]
which is different from (3.12). From this observation we notice that, for the global case $|y| \gg 2^{j+a}$, the direct application of the above $TT^*$ estimate does not lead us to have the bound of (3.12) independent of the size of $y$. In order to
overcome this difficulty, we shall apply the Cotlar–Stein almost orthogonality lemma.

By duality, the estimate (3.14) follows from

\[
\left\| \sum_{j,k \in \mathbb{Z}} [T^m_{j,k,N}]^* T^m_{j,k,N} \right\|_{op} \lesssim \log N
\]

where we regard \( T^m_{j,k,N} = 0 \) for \( k \notin D_N \). In proving (3.18), it suffices to prove that

\[
\left\| [T^m_{j,k_1,N}]^* T^m_{j,k_1,N} [T^m_{j,k_2,N}]^* T^m_{j,k_2,N} \right\|_{op} \lesssim \left| \frac{|k_1 - k_2|}{N} 2^{m+j} \lambda \right|^{-2} \min \left\{ \frac{1}{N} 2^{m+j} \lambda, \left| \frac{1}{N} 2^{m+j} \lambda \right|^4 \right\}
\]

since the Cotlar–Stein lemma with (3.19) yields that

\[
\left\| \sum_{j,k} [T^m_{j,k,N}]^* T^m_{j,k,N} \right\|_{op} \lesssim \sum_{j \in \mathbb{Z}} \left\| \sum_{k} [T^m_{j,k,N}]^* T^m_{j,k,N} \right\|_{op} \lesssim \log N \sum_{j \in \mathbb{Z}} \min \left\{ \left| \frac{1}{N} 2^{m+j} \lambda \right|^{-1}, \left| \frac{1}{N} 2^{m+j} \lambda \right|^4 \right\} \lesssim \log N.
\]

In proving (3.19), it suffices to show that

\[
\left\| [T^m_{j,k_1,N}]^* T^m_{j,k_1,N} [T^m_{j,k_2,N}]^* T^m_{j,k_2,N} \right\|_{op} \lesssim \left| \frac{|k_1 - k_2|}{N} 2^{m+j} \lambda \right|^{-2},
\]

and

\[
\left\| [T^m_{j,k_1,N}]^* T^m_{j,k_1,N} [T^m_{j,k_2,N}]^* T^m_{j,k_2,N} \right\|_{op} \lesssim \left| \frac{|k_1 - k_2|}{N} 2^{m+j} \lambda \right|^{-2} \left| \frac{1}{N} 2^{m+j} \lambda \right|^4.
\]

We now conclude the proof by showing (3.20) and (3.21).

**Proof of (3.20).** By (3.2), the integral kernel \( S(x, z) \) of \( S^m_{j,k_1,N} S^m_{j,k_2,N} \) is

\[
\int \exp \left( i \lambda \left( \frac{k_1}{N} P_\alpha(x, y) - \frac{k_2}{N} P_\alpha(z, y) \right) \right) \varphi_j(x - y) \varphi_j(z - y) \eta \left( \frac{y}{2^m} \right)^2 \ dy.
\]
The $y$-derivative of phase function in the oscillatory term is

$$
\frac{\lambda}{N} \left( -(k_1 - k_2)y - 2\alpha (k_1(x - y) - k_2(z - y)) \right).
$$

We assume $k_1 \geq k_2$ without loss of generality. We now show (3.20) by distinguishing two cases.

**Case 1.** \((k_1 - k_2)2^m \geq 10k_1(|\alpha| + 1)2^j\). For this case \(\frac{\lambda}{N}(k_1 - k_2)y\) is the dominating factor in (3.22), which enables us to apply integration by parts. We obtain that

$$
\|S_{m,k_1,N}[S_{m,k_2,N}]^*\|_{op} \lesssim \left| \frac{|k_1 - k_2|}{N}2^{m+j}\lambda \right|^{-2}.
$$

and this implies that

$$
\|T_{m,k_1,N}[T_{m,k_2,N}]^*\|_{op} \lesssim \left| \frac{|k_1 - k_2|}{N}2^{m+j}\lambda \right|^{-2}.
$$

Since

$$
\|T_{m,k_1,N}[T_{m,k_2,N}]^*\|_{op} \leq \|T_{m,k_1,N}\|_{op} \|T_{m,k_2,N}\|_{op},
$$

we obtain (3.20) from (3.23) for Case 1.

