MULTIPLE HILBERT TRANSFORMS ASSOCIATED WITH POLYNOMIALS

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ABSTRACT. Let $\Lambda = (\Lambda_1, \cdots, \Lambda_d)$ with $\Lambda_\nu \subset \mathbb{Z}_n^+$, and set $P_\Lambda$ the family of all vector polynomials,

$$P_\Lambda = \left\{ P_\Lambda : P_\Lambda(t) = \left( \sum_{m \in \Lambda_1} c^1_m t^m, \cdots, \sum_{m \in \Lambda_d} c^d_m t^m \right) \text{ with } t \in \mathbb{R}^n \right\}.$$ 

Given $P_\Lambda \in P_\Lambda$, we consider a class of multi-parameter oscillatory singular integrals,

$$I(P_\Lambda, \xi, r) = \text{p.v.} \int_{[-r, r]^n} e^{i \langle \xi, P_\Lambda(t) \rangle} \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} \text{ where } \xi \in \mathbb{R}^d, r \in \mathbb{R}_+^n.$$ 

When $n = 1$, the integral $I(P_\Lambda, \xi, r)$ for any $P_\Lambda \in P_\Lambda$ is bounded uniformly in $\xi$ and $r$. However, when $n \geq 2$, the uniform boundedness depends on each individual polynomial $P_\Lambda$. In this paper, we fix $\Lambda$ and find a necessary and sufficient condition on $\Lambda$ that

$$\text{for all } P_\Lambda \in P_\Lambda, \quad \sup_{\xi, r} \left| I(P_\Lambda, \xi, r) \right| \leq C_{P_\Lambda} < \infty.$$ 

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1. **Introduction**

Let $\mathbb{Z}_+^n$ denote the set of all nonnegative integers and let $\Lambda_\nu \subset \mathbb{Z}_+^n$ be the finite set of multi-indices for each $\nu = 1, \cdots, d$. Given $\Lambda = (\Lambda_1, \cdots, \Lambda_d)$, we set $P_\Lambda$ the family of all vector polynomials $P_\Lambda$ of the following form:

\[
P_\Lambda = \left\{ P_\Lambda : P_\Lambda(t) = \left( \sum_{m \in \Lambda_1} c_1^m t^m, \cdots, \sum_{m \in \Lambda_d} c_d^m t^m \right) \right\} \text{ with } t \in \mathbb{R}^n
\]

where $c_\nu^m$'s are nonzero real numbers. Given $P_\Lambda \in P_\Lambda$, $\xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d$ and $r = (r_1, \cdots, r_d) \in \mathbb{R}_+^d$, we define a multi-parameter oscillatory singular integral:

\[
I(P_\Lambda, \xi, r) = \text{p.v.} \int_{\prod[-r_j, r_j]} e^{i\langle \xi, P_\Lambda(t) \rangle} \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}
\]

where the principal value integral is defined by

\[
\lim_{\epsilon \to 0} \int_{\prod\{\epsilon_j < |t_j| < r_j\}} e^{i\langle \xi, P_\Lambda(t) \rangle} \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}
\]

where $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$ with $\epsilon_j > 0$. The existence of this limit follows by the Taylor expansion of $t \to e^{i\langle \xi, P_\Lambda(t) \rangle}$ and the cancelation property $\int dt_\nu/t_\nu = 0$ with $\nu = 1, \cdots, n$.

We see that whether $\sup_\xi |I(P_\Lambda, \xi, r)|$ is finite or not depends on

1. **Sets $\Lambda_\nu$ of exponents of monomials in $P_\Lambda(t)$**.
2. **Coefficients of polynomial $P_\Lambda(t)$**.
3. **Domain of integral $\prod[-r_j, r_j]$**.

(1) The dependence on set $\Lambda_\nu$ of exponents is observed in the following simple cases:

\[
\sup_{\xi \in \mathbb{R}} |I(P_\Lambda, \xi, (1, 1))| = \begin{cases} 
\sup_{\xi \in \mathbb{R}} \left| \int_{-1}^1 \int_{-1}^1 \sin(\xi t_1 t_2) \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right| = \infty & \text{if } \Lambda = \{(1, 1)\} \\
\sup_{\xi \in \mathbb{R}} \left| \int_{-1}^1 \int_{-1}^1 \sin(\xi t_1^2 t_2^2) \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right| = 0 & \text{if } \Lambda = \{(2, 1)\}
\end{cases}
\]
(2) The dependence on coefficients of polynomials $P_\Lambda$ first appeared in [12], later in [1] and [13]. Let $P_\Lambda(t) = t_1^3 t_2^3 - t_1 t_2^3$ and $Q_\Lambda(t) = t_1^2 t_2^3 + t_1^3 t_2^2$. Then these have the same exponent set $\Lambda$, but sup$_\xi \mathcal{I}(P_\Lambda, \xi, r) < \infty$ and sup$_\xi \mathcal{I}(Q_\Lambda, \xi, r) = \infty$. However, in this paper, we are not concerned with this coefficient dependence. We rather search for a condition on $\Lambda$ so that

\begin{equation}
\text{for all } P_\Lambda \in \mathcal{P}_\Lambda, \quad \sup_{\xi \in \mathbb{R}^d} |\mathcal{I}(P_\Lambda, \xi, r)| \leq C_{P_\Lambda} < \infty.
\end{equation}

(3) The dependence on the domain $\prod[-r_j, r_j]$ is observed for the case $\Lambda = \{(2,2), (3,3)\}$,

\begin{align*}
\sup_{\xi \in \mathbb{R}, 0 < r_1, r_2 < 1} & \left| \int_{-r_2}^{r_2} \int_{-r_1}^{r_1} e^{i\xi(t_1^2 t_2^2 + t_1^3 t_2^2)} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right| < \infty, \\
\sup_{\xi \in \mathbb{R}, 0 < r_1, r_2 < \infty} & \left| \int_{-r_2}^{r_2} \int_{-r_1}^{r_1} e^{i\xi(t_1^2 t_2^2 + t_1^3 t_2^2)} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right| = \infty.
\end{align*}

In the former integral, a monomial $t_1^2 t_2^2$ dominating $t_1^3 t_2^3$ with small $t_1, t_2$, makes the vanishing property $\int \frac{dt}{t_i} = 0$ effective. But in the latter integral, a monomial $t_1^3 t_2^3$, dominating $t_1^2 t_2^2$ with large $t_1, t_2$, weakens the cancellation effect of the integral $\int \frac{dt}{t_i}$. Knowing this dependence on whether $r_j$ is taken from a finite interval $(0, 1)$ or an infinite interval $(0, \infty)$, we set up our problem by first fixing the range of $r$ according to $S \subset N_n = \{1, \ldots, n\}$:

\begin{equation}
r \in I(S) = \prod_{j=1}^n I_j \quad \text{where } I_j = (0, 1) \text{ for } j \in S \text{ and } I_j = (0, \infty) \text{ for } j \in N_n \setminus S.
\end{equation}

Instead of (1.2), we shall find the necessary and sufficient condition on $\Lambda$ and $S$ that

\begin{equation}
\text{for all } P_\Lambda \in \mathcal{P}_\Lambda, \quad \sup_{\xi \in \mathbb{R}^d, r \in I(S)} |\mathcal{I}(P_\Lambda, \xi, r)| \leq C_{P_\Lambda} < \infty.
\end{equation}

For each Schwartz function $f$ on $\mathbb{R}^d$ and a vector polynomial $P_\Lambda \in \mathcal{P}_\Lambda$, the multiple Hilbert transform of $f$ associated to $P_\Lambda$ is defined to be

\begin{equation}
(H_r^{P_\Lambda} f)(x) = \text{p.v.} \int_{\prod_{j=1}^n [-r_j, r_j]} f(x - P_\Lambda(t)) \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}.
\end{equation}

Here $r_j = 1$ with $j \in S$ corresponds to a local Hilbert transform, and $r_j = \infty$ with $j \in N_n \setminus S$ corresponds to a global Hilbert transform. Since $\mathcal{I}(P_\Lambda, \xi, r)$ is the Fourier...
multiplier of the Hilbert transform $\mathcal{H}_F^\Lambda$, the boundedness (1.4) is equivalent to that
\[
\text{sup}_{r \in I(S)} \| \mathcal{H}_F^\Lambda \|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} \leq C_P \quad \text{where } p = 2.
\]
In this paper, we show (1.4) and (1.5) with $1 < p < \infty$ for all $n$ and $d$ when $S \subset N_n$. To determine the conditions that ensure that (1.4) and (1.5) hold, we study the concept of faces and their dual faces (cones) of the Newton Polyhedron associated with $\Lambda$ and $S \subset N_n$. It is noteworthy in advance that the necessary and sufficient condition of (1.4) is not determined by only faces but also by dual faces of the Newton polyhedron, which has not appeared explicitly in the graph case $\Lambda = (e_1, \cdots, e_n, \Lambda_{n+1})$ or low dimensional case $n \leq 2$.

**Organization.** In Section 2, we define some useful combinatorial notions. In Section 3, we state the main results. We prove combinatorial lemmas in Sections 4 and 5. Next, we show preliminary analytic estimates in Section 6. The bulk of the proof of Main Theorems are found in Sections 7-11.

**Notations.** For the sake of distinction, we shall use the notations
\[
i \cdot j = i_1 j_1 + \cdots + i_n j_n, \quad \langle x, y \rangle = x_1 y_1 + \cdots + x_d y_d
\]
for the inner products on $\mathbb{Z}^n$, $\mathbb{R}^d$, respectively. Note that a constant $C$ may be different on each line. As usual, the notation $A \lesssim B$ for two scalar expressions $A, B$ will mean $A \leq CB$ for some positive constant $C$ independent of $A, B$ and $A \approx B$ will mean $A \lesssim B$ and $B \lesssim A$.

2. **Definitions of Polyhedra, Their Faces and Cones**

In this section, we define basic notions of polyhedra, faces and their dual faces

**2.1. Polyhedron.**

**Definition 2.1.** Let $U \subset \mathbb{R}^n$ be a subspace endowed with an inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^n$. Then $V$ is called an affine subspace in $\mathbb{R}^n$ if $V = p + U$ for some $p \in \mathbb{R}^n$.

**Definition 2.2.** Let $V$ be an affine subspace in $\mathbb{R}^n$. A hyperplane in $V$ is a set
\[
\pi_{q,r} = \{ y \in V : \langle q, y \rangle = r \} \quad \text{where } q \in \mathbb{R}^n \text{ and } r \in \mathbb{R}.
\]
The corresponding closed upper half-space and lower half-space are
\[ \pi_{q,r}^+ = \{ y \in V : \langle q, y \rangle \geq r \} \quad \text{and} \quad \pi_{q,r}^- = \{ y \in V : \langle q, y \rangle \leq r \}. \]

The open upper half-space and lower half space are
\[ (\pi_{q,r}^+)^\circ = \{ y \in V : \langle q, y \rangle > r \} \quad \text{and} \quad (\pi_{q,r}^-)^\circ = \{ y \in V : \langle q, y \rangle < r \}. \]

**Definition 2.3** (Polyhedron in $V$). Let $V$ be an affine subspace in $\mathbb{R}^n$ and let $\Pi = \{ \pi_{q_j,r_j} \}_{j=1}^N$ be a set of hyperplanes in $V$. A polyhedron $P$ in $V$ is defined to be an intersection of closed upper half-spaces $\pi_{q_j,r_j}^+$:
\[ P = \bigcap_{j=1}^N \pi_{q_j,r_j}^+ = \bigcap_{j=1}^N \{ y \in V : \langle q_j, y \rangle \geq r_j \} \quad \text{for } 1 \leq j \leq N \}. \]

We call the above collection $\Pi = \Pi(P)$ the generator of $P$. We denote the polyhedron $P$ by $P(\Pi)$ indicating its generator $\Pi$. Sometimes, we also mean the generator $\Pi$ of $P$ to be the collection of normal vectors $\{ q_j \}_{j=1}^N$ instead of hyperplanes $\{ \pi_{q_j,r_j} \}_{j=1}^N$.

**Example 2.1.** A set of hyper-planes $\Pi = \{ \pi_{q_j,r_j} : 1 \leq j \leq 5 \}$ where $q_j = \frac{1}{2}(e_1 + e_2 + e_3 - e_j) - \frac{1}{3}(e_1 + e_2 + e_3)$ with $r_j = 1/6$ for $j = 1, 2, 3$ and $q_j = \pm(e_1 + e_2 + e_3)$ with $r_j = \pm1$ for $j = 4, 5$ makes a triangle $P(\Pi) = Ch(e_1, e_2, e_3)$ (two dimensional polyhedron) in $\mathbb{R}^3$.

**Definition 2.4.** Let $B = \{ q_1, \cdots, q_M \} \subset \mathbb{R}^n$. Then the span of $B$ is the set
\[ \text{Sp}(B) = \left\{ \sum_{j=1}^M c_j q_j : c_j \in \mathbb{R} \right\}. \]

The convex span of $B$ and its interior are defined by
\[ \text{CoSp}(B) = \left\{ \sum_{j=1}^M c_j q_j : c_j \geq 0 \right\} \quad \text{and} \quad \text{CoSp}^\circ(B) = \left\{ \sum_{j=1}^M c_j q_j : c_j > 0 \right\} \]
respectively. Finally the convex hull of $B$ is the set
\[ \text{Ch}(B) = \left\{ \sum_{j=1}^M c_j q_j : c_j \geq 0 \text{ and } \sum_{j=1}^M c_j = 1 \right\}. \]

If $B \subset \mathbb{R}^n$ is not a finite set, then the span of $B$ is defined by the collection of all finite linear combinations of vectors in $B$. 
Definition 2.5 (Ambient Space of Polyhedron). Let $\mathbb{P} \subset \mathbb{R}^n$ and $p, q \in \mathbb{P}$. Then
\[ \text{Sp}(\mathbb{P} - p) = \text{Sp}(\mathbb{P} - q) \text{ for all } p, q \in \mathbb{P}. \]

We denote the vector space $\text{Sp}(\mathbb{P} - p)$ by $V(\mathbb{P})$. The dimension of $\mathbb{P}$ is defined by
\[ \text{dim}(\mathbb{P}) = \text{dim}(V(\mathbb{P})). \]

From the fact $p - q \in V(\mathbb{P})$,
\[ V(\mathbb{P}) + p = V(\mathbb{P}) + q. \]

We call $V(\mathbb{P}) + p$ the ambient affine space of $\mathbb{P}$ in $\mathbb{R}^n$ and denote it by $V_{am}(\mathbb{P})$:
(2.1) \[ V_{am}(\mathbb{P}) = V(\mathbb{P}) + p, \]
which is the smallest affine space containing $\mathbb{P}$.

Example 2.2. Let $\mathbb{P}$ be the polyhedron defined in Example 2.1. Then $V(\mathbb{P}) = \{ y : \langle e_1 + e_2 + e_3, y \rangle = 0 \}$ and $V_{am} = \{ y : \langle e_1 + e_2 + e_3, y \rangle = 1 \}$.

Definition 2.6. Let $B \subset \mathbb{R}^n$. Then the rank of a set $B$ is the number of linearly independent vectors in $B$:
\[ \text{rank}(B) = \text{dim}(\text{Sp}(B)). \]

2.2. Faces of Polyhedron.

Definition 2.7 (Face). Let $V$ be an affine subspace in $\mathbb{R}^n$. Given a set $\Pi$ of hyperplane in $V$, let $\mathbb{P} = \mathbb{P}(\Pi)$ be a polyhedron in $V$. A subset $F \subset \mathbb{P}$ is a face if there exists a hyperplane $\pi_{q,r}$ in $V$ (which does not have to be in $\Pi$) such that
(2.2) \[ F = \pi_{q,r} \cap \mathbb{P} \text{ and } \mathbb{P} \setminus F \subset \pi_{q,r}^+. \]

We may replace $\mathbb{P} \setminus F$ by $\mathbb{P}$, or $\pi_{q,r}^+$ by $(\pi_{q,r}^+)^c$ in (2.2). Thus $F$ is a face of $\mathbb{P}$ if and only if there exists a vector $q \in \mathbb{R}^n$ and $r \in \mathbb{R}$ satisfying
(2.3) \[ \langle q, u \rangle = r < \langle q, y \rangle \text{ for all } u \in F \text{ and } y \in \mathbb{P} \setminus F. \]

When $F$ is a face of $\mathbb{P}$, it is denoted by $F \preceq \mathbb{P}$. The above hyperplane $\pi_{q,r}$ is called the supporting hyperplane of the face $F$. The dimension of a face $F$ of $\mathbb{P}$ is the dimension of an ambient affine space $V_{am}(F)$ of $F$ where $V_{am}(F)$ is defined in (2.1). We denote the set
of all $k$-dimensional faces of $\mathbb{P}$ by $\mathcal{F}^k(\mathbb{P})$, and $\bigcup \mathcal{F}^k(\mathbb{P})$ by $\mathcal{F}(\mathbb{P})$. By convention, an empty set is $-1$ dimensional face. Let $\dim(\mathbb{P}) = m$. Then we call, a face $F$ whose dimension is less than $m$, a proper face of $\mathbb{P}$ and denote it by $F \not\subseteq \mathbb{P}$.

**Definition 2.8** (Facet). Let $\mathbb{P} = \mathbb{P}(\Pi)$ be a polyhedron in an affine space $V$ with $\dim(\mathbb{P}) = \dim(V_{am}(\mathbb{P})) = m$. Then $m - 1$ dimensional face $F$ of $\mathbb{P}$ is called a facet of $\mathbb{P}$.

**Example 2.3.** Let $\mathbb{P}$ be the polyhedron defined in Example 2.1. Then for each fixed $k = 4, 5$, 

$$\mathbb{P} = \pi_{q_k,r_k} \cap \mathbb{P} = \bigcap_{j=4,5} (\pi_{q_j,r_j} \cap \mathbb{P})$$

is a 2 dimensional face of $\mathbb{P}$. For each fixed $k = 1, 2, 3$,

$$F_k = \pi_{q_k,r_k} \cap \mathbb{P} = \bigcap_{j=k,4} (\pi_{q_j,r_j} \cap \mathbb{P}) = \bigcap_{j=k,5} (\pi_{q_j,r_j} \cap \mathbb{P}) = \bigcap_{j=k,4,5} (\pi_{q_j,r_j} \cap \mathbb{P}) = \operatorname{Ch}(e_\ell, e_m)$$

with $\ell, m \neq k$ is a 1 dimensional face (edge) of $\mathbb{P}$. For each fixed $k, \ell \in \{1, 2, 3\}$ with $k \neq \ell$,

$$F_{k,\ell} = \bigcap_{j=k,\ell} (\pi_{q_j,r_j} \cap \mathbb{P}) = \bigcap_{j=k,\ell,4} (\pi_{q_j,r_j} \cap \mathbb{P}) = \bigcap_{j=k,\ell,5} (\pi_{q_j,r_j} \cap \mathbb{P}) = \bigcap_{j=k,\ell,4,5} (\pi_{q_j,r_j} \cap \mathbb{P}) = \{e_m\}$$

with $m \neq k, \ell$ is a 0 dimensional face (vertex) of $\mathbb{P}$.

**Definition 2.9.** Let $F$ be a face of a convex polyhedron $\mathbb{P}$. Then the boundary $\partial F$ of $F$ is defined to be $\bigcup G$, where the union is over all faces $G \not\subseteq F$. When $\dim(F) = k$,

$$\partial F = \bigcup_{\dim G = k-1, G \not\subseteq F} G,$$

since faces whose dimensions $< k - 1$ are contained on $k - 1$ dimensional faces of $F$. Note that $\partial F$ is the boundary of $F$ with respect to the usual topology of $V_{am}(F)$ in (2.1).

**Example 2.4.** Let a polyhedron $\mathbb{P}$ and its faces $F_k$ be defined as in Examples 2.1 through 2.3. Then the boundary $\partial \mathbb{P}$ of $\mathbb{P}$ is given by $\partial \mathbb{P} = \bigcup_{\dim F = 1, F \subseteq \mathbb{P}} F = F_1 \cup F_2 \cup F_3 \subseteq \bigcup_{\pi \in \Pi} \pi$.

**Definition 2.10.** Let $F$ be a face of a convex polyhedron $\mathbb{P}$. Then the interior $F^o$ of $F$ is defined to be $F^o = \mathbb{P} \setminus \partial F$. Note also that $F^o$ is the interior of $F$ with respect to the usual topology defined on $V_{am}(F)$ in (2.1).

**Example 2.5.** Observe that $\operatorname{CoSp}(p_1, \cdots, p_N)^o = \left\{ \sum_{j=1}^{N} \alpha_j p_j : \alpha_j > 0 \right\}$. 
2.3. A Cone (Dual Face) of Face.

Definition 2.11 (Dual face). Let $F$ be a face of a polyhedron $P$ in $\mathbb{R}^n$. Then the cone (dual face) $F^*$ of $F$ is defined by

$$
F^*|P = \{ q \in \mathbb{R}^n : \exists r \in \mathbb{R} \text{ such that } F \subset \pi_{q,r} \cap P \text{ and } P \setminus F \subset \pi_{q,r}^+ \} 
$$

(2.4)

$$
= \{ q \in \mathbb{R}^n : \exists r \in \mathbb{R} \text{ such that } \langle q, u \rangle = r \leq \langle q, y \rangle \text{ for all } u \in F, y \in P \setminus F \}.
$$

The interior of a cone (dual face) $F^*$ is the set of all nonzero normal vectors $q$ satisfying (2.2):

$$
(F^*)^o|P = \{ q \in \mathbb{R}^n : \exists r \in \mathbb{R} \text{ such that } F = \pi_{q,r} \cap P \text{ and } P \setminus F \subset \pi_{q,r}^+ \} 
$$

(2.5)

$$
= \{ q \in \mathbb{R}^n : \exists r \in \mathbb{R} \text{ such that } \langle q, u \rangle = r < \langle q, y \rangle \text{ for all } u \in F, y \in P \setminus F \}.
$$

We use the notation $F^*|(P, V)$ when we restrict $q$ in a given vector space $V$. Thus $F^*|P = F^*|(P, \mathbb{R}^n)$ in (2.4). If there is no ambiguity, we write just $F^*$ instead of $F^*|(P, \mathbb{R}^n)$. We note that $F^*$ itself is a polyhedron in $\mathbb{R}^n$ and $(F^*)^o$ is an interior of $F^*$.

Remark 2.1. To understand a cone $F^*$ as a dual face of $F$, one is likely to define a cone of $F$ by the collection of all normal vectors $q$ satisfying (2.2) as in (2.5). If so, the collection (2.5) is an open set, not a polyhedron anymore. To make $F^*$ itself a polyhedron, we define a cone (dual face) of $F$ by (2.4) instead of its interior (2.5).

Example 2.6. Let a polyhedron $P$ and its faces $F_k, F_{k,\ell}$ be defined as in Example 2.3. The dual face of one dimensional face $F_k$ is the two dimensional cone $F^*_k = \text{CoSp}(q_k, q_4, q_5)$ whereas $(F^*_k)^o = \text{CoSp}^o(q_k, q_4, q_5)$. The dual face of zero dimensional face $F_{k,\ell}$ is the three dimensional cone $F^*_k,\ell = \text{CoSp}(q_k, q_\ell, q_4, q_5)$ and $(F^*_k,\ell)^o = \text{CoSp}^o(q_k, q_\ell, q_4, q_5)$.