**Case 2.** \((k_1 - k_2)2^m < 10k_1(|\alpha| + 1)2^j\). By (3.1)

$$
|k_1 - k_2|2^{j+a} \leq |k_1 - k_2|2^m < 10k_12^{-10^2a+j}.
$$

Thus it follows that $k_1 - k_2 < k_1/2^5$. Therefore we have

$$
k_22^j \approx k_12^j \geq \frac{|k_1 - k_2|2^m}{10(|\alpha| + 1)}.
$$

Hence by (3.24) and (3.15),

$$
\|S_{m,k_2,2N}\|_{op} \lesssim \left| \frac{k_22^j}{N2^j}\lambda \right|^{-1/2} \lesssim \left| \frac{|k_1 - k_2|2^{m+j}\lambda}{N} \right|^{-1/2}
$$

and

$$
\|S_{m,k_2,2N}\|_{op} \lesssim \left| \frac{k_22^j}{N2^j}\lambda \right|^{-1/2} \lesssim \left| \frac{|k_1 - k_2|2^{m+j}\lambda}{N} \right|^{-1/2},
$$

to obtain (3.20) for Case 2. \(\square\)
Proof of (3.21). We write the integral kernel of $T_{m, k_1, N}^m[T_{m, k_2, N}]^*$ as

$$
\int \exp \left( i\lambda \left( \frac{k_1}{N} P_\alpha(x, y) - \frac{k_2}{N} P_\alpha(z, y) \right) \right) \left( \exp \left( i\lambda \left( -\frac{1}{2N} \right) P_\alpha(x, y) \right) - 1 \right) \left( \exp \left( i\lambda \left( \frac{1}{2N} \right) P_\alpha(z, y) \right) - 1 \right) \varphi_j(x - y) \varphi_j(z - y) \eta \left( \frac{y}{2^m} \right)^2 dy.
$$

Again we distinguish two cases.

Case 1. $(k_1 - k_2)2^m \geq 10k_1(|\alpha| + 1)2^j$. For this case $\frac{1}{N}(k_1 - k_2)y$ is the dominating factor in the phase function. This combined with (3.16) leads us to apply integration by parts and the mean value theorem to obtain that

$$
\| T_{m, k_1, N}^m[T_{m, k_2, N}]^* \|_{op} \lesssim \left| \frac{|k_1 - k_2|}{N} \right|^{2m+j} \lambda \left| \frac{1}{N^2} \right|^{2m+j} \lambda^2.
$$

(3.25)

We also combine (3.25) with the mean value estimate

$$
\| T_{m, k_1, N}^m \|_{op} + \| T_{m, k_2, N}^m \|_{op} \lesssim \left| \frac{1}{N^2} \right|^{2m+j} \lambda
$$

to obtain (3.21).

Case 2. $(k_1 - k_2)2^m < 10k_1(|\alpha| + 1)2^j$. By (3.24) and (3.17), if $k$ is either $k_1$ or $k_2$,

$$
\| T_{m, k, N}^m \|_{op} \lesssim \left| \frac{k}{N} \right|^{2j} \lambda \left| \frac{1}{N^2} \right|^{2m+j} \lambda \lesssim \left| \frac{|k_1 - k_2|}{N} \right|^{2m+j} \lambda \left| \frac{1}{N^2} \right|^{2m+j} \lambda
$$

to obtain (3.21).

We have now completed the proof of Proposition 1. The upper bound of Theorem 1 is obtained from the case $\alpha = -1/2$ of Proposition 1. The lower bound of Theorem 1 can be obtained by slight change of the Euclidean plane case. More precisely, set

$$
f_{12}(x_1, x_2) = \frac{1}{|\{(x_1, x_2)\}|} \chi_{\{10 \leq |(x_1, x_2)| \leq N\}}(x_1, x_2),
$$

where $\chi_B$ is a characteristic function supported on the set $B$. Then it is known that

$$
\left( \int_{|x| < N} |M_{R_N} f_{12}(x_1, x_2)|^2 dx \right)^{1/2}/\|f_{12}\|_{L^2(R^2)} \geq c \log N.
$$

(3.26)
Here we remind that $M_{\mathcal{R}_N}f_{12}$ is the 2D classical Nikodym maximal function defined as the maximal average of $f_{12}$ over all rectangles with side lengths $r$ and $r/N$ where note that (3.26) holds when we restrict one side length $r < N$. We choose

$$f(x_1, x_2, x_3) = f_{12}(x_1, x_2)\chi_{[-10N^2, 10N^2]}(x_3),$$

then, $\mathcal{M}_N^p f(x_1, x_2, x_3)$ in view of (1.3) is written as

$$\sup_{R \in \mathcal{R}_N} \frac{1}{|R|} \int_R f_{12}(x_1 + y_1, x_2 + y_2)\chi_{[-10N^2, 10N^2]}(x_3 + x_1y_2)dy_1dy_2.$$

For sufficiently large $N$ and $|(x_1, x_2)| < N$, we see that $|x_1y_2| < N^2$ in the above integral. Thus,

$$(4.1) \quad \mathcal{M}_N^p f(x_1, x_2, x_3) \geq \sup_{R \in \mathcal{R}_N} \frac{1}{|R|} \int_R f_{12}(x_1 + y_1, x_2 + y_2)\chi_{[-N^2, N^2]}(x_3)dy_1dy_2$$

$$= M_{\mathcal{R}_N}f_{12}(x_1, x_2)\chi_{[-N^2, N^2]}(x_3).$$

From this combined with (3.26),

$$\frac{\|\mathcal{M}_N^p f\|_{L^2(\mathbb{H}^1)}}{\|f\|_{L^2(\mathbb{H}^1)}} \geq c \log N.$$

4. Proof of lower bound of Theorem 2

Define the operator $\mathcal{N}_N$ by

$$(4.1) \quad \mathcal{N}_N f(x_1, x_2, x_3) = \sup_{R \in \mathcal{P}_N} \frac{1}{|R|} \int_R |f((x_1, x_2, x_3) \cdot (y_1, y_2, 0))| dy_1dy_2$$

where $\mathcal{P}_N$ is the family of the rectangles of $\mathcal{P}_N$ in (2.3), whose side lengths are fixed as 1 and $1/N$. Obviously

$$\|\mathcal{N}_N\|_{L^2(\mathbb{H}^1) \to L^2(\mathbb{H}^1)} \leq \|\mathcal{M}_N\|_{L^2(\mathbb{H}^1) \to L^2(\mathbb{H}^1)}.$$ 

We show that there exists $c > 0$ satisfying

$$(4.2) \quad cN^{1/4} \leq \|\mathcal{N}_N\|_{L^2(\mathbb{H}^1) \to L^2(\mathbb{H}^1)}.$$
Applying the change of variable $y_2 = y_2' + \frac{k}{N} y_1$, we rewrite an average of $f$ over $R$ in (4.1) as $S_k f(x_1, x_2, x_3)$ given by

$$N \int_{\tilde{R}} f \left( x_1 + y_1, x_2 + \left( \frac{k}{N} y_1 + y_2 \right), x_3 + \frac{1}{2} \left( x_1 \left( \frac{k}{N} y_1 + y_2 \right) - x_2 y_1 \right) \right) dy$$

where

$$\tilde{R} = \{(y_1, y_2) : |y_1| < 1, |y_2| < 1/N\}.$$ 

Then we have for $f \geq 0$,

$$N_N f(x) = \sup_{k \in D_N} S_k f(x) \text{ where } x = (x_1, x_2, x_3).$$

We let $\tilde{f}(x_1, x_2, x_3) = f(x_1, x_2, x_3 - x_1 x_2/2)$ and check that

$$S_k f(x) = N \int_{\tilde{R}} \tilde{f} \left( x_1 + y_1, x_2 + \left( \frac{k}{N} y_1 + y_2 \right), x_3 + \frac{1}{2} x_1 x_2 + P(x_1, y_1, y_2) \right) dy,$$

where

$$P(x_1, y_1, y_2) = \frac{1}{2} \left( \frac{k}{N} \left( (x_1 + y_1)^2 - x_1^2 \right) + y_1 y_2 \right) + x_1 y_2.$$

Here we note that $P_0(x, y) = x^2 - y^2$ in (2.11) comes from $(x_1 + y_1)^2 - x_1^2$ above.

It suffices to prove that there exists a constant $c > 0$ and a function $f \in L^2(\mathbb{R}^1)$ such that

$$\| \sup_{k \in D_N} \tilde{S}_k f \|_{L^2(\mathbb{R}^1)} \geq c N^{1/2} \| f \|_{L^2(\mathbb{R}^1)}$$

where

$$\tilde{S}_k f(x) = N \int_{\tilde{R}} f \left( x_1 + y_1, x_2 + \left( \frac{k}{N} y_1 + y_2 \right), x_3 + P(x_1, y_1, y_2) \right) dy.$$

Now we shall show (4.3). Let us define

$$(4.4) \quad f(x_1, x_2, x_3) = \chi_{[-1/\sqrt{N}, 1/\sqrt{N}]}(x_1) \chi_{[-10, 10]}(x_2) \chi_{[-100/N, 100/N]}(x_3).$$