2.4. Generalized Newton Polyhedron. For each $S \subset N_n = \{1, \ldots, n\}$, we define

$$
\mathbb{R}^S_+ = \{(u_1, \ldots, u_n) : u_j \geq 0 \text{ for } j \in S \text{ and } u_j = 0 \text{ for } j \in N_n \setminus S \}.
$$

Definition 2.12. Let $\Omega$ be a finite subset of $\mathbb{Z}_+^n$ and $S \subset N_n = \{1, \ldots, n\}$. We define a Newton polyhedron $N(\Omega, S)$ associated with $\Omega$ and $S$ by the convex hull containing $(\Omega + \mathbb{R}^S_+)$ in $\mathbb{R}^n$:

$$
N(\Omega, S) = \text{Ch} \left( \Omega + \mathbb{R}^S_+ \right).
$$
By \( \mathbb{R}_+^0 = \{0\} \) and \( \mathbb{R}_+^n = \mathbb{R}_+^n \), we see that \( N(\Omega, \emptyset) = \text{Ch}(\Omega) \), and \( N(\Omega, N_n) = \text{Ch} (\Omega + \mathbb{R}_+^n) \) that is the usual Newton Polyhedron denoted by \( N(\Omega) \). Note that \( N(\Omega, S) \) is a polyhedron in the sense of Definition 2.3. See Figure 2.1 in Section 2.

**Definition 2.13.** Let \( \Lambda = (\Lambda_\nu) \) with \( \Lambda_\nu \subset \mathbb{Z}_+^n \) and \( S \subset \{1, \cdots, n\} \). Then, the ordered \( d \)-tuple of Newton polyhedra \( N(\Lambda_\nu, S) \)'s is defined by

\[
\tilde{N}(\Lambda, S) = (N(\Lambda_\nu, S))_{\nu=1}^d.
\]

To indicate a given polynomial \( P = (P_\nu) \in \mathcal{P}_\Lambda \), we also denote \( \tilde{N}(\Lambda, S) \) by \( \tilde{N}(P, S) \).

**Definition 2.14.** Let \( \Lambda = (\Lambda_\nu) \) with \( \Lambda_\nu \subset \mathbb{Z}_+^n \) and \( S \subset \{1, \cdots, n\} \). We define the collection of \( d \)-tuples of faces \( F_\nu \in \mathcal{F}(N(\Lambda_\nu, S)) \) by

\[
\mathcal{F}(\tilde{N}(\Lambda, S)) = \{F = (F_1, \cdots, F_d) : F_\nu \in \mathcal{F}(N(\Lambda_\nu, S))\}.
\]

For each \( F \in \mathcal{F}(\tilde{N}(\Lambda, S)) \), we denote \( d \)-tuple of cones by \( F^* = (F_\nu^*) \). Let \( P_\Lambda \in \mathcal{P}_\Lambda \), To each \( d \)-tuple of faces \( F \in \mathcal{F}(\tilde{N}(\Lambda, S)) \), we assign a polynomial \( P_F \) whose exponents in \( \Lambda \).
are restricted to $F$:

$$P_F = \left( \sum_{m \in F_1 \cap \Lambda_1} c^1_m t^m, \ldots, \sum_{m \in F_d \cap \Lambda_d} c^d_m t^m \right)$$

2.5. Basic Decompositions According to Faces and Cones. Choose $\psi \in C^\infty_c([-2, 2])$ with $0 \leq \psi \leq 1$ and $\psi(u) = 1$ for $|u| \leq 1/2$. Put $\eta(u) = \psi(u) - \psi(2u)$ and $h(u) = \eta(u)/u$ for $u \neq 0$. Let $\Lambda = (\Lambda_1, \ldots, \Lambda_d)$ and $P_\Lambda \in P_\Lambda$. For each $F = (F_\nu) \in F(\bar{\mathbf{N}}(\Lambda, S))$ and $J \in \mathbb{Z}^n$, define

$$(2.6) \quad \mathcal{I}_J(P_F, \xi) = \int_{\mathbb{R}^n} \exp \left( i \sum_{\nu=1}^d \left( \sum_{m \in F_\nu \cap \Lambda_\nu} c^\nu_m 2^{-J \cdot m} t^m \right) \xi_\nu \right) \prod_{\ell=1}^n h(t_\ell) \, dt = \int_{\mathbb{R}^n} \exp \left( i \sum_{m \in \bigcup F_\nu \cap \Lambda_\nu} 2^{-J \cdot m} \langle \xi, c^\nu_m \rangle t^m \right) \prod_{\ell=1}^n h(t_\ell) \, dt$$

where $c^\nu_m = (c^\nu_m)$ defined in (1.1). We shall write $\mathcal{I}_J(P_\Lambda, \xi)$ instead of $\mathcal{I}_J(P_{\bar{\mathbf{N}}(\Lambda, S)}), \xi$. We also write $\mathcal{I}_J(P_F, \xi)$ as the Fourier multiplier of the operator

$$f \to H^{-P_F}_J * f. \quad (2.7)$$

Definition 2.15. Given $S \subset N_n = \{1, \ldots, n\}$, we define

$$1_S = (r_i) \text{ where } r_i = 1 \text{ for } i \in S \text{ and } r_i = \infty \text{ for } i \in N_n \setminus S$$

and

$$Z(S) = \prod_{i=1}^n Z_i \text{ where } Z_i = \mathbb{R}_+ \text{ if } i \in S \text{ and } Z_i = \mathbb{R} \text{ if } i \in N_n \setminus S.$$

Then by using $Z_i$ above, we write

$$\prod_{i \in S} \{-1 < t_i < 1\} \prod_{i \in N_n \setminus S} \{-\infty < t_i < \infty\} = \prod_{i=1}^n \left( \bigcup_{k_i \in Z_i \cap \mathbb{Z}} \{\lfloor k_i \rfloor \approx 2^{-k_i}\} \right),$$

and make the following dyadic decomposition:

$$(2.8) \quad I(P_\Lambda, 1_S) = \sum_{J \in Z(S) \cap \mathbb{Z}^n} \mathcal{I}_J(P_\Lambda, \xi) \quad \text{and} \quad H^{P_\Lambda}_{1_S} = \sum_{J \in Z(S) \cap \mathbb{Z}^n} H^{P_\Lambda}_J * f.$$
where we may replace $P_\Lambda$ with $P_\mathcal{F}$ in (2.8). As the name (dual face) tells, each $J \in \mathbb{F}_\nu \cap \mathbb{Z}^n$ can be understood as a linear functional mapping $n \in \mathbb{R}^n$ to $J \cdot n \in \mathbb{R}$ satisfying the following dominating property:

\begin{equation}
2^{-J \cdot m} = 2^{-r(J)} \geq 2^{-J \cdot n} \quad \text{for all } m \in \mathbb{F}_\nu \text{ and } n \in \mathbb{F}_\nu \setminus \mathbb{F}_\nu.
\end{equation}

Thus, for $J \in \bigcap_{\nu=1}^d \mathbb{F}_\nu^*$ in (2.8) with the property (2.9) in (2.6),

\begin{equation}
\exists \alpha \in \mathbb{Z}_+^n, \quad \left( \frac{\partial}{\partial t} \right)^\alpha \left( \sum_{\nu=1}^d \left( \sum_{m \in \Lambda_\nu} \ell_m^{\nu} 2^{-J \cdot m \ell_m^{\nu}} \xi_\nu \right) \right) \approx 2^{-J \cdot m_{\nu}} \xi_\nu \quad \text{for all } m_\nu \in \mathbb{F}_\nu \cap \Lambda_\nu.
\end{equation}

This combined with $Z(S) = \bigcup_{F=(F_\nu) \in \mathcal{F}(\mathcal{N}(\Lambda, S))} \left( \bigcap_{\nu=1}^d \mathbb{F}_\nu^* \right)$ in Section 4 suggests us to decompose:

$$
\sum_{J \in Z(S) \cap \mathbb{Z}^n} \mathcal{I}_J(P_\Lambda, \xi) = \sum_{F \in \mathcal{F}(\mathcal{N}(\Lambda, S))} \sum_{J \in \bigcap_{\nu=1}^d \mathbb{F}_\nu^* \cap \mathbb{Z}^n} \mathcal{I}_J(P_\Lambda, \xi).
$$

Next, we prove in Sections 7 that for each $F = (F_\nu) \in \mathcal{F}(\mathcal{N}(\Lambda, S))$,

\begin{equation}
\sum_{s=1}^N \sum_{J \in \bigcap_{\nu=1}^d \mathbb{F}_\nu \cap \mathbb{Z}^n} \left| \mathcal{I}_J(P_{F(s)}; \xi) - \mathcal{I}_J(P_{F(s-1)}; \xi) \right| + \sum_{J \in \bigcap_{\nu=1}^d \mathbb{F}_\nu \cap \mathbb{Z}^n} \left| \mathcal{I}_J(P_F; \xi) \right| \leq C.
\end{equation}

Here $F(s) = (F_\nu(s))$ will be chosen in a suitable way so that $F_\nu(s - 1) \succeq F_\nu(s)$ with $\nu = 1, \ldots, d$ where $\mathcal{N}(\Lambda, S) = \mathbb{F}(0)$ and $\mathbb{F}(N) = \mathbb{F}$ as in (3.13).

### 3. Main Theorem and Background

In order to state the main results, we first try to find an appropriate condition on an exponent set $\bigcup_{\nu=1}^d \mathbb{F}_\nu \cap \Lambda_\nu$ which guarantees $\mathcal{I}_J(P_F, \xi) \equiv 0$.

#### 3.1. Even Sets

Let $\bigcup_{\nu=1}^d \mathbb{F}_\nu \cap \Lambda_\nu = \{m_1, \ldots, m_N\}$. Suppose every vector $m$ of the form

$$
\alpha_1 m_1 + \cdots + \alpha_N m_N \text{ with } \alpha_j = 0 \text{ or } 1
$$

where we may replace $P_\Lambda$ with $P_\mathcal{F}$ in (2.8). As the name (dual face) tells, each $J \in \mathbb{F}_\nu \cap \mathbb{Z}^n$ can be understood as a linear functional mapping $n \in \mathbb{R}^n$ to $J \cdot n \in \mathbb{R}$ satisfying the following dominating property:

\begin{equation}
2^{-J \cdot m} = 2^{-r(J)} \geq 2^{-J \cdot n} \quad \text{for all } m \in \mathbb{F}_\nu \text{ and } n \in \mathbb{F}_\nu \setminus \mathbb{F}_\nu.
\end{equation}

Thus, for $J \in \bigcap_{\nu=1}^d \mathbb{F}_\nu^*$ in (2.8) with the property (2.9) in (2.6),

\begin{equation}
\exists \alpha \in \mathbb{Z}_+^n, \quad \left( \frac{\partial}{\partial t} \right)^\alpha \left( \sum_{\nu=1}^d \left( \sum_{m \in \Lambda_\nu} \ell_m^{\nu} 2^{-J \cdot m \ell_m^{\nu}} \xi_\nu \right) \right) \approx 2^{-J \cdot m_{\nu}} \xi_\nu \quad \text{for all } m_\nu \in \mathbb{F}_\nu \cap \Lambda_\nu.
\end{equation}

This combined with $Z(S) = \bigcup_{F=(F_\nu) \in \mathcal{F}(\mathcal{N}(\Lambda, S))} \left( \bigcap_{\nu=1}^d \mathbb{F}_\nu^* \right)$ in Section 4 suggests us to decompose:

$$
\sum_{J \in Z(S) \cap \mathbb{Z}^n} \mathcal{I}_J(P_\Lambda, \xi) = \sum_{F \in \mathcal{F}(\mathcal{N}(\Lambda, S))} \sum_{J \in \bigcap_{\nu=1}^d \mathbb{F}_\nu^* \cap \mathbb{Z}^n} \mathcal{I}_J(P_\Lambda, \xi).
$$

Next, we prove in Sections 7 that for each $F = (F_\nu) \in \mathcal{F}(\mathcal{N}(\Lambda, S))$,

\begin{equation}
\sum_{s=1}^N \sum_{J \in \bigcap_{\nu=1}^d \mathbb{F}_\nu \cap \mathbb{Z}^n} \left| \mathcal{I}_J(P_{F(s)}; \xi) - \mathcal{I}_J(P_{F(s-1)}; \xi) \right| + \sum_{J \in \bigcap_{\nu=1}^d \mathbb{F}_\nu \cap \mathbb{Z}^n} \left| \mathcal{I}_J(P_F; \xi) \right| \leq C.
\end{equation}

Here $F(s) = (F_\nu(s))$ will be chosen in a suitable way so that $F_\nu(s - 1) \succeq F_\nu(s)$ with $\nu = 1, \ldots, d$ where $\mathcal{N}(\Lambda, S) = \mathbb{F}(0)$ and $\mathbb{F}(N) = \mathbb{F}$ as in (3.13).

### 3. Main Theorem and Background

In order to state the main results, we first try to find an appropriate condition on an exponent set $\bigcup_{\nu=1}^d \mathbb{F}_\nu \cap \Lambda_\nu$ which guarantees $\mathcal{I}_J(P_F, \xi) \equiv 0$.

#### 3.1. Even Sets

Let $\bigcup_{\nu=1}^d \mathbb{F}_\nu \cap \Lambda_\nu = \{m_1, \ldots, m_N\}$. Suppose every vector $m$ of the form

$$
\alpha_1 m_1 + \cdots + \alpha_N m_N \text{ with } \alpha_j = 0 \text{ or } 1
$$

has at least one even component. Then, the Taylor expansion of the exponential function in (2.6) yields that

\[
I_J(\mathcal{F}, \xi) = \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \left( \sum_{m \in \bigcup \mathcal{F}_\nu \cap \Lambda_\nu} 2^{-J-m} \langle \xi, c_m \rangle t^m \right)^k k! \prod_{\ell=1}^{n} h(t_\ell) dt
\]

(3.1)

since \(h(t_\ell)\) is an odd function for each \(\ell = 1, \ldots, n\). This observation leads to the following notions of even and odd sets in \(\mathbb{Z}_+^n\). Let \(\Omega = \{ m_1, \cdots, m_N \} \subset \mathbb{Z}_+^n\) and let the set of the sum of vectors in \(\Omega\) be

\[
\Sigma(\Omega) = \{ \alpha_1 m_1 + \cdots + \alpha_N m_N : \alpha_j = 0 \text{ or } 1 \}.
\]

**Remark 3.1.** We see that \(\alpha_1 m_1 + \cdots + \alpha_N m_N\) has an even component if and only if the same sum with each \(\alpha_j\) replaced by \(\alpha_j \mod 2\) does.

**Definition 3.1.** A finite subset \(\Omega = \{ m_1, \cdots, m_N \}\) of \(\mathbb{Z}_+^n\) is said to be **odd** iff there exists at least one vector \(m \in \Sigma(\Omega)\) all of whose components are odd numbers such that

\[
m = (\text{odd}, \cdots, \text{odd}).
\]

**Definition 3.2.** A finite subset \(\Omega\) of \(\mathbb{Z}_+^n\) is said to be **even** iff \(\Omega\) is not odd, that is, every \(m = (m_1, \cdots, m_n) \in \Sigma(\Omega)\) has at least one even numbered component \(m_j\).

**Example 3.1.** In \(\mathbb{Z}_+^3\), let \(A = \{(1, 1, 0), (3, 2, 1)\}\), and \(B = \{(1, 1, 0), (0, 0, 3)\}\). Then \(A\) is an even set and \(B\) an odd set. Notice that \(A\) is an even set, though there is no \(k \in \{1, 2, 3\}\) such that \(k^{th}\) component of every vector in \(A\) is even.

In (3.1), we have proved the following proposition:

**Proposition 3.1.** Suppose that \(\bigcup_{\nu=1}^{d}(\mathcal{F}_\nu \cap \Lambda_\nu)\) is an even set. Then \(I_J(\mathcal{F}, \xi) \equiv 0\)

We shall perform the estimates (2.11) by using a full rank condition of \(\bigcup \mathcal{F}_\nu(s)\) (formulated in Proposition 6.1) or a vanishing property in Propositions 3.1. Thus, the evenness...
condition in Propositions 3.1 shall be imposed on the only faces contained in the subfamily \( \mathcal{A} \) of \( \mathcal{F}(\tilde{N}(\Lambda, S)) \) satisfying the following two conditions:

\[
\text{(3.3) Low Rank Condition: } \text{rank} \left( \bigcup_{\nu=1}^{d} F_{\nu} \right) \leq n - 1 \text{ for } F \in \mathcal{A},
\]

\[
\text{(3.4) Overlapping Cone Condition: } \bigcap_{\nu=1}^{d} (F_{\nu}^*)^\circ \neq \emptyset \text{ for } F \in \mathcal{A}
\]

where the overlapping cone condition comes from the decompositions in \( J \) and the dominating condition (2.9).

3.2. Statement of Main Results.

**Definition 3.3.** Given \( \tilde{N}(\Lambda, S) = (N(\Lambda_{\nu}, S))_{\nu=1}^{d} \), we set the collection of all \( d \)-tuples of faces satisfying both low rank condition (3.3) and overlapping (3.4) by

\[
\mathcal{F}_{\text{lo}}(\tilde{N}(\Lambda, S)) = \left\{ (F_{\nu}) \in \mathcal{F}(\tilde{N}(\Lambda, S)) : \text{rank} \left( \bigcup_{\nu=1}^{d} F_{\nu} \right) \leq n - 1 \text{ and } \bigcap_{\nu=1}^{d} (F_{\nu}^*)^\circ \neq \emptyset \right\}.
\]

We also let the collection of all \( d \)-tuples of faces satisfying only the low rank condition (3.3) by

\[
\mathcal{F}_{\text{l}}(\tilde{N}(\Lambda, S)) = \left\{ (F_{\nu}) \in \mathcal{F}(\tilde{N}(\Lambda, S)) : \text{rank} \left( \bigcup_{\nu=1}^{d} F_{\nu} \right) \leq n - 1 \right\}.
\]

Let \( P = (P_{\nu})_{\nu=1}^{d} \) be a vector polynomial. For each \( \nu = 1, \ldots, d \), we define a set \( \Lambda(P_{\nu}) \) to be a set of all exponents of the monomials in \( P_{\nu} \):

\[
\Lambda(P_{\nu}) = \left\{ m \in \mathbb{Z}_{+}^{n} : c_{m}^{\nu} \neq 0 \text{ in } P_{\nu}(t) = \sum_{m \in \Lambda((AP)^{\nu})} c_{m}^{\nu} t^{m} \right\}.
\]

Moreover, we denote a \( d \)-tuple \( (\Lambda(P_{\nu}))_{\nu=1}^{d} \) by \( \Lambda(P) \). Denote the set of \( d \times d \) invertible matrices by \( GL(d) \). For \( A \in GL(d) \) and \( P \in \mathcal{P}_{\Lambda} \) with \( P(t) = (P_{1}(t), \ldots, P_{d}(t)) \), we let \( AP \) be a vector polynomial given by the matrix multiplication

\[
AP(t) = \left( \sum_{m \in \Lambda((AP)^{\nu})} a_{m}^{\nu} t^{m} \right)_{\nu=1}^{d} \text{ with } a_{m}^{\nu} \neq 0
\]

where we regard \( P(t) \) and \( AP(t) \) above as column vectors. Then \( AP \in \mathcal{P}_{\tilde{\Lambda}} \) where \( \tilde{\Lambda} = \Lambda(AP) \). If \( A = I \) an identity matrix, \( \tilde{\Lambda} = (\Lambda((AP)^{\nu}))_{\nu=1}^{d} = (\Lambda(P_{\nu}))_{\nu=1}^{d} = \Lambda^{\nu} \).
Definition 3.4. Let $P \in \mathcal{P}_\Lambda$ where $\Lambda = (\Lambda_\nu)$ with $\Lambda_\nu \subset \mathbb{Z}_n^+$ and $S \subset \{1, \ldots, n\}$. Let $A \in \text{GL}(d)$. Given a vector polynomial $AP$, we consider the $d$-tuple of Newton polyhedrons

$$\vec{N}(AP, S) = (N(((AP)_\nu), S))_{\nu=1}^d$$

and $d$-tuple of their faces

$$\mathcal{F}(\vec{N}(AP, S)) = \{ \mathbb{F}_A = ((\mathbb{F}_A)_\nu, \ldots, (\mathbb{F}_A)_d) : (\mathbb{F}_A)_\nu \in \mathcal{F}(N(((AP)_\nu), S)) \}.$$ 

Main Theorem 3.1. Let $\Lambda = (\Lambda_1, \ldots, \Lambda_d)$ with $\Lambda_\nu \subset \mathbb{Z}_n^+$ and $S \subset N_n$. Let $1 < p < \infty$. For all $P \in \mathcal{P}_\Lambda$ there exists $C_P > 0$ such that

$$\sup_{r \in I(S)} \| \mathcal{H}_r^P \|_{L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)} \leq C_P$$

if and only if for all $A \in \text{GL}(d)$ and $P \in \mathcal{P}_\Lambda$,

$$(3.6) \bigcup_{\nu=1}^d (\mathbb{F}_A)_\nu \cap \Lambda(((AP)_\nu)$$

is an even set whenever $\mathbb{F}_A = ((\mathbb{F}_A)_\nu) \in \mathcal{F}_{lo}(\vec{N}(AP, S))$

where the set $\mathcal{F}_{lo}(\vec{N}(AP, S))$ is defined as in Definition 3.3.

As a special case, assume $\Lambda_\nu$'s are pairwise disjoint, that is, $\Lambda_\mu \cap \Lambda_\nu = \emptyset$ for any $\mu \neq \nu$.

Main Theorem 3.2. Let $\Lambda = (\Lambda_1, \ldots, \Lambda_d)$ with $\Lambda_\nu \subset \mathbb{Z}_n^+$ and $S \subset N_n$. Suppose that $\Lambda_\nu$'s are pairwise disjoint. Let $1 < p < \infty$. Then

for all $P \in \mathcal{P}_\Lambda$, there exists $C_P > 0$ such that

$$\sup_{r \in I(S)} \| \mathcal{H}_r^P \|_{L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)} \leq C_P$$

if and only if $\bigcup_{\nu=1}^d (\mathbb{F}_\nu \cap \Lambda_\nu)$ is an even set for $\mathbb{F} = (\mathbb{F}_\nu) \in \mathcal{F}_{lo}(\vec{N}(\Lambda, S))$.