Let

$$U = \left\{ (x_1, x_2, x_3) : \frac{1}{2} \leq x_1 \leq 1, \ 0 \leq x_2 \leq 1, \ \frac{1}{16} \leq x_3 \leq \frac{1}{8} \right\}.$$
Then we show that for each $x = (x_1, x_2, x_3) \in U$

\begin{equation}
\exists k \in \mathcal{D}_N \text{ depending on } x \text{ such that } \tilde{S}_k f(x) \geq \frac{1}{\sqrt{N}}.
\end{equation}

This implies that

\[ \sup_k \tilde{S}_k f(x) \geq \frac{1}{\sqrt{N}} \text{ on } U. \]

Therefore we have

\begin{equation}
\| \sup_{k \in \mathcal{D}_N} \tilde{S}_k f \|_{L^2(H^1)}^2 \geq \int_U | \sup_{k \in \mathcal{D}_N} \tilde{S}_k f(x) |^2 dx \geq \frac{1}{32N}.
\end{equation}

From (4.4)

\begin{equation}
\| f \|_{L^2(H^1)}^2 = \frac{8000}{N\sqrt{N}}.
\end{equation}

Hence (4.3) follows from (4.6) and (4.7). Now it suffices to show (4.5).

\textit{Proof of (4.5).} Let $V = \{(x_1, x_3) : \frac{1}{2} \leq x_1 \leq 1, \frac{1}{16} \leq x_3 \leq \frac{1}{8}\}$. Then we observe that for each $(x_1, x_3) \in V$, \n
\begin{equation}
\exists k = k(x_1, x_3) \in \mathcal{D}_N \text{ such that } \left| x_3 - \frac{1}{2N} x_1^2 \right| < \frac{5}{N}.
\end{equation}

This implies that on the region

\[ |x_1 + y_1| \leq 1/\sqrt{N}, |y_2| \leq 1/N, |y_1| < 1 \text{ and } (x_1, x_3) \in V, \]

we have

\begin{equation}
x_3 + \frac{1}{2} \left( \frac{k}{N} \left( -x_1^2 + (x_1 + y_1)^2 \right) + y_1 y_2 \right) + x_1 y_2 \leq \frac{20}{N}.
\end{equation}

Obviously we see that for any $k \in \mathcal{D}_N$, $|y_1| < 1$ and $|y_2| \leq 1/N$,

\begin{equation}
\left| x_2 + \left( \frac{k}{N} y_1 + y_2 \right) \right| \leq 10 \text{ for } 0 \leq x_2 \leq 1.
\end{equation}

Thus by (4.9) and (4.10), we check the support condition of (4.4) to obtain that for any $(x_1, x_2, x_3) \in U$ with $k$ in (4.8)

\[ f \left( x_1 + y_1, x_2 + \left( \frac{k}{N} y_1 + y_2 \right), x_3 + \frac{1}{2} \left( \frac{k}{N} ((x_1 + y_1)^2 - x_1^2) + y_1 y_2 \right) + x_1 y_2 \right) = 1, \]
on the region $R_x = \{ y \in \tilde{R} : |x_1 + y_1| \leq \frac{1}{\sqrt{N}} \}$. This combined with the measure estimate of the set $R_x$ yields (4.5).

\[ \square \]

5. Proof of upper bound of theorem 2

We are able to assume that the region of integral is restricted to $[0, r]$ in (1.2) as we did in Theorem 1. For each $j \in \mathbb{Z}$, $k \in D_N$, we define the measure $\nu_{j,k,N}$,

$$
\nu_{j,k,N}(f) = \int f(y_1, y_2, 0) \frac{1}{2^j} \varphi \left( \frac{y_1}{2^j} \right) N \frac{1}{2^j} \psi \left( N \left( y_2 + \frac{k}{N} y_1 \right) \right) dy_1 dy_2.
$$

Here the functions $\varphi$ and $\psi$ are presumably those introduced in Sections 2 and 3. By using (2.2) and (2.7),

$$
|\mathcal{M}_N f(x_1, x_2, x_3)| \lesssim \sup_{j \in \mathbb{Z}, k \in D_N} |\nu_{j,k,N} * f(x_1, x_2, x_3)|.
$$

By applying the same induction argument as in Section 2.2, it suffices to show that

$$
\left\| \left( \sum_{j \in \mathbb{Z}, k \in D_N} |(\nu_{j,2k-1,2N} - \nu_{j,2k,2N}) * f|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^1)} \lesssim N^{1/4} (\log N) \| f \|_{L^2(\mathbb{R}^1)}.
$$

By using formula (2.9),

$$
[\hat{\nu}_{j,k,N}(\lambda) h](x) = \int \exp(i \lambda \frac{k}{N} (x^2 - y^2)) \hat{\psi} \left( \frac{\lambda 2^j (x + y)}{2N} \right) \varphi_j(x - y) h(y) dy.
$$

Let

$$
U_{j,k,N}(\lambda) = \hat{\nu}_{j,2k-1,2N}(\lambda) - \hat{\nu}_{j,2k,2N}(\lambda).
$$

By Proposition 2, it suffices to prove that for some $C$ independent of $\lambda$,

$$
(5.1) \left\| \left( \sum_{j \in \mathbb{Z}, k \in D_N} |U_{j,k,N}(\lambda) h|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^1)} \leq C N^{1/4} (\log N) \| h \|_{L^2(\mathbb{R}^1)}.
$$

Fix $\lambda$ and choose $l \in \mathbb{Z}$ such that

$$
2^l \leq N^{1/2} < 2^{l+1}.
$$
For each $m \in \mathbb{Z}$, we let
\begin{equation}
U_{j,k,N}^m = V^m_{j,2k-1,2N} - V^m_{j,2k,2N}
\end{equation}
where
\begin{equation}
V_{j,k,N}^m h(x) = \int \exp \left( i \lambda \frac{k}{N} (x^2 - y^2) \right) \hat{\psi} \left( \frac{\lambda 2^j (x + y)}{2N} \right) \varphi_j(x - y) \eta \left( \frac{y}{2^m} \right) h(y) dy.
\end{equation}

Here the function $\eta$ is the same as the function $\eta$ introduced in Section 3. We split $U_{j,k,N}(\lambda) = U_{j,k,N}^{\text{loc}} + U_{j,k,N}^{\text{med}} + U_{j,k,N}^{\text{glo}}$ such that
\begin{equation}
\begin{aligned}
U_{j,k,N}^{\text{loc}} &= \sum_{m < j - \ell} U_{j,k,N}^m \\
U_{j,k,N}^{\text{med}} &= \sum_{j - \ell \leq m \leq j - 10} U_{j,k,N}^m \\
U_{j,k,N}^{\text{glo}} &= \sum_{j - 10 < m} U_{j,k,N}^m.
\end{aligned}
\end{equation}

Note each of the above three operators is defined according to the size of $|y|$, namely, the kernel of $U_{j,k,N}^{\text{loc}}$ is supported on $|y| \lesssim 2^{j-\ell}$, that of $U_{j,k,N}^{\text{med}}$ on $2^{j-\ell} \lesssim |y| \lesssim 2^{j-10}$, and $U_{j,k,N}^{\text{glo}}$ on $2^{j-10} \lesssim |y|$.

**Estimate of $U_{j,k,N}^{\text{loc}}$.** We show that
\begin{equation}
\left\| \left( \sum_{j \in \mathbb{Z}, k \in D_N} |U_{j,k,N}^{\text{loc}} h|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R})} \lesssim N^{1/4} \| h \|_{L^2(\mathbb{R}^1)}.
\end{equation}

Note
\begin{align*}
U_{j,k,N}^{\text{loc}} h(x) &= \int \left( \exp \left( i \lambda \frac{2k-1}{2N} (x^2 - y^2) \right) - \exp \left( i \lambda \frac{2k}{2N} (x^2 - y^2) \right) \right) \\
&\quad \times \hat{\psi} \left( \frac{\lambda 2^j (x + y)}{2N} \right) \varphi_j(x - y) \psi \left( \frac{y}{2^{j-\ell-1}} \right) h(y) dy.
\end{align*}

On the support of the integral
\begin{equation}
|x + y| \approx |x - y| \approx 2^j
\end{equation}
because \( |y| \ll 2^j \). Since \( \hat{\psi} \) is a Schwartz function,

\[
(5.6) \quad \left| \hat{\psi} \left( \frac{\lambda 2^j (x + y)}{2N} \right) \right| \leq C \min \left\{ \frac{N}{\lambda 2^j}, 1 \right\}.
\]

By using the mean value theorem,

\[
(5.7) \quad \left| \exp \left( i \lambda \frac{2k - 1}{2N} (x^2 - y^2) \right) - \exp \left( i \lambda \frac{2k}{2N} (x^2 - y^2) \right) \right| \lesssim \min \left\{ \frac{\lambda 2^j}{N}, 1 \right\}.
\]

From (5.5)-(5.7) combined with the support condition \( |y| \leq 2^j - l \),

\[
\| \mathcal{U}^\text{loc}_{j,k,N} \|_{\text{op}} \leq C 2^{-l/2} \min \left\{ \frac{\lambda 2^j}{N}, \frac{N}{\lambda 2^j} \right\}.
\]

This yields (5.4) because \( 2^l \approx N^{1/2} \).