Remark 3.2. Main Theorems 3.1 and 3.2 do not give a criteria for the boundedness with a given individual polynomial $P_\Lambda$, but enables us to determine the boundedness for universal polynomials $P_\Lambda$ with a set $\Lambda$ of exponents fixed. Also, Main Theorems 3.1 and 3.2 do not give a condition for the boundedness of $\| \mathcal{H}_r^P \|_{L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)}$ with fixed $r$, but for the boundedness of $\| \mathcal{H}_r^P \|_{L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)}$ uniformly in $r$. It is interesting to know if $\sup_{r \in I(S)} \| \mathcal{H}_r^P \|_{L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)}$ can be replaced by $\| \mathcal{H}_1^P \|_{L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)}$ in the above theorems where $1_S$ is defined in Definition 2.15.
Remark 3.3. For the case \(d = 1\), we observe that \( \mathcal{F}_\text{lo}(\tilde{N}(\Lambda, S)) = \mathcal{F}_1(\tilde{N}(\Lambda, S)) \) always holds. Thus, Main Theorem 3.2 tells that for \( \Lambda \subset \mathbb{Z}_+^n \) and \( S \subset N_n \) with \( d = 1 \) in (1.1),

\[
\text{for all } P \in \mathcal{P}_\Lambda, \exists C_P > 0 \text{ such that } \sup_{r \in I(S)} \| \mathcal{H}_P^r \|_{L^p(\mathbb{R}^1) \to L^p(\mathbb{R}^1)} \leq C_P
\]

if and only if for all \( F \in \mathcal{F}_1(\tilde{N}(\Lambda, S)) \), \( F \cap \Lambda \) is an even set. Here \( \mathcal{F}_1(\tilde{N}(\Lambda, S)) \) is defined in (3.5).

Let \( P_\Lambda \) be a form of a graph \((t_1, \cdots, t_n, P_{n+1}(t))\) so that \( \Lambda = (\{e_1\}, \cdots, \{e_n\}, \Lambda_{n+1}) \). For this case, we are able to show that the \( L^p \) boundedness of \( \mathcal{H}_{1S}^{P_\Lambda} \) and the uniform \( L^p \) boundedness of \( \mathcal{H}_r^{P_\Lambda} \) in \( r \in I(S) \) are equivalent. Moreover, we do not need the overlapping condition (3.4), since we can make the condition \( \bigcup_{\nu=1}^{d} (\mathbb{F}_\nu)^{\circ} \neq \emptyset \) always holds for the graph case.

Corollary 3.1. Let \( 1 < p < \infty \) and let \( \Lambda = (\{e_1\}, \cdots, \{e_n\}, \Lambda_{n+1}) \) and \( S \subset N_n \). Then

\[
\text{for all } P \in \mathcal{P}_\Lambda, \exists C_P > 0 \text{ such that } \| \mathcal{H}_1^{P} \|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} \leq C_P
\]

if and only if for all \( \mathbb{F}_{n+1} \in \mathcal{F}(\mathbb{N}(\Lambda_{n+1}, S)) \) and \( A \subset \{e_1, \cdots, e_n\} \) with \( \text{rank}(\mathbb{F}_{n+1} \cup A) \leq n - 1 \), the set \((\mathbb{F}_{n+1} \cap \Lambda_{n+1}) \cup A\) is an even set.

Remark 3.4. The above evenness condition in Corollary 3.1 is equivalent to

\[
\text{for all } F \in \mathcal{F}_1(\tilde{N}(\Lambda, S)), \text{ the set } \bigcup_{\nu=1}^{d} (\mathbb{F}_\nu \cap \Lambda_\nu) \text{ is an even set.}
\]

Remark 3.5. For the case \( n = 2 \) in Main Theorems 3.1 and 3.2, the family \( \mathcal{F}_\text{lo}(\tilde{N}(\Lambda, S)) \) can be replaced by \( \mathcal{F}_1(\tilde{N}(\Lambda, S)) \). For the case \( n = 2 \),

evenness condition for \( \mathcal{F}_\text{lo}(\tilde{N}(\Lambda, S)) \) ⇔ evenness condition for \( \mathcal{F}_1(\tilde{N}(\Lambda, S)) \).

It suffices to show \( \Rightarrow \). Suppose that \( \bigcup_{\nu=1}^{d} \mathbb{F}_\nu \cap \Lambda_\nu \) is an odd set with \( \text{rank} \left( \bigcup_{\nu=1}^{d} \mathbb{F}_\nu \right) \leq 1 \). Then there exists \( \mu \) such that \( \mathbb{F}_\mu \cap \Lambda_\mu \) has a point (odd,odd), because both of two points (even,odd),(odd,even) can not lie in the one line passing through the origin. Therefore, \( \mathcal{G} = (\mathcal{G}_\nu) \) defined by \( \mathcal{G}_\mu = \mathbb{F}_\mu \) and \( \mathcal{G}_\nu = \emptyset \) for \( \nu \neq \mu \) satisfies that \( \mathcal{G} \in \mathcal{F}_\text{lo}(\tilde{N}(\Lambda, S)) \) and \( \bigcup_{\nu=1}^{d} \mathcal{G}_\nu \cap \Lambda_\nu \) is an odd set.
Remark 3.6. For the case \( n \geq 3 \), the overlapping condition is crucial in Main Theorems 3.1 and 3.2. Moreover, we note that it is not just cones \( \bigcap_{\nu=1}^{d} F_{\nu} \), but their interiors \( \bigcap_{\nu=1}^{d} (F_{\nu})^{\circ} \) that satisfy the overlapping condition (3.4). The example 4.1 in Section 4 shows that there exists \( F \in \mathcal{F}(\tilde{N}(\Lambda, S)) \) such that \( \bigcap_{\nu=1}^{d} F_{\nu} \neq \emptyset \) and \( \bigcup_{\nu=1}^{d} (F_{\nu} \cap \Lambda_{\nu}) \) is an odd set, but

\[
\text{for all } P \in \mathcal{P}_{\Lambda} \quad \sup_{r \in I(S)} \|H_{r}^{P}\|_{L^{p}(\mathbb{R}^{d}) \rightarrow L^{p}(\mathbb{R}^{d})} \leq C_{P}.
\]

Remark 3.7. Together, Main Theorems 3.1 and 3.2 imply that the following combinatorial fact is true: If the sets \( \Lambda_{1}, \ldots, \Lambda_{d} \) are pairwise disjoint, then the following two statements are equivalent:

1. \( \bigcup_{\nu=1}^{d} (F_{\nu} \cap \Lambda_{\nu}) \) is an even set for \( F = (F_{\nu}) \in \mathcal{F}_{lo}(\tilde{N}(\Lambda, S)) \)
2. for all \( A \in \text{GL}(d) \) and \( P \in \mathcal{P}_{\Lambda} \), \( \bigcup_{\nu=1}^{d} (F_{A\nu} \cap \Lambda((AP)_{\nu})) \) is an even set whenever \( F_{A} = ((F_{A})_{\nu}) \in \mathcal{F}_{lo}(\tilde{N}(AP, S)) \).

3.3. Background. In the one parameter case \((n = 1)\), the operator \( H_{r}^{\Lambda} \) with \( r = (1, \ldots, 1) \) can be regarded as a particular instance of singular integrals along curves satisfying finite type condition in E. M. Stein and S. Wainger [21]. The \( L^{p} \) theory of those singular integrals has been developed quite well. For example, see M. Christ, A. Nagel, E. M. Stein and S. Wainger [6] for singular Radon transforms with the curvature conditions in a very general setting. See also M. Folch-Gabayet and J. Wright [8] for the case that phase functions \( P_{\Lambda} \) are given by rational functions.

In the multi-parameter case \((n \geq 2)\), it is A. Nagel and S. Wainger [12] who introduced the (global) multiple Hilbert transforms along surfaces having certain dilation invariance properties and obtained their \( L^{2} \) boundedness. In [17], F. Ricci and E. M. Stein established an \( L^{p} \) theorem for multi-parameter singular integrals whose kernels satisfy more general dilation structure. A special case of their results implies that if \( \Lambda = (\{e_{1}\}, \cdots, \{e_{n}\}, \{m\}) \) where at least \( n - 1 \) coordinates of \( m \) are even, then \( \|H_{1S}^{\Lambda}\|_{L^{p}(\mathbb{R}^{n+1}) \rightarrow L^{p}(\mathbb{R}^{n+1})} \) are bounded for \( 1 < p < \infty \). In [3], A. Carbery, S. Wainger and J. Wright obtained a necessary and sufficient condition for \( L^{p}(\mathbb{R}^{3}) \) boundedness of \( H_{1S}^{\Lambda} \) with \( S = \{1, 2\}, \Lambda = (\{e_{1}\}, \{e_{2}\}, \Lambda_{3}) \) where \( d = 3 \) and \( n = 2 \). Their theorem states that
Theorem 3.1 (Double Hilbert transform [3]). Let \( \Lambda = (\{e_1\}, \{e_2\}, \Lambda_3) \) and \( S = \{1, 2\} \) with \( n = 2 \) and \( d = 3 \). For \( 1 < p < \infty \), the local double Hilbert transform \( H^{P_S}_{1S} \) is bounded in \( L^p(\mathbb{R}^3) \) if and only if every vertex \( \text{m} \) in \( N(\Lambda_3, S) \) has at least one even component.

By using our terminology defined in Definition 3.2, the above condition is to be said that \( \{\text{m}\} \) is an even set. We shall often state the historical results by the terms defined in this paper. S. Patel [14] extends this result to \( S = \emptyset \) corresponding to the global Hilbert transform. The necessary and sufficient condition for this case is not described by the faces of the Newton polyhedron \( N(\Lambda_3) \) but by those of the convex hull \( \text{Ch}(\Lambda_3) = N(\Lambda_3, \emptyset) \):

for every face \( F \) of \( \text{Ch}(\Lambda_3) \) with \( \text{rank}(F) \leq 1 \), \( F \cap \Lambda_3 \) is an even set.

S. Patel [13] also studies the case \( n = 2 \) and \( d = 1 \). He has shown that the necessary and sufficient condition for the \( L^p \) boundedness of \( H^{P_S}_{1S} \) cannot be determined by only the geometry of \( N(\Lambda) \) but by coefficients of the given polynomial \( P_{\Lambda}(t) \). More precisely, the necessary and sufficient condition (3.7) is described in terms of not a single vertex \( \text{m} \) and its coefficient \( c_{\text{m}} \) in \( P_{\Lambda} \), but the sum of quantities associated with many vertices and their corresponding coefficients:

\[
(3.7) \quad \sum_{(m_j, n_j) = (\text{odd, odd}) \text{ vertices of } N(\Lambda)} \frac{\text{sgn}(a_{m_j n_j})}{m_j n_j} \begin{pmatrix} m_j & 0 \\ 0 & n_j \end{pmatrix} (n_j^1 - n_j^2) = 0.
\]

where \( (m_j, n_j) = \pi_{n_j^1, 1} \cap \pi_{n_j^2, 1} \) with \( \pi_{n_j^i, 1} \in \Pi(N(\Lambda)) \).

A. Carbery, S. Wainger and J. Wright [2] obtain the asymptotic behaviors of the oscillatory singular integrals associated with analytic phase functions \( P(t_1, t_2) \), which extends Theorem 3.1 to the class of analytic functions. They [2] also find an example of finite type surface \( (t_1, t_2, P(t_1, t_2)) \) with its formal Taylor series satisfying evenness hypothesis, however \( \mathcal{H}^P_f \) not bounded in \( L^2(\mathbb{R}^3) \). We also refer to [5] dealing with a certain class of flat surfaces \( (t_1, t_2, P(t_1, t_2)) \) without any curvature.

In the general setting of polynomial surfaces defining the Double Hilbert transform, M. Pramanik and C. W. Yang [16] obtain the \( L^p \) theorem for the case \( \Lambda = (\{e_1\}, \{e_2\}, \Lambda_1, \ldots, \Lambda_k) \) and \( S = \{1, 2\} \) with \( n = 2 \) and \( d = k + 2 \). The necessary and sufficient condition for the \( L^p \) boundedness of the Hilbert transform \( H^{P_S}_{1S} \) is that for every invertible \( k \times k \) matrix \( A \),

every vertex \( \text{m} \in N((AP)_\nu, S) \) has at least one even numbered component.
The triple Hilbert transforms $\mathcal{H}^{P}_1$ with $S = \{1, 2, 3\}$ and $\Lambda = (\{e_1\}, \{e_2\}, \{e_3\}, \Lambda_4)$ were studied in the two papers [1] [4] published in 2009. In [1], A. Carbery, S. Wainger and J. Wright have discovered a remarkable differences between the triple and the double Hilbert transforms. The $L^2$ boundedness of the triple Hilbert transform $\mathcal{H}^{P}_1$ depends on the coefficients of $P_\Lambda$ as well as the Newton polyhedron $N(\Lambda_4)$, whereas that of the double Hilbert transform depends only on the Newton polygon $N(\Lambda_3)$. They found a vector polynomial $P_\Lambda(t) = (t_1, t_2, t_3, P_{\Lambda_4}(t))$ such that the corresponding triple Hilbert transform $\mathcal{H}^{P}_1$ is bounded on $L^2(\mathbb{R}^4)$ although $N(\Lambda_4, S)$ breaks the above evenness condition. They establish two types of theorems. First one gives the necessary and sufficient condition that the operators $\mathcal{H}^{P}_1$ are bounded in $L^2$ for all polynomials $P_\Lambda \in \mathcal{P}(\Lambda)$ when $\Lambda$ is given. This theorem is called the universal theorem. The second theorem is to inform the necessary and sufficient condition that one individual operator $\mathcal{H}^{P}_1$ is bounded in $L^2$ when a polynomial $P_\Lambda$ is given. This theorem is called the individual theorem. The condition of the first theorem is expressed solely in terms of $N(\Lambda_4)$ but that of the second in terms of individual coefficients of the given polynomial $P_\Lambda$ in question. Indeed, in both theorems, they assume certain hypotheses related with non-degeneracy conditions of polynomials $P_\Lambda$ and find the necessary and sufficient condition for the $L^2$ boundedness of $\mathcal{H}^{P}_1$. In [4], Y.K. Cho, H. Hong, C.W. Yang and the author obtain the universal theorem of [1] without any non-degeneracy hypothesis on $\Lambda$: The necessary and sufficient condition for the $L^p$ boundedness of $\mathcal{H}^{P}_1$ is that

for all $F_4 \in \mathcal{F}(N(\Lambda_4))$ and $A \subset \{e_1, e_2, e_3\}$ with rank $(F_4 \cup A) \leq 2$,

the set $(F_{n+1} \cap \Lambda_4) \cup A$ is an even set.

Remark 3.8. The necessary and sufficient conditions in [1],[3],[4],[14] are same as those of Corollary 3.1 for $n = 2, 3$.

As a variable coefficient version, we define $\mathcal{H}^P(f)(x)$ by

$$\int_{\prod_{j=1}^n [-r_j, r_j]} f(x_1 - t_1, \ldots, x_n - t_n, x_{n+1} - P(x_1, \ldots, x_n, t_1, \ldots, t_n)) \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}$$
whose corresponding oscillatory singular integral operator is given by

\[ T^P_\lambda(f)(x) = \text{p.v.} \int_{\{y:|x_j-y_j|<r_j\}} \frac{e^{i\lambda P(x,y)}}{(x_1-y_1)\cdots(x_n-y_n)} f(y)dy_1\cdots dy_n. \]

In view of the analogy between the integral operators of D. H. Phong and E. M. Stein [15] and the scalar valued integral of Varchenko [23], one may find the criteria for determining the uniform \( L^2 \) boundedness \( T^P_\lambda \) in \( \lambda \) in terms of the Newton polyhedron associated with the polynomial \( P(x,y) \). A more generalized version is the multi-parameter singular Radon transform:

\[ R^{P_\Lambda}(f)(x) = \text{p.v.} \int_{\prod_{j=1}^n [-r_j,r_j]} f(P_\Lambda(x,t)) \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}. \]

Recently, E. M. Stein and B. Street in [22, 19, 20] obtain the \( L^p \) boundedness for a certain class of multi-parameter singular Radon transform. A. Nagel, F. Ricci, E. M. Stein and S. Wainger in [10] study the singular integral operators on a homogeneous nilpotent Lie group that are given by convolution with flag kernels. Here flag kernels are product type singular kernels that are generalized versions of our kernel \( \frac{1}{t_1} \cdots \frac{1}{t_n} \). This result was preceded by [9] that proves the \( L^p \) boundedness of convolution operators with some special types of flag kernels. This result applies to obtain \( L^p \) regularity for the solutions of Cauchy-Riemann equations on CR manifolds.

Under the same setting as in the definition of multiple Hilbert transforms (1.1), we consider the multi-parameter maximal function

\[ \mathcal{M}_\Lambda f(x) = \sup_{r_1,\ldots,r_n>0} \frac{1}{r_1 \cdots r_n} \int_{-r_1}^{r_1} \cdots \int_{-r_n}^{r_n} |f(x-P_\Lambda(t))| \, dt \]

defined for each locally integrable function \( f \) on \( \mathbb{R}^d \).

**Theorem 3.2.** For \( 1 < p \leq \infty \), \( \mathcal{M}_\Lambda \) is a bounded operator from \( L^p(\mathbb{R}^d) \) into itself and there exists a bound \( C_p \) depending only on \( p, n, d \) and the maximal degree of the polynomials \( P_\nu \) such that

\[ \| \mathcal{M}_\Lambda f \|_{L^p(\mathbb{R}^d)} \leq C_p \| f \|_{L^p(\mathbb{R}^d)}. \]

**Remark 3.9.** This result can be proved by combining a theorem of Ricci and Stein ([17], Theorem 7.1) and the so-called lifting argument by Proposition 1 of XI.2 in [18] and iterating.
Remark 3.10. B. Street in [20] showed the $L^p$ boundedness for a variable coefficient version of $M_{\Lambda}$ associated with analytic functions. Furthermore, A. Nagel and M. Pramanik in [11] obtain the $L^p$ boundedness for a different kind of multi-parameter maximal operators, that were motivated by the study of several complex variables. This maximal average is taken over a family of sets (balls) that are defined by a finite number of monomial inequalities. In particular, to establish the $L^p$ theory in [11], the geometric properties of the associated polyhedra are also systematically studied.

Scheme for Proof of Main Theorems. As a motive for this problem, we remark the result of A. Carbery, S. Wainger and J. Wright in [3]: Given a polynomial $P_{\Lambda} \in \mathcal{P}_{\Lambda}$ with $\Lambda = (\{e_1\}, \{e_2\}, \Lambda_3)$ with $n = 2, d = 3$ and $S = \{1, 2\}$, a necessary and sufficient condition for
\[
\left\| H_r^{P_{\Lambda}} \right\|_{L^p(\mathbb{R}^3) \rightarrow L^p(\mathbb{R}^3)} \leq C \quad \text{where } r = (1, 1)
\]
is that every vertex $m$ in a Newton polyhedron $N(\Lambda_3) = \text{Ch}(\Lambda_3 + \mathbb{R}^2_+)$ has at least one even component. The idea of the proof in [3] is to split the sum of dyadic pieces $H_r^{P_{\Lambda}} = \sum_{J \in \mathbb{Z}^2_+} H_J^{P_{\Lambda}}$ into finite sums of cones $\{J \in m^*\}$ associated with vertices $m$ of $N(\Lambda_3)$:
\[
(3.9) \quad \sum_{m \text{ is a vertex of } N(\Lambda_3)} \left( \sum_{J \in m^*} H_J^{P_{\Lambda}} \right) \text{ with } m^* = \{\alpha_1 q_1 + \alpha_2 q_2 : \alpha_1, \alpha_2 \geq 0\}
\]
where $q_j$ is a normal vector of the supporting line $\pi_{q_j}$ of an edge $F_j$ of $N(\Lambda_3)$ such that $m = \bigcap_{j=1}^2 F_j$. They proved that for $\Lambda' = (\{e_1\}, \{e_2\}, \{m\})$,
\[
(3.10) \quad \left\| \sum_{J \in m^*} \left( H_J^{P_{\Lambda}} - H_J^{P_{\Lambda'}} \right) \right\|_{L^p(\mathbb{R}^3) \rightarrow L^p(\mathbb{R}^3)} + \left\| \sum_{J \in m^*} H_J^{P_{\Lambda'}} \right\|_{L^p(\mathbb{R}^3) \rightarrow L^p(\mathbb{R}^3)} \leq C
\]
by using the vertex dominating property (1) and the vanishing property (2):

1. vertex dominating property: $J \in m^* \Rightarrow 2^{-J \cdot m} \geq 2^{-J \cdot n}$ for $n \in N(\Lambda_3) \setminus \{m\}$,
2. vanishing property: at least one component of $m$ is even, that implies $H_J^{P_{(\emptyset, \emptyset, m)}} \equiv 0$.

For the case $n \geq 3$, we shall establish the corresponding cone type decomposition (3.9) and the reduction estimate (3.10) together with (1) and (2). As an analogue of (3.9), we
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\[
\mathcal{H}_r^{P_\Lambda} = \sum_{J \in \mathbb{Z}_+^d} \mathcal{H}_J^{P_\Lambda}
\]
with \( r = (1, \cdots, 1) \) into

\[
(3.11) \quad \sum_{(F_\nu); F_\nu \text{ is a face of } N(A_\nu)} \left( \sum_{J \in \bigcap_{\nu=1}^d F_\nu} \mathcal{H}_J^{P_\Lambda} \right) \text{ with } F_\nu^* = \{ \sum_{j=1}^{N_\nu} \alpha_j q_j : \alpha_j \geq 0 \}.
\]

Here \( q_j \) is a normal vector of the supporting plane \( \pi_{q_j} \) of a face \( F_\nu \) in the Newton polyhedron \( N(A_\nu) \), where \( F_\nu = \bigcap_{j=1}^{N_\nu} \pi_{q_j} \). For this purpose, we provide the properties of faces and their dual faces (cones) related with their representations, next obtain (3.11) in Sections 4. As an analogue of (3.10), we prove in Section 7 that

\[
(3.12) \quad \left\| \sum_{J \in \bigcap_{\nu=1}^d F_\nu^*} \left( \mathcal{H}_J^{P_\Lambda} - \mathcal{H}_J^{P_{\Lambda'}} \right) \right\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} + \left\| \sum_{J \in \bigcap_{\nu=1}^d F_\nu^*} \mathcal{H}_J^{P_{\Lambda'}} \right\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} \leq C
\]

where \( \Lambda' = (F_\nu \cap A_\nu)_{\nu=1}^d \). To show (3.12), we use the dominating and vanishing properties:

1. If \( J \in \bigcap_{\nu=1}^d F_\nu^* \), then \( 2^{-J \cdot m} \geq 2^{-J \cdot n} \) where \( m \in F_\nu \) and \( n \in N(A_\nu) \setminus F_\nu \).
2. If sum of elements in \( \bigcup_{\nu=1}^d F_\nu \cap \Lambda_\nu \) has at least one even component, \( \mathcal{H}_J^{P_{\Lambda'}} \equiv 0 \).