**Estimate of \( \mathcal{U}^\text{med}_{j,k,N} \).** We show that

\[
\left\| \left( \sum_{j \in \mathbb{Z}, k \in D_N} | \mathcal{U}^\text{med}_{j,k,N} h \|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^1)} \lesssim N^{1/4} (\log N) \| h \|_{L^2(\mathbb{R}^1)}.
\]

For this estimate it suffices to prove that for fixed \( m \) with \( j - \ell - 2 \leq m \leq j - 8 \),

\[
(5.8) \quad \left\| \left( \sum_{j \in \mathbb{Z}, k \in D_N} | \mathcal{U}^m_{j,k,N} h \|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^1)} \lesssim N^{1/4} \| h \|_{L^2(\mathbb{R}^1)},
\]

since \( \ell \approx \log N \). This can be obtained from the dual estimate

\[
(5.9) \quad \left\| \sum_{j \in \mathbb{Z}, k \in D_N} [\mathcal{U}^m_{j,k,N}]^* \mathcal{U}^m_{j,k,N} \right\|_{\text{op}} \lesssim N^{1/2}.
\]

In proving (5.9), it suffices to prove that

\[
(5.10) \quad \left\| [\mathcal{U}^m_{j,k_1,N}]^* \mathcal{U}^m_{j,k_1,N} [\mathcal{U}^m_{j,k_2,N}]^* \mathcal{U}^m_{j,k_2,N} \right\|_{\text{op}} \lesssim \left| \frac{|k_1 - k_2| 2^j \lambda}{N} \right|^{-1} \min \left\{ 1, \frac{1}{N} \lambda 2^j \lambda \right\}^2
\]

since the Cotlar–Stein lemma with (5.10) yields that
\[
\left\| \sum_{j,k} U_{j,k,N}^m [U_{j,k,N}^m]^* \right\|_{op} \lesssim \sum_{j,k} \left\| \sum_k U_{j,k,N}^m [U_{j,k,N}^m]^* \right\|_{op}
\lesssim \sum_{j\in \mathbb{Z}} \left( \sum_{k=1}^N \frac{1}{\sqrt{k}} \right) \min \left\{ \left| \frac{1}{N} 2^{2j} \lambda \right|^{-1/2}, \left| \frac{1}{N} 2^{2j} \lambda \right|^{1/2} \right\}
\lesssim N^{1/2}.
\]

We now show (5.10). Note
\[
U_{j,k,N}^m h(x) = \int \left( \exp \left( i \lambda \frac{2k - 1}{2N} (x^2 - y^2) \right) - \exp \left( i \lambda \frac{2k}{2N} (x^2 - y^2) \right) \right) \hat{\psi} \left( \frac{\lambda 2^j (x+y)}{2N} \right) \varphi_j (x-y) \eta \left( \frac{y}{2^m} \right) h(y) dy.
\]
(5.11)
Let \( K(x,y) \) be the integral kernel of the above operator. By using (5.7) combined with the support condition \(|y| \approx 2^m\), we obtain the Hilbert–Schmidt norm,
\[
\left\| U_{j,k,N}^m \right\|_{op} \leq \sqrt{\int |K(x,y)|^2 dy dx} \leq C 2^{(m-j)/2} \min \left\{ \frac{\lambda 2^j}{N}, 1 \right\}.
\]
(5.12)
Next we show that
\[
\left\| U_{j,k,1,N}^m [U_{j,k,2,N}^m]^* \right\|_{op} \lesssim 2^{j-m} \left| \frac{k_1 - k_2}{N} \right| 2^{2j} \lambda^{-1}. \quad (5.13)
\]
We see that (5.10) follows from (5.12) and (5.13).