The main feature emerging in the general case \( n \geq 3 \) is that the evenness hypothesis of (2) needs to be satisfied only if the following overlapping and low rank conditions hold

\[
\bigcap_{\nu=1}^d (F_\nu^*)^\circ \neq \emptyset \quad \text{and} \quad \text{rank} \left( \bigcup_{\nu=1}^d F_\nu \right) \leq n - 1.
\]

Note that the dual faces \( F_\nu^* \) as well as faces \( F_\nu \) of the Newton polyhedra associated with \( A_\nu \) are involved in determining (1.4). Thus, a difficulty in showing (3.12) is to keep the above cone overlapping condition until the low ranked faces occurs. For this purpose, we construct in Section 5 a sequence of faces and cones such that

\[
N(A_\nu) = F_\nu(0) \supset \cdots \supset F_\nu(s) \supset \cdots \supset F_\nu(N) = F_\nu,
\]

\[
(3.13) \quad N^*(A_\nu) = F_\nu^*(0) \subset \cdots \subset F_\nu^*(s) \subset \cdots \subset F_\nu^*(N) = F_\nu^*.
\]

This sequence plays crucial roles to keep \( \bigcap_{\nu=1}^d (F_\nu^*(s))^\circ \neq \emptyset \) with \( s = 1, \cdots, N \) and gives an efficient size control of \( J \cdot m \) with \( J \in \bigcap_{\nu=1}^d F_\nu^* \) and \( m \in F_\nu(s) \).
4. Combinatorial Lemmas

As we have seen in Examples 2.3 and 2.6, we prove the representation formula for a face \( F \) of \( P = P(\Pi) \) and its cone (dual face) \( P^* \) given by

\[
F = \bigcap_{\pi_q \in \Pi; P \subset \pi_q} (\pi_q \cap P) \quad \text{and} \quad P^* = \operatorname{CoSp}\{q : F \subset \pi_q \in \Pi\}.
\]

Next, we make a cone decomposition such as \( Z(S) = \bigcup_{F \preceq N(\Lambda,S)} F \) where \( Z(S) \) is defined in Definition 2.15. Moreover, we give some properties of faces and dual faces which are used for the proof of Main Theorem. For further study, we refer readers to [7].

4.1. Elementary properties of faces and dual faces.

**Lemma 4.1.** Let \( P = P(\Pi) \) be a polyhedron in an affine space \( V \). Then

(1) If \( F, G \preceq P \) and \( G \subset F \), then \( G \preceq F \).

(2) Let \( \pi_q, r \in \Pi \). Then \( F = \pi_q, r \cap P \) is a face of \( P \).

(3) Let \( \Delta \subset \Pi \). Then \( F = \bigcap_{\pi_q, r \in \Delta} F_{q, r} \) with \( F_{q, r} = \pi_q, r \cap P \), is a face of \( P \).

**Proof of Lemma 4.1.** Since \( G \preceq P \), there exist \( q, r \) satisfying (2.3) where \( F \) replaced by \( G \). Next we can also replace \( P \) by \( F \). This proves (1). By Definitions 2.2 and 2.3 for \( \pi_q, r \) and \( P \),

\[
\langle q, u \rangle = r < \langle q, y \rangle \quad \text{for all } u \in F = \pi_q, r \cap P \text{ and } y \in P \setminus F
\]

which shows (2.3). Thus (2) is proved. Let \( \Delta = \{\pi_{q_j, r_j} : 1 \leq j \leq M\} \subset \Pi \). Then by (2), for every \( j = 1, \cdots, M \)

\[
\langle q_j, u \rangle = r_j \leq \langle q_j, y \rangle \quad \text{for all } u \in F = \bigcap_{j=1}^{M} F_{q_j, r_j} \text{ and } y \in P \setminus F.
\]

For \( y \in P \setminus F = \bigcup_{j=1}^{M} (P \setminus F_{q_j, r_j}) \) above, there exists \( j = \ell \) such that

\[
y \in P \setminus F_{q_\ell, r_\ell}.
\]

Thus \( \leq \) in (4.1) is replaced by \( < \) for \( j = \ell \). Hence we sum (4.1) in \( j \) to obtain that

\[
\langle \sum_{j=1}^{M} c_j q_j, u \rangle = \sum_{j=1}^{M} c_j r_j < \langle \sum_{j=1}^{M} c_j q_j, y \rangle \quad \text{for all } u \in F \text{ and } y \in P \setminus F
\]
where \( q = \sum_{j=1}^{M} c_j q_j \) and \( r = \sum_{j=1}^{M} c_j r_j \). Hence this with (2.3) yields (3). \( \square \)

**Lemma 4.2.** Let \( P \) be a polyhedron and \( F, G \in \mathcal{F}(P) \). Then \( F \preceq G \) if and only if \( G^* \preceq F^* \).

**Proof of Lemma 4.2.** We first show \( G^* \preceq F^* \). If \( F = G \), we are done. Let \( F \nsubseteq G \). It suffices to show that there exists \( q \in \mathbb{R}^n \) and \( r \in \mathbb{R} \) such that

\[
(q, u) = r < (q, v) \quad \text{for all } u \in G^* \text{ and } v \in F^* \setminus G^*,
\]

which means that \( G^* \preceq F^* \) by (2.3) in Definition 2.7. Choose \( q = n - m \) with \( n \in G \setminus F \) and \( m \in F \). Then \( (q, u) = 0 \) because \( m, n \in G \) and \( u \in G^* \). By \( v \in F^* \setminus G^* \) with \( m \in F \) and \( n \in G \setminus F \), \( (q, v) > 0 \) in view of Definition 2.11. Therefore (4.3) is proved. We next show that \( G^* \preceq F^* \) implies that \( F \preceq G \). Observe that if \( q \in G^* \), then there exists unique \( \rho = \inf \{ (x, q) : x \in P \} \) such that \( \pi_{q, \rho} \) is a supporting plane of a face containing \( G \). Since \( G \) is a face, there exists \( q \in (G^*)^c \subset G^* \). By Definition 2.11, \( \pi_{q, \rho} \cap P = G \). From \( q \in G^* \subset F^* \), it follows that \( F \subset \pi_{q, \rho} \cap P = G \), which yields \( F \preceq G \) by (1) of Lemma 4.1. \( \square \)

**4.2. Low Dimensional Polyhedron in \( \mathbb{R}^n \).** A polyhedron \( P = P(\Pi) \) in \( \mathbb{R}^n \) with \( \dim(P) = m \leq n \) is regarded as a \( m \) dimensional polyhedron in the affine space \( V_{am}(P) \) of dimension \( m \) defined in (2.1). Since \( V_{am}(P) \) itself is a polyhedron in \( \mathbb{R}^n \), we shall choose the generator \( \Pi \) of \( P \) split into two parts \( \Pi = \Pi_a \cup \Pi_b \); \( \Pi_b \) which is a generator of \( V_{am}(P) \) called the background generator, and \( \Pi_a \) a generator of \( P \) in \( V_{am}(P) \) called the main generator. See the left picture in Figure 4.1.

**Lemma 4.3.** Let \( P \subset \mathbb{R}^n \) be a polyhedron with \( \dim(P) = \dim(V_{am}(P)) = m \leq n \). Then \( P = P(\Pi_a \cup \Pi_b) \) such that

\[
(4.4) V_{am}(P) = P(\Pi_b) \text{ in } \mathbb{R}^n \text{ with } \Pi_b = \{ \pi_{\pm n_i, \pm s_i} \}_{i=1}^{n-m} \text{ with } n_i \perp n_j \text{ for } i \neq j, \\
(4.5) P = P(\Pi_a') \text{ in } V_{am}(P) \text{ with } \Pi_a = \{ \pi_{q_j, r_j} \}_{j=1}^{M} \text{ with } q_j \in V(P) = Sp^\perp(\{n_i\}_{i=1}^{n-m})
\]

where \( \Pi_a' = \{ \pi_{q_j, r_j} \cap V_{am}(P) \}_{j=1}^{M} \). There are infinitely many different generators \( \Pi \) expressing the same polyhedron \( P \). However, we shall use only generators of the form \( \Pi = \Pi_a \cup \Pi_b \) satisfying (4.4) and (4.5).
Proof. If \( n = m \), then let \( \Pi_a = \Pi \) and \( \Pi_b = \emptyset \) so that \( \mathbb{P}(\Pi_b) = \mathbb{R}^n \). Let \( m < n \). There exist \( n - m \) orthonormal vectors \( \mathbf{n}_i \)'s and some constants \( s_i \)'s such that

\[
V(\mathbb{P}) = \bigcap_{i=1}^{n-m} \pi_{\mathbf{n}_i,0} \quad \text{so that} \quad V_{am}(\mathbb{P}) = \bigcap_{i=1}^{n-m} \pi_{\mathbf{n}_i,s_i}.
\]

where \( V(\mathbb{P}) = \text{Sp}^\perp(\{\mathbf{n}_i\}_{i=1}^{n-m}) \). By (2.1), \( V_{am}(\mathbb{P}) = V(\mathbb{P}) + \mathbf{r} \) with \( \mathbf{r} \in \mathbb{P} \) and \( s_i = \mathbf{r} \cdot \mathbf{n}_i \). This combined with \( \pi_{\mathbf{n}_i,s_i} \cap \pi_{-\mathbf{n}_i,-s_i} = \pi_{\mathbf{n}_i,s_i} \) implies

\[
V_{am}(\mathbb{P}) = \mathbb{P}(\Pi^b) \quad \text{with} \quad \Pi_b = \{\pi_{\pm\mathbf{n}_i,\pm s_i} : i = 1, \cdots, n-m \},
\]

which yields (4.4). By Definition 2.3, there are \( \mathbf{p}_j \in \mathbb{R}^n \) such that

\[
(4.6) \quad \mathbb{P} = \bigcap_j \{ \mathbf{x} \in V_{am}(\mathbb{P}) : \langle \mathbf{p}_j, \mathbf{x} \rangle \geq \rho_j \}.
\]
Let \( x \in \mathbb{P} \) and \( P_{V(\mathbb{P})} \) be a projection map to the vector space \( V(\mathbb{P}) \). Then from \( x - \tau \in V(\mathbb{P}) \),

\[
\langle p_j, x \rangle = \langle P_{V(\mathbb{P})}(p_j), x \rangle + \langle P_{V^\perp(\mathbb{P})}(p_j), x \rangle = \langle P_{V(\mathbb{P})}(p_j), x \rangle + \langle \text{Proj}_{V^\perp(\mathbb{P})}(p_j), \tau \rangle.
\]

We put \( \text{Proj}_{V(\mathbb{P})}(p_j) = q_j \) and \( r_j = p_j - \langle \text{Proj}_{V^\perp(\mathbb{P})}(p_j), \tau \rangle \) and rewrite (4.6) as

\[
\mathbb{P} = \bigcap_{j=1}^{M} \{ x \in V_{am}(\mathbb{P}) : \langle q_j, x \rangle \geq r_j \} \text{ where } q_j = \text{Proj}_{V(\mathbb{P})}(p_j) \in V(\mathbb{P}) = \text{Sp}^\perp(\{v_i\}_{i=1}^{n-m}).
\]

This proves (4.5). Finally, \( \mathbb{P} = \mathbb{P}(\Pi_a \cup \Pi_b) \) follows from (4.4) and (4.5). \( \square \)

4.3. Properties of Boundary and Interior of Face.

**Lemma 4.4.** Let \( \mathbb{P} = \mathbb{P}(\Pi) \) be a polyhedron. Then \( \partial \mathbb{P} \subset \bigcup_{\pi \in \Pi} \pi \).

**Proof of Lemma 4.4.** Let \( x \in \partial \mathbb{P} \). Assume \( x \in \bigcap_{\pi \in \Pi} (\pi_+)^o \). Then a ball \( B(x, \epsilon) \) with some \( \epsilon > 0 \) is contained in \( \bigcap_{\pi \in \Pi} (\pi_+)^o \subset \bigcap_{\pi \in \Pi} (\pi_+) = \mathbb{P} \) in \( V_{am}(\mathbb{P}) \). Thus, \( x \notin \partial \mathbb{P} \) because \( \partial \mathbb{P} \) is a boundary of \( \mathbb{P} \) with respect to the usual topology of \( V_{am}(\mathbb{P}) \). Hence \( x \notin \bigcap_{\pi \in \Pi} (\pi_+)^o \). Combined with \( x \in \partial \mathbb{P} \subset \mathbb{P} = \bigcap_{\pi \in \Pi} \pi_+ \), we have \( x \in \bigcup_{\pi \in \Pi} \pi \). \( \square \)

**Lemma 4.5.** Let \( \mathbb{P} \) be a \( m \) dimensional polyhedron with a generator \( \Pi = \Pi(\mathbb{P}) \) in Lemma 4.3. Suppose that \( \mathbb{B} \subset \partial \mathbb{P} \) is a convex set. Then there is \( \pi \in \Pi \) such that \( \mathbb{B} \subset \mathbb{F}_\pi = \pi \cap \mathbb{P} \) where \( \mathbb{F}_\pi \) is \( m - 1 \) dimensional face of \( \mathbb{P} \).

**Proof of (4.5).** By Definition 2.9 and by Lemma 4.4,

(4.7) \[
\mathbb{B} \subset \partial \mathbb{P} \subset \bigcup_{\pi \in \Pi} \pi \cap \mathbb{P}.
\]

By (4.5) of Lemma 4.3, we may take \( \Pi = \Pi'_a \) in (4.7). Thus from (2) of Lemma 4.1, each \( \mathbb{F}_\pi = \pi \cap \mathbb{P} \) with \( \pi \in \Pi \) is a face of \( \mathbb{P} \) with dimension \( m - 1 \). Therefore by Definition 2.9, the second \( \subset \) in (4.7) is replaced with \( = \). We next show that

(4.8) \[
\mathbb{B} \subset \mathbb{F}_\pi \text{ for some } \pi \in \Pi \text{ where } \mathbb{B} \subset \partial \mathbb{P} = \bigcup_{\pi \in \Pi} \mathbb{F}_\pi.
\]

In order to show (4.8), we shall use an elementary geometric property: Given a hyperplane \( \tilde{\pi} \) and a line segment \( \text{Ch}(p_1, p_2) \) with \( \frac{p_1 + p_2}{2} \in \tilde{\pi} \), we have only two cases:

(4.9) \[
(1) \text{Ch}(p_1, p_2) \subset \tilde{\pi}, \text{ or } (2) p_1 \in (\tilde{\pi}_+)^o \text{ and } p_2 \in (\tilde{\pi}_-)^o.
\]
where \( p_1, p_2 \) may be switched. Assume the contrary to (4.8). Then

\[
\text{(4.10) there exists } p_1, p_2 \in B \text{ such that } \text{Ch}(p_1, p_2) \nsubseteq F_{\pi} \text{ for any } F_{\pi} \text{ in (4.8).}
\]

We shall find a contradiction to (4.10). It suffices to show that

\[
p_1, p_2 \subset B \text{ implies that } \text{Ch}(p_1, p_2) \nsubseteq F_{\pi} \text{ for some face } F_{\pi} \text{ in (4.8).}
\]

By \( \text{Ch}(p_1, p_2) \subset B \subset \partial P = \bigcup_{\pi \in \Pi} F_{\pi} \),

there exists \( \pi \in \Pi \) such that \( (p_1 + p_2)/2 \in F_{\pi} = \pi \cap P \subset \pi \).

Combined with \( p_1, p_2 \in B \subset P \subset \pi^+ \), we apply this \( \pi \) to \( \tilde{\pi} \) in (4.9). Then (2) in (4.9) is impossible. We have (1) in (4.9), that is, \( \text{Ch}(p_1, p_2) \subset \pi \). Thus \( \text{Ch}(p_1, p_2) \subset \pi \cap P = F_{\pi} \). \( \square \)

**Lemma 4.6.** Let \( P \) be a polyhedron and \( F \preceq P \) with \( \dim(F) = k \). Suppose that \( B \subset \partial F \) is a convex set. Then there is a \( k-1 \) dimensional face \( G \) such that \( B \subset G \preceq F \).

**Proof.** Apply Lemma 4.5 to a face \( F \) that is always a polyhedron. \( \square \)

**Lemma 4.7.** Let \( P = P(\Pi) \) with \( \dim(V_{am}(P)) = m \) and \( F \) be a proper face of \( P \). Then

\[
\exists \pi \in \Pi \text{ such that } F \subset \pi.
\]

**Proof.** Apply Lemma 4.5 with \( B = F \subset \partial P \). \( \square \)

4.4. **Representations of Faces.** We now prove that every proper face \( F \) is represented as the intersection of its facets containing \( F \).

**Lemma 4.8.** Let \( P = P(\Pi) \) be a polyhedron in an affine space \( V \) such that \( \dim(P) = \dim(V_{am}(P)) = m \) where \( \Pi = \Pi_a \cup \Pi_b \) as in Lemma 4.3. Then every facet \( F \) of \( P \) is expressed as

\[
F = \pi \cap P \text{ and } P \setminus F \subset (\pi^+)^{\circ} \text{ for some } \pi \in \Pi_a \subset \Pi.
\]

**Proof.** By Lemma 4.3, we regard \( P = P(\Pi') \) as a polyhedron in the \( m \) dimensional affine space \( V_{am}(P) \). Here \( \pi' = \pi \cap V_{am}(P) \in \Pi'_a \) is a \( m-1 \) dimensional hyperplane in \( V_{am}(P) \). By Lemma 4.7,

\[
\exists \pi' \in \Pi'_a \text{ such that } F \subset \pi' = \pi \cap V_{am}(P) \text{ where } \pi \in \Pi_a.
\]
On the other hand, by Definition 2.7, there exists an $m - 1$ dimensional hyperplane $\pi_{q,r}$ in $V_{am}(\mathcal{P})$ such that

\[(4.12) \quad F = \pi_{q,r} \cap \mathcal{P} \text{ and } \mathcal{P} \setminus F \subset (\pi_{q,r}^+)\circ.\]

In view of (4.11) and (4.12), both $m - 1$ dimensional hyperplanes $\pi'$ and $\pi_{q,r}$ in $V_{am}(\mathcal{P})$ contain the $m - 1$ dimensional polyhedron $F$. Thus $\pi' = \pi_{q,r}$. By this and (4.12),

\[F = \pi_{q,r} \cap \mathcal{P} = \pi' \cap \mathcal{P} = (\pi \cap V_{am}(\mathcal{P})) \cap \mathcal{P} = \pi \cap \mathcal{P} \text{ where } \pi \in \Pi_a\]

and $\mathcal{P} \setminus F \subset (\pi_{q,r}^+)\circ = ((\pi')^+)\circ \subset (\pi^+)\circ$. \hfill $\square$

**Proposition 4.1 (Face Representation).** Let $\mathcal{P} = \mathcal{P}(\Pi)$ be a polyhedron in $\mathbb{R}^n$ where $\Pi = \Pi_a \cup \Pi_b$ as in Lemma 4.3. Let $\dim(\mathcal{P}) = m \leq n$. Let $\mathcal{F}$ be a face of $\mathcal{P}$ with $\dim(\mathcal{F}) \leq n - 1$ and let $\Pi(\mathcal{F}) = \{ \pi \in \Pi : \mathcal{F} \subset \pi \cap \mathcal{P} \}$. Then $\mathcal{F}$ has the following representation:

\[(4.13) \quad \mathcal{F} = \bigcap_{\pi \in \Pi(\mathcal{F})} \mathcal{F}_\pi \text{ where } \mathcal{F}_\pi = \pi \cap \mathcal{P} .\]

Here $\Pi(\mathcal{F})$ splits $\Pi(\mathcal{F}) = \Pi_a(\mathcal{F}) \cup \Pi_b(\mathcal{F})$ where

\[(4.14) \quad \Pi_a(\mathcal{F}) = \Pi_a \cap \Pi(\mathcal{F}) = \{ \pi_{q_j} \}_{j=1}^\ell \quad \text{and} \quad \Pi_b(\mathcal{F}) = \Pi_b = \{ \pi_{\pm n_i} \}_{i=1}^{n-m}\]

as in (4.4) and (4.5). See the left side of Figure 4.1.

**Remark 4.1.** There can be different $\bar{\Pi}_1, \bar{\Pi}_2 \subset \Pi(\mathcal{P})$ such that $\mathcal{F} = \bigcap_{\pi \in \bar{\Pi}_1} \mathcal{F}_\pi = \bigcap_{\pi \in \bar{\Pi}_2} \mathcal{F}_\pi$ as in Example 2.3. But $\Pi(\mathcal{F})$ is defined as the maximal $\bar{\Pi}$ satisfying $\mathcal{F} = \bigcap_{\pi \in \bar{\Pi}} \mathcal{F}_\pi$ because $\Pi(\mathcal{F})$ is the collection of all $\pi \in \Pi(\mathcal{P})$ such that $\mathcal{F} \subset \pi \cap \mathcal{P}$.

We call $\Pi(\mathcal{F}) = \{ \pi \in \Pi : \mathcal{F} \subset \pi \cap \mathcal{P} \}$ the full generator of $\mathcal{F}$.

**Remark 4.2.** Sometimes we denote by $\Pi(\mathcal{F})$ the set of only normal vectors $\{ p_j \}_{j=1}^N$ instead of whole planes $\{ \pi_{p_j} \}_{j=1}^N = \Pi(\mathcal{F})$ without any ambiguity.

**Proof of Proposition 4.1.** Let $\dim(\mathcal{F}) = m \leq n - 1$. An improper face $\mathcal{F} = \mathcal{P}$ has an expression

\[\mathcal{P} = \bigcap_{\pi \in \Pi_b} \pi \cap \mathcal{P} \text{ where } \pi \in \Pi_b.\]
It suffices to show that each proper face \( F \) of \( \mathcal{P}(\Pi) \) has at least one expression:

\[
F = \bigcap_{j=1}^{M} F_j \quad \text{where} \quad F_j = \pi_j \cap \mathcal{P} \quad \text{with} \quad \pi_j \in \Pi_a \subset \Pi \quad \text{are facets of} \quad \mathcal{P}.
\]

To show (4.15), we first let \( F \) be a face of codimension 1 of the \( m \)-dimensional ambient affine space \( V_{am}(\mathcal{P}) \). Then \( F \) itself is a facet of \( \mathcal{P} \). So, \( F = \pi \cap \mathcal{P} \) with \( \pi \in \Pi_a \subset \Pi \) by Lemma 4.8. Let \( F \) be a face of codimension 2 of the \( m \)-dimensional ambient affine space \( V_{am}(\mathcal{P}) \). By Lemma 4.7,

\[
\pi_{q,r} \in \Pi_a \quad \text{such that} \quad F \subset \pi_{q,r}.
\]

By (2) of Lemma 4.1,

\[
\mathcal{P}' = \pi_{q,r} \cap \mathcal{P} \text{ is a facet of } \mathcal{P} \text{ such that } \dim(\mathcal{P}') = m - 1.
\]

Moreover, observe that \( \mathcal{P}' \) itself is an \( m - 1 \) dimensional polyhedron with

\[
\Pi_a(\mathcal{P}') \subset \{ \pi_{q,r} \cap \pi : \pi \in \Pi_a(\mathcal{P}) \}.
\]

By \( F \subset \mathcal{P}' \) and (1) of Lemma 4.1, \( m - 2 \) dimensional face \( F \) of \( \mathcal{P} \) is a facet of an \( m - 1 \) dimensional polyhedron \( \mathcal{P}' \). Hence, by Lemma 4.8 there exists \( \pi' \in \Pi_a(\mathcal{P}') \) in (4.17) such that \( F = \pi' \cap \mathcal{P}' \). Thus, by (4.17) there exists \( \pi \in \Pi_a(\mathcal{P}) \) such that \( \pi' = \pi_{q,r} \cap \pi \) and

\[
F = \pi' \cap \mathcal{P}' = (\pi_{q,r} \cap \pi) \cap \mathcal{P}' = (\pi_{q,r} \cap \mathcal{P}) \cap (\pi \cap \mathcal{P}) = F_{\pi_{q,r}} \cap F_{\pi}.
\]

where \( F_{\pi} \) and \( F_{\pi_{q,r}} \) are facets of \( \mathcal{P} \). We finish the proof of (4.15) inductively. \( \square \)

4.5. Representations of Cones (Dual Faces).

**Proposition 4.2** (Cone representation). Suppose a face \( F \) of a polyhedron \( \mathcal{P} \) is given by

\[
F = \bigcap_{\pi \in \Pi(\mathcal{F})} F_{\pi} = \left( \bigcap_{\pi \in \Pi_a(\mathcal{F})} F_{\pi} \right) \cap \left( \bigcap_{\pi \in \Pi_b(\mathcal{F})} F_{\pi} \right)
\]

with \( \Pi(\mathcal{F}) \) the collection of all \( \pi \in \Pi \) such that \( F \subset \pi \cap \mathcal{P} \) as in Proposition 4.1 where

1. \( \Pi_a(\mathcal{F}) = \Pi_a \cap \Pi(\mathcal{F}) = \{ \pi_{q,j} \}_{j=1}^{l} \) and \( F_{\pi} = \pi \cap \mathcal{P} \) with \( \pi \in \Pi_a \),
2. \( \Pi_b(\mathcal{F}) = \Pi_b = \{ \pi_{s,j} \}_{i=1}^{n-m} \) where \( V(\mathcal{P}) = Sp_{1}(\{ n_{i} \}_{i=1}^{m-n}) \), and \( F_{\pi} = \pi \cap \mathcal{P} = \mathcal{P} \).
Then $F$ has a cone (dual face) of the following form:

$$(F^*)^\circ | P = \text{CoSp}(\{q_j : j = 1, \ldots, \ell\}) \oplus V(P)^\perp = \text{CoSp}(\{q_j : j = 1, \ldots, \ell\}) \oplus \text{CoSp}(\{n_i, -n_i\}_{i=1}^{n-m}).$$

In conclusion, every proper face $F \preceq P = P(\Pi)$ with $\Pi(F) = \{p_j\}_{j=1}^N = \{q_j\}_{j=1}^\ell \cup \{\pm n_i\}_{i=1}^{n-m}$ as above takes its cone (dual face) of the form:

$$(F^*)^\circ | P = (F^*)^\circ | (P, \mathbb{R}^n) = \text{CoSp}(\{p_i : i = 1, \ldots, N\}).$$

Here $F^*|P = F^*|(P, \mathbb{R}^n) = \text{CoSp}(\{p_i : i = 1, \ldots, N\})$ similarly.

**Remark 4.3.** See the right side of Figure 4.1, which elucidates the relation between faces and their cones (dual faces). In the above proposition, $F^*|(P, \mathbb{R}^n) = F^*|(P, V(P)) \oplus V(P)^\perp$ where $F^*|(P, V(P)) = \text{CoSp}(\{p_i : p_i \in \Pi_a(F)\})$ with $\Pi_a(F)$ in (4.14). From this, we also obtain that $\dim(F) + \dim(F^*|(P, \mathbb{R}^n)) = n$ whereas $\dim(F) + \dim(F^*|(P, V(P)) = \dim(V(P))$.

In order to prove Proposition 4.2, we need the following three lemmas.

**Lemma 4.9.** Let $P = P(\Pi)$ be a polyhedron in an inner product space $V$ with $\dim(P) = \dim(V) = n$. Let $F \in F(P)$ be a facet expressed as

$$(4.19) \quad F = \pi_{q, r} \cap P \quad \text{where} \quad \pi_{q, r} \in \Pi \quad \text{and} \quad P \setminus F \subset (\pi_{q, r})^\circ.$$

Then

$$(F^*)^\circ | (P, V) = \text{CoSp}(q).$$

**Proof.** Let $q' = cq \subset \text{CoSp}(q)$ with $q$ in (4.19) and $c > 0$. Then $q'$ satisfies (2.5) in Definition 2.11. So $q' \in (F^*)^\circ$. Thus $\text{CoSp}(q) \subset (F^*)^\circ$. Let $p \in (F^*)^\circ$. Then by (2.5), $\pi_{p, r} \cap P = F$. This combined with (4.19) implies that $F = (\pi_{p, r} \cap P) \cap F = \pi_{p, r} \cap (\pi_{q, r} \cap P)$. Then $F \subseteq \text{CoSp}(p)$, which proves $(F^*)^\circ \subset \text{CoSp}(q)$. 

**Lemma 4.10.** Let $P = P(\Pi)$ be a polyhedron in an inner product space $V$ with

$$\dim(P) = \dim(V) = n.$$
Let $G \in \mathcal{F}(\mathbb{P})$ be a proper face of $\mathbb{P}$ with the full generator $\Pi(G) = \{ \pi \in \Pi : G \subset \pi \cap \mathbb{P} \} = \{ \pi_{q_j} \}^M_{j=1}$ so that

$$G = \bigcap_{j=1}^{M} F_j$$

where $F_j$ is of the form $F_j = \pi_{q_j} \cap \mathbb{P}$ and $(F_j^*)^\circ = \text{CoSp}^\circ(\{q_j\})$ for $j = 1, \cdots, M$. Then,

$$(4.20) \quad (G^*)^\circ(\mathbb{P}, V) = \text{CoSp}^\circ(\{q_j\}^M_{j=1}) \text{ and } (G^*| \mathbb{P}, V) = \text{CoSp}(\{q_j\}^M_{j=1}).$$

Proof. We shall show its proof in Appendix at the end of this paper, where we adapt the proof in [7] (for the cases of cones) to the cases of polyhedra. □

Next, we note that $F^*$ is translation-invariant in the following sense.

**Lemma 4.11.** Let $m \in V$. Then $[(F + m)^*]^\circ(\mathbb{P} + m, V) = (F^*)^\circ(\mathbb{P}, V)$.

Proof. Note that $q \in [(F + m)^*]^\circ(\mathbb{P} + m, V)$ if and only if there exists $\rho$ such that

$$\langle q, u + m \rangle = \rho < \langle q, y + m \rangle \text{ for } u + m \in F + m \text{ and } y + m \in (\mathbb{P} + m) \setminus (F + m),$$

that is equivalent to the following:

$$\exists \rho' = \rho - \langle q, m \rangle \text{ such that } \langle q, u \rangle = \rho' < \langle q, y \rangle \text{ for } u \in F \text{ and } y \in \mathbb{P} \setminus F$$

which means that $q \in (F^*)^\circ(\mathbb{P}, V)$. □

**Proof of Proposition 4.2.** We claim that $F$ has a cone (dual face) of the following form:

$$(F^*)^\circ | \mathbb{P} = \text{CoSp}^\circ(\{q_j : j = 1, \cdots, \ell \}) \oplus V(\mathbb{P})^\perp.$$ 

By (2.1),

$$\exists m \in V \text{ such that } V_{am}(\mathbb{P}) = m + V(\mathbb{P}).$$

We first work with $m = 0$. By (4.5) of Lemma 4.3, we regard $\mathbb{P}$ as a polyhedron $\mathbb{P}(\Pi'_a)$ defined in $V_{am}(\mathbb{P})$. Thus by (1) of (4.18) and Lemma 4.10,

$$(F^*)^\circ | \mathbb{P}, V(\mathbb{P})) = \text{CoSp}^\circ(\{q_j : j = 1, \cdots, \ell \}).$$

This means that $q \in \text{CoSp}^\circ(\{q_j : j = 1, \cdots, \ell \})$ if and only if $q \in (F^*)^\circ(\mathbb{P}, V(\mathbb{P}))$, that is,

$$q \in V(\mathbb{P}) \text{ and } r \text{ such that } \langle q, u \rangle = r < \langle q, y \rangle \text{ for all } u \in F, y \in \mathbb{P} \setminus F.$$
By this combined with $\langle n, u \rangle = \langle n, y \rangle = 0$ for all $n \in V(\mathbb{P})^\perp$ and $u, y \in V(\mathbb{P})$, we see that

$$q \in \text{CoSp}^\circ(\{q_j : j = 1, \ldots, \ell\}) \oplus V(\mathbb{P})^\perp$$

if and only if

$$\exists q \in V(\mathbb{P}) \oplus V(\mathbb{P})^\perp = \mathbb{R}^n \text{ and } r \text{ such that } \langle q, u \rangle = r < \langle q, y \rangle \text{ for all } u \in F, y \in \mathbb{P} \setminus F.$$ 

Hence we have for a proper face $F$,

$$F^*|(_{F, \mathbb{R}^n}) = \text{CoSp}^\circ(\{q_j : j = 1, \ldots, \ell\}) \oplus V^\perp(\mathbb{P}).$$

The case $m \neq 0$ follows from the case $m = 0$ in (4.21) and Lemma 4.11. Similarly,

$$F^*|(_{F, \mathbb{R}^n}) = \text{CoSp}(\{q_j : j = 1, \ldots, \ell\}) \oplus V^\perp(\mathbb{P}).$$

We finished the proof of Proposition 4.2.

\[\square\]

**Remark 4.4.** By (2) of (4.18), an improper face $\mathbb{P}$ has the expression that

$$\mathbb{P} = \bigcap_{\pi \in \Pi_b} F_\pi = \bigcap_{\pi \in \Pi_b} \pi \cap \mathbb{P} \text{ where } \Pi_b = \{\pi_{\pm n_i}\}_{i=1}^{n-m}.$$ 

Then we see that

$$\mathbb{P}^*|\mathbb{P} = V^\perp(\mathbb{P}) = \text{CoSp}(\{\pm n_i\}_{i=1}^{n-m}) \text{ and } (\mathbb{P}^*)^\circ|\mathbb{P} = V^\perp(\mathbb{P}) \setminus \{0\}.$$ 

This accords with Definition 2.11 together with (4.21) and (4.22). Finally, when $\mathbb{P} = \mathbb{P}(\Pi)$ with $\Pi = \{p_j\}_{j=1}^N$, we take $F^* = \text{CoSp}(\{p_j\}_{j=1}^N)$ if $F = \emptyset$.

In Example 4.1, we construct the faces and cones for the Newton Polyhedrons $\mathbf{N}(\Lambda_1)$ and $\mathbf{N}(\Lambda_2)$ associated with a polynomial $P_\Lambda(t) = (P_{\Lambda_1}(t_1, t_2, t_3), P_{\Lambda_2}(t_1, t_2, t_3))$ and check the hypotheses of Main Theorem 3.2 for $n = 3$ and $d = 2$ with $S = \{1, 2, 3\}$.

**Example 4.1.** Consider the polynomial $P_\Lambda(t) = (c_{m_1}^{m_1}t^{m_1} + c_{m_2}^{m_2}t^{m_2} + c_{n_2}^{n_2}t^{n_2})$ where

$$\Lambda_1 = \{m_1 = (0, 0, 2), n_1 = (3, 3, 0)\},$$

$$\Lambda_2 = \{m_2 = (0, 0, 3), n_2 = (3, 2, 1)\}.$$ 

Normal vectors $\{q_j^\nu\}_{j=1}^5$ of facets of $\mathbf{N}(\Lambda_\nu)$ for $\nu = 1, 2$ are

$$q_j^\nu = e_j \text{ for } j = 1, 2, 3, q_4^\nu = \frac{2, 0, 3}{\sqrt{13}}, q_5^\nu = \frac{0, 2, 3}{\sqrt{13}}, \text{ and } q_3^\nu = \frac{0, 1, 1}{\sqrt{2}}.$$
See Figure 4.2, where normal vectors $q^v_j$ are written without the superscript $\nu = 1$ for simplicity. All the faces of $\mathbf{N}(\Lambda_\nu)$ for $\nu = 1, 2$ are written as

\[ \mathcal{F}^2(\mathbf{N}(\Lambda_\nu)) = \left\{ \mathbb{F}(q^\nu_j) = \pi q^\nu_j \cap \mathbf{N}(\Lambda_\nu) : j = 1, \ldots, 5 \right\}, \]

\[ \mathcal{F}^1(\mathbf{N}(\Lambda_\nu)) = \left\{ \mathbb{F}(q^\nu_i) \cap \mathbb{F}(q^\nu_j) : (i, j) = (1, 2), (1, 4), (2, 5), (3, 4), (3, 5), (4, 5) \right\}, \]

\[ \mathcal{F}^0(\mathbf{N}(\Lambda_\nu)) = \left\{ \mathbb{F}(q^\nu_1) \cap \mathbb{F}(q^\nu_2) \cap \mathbb{F}(q^\nu_4) \cap \mathbb{F}(q^\nu_5), \mathbb{F}(q^\nu_3) \cap \mathbb{F}(q^\nu_4) \cap \mathbb{F}(q^\nu_5) \right\} = \{ m_\nu, n_\nu \}. \]

Cones of 0-faces (vertices) are

\[ (\mathcal{F}^0(\mathbf{N}(\Lambda_\nu)))^* = \{ m_\nu^* = \text{CoSp}(q^\nu_1, q^\nu_2, q^\nu_4, q^\nu_5) \} \quad \text{and} \quad n_\nu^* = \text{CoSp}(q^\nu_3, q^\nu_4, q^\nu_5). \]

Cones of 1-faces (edges) are

\[ (\mathcal{F}^1(\mathbf{N}(\Lambda_\nu)))^* = \{ [\mathbb{F}(q^\nu_i) \cap \mathbb{F}(q^\nu_j)]^* = \text{CoSp}(q^\nu_i, q^\nu_j) \}. \]
Cones of 2-faces are

\[ (\mathcal{F}^2(N(A_\nu)))^* = \{ (\mathcal{F}(q^\nu_j))^* = \text{CoSp}(q^\nu_j) \} \].

All possible combinations \( \bigcup \nu F_\nu \cap \Lambda_\nu \) with \( \text{rank}(\bigcup \nu F_\nu) \leq 2 \) are even sets except the following two odd set:

1. odd set \( \{ n_1, m_2 \} \),
2. odd set \( \{ m_1, n_1, m_2 \} \).

We can check the following in view of Figure 4.2, where we add the cone \((\mathcal{F}(q^2_5))^* = \text{CoSp}(q^2_5)\) for the face \(\mathcal{F}(q^2_5)\) of \(N(A_2)\).

From \( \{ n_1, m_2 \} \) where \( n_1^* = \text{CoSp}(q^1_3, q^1_4, q^1_5) \) and \( m_2^* = \text{CoSp}(q^2_1, q^2_2, q^2_3, q^2_5) \),

\[ (n_1^*)^o \cap (m_2^*)^o = \emptyset \text{ and } n_1^* \cap m_2^* = \text{CoSp}\left(\frac{(2, 0, 3)}{\sqrt{13}}\right) \] .

From \( \{ m_1 n_1, m_2 \} \) where \( \overline{m_1 n_1}^* = \text{CoSp}(q^1_4, q^1_3) \) and \( m_2^* = \text{CoSp}(q^2_1, q^2_3, q^2_5) \),

\[ (\overline{m_1 n_1})^o \cap (m_2^*)^o = \emptyset \text{ and } \overline{m_1 n_1}^* \cap m_2^* = \text{CoSp}\left(\frac{(2, 0, 3)}{\sqrt{13}}\right) \] .

As we point out in Remark 3.6, it is not just cones \( \bigcap F^*_\nu \), but their interiors \( \bigcap(F^*_\nu)^o \) that satisfy the overlapping condition (3.4). Thus even if \( \{ n_1, m_2 \} \) and \( \{ m_1, n_1, m_2 \} \) are odd sets, it does not prevent the uniform boundedness of the integrals:

\[
\sup_{r_j \in (0, 1), \xi \in \mathbb{R}^2} \left| \int_{\Pi(-r_j, r_j)} e^{i \xi_1 (c_{m_1}^1 t_1 + c_{n_1}^1 t_1)} + e^{i \xi_2 (c_{m_2}^2 t_2 + c_{n_2}^2 t_2)} \frac{dt_1 dt_2 dt_3}{t_1 t_2 t_3} \right| \leq C.
\]

4.6. Cone Decompositions. Recall that \( Z(S) = \prod Z_i \) with \( Z_i = \mathbb{R}_+ \) for \( i \in S \) and \( Z_i = \mathbb{R} \) for \( i \in N_n \setminus S \) as in Definition 2.15. We decompose \( Z(S) \) into finite number of different cones that appears in (2.8) and (2.11) as follows:

**Proposition 4.3.** Let \( \Lambda = (\Lambda_\nu) \) with \( \Lambda_\nu \subseteq \mathbb{Z}^n_+ \) and \( S \subseteq \{1, \cdots, n\} \). Then,

\[
\bigcup_{F \in \mathcal{F}(N(A, S))} \text{Cap}(F^*) = Z(S) \text{ where } \text{Cap}(F^*) = \bigcap_{\nu=1}^d F^*_\nu \text{ for } F = (F_\nu) \in \mathcal{F}(N(A, S)).
\]
Moreover,
\[
\bigcup_{F \in \mathcal{F}(\mathcal{N}(\Lambda, S))} \text{Cap}((F^*)^\circ) = Z(S) \setminus \{0\} \text{ where } \text{Cap}((F^*)^\circ) = \bigcap_{\nu=1}^{d} (F_{\nu}^*)^\circ.
\]

**Lemma 4.12.** Let \( \mathbb{P} = \mathbb{P}(\Pi) \) with \( \Pi = \{ \pi_{q_j}r_j : j = 1, \cdots, N \} \) be a polyhedron. Then
\[
\inf\{\langle x, e \rangle : x \in \mathbb{P} \} > -\infty \text{ if and only if } e \in \text{CoSp}(\{q_j : j = 1, \cdots, N\}).
\]

**Proof.** Let \( \rho = \inf\{\langle x, e \rangle : x \in \mathbb{P} \} > -\infty \) and set the plane \( \pi_{e, \rho} = \{x \in V : \langle x, e \rangle = \rho\} \). Since \( \mathbb{P} \) is a closed set, \( F \) defined by \( \pi_{e, \rho} \cap \mathbb{P} \) is a non-empty closed set. From \( \rho \leq x \cdot e \) for all \( x \in \mathbb{P} \) and \( F = \pi_{e, \rho} \cap \mathbb{P} \),
\[
(4.24) \quad \mathbb{P} \setminus F \subset (\pi_{e, \rho}^+)\circ.
\]
Thus \( \mathbb{F} \leq \mathbb{P} \) and \( e \in \mathbb{F}^* \subset \text{CoSp}(\{q_j : j = 1, \cdots, N\}) \) by using Propositions 4.1 and 4.2. To show the other direction, let \( e = \sum_{j=1}^{N} c_j q_j \in \text{CoSp}(\{q_j : j = 1, \cdots, N\}) \). Then
\[
\langle e, x \rangle = \sum_{j=1}^{N} c_j \langle q_j, x \rangle \geq \sum_{j=1}^{N} c_j r_j > -\infty
\]
for all \( x \in \mathbb{P} = \bigcap_{j=1}^{N} \{\langle q_j, x \rangle \geq r_j : j = 1, \cdots, N\} \). \( \Box \)

**Lemma 4.13.** Let \( \mathbb{P} = \mathbb{P}(\Pi) \) with \( \Pi = \{ \pi_{q_j}r_j : j = 1, \cdots, N \} \) be a polyhedron. Then
\[
\text{CoSp}(\{q_j : j = 1, \cdots, N\}) = \bigcup_{\mathbb{F} \leq \mathbb{P}} \mathbb{F}^*.
\]

**Proof.** We first show \( \subset \). Let \( e = \sum_{j=1}^{N} c_j q_j \in \text{CoSp}(\{q_j : j = 1, \cdots, N\}) \) and let
\[
\rho = \inf\left\{ \langle e, x \rangle = \sum_{j=1}^{N} c_j \langle q_j, x \rangle : x \in \mathbb{P} \right\}
\]
that exits from Lemma 4.12. Set \( F = \pi_{e, \rho} \cap \mathbb{P} \). By \( (4.24) \), \( F \) is a face of \( \mathbb{P} \) with a supporting plane \( \pi_{e, \rho} \). Thus \( e \in \mathbb{F}^* \). The other direction \( \supset \) follows from Propositions 4.1 and 4.2. \( \Box \)

**Lemma 4.14.** Let \( u = (u_1, \cdots, u_n) \in \mathbb{F}^* \) and \( F \in \mathcal{F}(\mathcal{N}(\Lambda, S)) \). Then \( u_j \geq 0 \) for all \( j \in S \).

**Proof.** Let \( m \in \mathbb{F} \) and \( j \in S \). Then \( m + re_j \in \mathcal{N}(\Lambda, S) \) for all \( r \geq 0 \). By Definition 2.4 and \( u \in \mathbb{F}^* \), we have \( u_j r = \langle u, m + re_j - m \rangle \geq 0 \). Thus \( u_j \geq 0 \). See Figure 2.1. \( \Box \)
Lemma 4.15. Let $S \subset N_n$ and $\Omega \subset \mathbb{Z}_+^n$ be a finite set. Suppose that $\mathcal{P} = \mathcal{P}(\Pi)$ with $\Pi = \{\pi_{q,j} : j = 1, \ldots, N\}$ is a polyhedron given by $\mathcal{N}(\Omega, S)$. Then

$$\bigcup_{\mathcal{F} \in \mathcal{F}(\mathcal{N}(\Omega, S))} \mathcal{F}^* = Z(S).$$

Moreover, $\bigcup_{\mathcal{F} \in \mathcal{F}(\mathcal{N}(\Omega, S))} (\mathcal{F}^*)^\circ = Z(S) \setminus \{0\}$.

Proof. It follows $\subset$ from Lemma 4.14. We next show $\supset$. Put

$$m_k = \min \{u_k : u = (u_1, \ldots, u_n) \in \Omega\},$$
$$M_k = \max \{u_k : u = (u_1, \ldots, u_n) \in \Omega\}.$$

By $\mathcal{N}(\Omega, S) = \text{Ch}\{u + \mathbb{R}_+^S : u \in \Omega\}$,

1. if $k \in N_n \setminus S$, then $m_k \leq x_k \leq M_k$ for all $x = (x_1, \ldots, x_n) \in \mathcal{N}(\Omega, S)$,
2. if $k \in S$, then $m_k \leq x_k$ for all $x = (x_1, \ldots, x_n) \in \mathcal{N}(\Omega, S)$.

Thus, if $k \in N_n \setminus S$, then $e_k, -e_k \in \text{CoSp}(\{q_j : j = 1, \ldots, N\})$ by Lemma 4.12 and (1) above. If $k \in S$, then $e_k \in \text{CoSp}(\{q_j : j = 1, \ldots, N\})$ by Lemma 4.12 and (2) above. Hence

$$Z(S) = \text{CoSp}(\{\pm e_k : k \in N_n \setminus S\} \cup \{e_k : k \in S\}) \subset \text{CoSp}(\{q_j : j = 1, \ldots, N\}).$$

By Lemma 4.13, $Z(S) \subset \bigcup_{\mathcal{F} \in \mathcal{F}(\mathcal{N}(\Omega, S))} \mathcal{F}^*$. By Definitions 2.9-2.11 together with (4.23),

(4.25) $$\bigcup_{\mathcal{F} \in \mathcal{F}(\mathcal{N}(\Omega, S))} (\mathcal{F}^*)^\circ = \bigcup_{\mathcal{F} \in \mathcal{F}(\mathcal{N}(\Omega, S))} \mathcal{F}^* \setminus \{0\}.$$ 

This implies the last statement. \hfill \Box

Proof of Proposition 4.3. By Lemma 4.15,

$$\bigcup_{\mathcal{F}_\nu \in \mathcal{F}(\mathcal{N}(\Lambda_\nu, S))} \mathcal{F}_\nu^* = Z(S) \text{ for every } \nu = 1, \ldots, d.$$ 

Hence, by taking an intersection for $\nu = 1, \ldots, d$,

$$\bigcup_{\mathcal{F} \in \mathcal{F}(\mathcal{N}(\Lambda, S))} \text{Cap}(\mathcal{F}^*) = \bigcap_{\nu=1}^d \bigcup_{\mathcal{F}_\nu \in \mathcal{F}(\mathcal{N}(\Lambda_\nu, S))} \mathcal{F}_\nu^* = Z(S).$$

The last statement follows from (4.25). \hfill \Box
4.7. Projective Cone; Boundary Deleted Neighborhood. We consider the projection of a cone $F^*$ to the sphere $S^{n-1}$, which will be used for proving Proposition 7.1.