Proof of (5.13). The integral kernel \( V(x,z) \) of the operator \( \Psi_{j,k,1,N}^m [\Psi_{j,k,2,N}^m]^* \) is
\[
V(x,z) = \int \exp \left( i \lambda \left( \frac{k_1}{N} (x^2 - y^2) - \frac{k_2}{N} (z^2 - y^2) \right) \right) \Theta(x,y,z) dy
\]
(5.14)
where \( \Theta(x,y,z) \) is
\[
\hat{\psi} \left( \frac{\lambda 2^j (x+y)}{2N} \right) \hat{\psi} \left( \frac{\lambda 2^j (z+y)}{2N} \right) \varphi_j (x-y) \varphi_j (z-y) \eta \left( \frac{y}{2^m} \right)^2.
\]
The derivative of the phase of the oscillatory term with respect to the \( y \) variable is given by

\[(5.15) \quad -2 \frac{\lambda}{N} (k_1 - k_2)y.\]

Note that

\[
\left| \frac{\partial}{\partial y} \left( \frac{\lambda 2^{j} (x + y)}{N} \right) \right| = \frac{\lambda 2^{j}}{N} \left| \left( \frac{\lambda 2^{j} (x + y)}{N} \right)' \right| \lesssim \frac{1}{|x + y|}
\]

(5.16)

and

\[
\left| \frac{\partial}{\partial y} \left( \eta \left( \frac{y}{2^m} \right) \right) \right| \lesssim \frac{1}{2^m}.
\]

(5.17)

By the support condition that \(|y| \approx 2^m \leq 2^{j-10}\) and \(2^{j-1} \leq |x - y| \leq 2^{j+1}\), we observe that \(|x + y| \approx 2^j\) in (5.16). Thus the derivative of the amplitude \(\Theta(x, y, z)\) is dominated by

\[(5.18) \quad \left| \frac{\partial}{\partial y} \Theta(x, y, z) \right| \leq \frac{C}{2^{2j+m}}.\]

By using (5.15) and (5.18), we apply integration by parts in (5.14) to obtain that

\[|V(x, z)| \lesssim \frac{N}{\lambda|k_1 - k_2|2^{j+m}}.\]

This yields that

\[
\left\| \mathcal{V}_{j,k,N}^m \mathcal{V}_{j,k_2,N}^m \right\|_{\text{op}} \lesssim \left| \frac{\lambda |k_1 - k_2|2^{j+m}}{N} \right|^{-1},\]

which combined with (5.2) implies (5.13). \(\square\)

**Estimate of** \(U_{j,k,N}^{glo}\). We show that

\[(5.19) \quad \left\| \left( \sum_{j \in \mathbb{Z}, k \in D_N} |U_{j,k,N}^{glo}|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^1)} \lesssim \| h \|_{L^2(\mathbb{R}^1)}.\]

We split again

\[U_{j,k,N}^{glo} = U_{j,k,N}^{glo,1} + U_{j,k,N}^{glo,2}\]
where

\[ U_{j,k,N}^{\text{glo},1} = \sum_{j+10 \leq m} U_{j,k,N}^m \]
\[ U_{j,k,N}^{\text{glo},2} = \sum_{j-10 < m < j+10} U_{j,k,N}^m \]

For the proof of (5.19), we show that

(5.20) \[ \left\| \sum_{j \in \mathbb{Z}, k \in D_N} [U_{j,k,N}^{\text{glo},1}]^* U_{j,k,N}^{\text{glo},1} \right\|_{op} \lesssim 1 \]

and

(5.21) \[ \left\| \sum_{j \in \mathbb{Z}, k \in D_N} [U_{j,k,N}^{\text{glo},2}]^* U_{j,k,N}^{\text{glo},2} \right\|_{op} \lesssim 1. \]

Proof of (5.20). For the case \( |y| \approx 2^m > 2^{j+10} \) in (5.11), it suffices to replace \( U_{j,k,N}^{\text{glo},1} \) in (5.20) by only one piece \( U_{j,k,N}^m \),

\[ \left\| \sum_{j \in \mathbb{Z}, k \in D_N} [U_{j,k,N}^m]^* U_{j,k,N}^m \right\|_{op} \lesssim 1. \]

For this, it suffices to show that

(5.22) \[ \left\| [U_{j,k_1,N}^m]^* U_{j,k_1,N}^m [U_{j,k_2,N}^m]^* U_{j,k_2,N}^m \right\|_{op} \lesssim \left| \frac{|k_1 - k_2|}{N} 2^{j+m} \lambda \right|^{-5/2} \min \left\{ 1, \left| \frac{1}{N} 2^{j+m} \lambda \right|^{3} \right\} \]