**Definition 4.1.** In stead of working with the cone $F^*$ of a face $F$, it is sometimes convenient to work with its intersection $F^* \cap S^{n-1}$ with the sphere. We denote it and its boundary by

$$S[F^*] = F^* \cap S^{n-1} \quad \text{and} \quad \partial S[F^*] = (\partial F^*) \cap S^{n-1}.$$ 

Given $K \subset S^{n-1}$, we define the $\epsilon$-neighborhood of $K$ by

$$N_\epsilon(K) = \{ x \in S^{n-1} : |x - y| < \epsilon \text{ for some } y \in K \}.$$ 

**Definition 4.2.** [Boundary Deleted $\epsilon$-neighborhood of $S[F^*]$] Let $P$ be a polyhedron in $\mathbb{R}^n$ with $\dim(P) = m < n$ and let $\mathcal{F}^k(P)$ denote the set of all $k$ dimensional face as in Definition 2.7. Then for each $F \in \mathcal{F}^{m-k}(P)$ where $k = 0, \ldots, m$, we define a boundary deleted $\epsilon$-neighborhood $S_\epsilon[F^*]$ by

$$S_\epsilon[F^*] = N_{\epsilon/M^k}(S[F^*]) \quad \text{when } F = P \in \mathcal{F}^m(P),$$

$$S_\epsilon[F^*] = N_{\epsilon/M^{k+1/3}}(S[F^*] \setminus N_{\epsilon/M^{k-1/3}}(\partial S[F^*])) \quad \text{when } F = P \in \mathcal{F}^{m-k}(P), \quad 1 \leq k \leq m,$$

where $F^* = \text{CoSp}(\Pi_a(F)) \oplus V_\perp(P) = \text{CoSp}(\{q_j\}_{j=1}^\ell) \oplus V_\perp(P)$ as in (4.22). Here $M$ will be chosen to be a large positive number. For the case that $\dim(P) = n$ with $F \in \mathcal{F}^{n-k}(P)$ where $k = 1, \ldots, n$, we define a boundary deleted $\epsilon$-neighborhood $S_\epsilon[F^*]$ by

$$S_\epsilon[F^*] = N_{\epsilon/M^k}(S[F^*]) \quad \text{when } F \in \mathcal{F}^{n-1}(P),$$

$$S_\epsilon[F^*] = N_{\epsilon/M^{k+1/3}}(S[F^*] \setminus N_{\epsilon/M^{k-1/3}}(\partial S[F^*])) \quad \text{when } F \in \mathcal{F}^{n-k}(P), \quad 2 \leq k \leq m$$

where $F^* = \text{CoSp}(\{q_j\}_{j=1}^\ell)$. See $S_\epsilon[F^*(q_\ell)]$ and $S_\epsilon[F^*(q_1, q_\ell)]$ in the right side of Figure 4.2.

**Lemma 4.16.** Let $\Omega \subset \mathbb{Z}^n_+$ and $P = N(\Omega, S) \subset \mathbb{R}^n$ be a polyhedron. Then

$$\bigcup_{F \in \mathcal{F}(N(\Omega, S))} S[F^*] \subset \bigcup_{F \in \mathcal{F}(N(\Omega, S))} S_\epsilon[F^*].$$

**Proof.** We prove the case $\dim(P) = m < n$. By Definition 4.2,

$$\bigcup_{F \in \mathcal{F}^m(S(\Omega, S))} S[F^*] \subset \bigcup_{F \in \mathcal{F}^m(S(\Omega, S))} S_\epsilon[F^*] = \bigcup_{F \in \mathcal{F}^m(S(\Omega, S))} N_{\epsilon/M^k}(S[F^*]).$$
Using this and Definition 4.2,
\[
\bigcup_{F \in \mathcal{F}^{m-1}(\mathbf{N}(\Omega,S))} S[F^*] \subset \left( \bigcup_{F \in \mathcal{F}^{m-1}(\mathbf{N}(\Omega,S))} S_\epsilon[F^*] \right) \cup \left( \bigcup_{F \in \mathcal{F}^m(\mathbf{N}(\Omega,S))} S_\epsilon[F^*] \right).
\]

Inductive applications of this inclusion complete the proof. □

Note that by Proposition 4.3,
\[
\bigcup_{F \in \mathcal{F}} = (\mathbf{F} \nu) \in \mathcal{F}(\mathbf{N}(P,S)) \quad \nu = 1, \ldots, d.
\]

By Lemma 4.16 together with (4.26), we have

**Lemma 4.17.** Let \( \Lambda = (\Lambda_\nu) \) with \( \Lambda_\nu \subset \mathbb{Z}_+^n \) and \( \mathbf{N}(\Lambda, S) = (\mathbf{N}(\Lambda_\nu, S)) \). Then
\[
Z(S) \cap \mathbb{S}^{n-1} \subset \bigcup_{F = (\mathbf{F}_\nu) \in \mathcal{F}(\mathbf{N}(\Lambda, S))} \bigcap_{\nu=1}^d S_\epsilon[F^*_\nu].
\]

Using this we can decompose for sufficiently small \( \epsilon > 0 \),
\[
(4.27) \quad \sum_{J \in Z(S)} H^P_J = \sum_{F \in \mathcal{F}(\mathbf{N}(\Lambda, S))} \sum_{J \in Z(S), J \cup |J|}^d \sum_{\nu=1}^d S_\epsilon[F^*_\nu].
\]

**Remark 4.5.** Note that we use the notation \( J \) for expressing only the elements of \( \mathbb{Z}^n \), where \( Z(S) \) means \( Z(S) \cap \mathbb{Z}^n \) without any ambiguity.

In order to check the overlapping condition (3.4), we need the following lemma.

**Lemma 4.18.** Let \( \mathbf{F}_\nu \in \mathcal{F}(\mathbf{N}(\Lambda_\nu, S)) \) for \( \nu = 1, \ldots, d \). Then for some sufficiently small \( \epsilon > 0 \), we have the property that \( \bigcap_{\nu=1}^d S_\epsilon[F^*_\nu] \neq \emptyset \) implies that \( \bigcap_{\nu=1}^d S[(F^*_\nu)] \neq \emptyset \).

**Proof of lemma 4.18.** We prove the case \( \dim(P) = m < n \). It suffices to find an \( \epsilon > 0 \) such that
\[
\bigcap_{\nu=1}^d S_\epsilon[F^*_\nu] = \emptyset \quad \text{implies that} \quad \bigcap_{\nu=1}^d S_\epsilon[F^*_\nu] = \emptyset.
\]

Suppose that \( d \)-tuple \( (\mathbf{F}_\nu) \) of faces are given so that
\[
(4.28) \quad \bigcap_{\nu=1}^d S_\epsilon[F^*_\nu] = \emptyset.
\]
Note that $S[F^*_h] \setminus N^\epsilon (\partial S[F^*_h]) \subset S[(F^*_h)^\circ]$ for any positive number $\epsilon > 0$ and $S[F^*_h] = S[(F^*_h)^\circ]$ for $\dim(F^*_h) = m$. From this, we split (4.28) into two smaller parts:

$$
(4.29) \bigcap_{\nu; \dim(F^*_h) \leq m-1} \left( S[F^*_h] \setminus N^\epsilon / M^{k(\nu)-1/3}(\partial S[F^*_h]) \right) \bigcap \bigcap_{\nu; \dim(F^*_h) = m} S[F^*_h] \subset \bigcap_{\nu=1}^d S[(F^*_h)^\circ] = \emptyset.
$$

Since $S[F^*_h]$ and $S[F^*_h] \setminus N^\epsilon / M^{k(\nu)-1/3}(\partial S[F^*_h])$ are closed sets in $S^{n-1}$ in (4.29), we take a little bit thicker intersection in $\nu = 1, \cdots, d$ with some large $M$ and small $\epsilon$ to obtain that

$$
\bigcap_{\nu; \dim(F^*_h) \leq m-1} N^\epsilon / M^{k(\nu)+1/3} \left( S[F^*_h] \setminus N^\epsilon / M^{k(\nu)-1/3}(\partial S[F^*_h]) \right) \bigcap \bigcap_{\nu; \dim(F^*_h) = m} S_{\epsilon/M}[F^*_h] = \emptyset.
$$

By Definition 4.2, we have

$$
\bigcap_{\nu} S_{\epsilon}[F^*_h] = \emptyset.
$$

This proves Lemma 4.18. The case $\dim(P) = n$ follows similarly. \qed

**Lemma 4.19.** Let $F$ be a face of $P = N(\Omega, S)$ with $\dim(P) = m < n$. Suppose that $\tilde{m} \in F \cap \Omega$ and $m \in \Omega \setminus F$. Then for all $p \in S_{\epsilon}[F^*_h]$ with $\dim(F) = m - k$ where $k = 1, \cdots, m$,

$$
(4.30) \quad p \cdot (m - \tilde{m}) \geq c > 0 \quad \text{where} \quad c \text{ is independent of } p.
$$

**Remark 4.6.** We shall use Lemma 4.19 for the estimate of the difference $I_{J}(P_{\Omega}, \xi) - I_{J}(P_{F}, \xi)$ where $|J|/|J| = p$. We do not need Lemma 4.19 if $\dim(F) = m - k$ with $k = 0$, since $F = N(\Omega, S)$ for the case $\dim(F) = m$ so that $I_{J}(P_{\Omega}, \xi) - I_{J}(P_{F}, \xi) = 0$. For the case $\dim(P) = n$, (4.30) also holds for all $S_{\epsilon}[F^*_h]$ with $\dim(F) = n - k$ where $k = 1, \cdots, n$.

**Proof of Lemma 4.19.** By Proposition 4.2,

$$
F^* = F^* | P = \text{CoSp}(\{ q_j \}_{j=1}^\ell \cup \{ \pm n_i \}_{i=1}^{n-m})
$$

where $\{ q_j \}_{j=1}^\ell$ and $\{ \pm n_i \}_{i=1}^{n-m}$ is defined as in (4.14). Here we can take $q_j \in S^{n-1}$. Then

$$
S[F^*] = \text{CoSp}(\{ q_j \}_{j=1}^\ell \cup \{ \pm n_i \}_{i=1}^{n-m}) \cap S^{n-1}
$$

$$
= \left\{ q \in S^{n-1} : q = \sum_{j} c_j q_j + \tau \text{ where } c_j > 0 \text{ and } \tau = \sum_{i=1}^{n-m} c_i, (\pm n_i) \in V(P) \right\}.
$$
Thus, for sufficiently large $M$,

\[
S[F^*] \setminus N_{\epsilon/M^{k-1/3}}(\partial S[F^*]) \subset \left\{ q \in \mathbb{S}^{n-1} : q = \sum_{j} c_j q_j + r \text{ with } c_j > \frac{\epsilon}{M^{k-1/4}} \text{ and } r \in V^\perp(\mathbb{P}) \right\}.
\]

By (4.18),

\[
F = \bigcap_{j} F_j \text{ with } F_j = \pi_{q_j} \cap \mathbb{P}.
\]

From $\tilde{m} \in F$ and $m \in \Omega \setminus F$,

\[
m \in \mathbb{P} \setminus F_k \quad \text{for some } k \in \{1, \cdots, \ell\} \text{ and } \tilde{m} \in F \subset F_k.
\]

Thus, by Definition 2.11,

\[
q_k \cdot (m - \tilde{m}) > \eta_k > 0 \text{ and } q_j \cdot (m - \tilde{m}) \geq 0 \text{ for } j = 1, \cdots, \ell
\]

where $\eta_k$ depends on $\Omega$. Let $q \in S[F^*] \setminus N_{\epsilon/M^{k-1/3}}(\partial S[F^*])$. Then by (4.31),

\[
q = \sum_{j=1}^{\ell} c_j q_j + r \text{ where } c_j \geq \epsilon/M^{k-1/4} \text{ and } r \in V^\perp(\mathbb{P}).
\]

Thus, we use (4.32) and the fact $r \cdot (m - \tilde{m}) = 0$ (which follows from $r \in V^\perp(\mathbb{P})$) to have

\[
q \cdot (m - \tilde{m}) = \sum_{j=1}^{\ell} c_j q_j \cdot (m - \tilde{m})
\]

\[
\geq c_k q_k \cdot (m - \tilde{m}) + \sum_{j=1, j \neq k}^{\ell} c_j q_j \cdot (m - \tilde{m})
\]

\[
\geq c_k q_k \cdot (m - \tilde{m}) + 0 \geq (\epsilon/M^{k-1/4}) \eta_k \geq \epsilon \eta/M^{k-1/4} > 0
\]

where $\eta = \min\{\eta_k : k = 1, \cdots, \ell\}$. Finally, let

\[
p \in S_{\epsilon}[F^*] = N_{\epsilon/M^{k+1/3}} \left( S[F^*] \setminus N_{\epsilon/M^{k-1/3}}(\partial S[F^*]) \right).
\]

Then there exists $q \in S[F^*] \setminus N_{\epsilon/M^{k-1/3}}(\partial S[F^*])$ satisfying (4.33) and $|p - q| < \epsilon/M^{k+1/3}$. For sufficiently large $M > 0$, we have $p \cdot (m - \tilde{m}) \geq \epsilon \eta/(2M^{k-1/4})$, which proves (4.30). □

Lemma 4.20. Let \( \Lambda \subset \mathbb{Z}_+^n \) and \( S_0 \subset S \subset N_n \). Suppose that \( F \in \mathcal{F}_n(\Lambda, S) \) and that there exits \( q = (q_j) \in (\mathbb{F}^*)^n|N(\Lambda, S) \) such that \( q_j = 0 \) if \( j \in S_0 \) and \( q_j > 0 \) if \( j \in S \setminus S_0 \). Then,

\[
(4.34) \quad F = F + \mathbb{R}_+^{S_0},
\]

and

\[
(4.35) \quad F = N(\Lambda \cap F, S_0).
\]

Here \( S_0 \) can be an empty set.

Proof of (4.34). Since \( 0 \in \mathbb{R}_+^{S_0} \), we have \( F \subset F + \mathbb{R}_+^{S_0} \). Let \( m + \sum_{j \in S_0} a_j e_j \in F + \mathbb{R}_+^{S_0} \) where \( m \in F \). Assume that \( m + \sum_{j \in S_0} a_j e_j \in N(\Lambda, S) \setminus F \). By Definition 2.11,

\[
\langle m, q \rangle < \left( m + \sum_{j \in S_0} a_j e_j, q \right) \quad \text{for} \quad q \in (\mathbb{F}^*)^n|N(\Lambda, S),
\]

which is impossible because \( q_j = 0 \) for \( j \in S_0 \) in the hypothesis. Thus

\[
m + \sum_{j \in S_0} a_j e_j \in F.
\]

This implies that \( F + \mathbb{R}_+^{S_0} \subset F \).

Proof of (4.35). By definition, \( N(\Lambda \cap F, S_0) \) is the smallest convex set containing \( (\Lambda \cap F) + \mathbb{R}_+^{S_0} \). In view of (4.34), the face \( F \) contains the set set \( (\Lambda \cap F) + \mathbb{R}_+^{S_0} \). Thus,

\[
N(\Lambda \cap F, S_0) \subset F.
\]

Next, we show that \( F \subset N(\Lambda \cap F, S_0) \). Let \( x \in F \subset N(\Lambda, S) = \text{Ch}(\Lambda + \mathbb{R}_+^S) \). Then,

\[
x = \sum_{m \in \Omega} c_m m \quad \text{with} \quad \Omega \text{ is a finite subset of} \quad \Lambda + \mathbb{R}_+^S
\]

where \( \sum_{m \in \Omega} c_m = 1 \) and \( c_m > 0 \). Assume that \( m \in \Omega \cap F^c \neq \emptyset \). Then by Definition 2.11, for \( q = (q_j) \in (\mathbb{F}^*)^n|N(\Lambda, S) \), we have \( \langle m, q \rangle \neq 0 \). Thus

\[
\langle x, q \rangle = \sum_{m \in \mathbb{F}^c \cap \Omega} c_m (m, q) + \sum_{m \in \mathbb{F}^c \cap \Omega} c_m (m, q) > \left( \sum_{m \in \Omega} c_m \right) \langle x, q \rangle = \langle x, q \rangle
\]
which is a contradiction. So $\Omega \cap F^c = \emptyset$. Hence

\[(4.36) \quad \mathbf{x} = \sum_{m \in \Omega} c_m \mathbf{m} \text{ where } \Omega \subset F \cap (\Lambda + R^S_+) . \]

Here each $\mathbf{m} \in \Omega \subset F \cap (\Lambda + R^S_+)$ above is expressed as

\[(4.37) \quad \mathbf{m} = \mathbf{z} + \sum_{j \in S} a_j \mathbf{e}_j \in F \text{ where } \mathbf{z} \in \Lambda \text{ and } a_j \geq 0. \]

By Definition 2.11, for $q = (q_j) \in (F^*)^N|\text{N}(\Lambda, S)$, $\mathbf{m} \in F$ and $\mathbf{z} \in \text{N}(\Lambda, S)$,

\[(4.38) \quad \left\langle \mathbf{z} + \sum_{j \in S} a_j \mathbf{e}_j, q \right\rangle \leq \langle \mathbf{z}, q \rangle. \]

If $\mathbf{z} \in \text{N}(\Lambda, S) \setminus F$, then the inequality in (4.38) is strict. This is impossible because $q_j$ with $j \in S$ in $q$ is nonnegative in the above hypothesis. Thus $\mathbf{z} \in F$ in (4.37). Moreover $a_j = 0$ for $j \in S \setminus S_0$ in (4.38) because $q_j > 0$ for $j \in S \setminus S_0$. Therefore $\mathbf{z} \in F \cap \Lambda$ and $j \in S_0$ in (4.37). Hence in (4.36),

$$\mathbf{x} = \sum_{m \in \Omega} c_m \mathbf{m} \text{ where } \Omega \subset (F \cap \Lambda) + R^S_+ \text{ and } \sum_{m \in \Omega} c_m = 1,$$

that is $\mathbf{x} \in \text{Ch}((F \cap \Lambda) + R^S_+) = \text{N}(F \cap \Lambda, S_0)$, which implies $F \subset \text{N}(\Lambda \cap F, S_0)$. \hfill \qed

4.9. Essential Faces. In constructing a sequence $\{F^*(s)\}^N_{s=0}$ in (3.13), we need the following concept of faces.

**Definition 4.3.** Let $\mathcal{P}$ be a polyhedron such that $\mathcal{B} \subset \mathcal{P}$. Then a set $F(\mathcal{B}|\mathcal{P})$ is defined to be the smallest face of $\mathcal{P}$ containing $\mathcal{B}$ in the sense that

$$\mathcal{B} \subset F(\mathcal{B}|\mathcal{P}) \preceq \mathcal{P} \text{ and } \mathcal{B} \nsubseteq G \text{ for any } G \nsubseteq F(\mathcal{B}|\mathcal{P}).$$

We call $F(\mathcal{B}|\mathcal{P})$ the essential face of $\mathcal{P}$ containing $\mathcal{B}$. See the first and second pictures in Figure 4.3.

**Lemma 4.21.** Let $\mathcal{P}$ be a polyhedron such that $\mathcal{B} \cap \mathcal{P}^o \neq \emptyset$. Then

$$F(\mathcal{B}|\mathcal{P}) = \mathcal{P}. \quad \text{(4.39)}$$

**Proof.** We see that $F(\mathcal{B}|\mathcal{P}) \preceq \mathcal{P}$. Assume that $F(\mathcal{B}|\mathcal{P}) \nsubseteq \mathcal{P}$. Then $\mathcal{B} \subset F(\mathcal{B}|\mathcal{P}) \subset \partial \mathcal{P}$. This is a contradiction to the hypothesis $\mathcal{B} \cap \mathcal{P}^o \neq \emptyset$. \hfill \qed
Lemma 4.22. Let $\mathbb{P}$ be a polyhedron such that $\mathbb{B} \subset \mathbb{P}$. Then

$$(F(B|\mathbb{P}))^\circ \cap \text{Ch}(\mathbb{B}) \neq \emptyset.$$  

Proof. If not, $\text{Ch}(\mathbb{B}) \subset \partial F(B|\mathbb{P})$. By Lemma 4.6, $\text{Ch}(\mathbb{B}) \subset \mathcal{G} \nsubseteq F(B|\mathbb{P})$, which is impossible by Definition 4.3. \qed

Lemma 4.23. Let $\mathbb{P}$ be a polyhedron in $\mathbb{R}^n$ and let $\mathbb{B} \subset \mathbb{P}$ be a convex set. Then

$$\mathbb{B}^\circ \subset F(B|\mathbb{P})^\circ.$$  

Proof. Let $V$ be an affine space with $\dim(V) = k$ and consider two affine spaces $V_1 \subset V$ with $\dim(V_1) \leq k - 1$ and $V_2 \subset V$ with $\dim(V_2) = k - 1$. Then, $V_2$ regarded as an hyperplane in $V$ is expressed as $\{x \in V : \langle q, x \rangle = r\}$ for some $q \in V$ and $r \in \mathbb{R}$. In view
of Definition 2.2,

\[(V_2^\circ)^o = \{x \in V : \langle q, x \rangle > r \} \text{ and } (V_2^-)^o = \{x \in V : \langle q, x \rangle < r \}.
\]

We need the following observation: If \(V_1\) and \(V_2\) meet at \(z \in V_1 \cap V_2\) with \(V_1 \not\subseteq V_2\) (transversally), then

\[(4.39) \quad B_{V_1}(z, \epsilon) \cap (V_2^\circ)^o \neq \emptyset \text{ and } B_{V_1}(z, \epsilon) \cap (V_2^-)^o \neq \emptyset \text{ for any } \epsilon > 0
\]

where \(B_{V_1}(z, \epsilon) = \{v \in V_1 : |v - z| < \epsilon\}\) is an \(\epsilon\)-neighborhood of \(z\) in \(V_1\). Let \(z \in \mathbb{B}^o\). Then we shall show that \(z \in (F(\mathbb{B}|\mathbb{P}))^o\). Since \(z \in \mathbb{B}^o \subset \mathbb{B} \subset F(\mathbb{B}|\mathbb{P})\), it suffices to prove that \(z \in \partial F(\mathbb{B}|\mathbb{P})\) leads to a contradiction that \(z \not\in \mathbb{B}^o\). If \(z \in \partial F(\mathbb{B}|\mathbb{P})\), then by Definition 2.9, \(z \in \mathbb{G} \not\subseteq F(\mathbb{B}|\mathbb{P})\). Let \(V_{am}(\mathbb{G})\) be the plane containing \(\mathbb{G}\) with

\[(4.40) \quad \dim(V_{am}(\mathbb{G})) = k - 1 \leq k = \dim(F(\mathbb{B}|\mathbb{P})) \text{ and } F(\mathbb{B}|\mathbb{P}) \subset V_{am}^+(\mathbb{G}).
\]

By Definition 4.3 and Lemma 4.22, \(\mathbb{B} \cap F(\mathbb{B}|\mathbb{P})^o \neq \emptyset\), that is, \(V_{am}(\mathbb{B}) \not\subset V_{am}(\mathbb{G})\). From \(z \in \mathbb{B}^o\) and \(z \in \mathbb{G}\), it follows that \(z \in V_{am}(\mathbb{B}) \cap V_{am}(\mathbb{G})\). By (4.39) and (4.40) together with \(V = V_{am}(F(\mathbb{B}|\mathbb{P}))\), we see that \(V_1 = V_{am}(\mathbb{B}) \subset V\) and \(V_2 = V_{am}(\mathbb{G}) \subset V\),

\[B_{V_{am}(\mathbb{B})}(z, \epsilon) \not\subseteq F(\mathbb{B}|\mathbb{P}), \text{ which implies that } B_{V_{am}(\mathbb{B})}(z, \epsilon) \not\subseteq \mathbb{B} \text{ for any } \epsilon > 0.
\]

This means that \(z \not\in \mathbb{B}^o\). 

**4.10. Invariance Property under Isomorphism.**

**Lemma 4.24.** Let \(T : V \to W\) be an isomorphism where \(V, W\) be inner product spaces in \(\mathbb{R}^n\). Let \(\mathbb{P} = \mathbb{P}(\Pi)\) with \(\Pi = \{\pi_{q_1,r_1}, \ldots, \pi_{q_N,r_N}\}\) be a polyhedron in \(V\). Then

1. \(T(\mathbb{P})\) is a polyhedron \(\mathbb{P}(\Pi_T)\) with \(\Pi_T = \{\pi_{(T^{-1})^t(q_1,r_1), \ldots, \pi_{(T^{-1})^t(q_N,r_N)}\}\}.
2. If \(\mathbb{P} \in \mathcal{F}^k(\mathbb{P})\), then \(T(\mathbb{P}) \in \mathcal{F}^k(T(\mathbb{P}))\) for all \(k \geq 0\).
3. \((T(\mathbb{P}))^o) \cap (T(\mathbb{P}), W) = (T^{-1})^t((\mathbb{P})^o \cap (\mathbb{P}, V))\) where \(T^t\) denotes a transpose of \(T\).
4. For any set \(B \subset V\), we have \(T(\text{Ch}(B)) = \text{Ch}(T(B))\).

**Proof.** Our proof is based on

\[(4.41) \quad \langle (T^{-1})^t \mathbf{p}, T(x) \rangle_W = \langle \mathbf{p}, T^{-1}T(x) \rangle_V = \langle \mathbf{p}, x \rangle_V \quad \text{for } \mathbf{p}, x \in V.
\]
By (4.41),
\[ T(\pi_{q_j,r_j}) = \{ T(x) : \langle q_j, x \rangle = r_j \} = \pi_{(T^{-1})^t(q_j),r_j} \quad \text{and} \quad T(\pi^+_{q_j,r_j}) = \pi^+_{(T^{-1})^t(q_j),r_j}. \]
Thus \( T(\mathbb{P}) = \mathbb{P}(\Pi_T) \) is a polyhedron because of Definition 2.3 and
\[ T(\mathbb{P}) = T \left( \bigcap_{q_j, r_j} \pi^+_{q_j,r_j} \right) = \bigcap_{q_j, r_j} \pi^+_{(T^{-1})^t(q_j),r_j}. \]
Hence (1) is proved. If \( F \in F(\mathbb{P}), \) by (2.2),
\[ F = \pi_{q_j,r_j} \cap \mathbb{P} \quad \text{and} \quad \mathbb{P} \setminus F \subset (\pi^+_{q_j,r_j})^c. \]
So,
\[ T(F) = \pi_{(T^{-1})^t(q_j),r_j} \cap T(\mathbb{P}) \quad \text{and} \quad T(\mathbb{P}) \setminus T(F) = T(\mathbb{P} \setminus F) \subset (\pi^+_{(T^{-1})^t(q_j),r_j})^c. \]
This means \( T(F) \in F(T(\mathbb{P})). \) Moreover \( V(F) \) and \( V(T(F)) \) are isomorphic. Hence \( T(F) \in F^k(T(\mathbb{P})). \) So (2) is proved. Next, (4.41) yields that
\[ (T(F)^*\circ)(T(\mathbb{P}),W) = \{ q \in W : \exists \rho \quad \text{such that} \quad \langle q, T(x) \rangle_W = \rho < \langle q, T(y) \rangle_W \quad \text{for all} \quad x \in F, \ y \in \mathbb{P} \setminus F \} = \{ (T^{-1})^t(p) \quad \text{with} \quad p \in V : \exists \rho \quad \text{such that} \quad \langle p, x \rangle_V = \rho < \langle p, y \rangle_V \quad \text{for all} \quad x \in F, \ y \in \mathbb{P} \setminus F \} = (T^{-1})^t((F^*)\circ)(\mathbb{P},V)). \]
This proves (3). Finally,
\[ T(\text{Ch}(B)) = \left\{ T\left( \sum_{j=1}^{N} c_j x_j \right) : x_j \in B \quad \text{and} \quad \sum_{j=1}^{N} c_j = 1 \quad \text{with} \quad c_j \geq 0 \right\} \]
\[ = \left\{ \sum_{j=1}^{N} c_j T(x_j) : T(x_j) \in T(B) \quad \text{and} \quad \sum_{j=1}^{N} c_j = 1 \quad \text{with} \quad c_j \geq 0 \right\} \]
\[ = \text{Ch}(T(B)) \]
which proves (4). \( \square \)

5. DESCENDING FACES V.S. ASCENDING CONES

By the cone decomposition in Proposition 4.3 and Remark 4.5, we shall prove that for each \( F \in F(\tilde{\mathbb{N}}(\Lambda, S)), \)
\[ \sup_{\xi \in \mathbb{R}^d} \sum_{J \in \text{Cap}(F^*)} |I_J(P_{\Lambda},\xi)| \leq C \quad \text{where} \quad \text{Cap}(F^*) = \bigcap_{\nu=1}^{d} F^*_\nu. \]
Suppose that we are given a face $F = (F_{\nu}) \in \mathcal{F}(P)$ with $P = (P_{\nu})$ where $P_{\nu} = N(\Lambda_{\nu}, S)$. 

To establish (5.1), as we have planned in (3.12), (3.13) and (2.11), we shall choose an appropriate descending chain \( \{F(s) : s = 0, \ldots, N\} \) in $\mathcal{F}(P)$ such that \( P = F(0) \succeq \cdots \succeq F(s) \succeq \cdots \succeq F(N) = F(\nu(s - 1) \succeq F\nu(s) \) for each $\nu$).

We shall make the estimates:

\[
\sum_{J \in \text{Cap}(F^*)} \left| \mathcal{I}_J(P_{F(s-1)}, \xi) - \mathcal{I}_J(P_{F(s)}, \xi) \right| \leq C \quad \text{for } s = 1, \ldots, N. \tag{5.3}
\]

To perform this estimates successfully, we need to have the full rank condition for applying Proposition 6.1:

\[
\text{rank} \left( \bigcup_{\nu=1}^{d} F_{\nu}(s - 1) \right) = n. \tag{5.4}
\]

Without the full rank condition, we need to have the overlapping condition for applying Proposition 3.1:

\[
\text{Cap}(F^*(s)^\circ) = \bigcap_{\nu=1}^{d} (F_{\nu}(s))^{\circ} \neq \emptyset. \tag{5.5}
\]

The following technical difficulty arises for each (5.4) and (5.5).

**Difficulty satisfying overlapping property (5.5).** By Lemma 4.2, we see that

\[
\text{Cap}(F^*(s-1)^\circ) \neq \emptyset \Rightarrow \text{Cap}(F^*(s)^\circ) \neq \emptyset \quad \text{whenever } F_{\nu}(s - 1) \succeq F_{\nu}(s) \text{ for all } \nu.
\]

However, $\text{Cap}(F^*(s-1)^\circ) \neq \emptyset \Rightarrow \text{Cap}(F^*(s)^\circ) \neq \emptyset$ is not always true even if $F_{\nu}(s - 1) \succeq F_{\nu}(s)$ for all $\nu$. To keep (5.5), we construct (5.2) in Definition 5.2 so that $\text{Cap}(F^*(s)^\circ)$ for every $s = 1, \ldots, N$ contains some common portion of $\text{Cap}(F^*)$ in (5.6). For this purpose, we facilitate the concept of the essential faces defined in Definition 4.3.

**Difficulty satisfying the full rank condition (5.4).** Even if we have (5.4), we might have

\[
\text{rank} \left( \bigcup_{\nu=1}^{d} F_{\nu}(s - 1) \cap \Lambda_\nu \right) \leq n - 1.
\]

For this case, in order to satisfy (6.5) in Proposition 6.1, $|\mathcal{I}_J(P_{F(s-1)}, \xi) - \mathcal{I}_J(P_{F(s)}, \xi)|$ in (5.3) must be dominated by $|2^{-J^m}\xi_\nu|^c$ not only with $m \in F_{\nu}(s - 1) \cap \Lambda_\nu$ exponents of
polyynomial $P_\Lambda$, but also with $m \in \mathbb{F}_\nu(s - 1)$ not exponents of that polynomial. To fulfill this requirement, we shall make an efficient size control tool for
\[
\left\{ 2^{-\sum_{j=1}^N J_n m_j} : m \in \mathbb{F}_\nu(s) \right\}_{s=1}^N \text{ with } J \in \text{Cap}(\mathbb{F}^*) \text{ fixed},
\]
in Proposition 5.2.

5.1. Construction of Descending Faces and Ascending Cones. Given a a face $F = (F_\nu) \in \mathcal{F}(\mathbb{P})$, an intersection $\bigcap_{\nu=1}^d F_\nu^*$ of cones is itself a cone type polyhedron. Thus there exist $p_1, \ldots, p_N$ in $\bigcap_{\nu=1}^d F_\nu^*$ forming
\[
(5.6) \quad \text{Cap}(\mathbb{F}^*) = \bigcap_{\nu=1}^d F_\nu^* = \text{CoSp}(p_1, \ldots, p_N).
\]
In order to show (7.4) and (7.5), we first split Cap($\mathbb{F}^*$) as
\[
\text{Cap}(\mathbb{F}^*) = \bigcup \text{Cap}(\mathbb{F}^*) (\sigma)
\]
where union is over all permutations $\sigma : \{1, \ldots, N\} \to \{1, \ldots, N\}$ and
\[
\text{Cap}(\mathbb{F}^*) (\sigma) = \{ \alpha_1 p_1 + \cdots + \alpha_N p_N \in \text{Cap}(\mathbb{F}^*) : \alpha_{\sigma(1)} \geq \alpha_{\sigma(2)} \geq \cdots \geq \alpha_{\sigma(N)} \geq 0 \}.
\]
To prove (7.4), it suffices to show for each $\sigma$,
\[
\sum_{J \in \text{Cap}(\mathbb{F}^*) (\sigma)} |\mathcal{I}_J(P_\Lambda, \xi)| \leq C_2.
\]
Since the order of $p_1, \ldots, p_N$ is random, it suffices to work with only $\sigma = \text{id}$ where
\[
(5.7) \quad \text{Cap}(\mathbb{F}^*) (\text{id}) = \{ \alpha_1 p_1 + \cdots + \alpha_N p_N \in \text{Cap}(\mathbb{F}^*) : \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_N \geq 0 \}.
\]

Definition 5.1. [Intersection of Cones] Let $\mathbb{P} = (\mathbb{P}_\nu)$ where $\mathbb{P}_\nu = \mathbb{P}(\Pi^\nu)$ is a polyhedron in $\mathbb{R}^n$ and $\dim(\mathbb{P}_\nu) = \dim(V(\mathbb{P}_\nu)) = m_\nu \leq n$. Suppose that $\Pi^\nu = \Pi^\nu_a \cup \Pi^\nu_b$ where $\Pi^\nu_a = \{ q_j^\nu \}_{j=1}^{L_\nu}$ is a generator for $\mathbb{V}_a(\mathbb{P}_\nu)$ in $V_{am}(\mathbb{P})$, and $\Pi^\nu_b = \{ \pm n_i^\nu \}_{i=1}^{n-m_\nu} \mathbb{F}_\nu$ is a generator for $V_{am}(\mathbb{P}_\nu)$ in $\mathbb{R}^n$ as in Lemma 4.3. By Propositions 4.1 and 4.2 with Remark 4.4, a face $F_\nu$ having an expression:
\[
\mathbb{F}_\nu = \bigcap_{j=1}^{N_\nu} \pi_{p_{\nu j}} \cap \mathbb{P}_\nu \text{ where } \Pi(\mathbb{F}_\nu) = \{ p_{\nu j}^\nu \}_{j=1}^{N_\nu} = \{ q_j^\nu \}_{j=1}^{L_\nu} \cup \{ \pm n_i^\nu \}_{i=1}^{n-m_\nu}
\]
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has its dual face (cone) of the form:

\[ F_\nu^* = \text{CoSp} \left( \{ p_\nu^j \}_{j=1}^{N_\nu} \right) \text{ where } \Pi(F_\nu) = \{ p_\nu^j \}_{j=1}^{N_\nu}. \]

Here we remind that

\[ \text{CoSp} \left( \{ \pm n_\nu^i \}_{i=1}^{n-m_\nu} \right) = V_{\perp}(P_\nu) \quad \text{and} \quad \{ q_\nu^j \}_{j=1}^{l_\nu} \subset V(P_\nu). \]

(5.8)

Lemma 5.1. In proving (5.1), we may assume that

\[ \text{Cap}(F^*) \cap (F^*_\nu)^o \neq \emptyset \quad \text{for all } \nu. \]

Proof. If the cone \( \text{Cap}(F^*) \) is given by \( \bigcap_{\nu=1}^{d} F_\nu^* = \{0\} \), the proof of (5.1) is done since there is only one term \( J = 0 \) in the summation of (5.1). Thus we assume that the cone \( \text{Cap}(F^*) \) is not \( \{0\} \), that is, \( \bigcap_{\nu=1}^{d} F_\nu^* \cap (F^*_1)^o \neq \emptyset \).

(5.9)

Let \( \left( \bigcap_{\nu=1}^{d} F_\nu^* \right) \cap (F^*_1)^o = \emptyset \), say \( \nu = 1 \). Then from \( \bigcap_{\nu=1}^{d} F_\nu^* \subset (F^*_1) \) and Definitions 2.9 and 2.10, we have \( \bigcap_{\nu=1}^{d} F_\nu^* \subset \partial F^*_1 \). By Lemma 4.6 there exists \( F^*_{1,1} \nsubseteq F^*_1 \) (so \( F_1 \nsubseteq F_{1,1} \)) such that \( \bigcap_{\nu=1}^{d} F_\nu^* \subset F^*_1 \). So we replace \( F^*_1 \) in \( \bigcap_{\nu=1}^{d} F_\nu^* \) by \( F^*_1 \) with keeping (5.10). If \( \left( \bigcap_{\nu=1}^{d} F_\nu^* \right) \cap (F^*_1)^o \neq \emptyset \) where \( F_1 = F_{1,1} \), we stop. Otherwise \( \left( \bigcap_{\nu=1}^{d} F_\nu^* \right) \cap (F^*_1)^o = \emptyset \) where \( F_1 = F_{1,1} \), we repeat this process until we have \( \left( \bigcap_{\nu=1}^{d} F_\nu^* \right) \cap (F^*_1)^o \neq \emptyset \) satisfying (5.10) where \( F_1 \) is taken as new \( F^*_{1,k} \) such that

\[ F^*_{1,k} \nsubseteq \cdots \nsubseteq F^*_{1,1} \nsubseteq F^*_1 \quad (F_1 \nsubseteq F_{1,2} \nsubseteq \cdots \nsubseteq F_{1,k}). \]

Assume we arrive at the final round with \( F_{1,k} = P_1 \) in (5.10). By (5.10) and Remark 4.4 that tells \( (P^*_1)^o = P^*_1 \setminus \{0\} \),

\[ \left( P^*_1 \cap \left( \bigcap_{\nu=2}^{d} F_\nu^* \right) \right) \cap (P^*_1)^o = \left( P^*_1 \cap \left( \bigcap_{\nu=2}^{d} F_\nu^* \right) \right) \cap P^*_1 \setminus \{0\} \neq \emptyset. \]

Hence our process ends up with \( \left( \bigcap_{\nu=1}^{d} F_\nu^* \right) \cap (F^*_1)^o \neq \emptyset \) satisfying (5.10) where \( F_1 \) is taken possibly as a face between the original \( F_1 \) and the entire \( P_1 \). By repeating the same argument to \( \nu = 2, \ldots, d \), we finish the proof. \( \square \)
Remark 5.1. Lemma 5.1 combined with Lemma 4.21 tells us $\text{Cap}(\mathbb{F}^*)$ is an essential part of $\mathbb{F}^*_\nu$ in the sense that $F(\text{Cap}(\mathbb{F}^*))|\mathbb{F}^*_\nu = \mathbb{F}^*_\nu$. This is used for proving Lemma 5.3.

Definition 5.2. [Essential Cone] Suppose that polyhedrons $P = (P_\nu)$ and faces $F = (F_\nu)$ are given as in Definition 5.1. Fix the order of $\{p_j\}_{j=1}^N$ in (5.6) and note that $\{\pm n_\nu i\}_{i=1}^{n-m_\nu} \subset F_\nu^*$. Then for each $\nu = 1, \cdots, d$ and $s = 0, 1, \cdots, N$, define a cone:

$$C_\nu(s) = \text{CoSp} (\{\pm n_\nu i\}_{i=1}^{n-m_\nu} \cup \{p_j\}_{j=1}^s) \subset \mathbb{F}^*_\nu$$

where from (5.9) and (5.6)

$$C_\nu(0) = \text{CoSp} (\{\pm n_\nu i\}_{i=1}^{n-m_\nu}) = V^\perp (P_\nu),$$

$$C_\nu(N) = \text{CoSp} (\{\pm n_\nu i\}_{i=1}^{n-m_\nu} \cup \{p_j\}_{j=1}^N) \supset \text{CoSp} (\{p_j\}_{j=1}^N) = \text{Cap}(\mathbb{F}^*).$$

In view of Definition 4.3, we define for each $s = 0, \cdots, N$ and $\nu = 1, \cdots, d$, the smallest face of $\mathbb{F}^*_\nu$ containing $C_\nu(s)$ by

$$F (F_\nu(C_\nu(s)|F^*_\nu)\),$$

and call it the essential cone of $\mathbb{F}^*_\nu$ containing $C_\nu(s)$. See the third picture of Figure 4.3.

Lemma 5.2. Suppose that the $F (F_\nu(C_\nu(s)|F^*_\nu))$ are defined above. Then for $s = 1, \cdots, N$,

$$\bigcap_{\nu=1}^d (F (F_\nu(C_\nu(s)|F^*_\nu)) )^\circ \neq \emptyset.$$

Proof. Fix $s \in \{1, \cdots, N\}$. By $C_\nu(s) = \text{CoSp} (\{\pm n_\nu i\}_{i=1}^{n-m_\nu} \cup \{p_j\}_{j=1}^s)$ in (5.11),

$$p_1 + \cdots + p_s = \sum_{j=1}^s p_j + \left(\sum_{i=1}^{n-m_\nu} n_\nu i + \sum_{i=1}^{n-m_\nu} -n_\nu i\right) \in C_\nu(s)^\circ$$

for each $\nu = 1, \cdots, d$. By Lemma 4.23,

$$C_\nu(s)^\circ \subset F (F_\nu(C_\nu(s)|F^*_\nu)) ^\circ.$$

By (5.15) and (5.16), we have $p_1 + \cdots + p_s \in \bigcap_{\nu=1}^d (F (F_\nu(C_\nu(s)|F^*_\nu)) )^\circ$ for $s \geq 1$. 

Lemma 5.3. For each $\nu = 1, \cdots, d$,

$$F (C_\nu(N)|F^*_\nu) = \mathbb{F}^*_\nu \text{ and } F (C_\nu(0)|F^*_\nu) = V^\perp (P_\nu).$$
Proof. The first identity follows from Lemmas 4.21 and 5.1 together with (5.13). The second follows from (5.12) with $V^\perp(P_\nu) = P_\nu^*$ in Remark 4.4 and $P_\nu^* \preceq F_\nu^*$ in Lemma 4.2. □

Since $F(C_\nu(s)|P_\nu^*)$ is a face of a cone $F_\nu^* = \text{CoSp}([p_\nu^*_j]_{j=1}^{N_\nu})$, we see that $F(C_\nu(s)|P_\nu^*)$ itself is also a cone expressed as:

$$(5.17) F(C_\nu(s)|P_\nu^*) = \text{CoSp}([p_\nu^*_j]_{j=1}^{N_\nu})$$

By $C_\nu(s-1) \subset C_\nu(s)$ in (5.11), the ascending condition holds:

$$(5.18) F(C_\nu(s-1)|P_\nu^*) \preceq F(C_\nu(s)|P_\nu^*).$$

Thus, we can make

$$(5.19) B_\nu^{s-1} \subset B_\nu^s \text{ in (5.17).}$$

For the case $s = 0, N$, by Lemma 5.3 with (5.8) and (5.9),

$$(5.20) F(C_\nu(N)|P_\nu^*) = \text{CoSp}([p_\nu^*_j]_{j=1}^{N_\nu}) \text{ and } F(C_\nu(0)|P_\nu^*) = \text{CoSp}(\pm n_\nu^*)_{i=1}^{n-m_\nu}.$$ 

Definition 5.3. By (5.17) and (3) of Lemma 4.1, we assign to each $F(C_\nu(s)|P_\nu^*)$, a face $F_\nu(s)$ of $P_\nu$ defined by

$$(5.21) F_\nu(s) = \bigcap_{j \in B_\nu^s} \left( \pi_{p_\nu^*_j} \cap P_\nu \right).$$

By (5.20),

$$(5.22) F_\nu(0) = \bigcap_{i=1}^{n-m_\nu} \left( \pi_{\pm n_\nu^*} \cap P_\nu \right) = P_\nu \text{ and } F_\nu(N) = \bigcap_{j \in B_\nu^N} \left( \pi_{p_\nu^*_j} \cap P_\nu \right) = F_\nu.$$

Definition 5.4. We denote the dual face (cone) of $F_\nu(s)$ by $F_\nu^*(s)$ as in Definition 2.11.