since the Cotlar–Stein lemma with (5.22) yields that

\[ \left\| \sum_{j,k} U_{j,k,N}^m [U_{j,k,N}^m]^* \right\|_{op} \lesssim \sum_{j \in \mathbb{Z}} \left\| \sum_{k} U_{j,k,N}^m [U_{j,k,N}^m]^* \right\|_{op} \]
\[ \lesssim \sum_{j \in \mathbb{Z}} \left( \sum_{k=1}^{\infty} \frac{1}{k^{5/4}} \right) \min \left\{ \left| \frac{1}{N} 2^{j+m} \lambda \right|^{-5/4}, \left| \frac{1}{N} 2^{j+m} \lambda \right|^{1/4} \right\} \]
\[ \lesssim 1. \]
By using the support condition $|y| \approx 2^m > 2^{j+10}$ in (5.11),

$$\left| \exp \left( i \lambda \frac{2k}{2N} (x^2 - y^2) \right) - \exp \left( i \lambda \frac{2k}{2N} (x^2 - y^2) \right) \right| \leq C \min \left\{ \frac{\lambda 2^j m}{N}, 1 \right\}.$$ 

So, we get

(5.23) \[ \| U_{j,k,N}^{m} \|_{op} \leq C \min \left\{ \frac{\lambda 2^j m}{N}, 1 \right\}. \]

Next we show that

(5.24) \[ \| U_{j,k1,N}^{m} [ U_{j,k2,N}^{m} ]^* \|_{op} \lesssim \left| \frac{|k_1 - k_2| 2^{j+m} \lambda}{N} \right|^{-5}. \]

If (5.24) is true, then by interpolating (5.24) and (5.23),

(5.25) \[ \| U_{j,k1,N}^{m} [ U_{j,k2,N}^{m} ]^* \|_{op} \lesssim \left| \frac{|k_1 - k_2| 2^{j+m} \lambda}{N} \right|^{-5/2} \min \left\{ \frac{\lambda 2^j m}{N}, 1 \right\}. \]

We see that (5.22) follows from (5.23) and (5.25). In proving (5.24), from (5.2) it suffices to show that

(5.26) \[ \| V_{j,k1,N}^{m} [ V_{j,k2,N}^{m} ]^* \|_{op} \lesssim \left| \frac{|k_1 - k_2| 2^{j+m} \lambda}{N} \right|^{-5}. \]

The integral kernel $S(x, z)$ of the operator $V_{j,k1,N}^{m} [ V_{j,k2,N}^{m} ]^*$ is in (5.14). From the observation that $|x + y| \approx |z + y| \approx 2^m > 2^{j+10}$ in (5.16) and (5.17), we obtain that

(5.27) \[ \left| \frac{\partial}{\partial y} \Theta(x, y, z) \right| \leq \frac{C}{2^{3j}}. \]

By using (5.15) and (5.27) with the support condition $|y| \approx 2^m > 2^{j+10}$, we apply integration by parts on the integral (5.14) to obtain that

$$|V(x, z)| \leq \left| \frac{CN}{\lambda |k_1 - k_2| 2^{j+m}} \right|^{5} 2^{-j}$$

which yields (5.26). The proof of (5.20) is finished. \[ \square \]
**Proof of (5.21).** Note that our difficulty here comes from the fact that for the case \( j - 10 < m < j + 10, |x + y| \) can vanish in (5.16), which prevent us from having (5.27). Instead, we have some bigger bound,

\[
\left| \frac{\partial}{\partial y} \Theta(x, y, z) \right| \leq \frac{C}{2^j} \max\left\{ \frac{1}{2^j}, \frac{\lambda 2^i}{N} \right\}.
\]

(5.28)

However, we can overcome this obstacle by the following observation. The integral kernel \( K(x, y) \) of the operator

\[
\left[ \mathcal{U}^{glo, 2}_{j, k, N} \right]^{*} \mathcal{U}^{glo, 2}_{j, k, N} h \]

is supported on the set

\[
\left\{ (x, y) : 2^j - 15 < |x| < 2^j + 15, 2^{j-15} < |y| < 2^{j+15} \right\}.
\]

Thus it suffices to prove for fixed \( j \) and \( m \) where \( j - 10 < m < j + 10, \)

\[
\left\| \sum_{k \in D_N} \mathcal{U}^{m}_{j, k, N} \right\|_{op} \lesssim 1.
\]

For proving this we show that

\[
\left\| \mathcal{U}^{m}_{j, k_1, N} \right\|_{op} \lesssim 1
\]

(5.29) since the Cotlar–Stein lemma with (5.29) yields that

\[
\left\| \sum_{k} \mathcal{U}^{m}_{j, k, N} \right\|_{op} \lesssim 1.
\]

For the proof of (5.29), we replace (5.27) by (5.28) and apply the same estimation of (5.22) with the support condition \(|y| \approx 2^j\) instead of \(2^m\). □

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