Remark 5.2. If $\Pi(F_\nu(s)) = [p_\nu^*_j]_{j \in B_\nu^s}$, then $F_\nu^*(s)$ and $F(C_\nu(s)|P_\nu^*)$ coincide because $F_\nu^*(s) = \text{CoSp}([p_\nu^*_j]_{j \in B_\nu^s}) = F(C_\nu(s)|P_\nu^*)$ by (5.17) and Propositions 4.2. But it might happen that $[p_\nu^*_j]_{j \in B_\nu^s} \subset \Pi(F_\nu(s))$, though $\Pi(F_\nu(s))$ is the maximal set (full generator) satisfying (5.21) by Propositions 4.1 and Remark 4.1.

In general, we have
Lemma 5.4. For $s = 0, \cdots, N$,

\[(5.23) \quad F(C_\nu(s)|F_\nu^*) \subseteq F_\nu(s) \quad \text{and} \quad F(C_\nu(s)|F_\nu^*)^\circ \subseteq (F_\nu(s))^\circ\]

and for $s = 0, N$,

\[(5.24) \quad F(C_\nu(0)|F_\nu^*) = F_\nu(0) \quad \text{and} \quad F(C_\nu(N)|F_\nu^*) = F_\nu(N).\]

Proof. Let $m \in F_\nu \setminus F_\nu(s)$ and $m_0 \in F_\nu(s) = \bigcap_{j \in B_\nu^s} (\pi_{\nu_j} \cap F_\nu)$. Then there exists at least one $k \in B_\nu^s$ such that $m \notin \pi_{\nu_j} \cap F_\nu$, which together with $\{p_{\nu_j}\}_{j = 1}^N = \Pi(F_\nu) \subseteq \Pi(F_\nu)$ in (5.17) implies that

\[p_{\nu_j} \cdot (m - m_0) > 0 \quad \text{and} \quad p_{\nu_j} \cdot (m - m_0) \geq 0 \quad \text{for all} \quad j \in B_\nu^s.\]

Let $q = \sum c_j p_{\nu_j} \in F(C_\nu(N)|F_\nu^*) = \text{CoSp}(p_{\nu_j}^\nu)_{j \in B_\nu^s}$ with $c_j \geq 0$ for all $j$. Then $q \cdot (m - m_0) \geq 0$. Thus $q \in F_\nu^*(s)$, which proves the first inequality of (5.23). The second inequality of (5.23) follows from the condition $c_j > 0$ for all $j$ in $q = \sum c_j p_{\nu_j} \in F(C_\nu(N)|F_\nu^*)^\circ$. Lastly, (5.24) follow from Lemma 5.3 and (5.22).

\[\square\]

Lemma 5.5. Suppose that the $F(C_\nu(s)|F_\nu^*)$ are defined above. Then for $s = 1, \cdots, N$,

\[\bigcap_{\nu=1}^d (F_\nu(s))^\circ \neq \emptyset.\]

Proof. It follows from Lemmas 5.2 and 5.4.

\[\square\]

Remark 5.3. Lemma 5.5 does not hold for the case $s = 0$. When we deal with this case $(F_\nu(0))_{\nu=1}^d = (N(A_\nu, S))_{\nu=1}^d$ in Proposition 7.1, we shall not use the above overlapping condition of Lemma 5.5.

Proposition 5.1. Suppose for each $\nu = 1, \cdots, d$ we have the ascending cones in (5.18):

\[V^\perp(P_\nu) = F(C_\nu(0)|F_\nu^*) \leq F(C_\nu(1)|F_\nu^*) \leq \cdots \leq F(C_\nu(N)|F_\nu^*) = F_\nu^*.\]

Then, we have a descending sequence $\{F_\nu(s)\}_{s=0}^N$ of faces and an ascending sequence of dual faces $\{F_\nu^*(s)\}_{s=0}^N$:

\[(5.25) \quad P_\nu = F_\nu(0) \geq F_\nu(1) \geq \cdots \geq F_\nu(N) = F_\nu,\]

\[(5.26) \quad V^\perp(P_\nu) = F_\nu^*(0) \leq F_\nu^*(1) \leq \cdots \leq F_\nu^*(N) = F_\nu^*.\]
Proof. By (5.19) and (5.21), we have
\[ F_\nu(s - 1) = \bigcap_{j \in B_\nu} \left( \pi p_j \cap P_\nu \right) \supset \bigcap_{j \in B_\nu} \left( \pi p_j \cap P_\nu \right) = F_\nu(s). \]
This and (1) of Lemma 4.1 yield (5.25). This and Lemma 4.2 yield (5.26). The cases \( s = 0, N \) are in Lemma 5.3 and (5.22). \( \square \)

5.2. Size Control Number. Before showing
\[ \sum_{J \in \text{Cap}(\mathbb{F}^*)} |I_J(P(F(s - 1), \xi) - I_J(P(F(s), \xi))| \leq C \]
in Section 7, we shall investigate the size of \( 2^{-Jm} \) with \( m \in F_\nu(s - 1) \setminus F_\nu(s) \) and
\[ J \in \text{Cap}(\mathbb{F}^*)(id) = \left\{ J = \sum_{j=1}^N \alpha_j p_j : \alpha_1 \geq \cdots \geq \alpha_s \geq \cdots \geq \alpha_N \geq 0 \right\}. \]
We assert in Proposition 5.2 that \( \alpha_s (s = 1, \cdots, N) \) is the key number controlling sizes:
\[ 2^{-C_2\alpha_s} \leq \frac{2^{-mJ}}{2^{-mJ}} = \frac{\text{Effect of Mean Value Property}}{\text{Effect of Decay Property}} \leq 2^{-C_1\alpha_s} \]
where \( m \in F_\nu(s - 1) \setminus F_\nu(s) \) and \( \tilde{m} \in F_\nu \). Here \( C_1, C_2 > 0 \) are independent of \( J \).

Definition 5.5. Given a cone \( \text{CoSp}(p_1, \cdots, p_N) \), its \( r \)-neighborhood is defined by
\[ D_r(\text{CoSp}(p_1, \cdots, p_N)) = \left\{ \sum_{j=1}^N c_j p_j \in \mathbb{F}^* : c_j > r > 0 \right\}. \]
We shall use the following three lemmas to prove (5.27) in Proposition 5.2.

Lemma 5.6. Suppose that \( \text{CoSp}(p_1, \cdots, p_k)^o \subset \text{CoSp}(q_1, \cdots, q_N)^o \). Then there exists \( c > 0 \) depending only on \( p_i, q_j \) with \( 1 \leq i \leq k \) and \( 1 \leq j \leq N \) such that
\[ D_r(\text{CoSp}(p_1, \cdots, p_k)) \subset D_{cr}(\text{CoSp}(q_1, \cdots, q_N)). \]
Proof. From \( p_1 + \cdots + p_k \in \text{CoSp}(p_1, \cdots, p_k)^o \subset \text{CoSp}(q_1, \cdots, q_N)^o \), we see that
\[ p_1 + \cdots + p_k = \sum_{j=1}^N c_j q_j \text{ where } c_j > 2c \text{ with } c \text{ depending on } p_i \text{'s.} \]
Let $p \in \mathcal{D}_r(\text{CoSp}(p_1, \cdots, p_k))$. By using (5.28), we split $p$ into two parts

\begin{equation}
(5.29) \quad p = \sum_{j=1}^{k} \alpha_j p_j = \sum_{j=1}^{k} \left( \alpha_j - \frac{r}{2} \right) p_j + \frac{r}{2} \sum_{j=1}^{N} c_j q_j \quad \text{where } \alpha_j > r.
\end{equation}

Since $\alpha_j - r/2 \geq r/2 > 0$, the first term on the right hand side of (5.29) is

$$
\sum_{j=1}^{k} \left( \alpha_j - \frac{r}{2} \right) p_j \in \text{CoSp}(p_1, \cdots, p_k) \subset \text{CoSp}(q_1, \cdots, q_N).
$$

The second term on the right hand side of (5.29) is

$$
\frac{r}{2} \sum_{j=1}^{N} c_j q_j \in \mathcal{D}_{cr}(\text{CoSp}(q_1, \cdots, q_N))
$$

because $(r/2)c_j \geq rc$. So, $p \in \mathcal{D}_{cr}(\text{CoSp}(q_1, \cdots, q_N))$. \hfill \Box

**Lemma 5.7.** Let $\mathbb{P}$ be a polyhedron and let $F$ be an proper face of $\mathbb{P}$. Suppose that $\tilde{m} \in F$ and $m \in \mathbb{P} \setminus F$. Then for all $p \in \mathcal{D}_r(F^*)$,

$$
p \cdot (m - \tilde{m}) \geq c > 0 \quad \text{where } c \text{ depends on } r, m, \tilde{m}.
$$

**Remark 5.4.** This lemma is needed only when $F \not\subseteq \mathbb{P}$ for the same reason in Remark 4.6. The proof of this lemma is also similar to that of Lemma 4.19.

**Proof.** Let $\Pi(F) = \{p_j\}^{N}_{j=1} = \{q_j\}^{\ell}_{j=1} \cup \{\pm n_i\}^{n-m}_{i=1}$ where $\{q_j\}^{\ell}_{j=1} = \Pi_a(F) \subset \Pi_a$ and $\{\pm n_i\}^{n-m}_{i=1} = \Pi_b$ so that

\begin{equation}
(5.30) \quad F^*|\mathbb{P} = \text{CoSp}(\{q_j\}^{\ell}_{j=1} \cup \{\pm n_i\}^{n-m}_{i=1}).
\end{equation}

By (4.18),

$$
F = \bigcap_{j=1}^{\ell} F_j \quad \text{with } F_j = \pi_{q_j} \cap \mathbb{P}.
$$

Since $\tilde{m} \in F$ and $m \in \mathbb{P} \setminus F$,

\begin{equation}
(5.31) \quad m \in \mathbb{P} \setminus F_k \quad \text{for some } k \in \{1, \cdots, \ell\} \text{ and } \tilde{m} \in F \subset \mathbb{F}_k.
\end{equation}

Thus by (5.31) and (2.5) in Definition 2.11,

\begin{equation}
(5.32) \quad q_k \cdot (m - \tilde{m}) > \eta > 0 \quad \text{and } q_j \cdot (m - \tilde{m}) \geq 0 \quad \text{for } j = 1, \cdots, \ell.
\end{equation}
where $\eta$ depends on $m, \tilde{m}$. Let $p \in \mathcal{D}_r(\mathbb{F}^*)$. Then $p = \sum_{j=1}^{\ell} c_j q_j + r$ where $c_j \geq r$ and $r \in V^+(\mathbb{F})$ according to Definition 5.5 and (5.30). Thus, by using (5.32) and $r \cdot (m - \tilde{m}) = 0$,

$$p \cdot (m - \tilde{m}) = \sum_{j=1}^{\ell} c_j q_j \cdot (m - \tilde{m})$$

$$= c_k q_k \cdot (m - \tilde{m}) + \sum_{j=1, j \neq k}^{\ell} c_j q_j \cdot (m - \tilde{m})$$

$$\geq c_k q_k \cdot (m - \tilde{m}) + 0 \geq c_k q_k \cdot (m - \tilde{m}) + 0 \geq r\eta > 0$$

where $c = r\eta$ is depending on $r, m, \tilde{m}$ and independent of $p$.

**Lemma 5.8.** Let $F = (F_\nu) \in F(\mathbb{N}(\Lambda, S))$ and let $F_\nu(s)$ where $s = 1, \cdots, N$ be defined as in Definition 5.2. Then for each $s = 1, \cdots, N$, every vector $p = \sum_{j=1}^{N} \alpha_j p_j \in \text{Cap}(\mathbb{F}^*)(\text{id}) = \left\{ \sum_{j=1}^{N} \alpha_j p_j : \alpha_1 \geq \cdots \geq \alpha_N \geq 0 \right\}$ defined in (5.7) is expressed as

$$p = r_1(s) + r_2(s) + r_3(s)$$

where for each $\nu = 1, \cdots, d$,

1. $r_1(s) \in F_\nu(s-1)$,
2. $r_2(s) = \alpha_s u(s)$ where $u(s) \in \mathcal{D}_r(F_\nu(s))$ for $r > 0$ independent of $\alpha_1, \cdots, \alpha_N$,
3. $r_3(s) = \alpha_{s+1} p_{s+1} + \cdots + \alpha_N p_N \in F_\nu(N)$.

**Proof.** We express $p = \sum_{j=1}^{N} \alpha_j p_j$ as $r_1(s) + r_2(s) + r_3(s)$ where

$$r_1(s) = (\alpha_1 - \alpha_s) p_1 + \cdots + (\alpha_{s-1} - \alpha_s) p_{s-1},$$

$$r_2(s) = \alpha_s (p_1 + \cdots + p_s),$$

$$r_3(s) = \alpha_{s+1} p_{s+1} + \cdots + \alpha_N p_N.$$

By (5.11), (5.23) and (5.33),

$$r_1(s) \in F_\nu(s-1)$$

proving (1). We next show (2). By $C_\nu(s) = \text{CoSp} \left( \{ \pm n_i^{\nu} \}_{i=1}^{n-m-\nu} \cup \{ p_j \}_{j=1}^{s} \right)$ in (5.11),

$$p_1 + \cdots + p_s = \sum_{j=1}^{s} p_j + \left( \sum_{i=1}^{n-m} n_i^{\nu} + \sum_{i=1}^{n-m} -n_i^{\nu} \right) \in \mathcal{D}_t(C_\nu(s)) \text{ for } t = 1.$$
By (5.17) and (5.23),
\[(5.37)\quad \text{CoSp}(\{p_j^\nu\}_{j \in B^\nu}) = F(C^\nu(s)|F^*) \subset F^\nu_\nu(s).\]

By Lemmas 4.23,
\[(5.38)\quad C^\nu(s)^\circ \subset F(C^\nu(s)|F^*\nu)^\circ = \text{CoSp}(\{p_j^\nu\}_{j \in B^\nu})^\circ.\]

By using (5.37),(5.38) and Lemmas 5.6,
\[D_t(C^\nu(s)) \subset D_{ct}(\text{CoSp}(\{p_j^\nu\}_{j \in B^\nu})) \subset D_{ct}(F^\nu_\nu(s))\]
for some \(c > 0\) depending on \(p_j^\nu\)'s. By this and (5.36), put
\[u(s) = p_1 + \cdots + p_s \in D_{ct}(F^\nu_\nu(s)).\]

Set \(r = ct > 0\). Note (2) follows from \(r_2(s) = \alpha_s u(s)\) in (5.34). Finally (3) follows from (5.11),(5.14),(5.23) and (5.35). \(\square\)

By Lemmas 5.7 and 5.8, we obtain the very useful proposition:

**Proposition 5.2.** Let \(F^\nu_\nu(s)\) and \(F^\nu_\nu(s)\) where \(\nu = 1, \cdots, d\) and \(s = 1, \cdots, N\) be defined as in Definition 5.3. Suppose that
\[p = \sum_{j=1}^N \alpha_j p_j \in \text{Cap}(F^*)(id) = \left\{ \sum_{j=1}^N \alpha_j p_j : \alpha_1 \geq \cdots \geq \alpha_N \geq 0 \right\},\]
and \(\tilde{m} \in F^\nu_\nu(N) = F^\nu_\nu\). Then for \(s = 1, \cdots, N\), there exist \(C_1, C_2 > 0\) such that
\[(5.39)\quad p \cdot (m - \tilde{m}) \geq C_1 \alpha_s \quad \text{for} \quad m \in F^\nu_\nu(s - 1) \setminus F^\nu_\nu(s),\]
\[(5.40)\quad p \cdot (n - \tilde{m}) \leq C_2 \alpha_s \quad \text{for} \quad n \in F^\nu_\nu(s - 1)\]
where \(C_1, C_2 > 0\) are independent of \(p \in \text{Cap}(F^*)(id)\), but may depend on \(m, \tilde{m}\) and \(n\).

**Proof.** By Lemma 5.8, we have \(p = r_1(s) + r_2(s) + r_3(s)\) satisfying (1),(2) and (3). Since \(m \in F^\nu_\nu(s - 1) \setminus F^\nu_\nu(s)\) and \(\tilde{m} \in F^\nu_\nu(N) \subset F^\nu_\nu(s)\), the property (2) of Lemma 5.8 combined with Lemma 5.7 yields that \(u(s) \cdot (m - \tilde{m}) > c > 0\), that is
\[(5.41)\quad r_2(s) \cdot (m - \tilde{m}) \geq c \alpha_s \quad \text{where} \quad c \text{ is independent of } p\.]
Since \( r_1(s) + r_3(s) \in \mathbb{F}_\nu^*(N) \) where \( \mathbb{F}_\nu^*(1) \subseteq \mathbb{F}_\nu^*(N) \) and \( \tilde{m} \in \mathbb{F}_\nu(N) \),
\[
(5.42) \quad (r_1(s) + r_3(s)) \cdot (m - \tilde{m}) \geq 0.
\]
Thus (5.39) follows from (5.41) and (5.42). Finally, the property (1) of Lemma 5.8 together
with the fact \( n \in \mathbb{F}_\nu(s - 1) \) and \( \tilde{m} \in \mathbb{F}_\nu(N) \subseteq \mathbb{F}_\nu(s - 1) \) yields
\[
(5.40) \quad \alpha_s \geq \alpha_{s+1} \geq \cdots \geq \alpha_N \text{ in } (5.34) \text{ and } (5.35),
\]
which proves (5.40).

6. Preliminary Results of Analysis

In this section, we prove Proposition 6.1 by facilitating the finite type condition in the
same spirit of [6] and [21]. Proposition 6.1 and Proposition 3.1 are two basic \( L^p \) estimation
tools used for the proof of sufficiency parts of Main Theorems 3.1 and 3.2.

6.1. Preliminary Inequalities. Recall a cutoff function \( \psi \in C^\infty_c([-2, 2]) \) with \( 0 \leq \psi \leq 1 \) \( \text{and } \psi(u) = 1 \text{ for } |u| \leq 1/2 \). Put \( \eta(u) = \psi(u) - \psi(2u) \). Given an integer \( k \in \mathbb{Z} \) and
\( \alpha, \beta, \gamma \in \{1, \cdots, n\} \), we consider the measures \( A^\alpha_\beta \) and \( P^\gamma \) defined in terms of Fourier transforms
\[
(6.1) \quad (A^\alpha_\beta)^\gamma(\xi) = \psi \left( \frac{\xi_\alpha}{2^k \xi_\beta} \right), \quad (P^\gamma)^\alpha(\xi) = \eta \left( 2^k \xi_\gamma \right).
\]

**Lemma 6.1.** Suppose that \( \{m_k\}_{k=1}^M, \{q_j\}_{j=1}^N \subset \mathbb{Z}^n \) where \( \text{rank } \left[ \{q_j\}_{j=1}^N \right] = n \). Given
\( \alpha_k, \beta_k, \gamma_j \in \{1, \cdots, n\} \), define
\[
(6.2) \quad A_J = A^\alpha_1 \cdots A^\alpha_M, \quad P_J = P^\gamma_1 \cdots P^\gamma_N
\]
for each \( J \in \mathbb{Z}^n \). Then for \( 1 < p < \infty \),
\[
(6.3) \quad \left\| \left( \sum_{J \in \mathbb{Z}^n} |P_J * f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)},
\]
and

\[ (6.4) \quad \left\| \left( \sum_{J \in \mathbb{Z}^n} |A_J \ast P_J \ast f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}. \]

Proof. It suffices to deal with the sum over \( \mathbb{Z}^n_+ \). We show (6.4). With \((r_J(t))\) denoting the Rademacher functions of product form,

\[
\left\| \left( \sum_{J \in \mathbb{Z}^n_+} |A_J \ast P_J \ast f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \approx \int_U \left\| \sum_{J \in \mathbb{Z}^n_+} r_J(t) A_J \ast P_J \ast f \right\|_{L^p(\mathbb{R}^d)} dt
\]

where \( U = [0,1]^n \). Consider the symbol

\[ m(\xi) = \left( \sum_{J \in \mathbb{Z}^n_+} r_J(t) A_J \right) \hat{\cdot}(\xi). \]

By using the full rank condition for the \( \alpha_j \) and the support conditions, we have that \( m \) satisfies

\[ |\partial_{\nu_1} \cdots \partial_{\nu_\ell} m(\xi_1, \cdots, \xi_d)| \leq C_{\ell} \frac{1}{|\xi_{\nu_1}| \cdots |\xi_{\nu_\ell}|} \]

for every \( \ell = 1, \cdots, d \), where \( 1 \leq \nu_1 < \cdots < \nu_\ell \leq d \). Thus the desired conclusion follows by the multi-parameter Marcinkiewicz multiplier theorem. Next (6.3) follows similarly. \( \square \)

Lemma 6.2. Let \( (\sigma_J)_{J \in \mathbb{Z}^n} \) be a sequence of positive measures on \( \mathbb{R}^d \). Suppose that

(i) \( \left\| \sigma_J \ast f \right\|_{L^1(\mathbb{R}^d)} \leq C \|f\|_{L^1(\mathbb{R}^d)} \quad (J \in \mathbb{Z}^n) \) for every \( f \in L^1(\mathbb{R}) \),

(ii) \( \left\| \sup_{J \in \mathbb{Z}^n} |\sigma_J \ast f| \right\|_{L^{p_0}(\mathbb{R}^d)} \leq C \|f\|_{L^{p_0}(\mathbb{R}^d)} \) for every \( f \in L^{p_0}(\mathbb{R}) \)

for some \( 1 < p_0 \leq 2 \) where the above constant \( C \) is independent of \( J \). Then

\[ \left\| \left( \sum_{J \in \mathbb{Z}^n} |\sigma_J \ast f_J|^2 \right)^{1/2} \right\|_{L^{p_1}(\mathbb{R}^d)} \lesssim \left( \sum_{J \in \mathbb{Z}^n} |f_J|^2 \right)^{1/2} \left\| f_J \right\|_{L^{p_1}(\mathbb{R}^d)} \quad \text{for } (f_J) \in L^{p_1}(\ell^2(\mathbb{R}^d)) \]

where \( p_1 \) determined by \( 1/p_1 \leq 1/2 (1 + 1/p_0) \).

Proof. For \( 1 \leq p, q \leq \infty \), consider the operator \( T \) defined by \( T[(f_J)] = (\sigma_J \ast f_J) \) on the mixed-norm spaces \( L^p(\ell^q) \). The condition (i) implies that \( T \) maps \( L^1(\ell^1) \) boundedly into itself. The condition (ii) and the positivity of each \( \sigma_J \) imply that \( T \) maps \( L^{p_0}(\ell^\infty) \)
boundedly into itself. It follows from the vector-valued Riesz-Thorin interpolation theorem
that \( T \) maps \( L^p(\ell^2) \) boundedly into itself.

6.2. Basic \( L^p \) estimates.

**Proposition 6.1.** Let \( \{H_J\}_{J \in \mathbb{Z}^d} \) be a set of measures and let \( \hat{H}_J \) be the Fourier multiplier of \( H_J \). Suppose that

\[
\left| \hat{H}_J(\xi) \right| \leq C \min \left\{ \left| 2^{-J q_i} \xi_{\nu_i} \right|^{\delta_1}, \left| 2^{-J q_i} \xi_{\nu_i} \right|^{\delta_2} : i = 1, \ldots, N \right\}
\]

where

\[
\text{rank} \{q_i : i = 1, \ldots, N\} = n.
\]

Then for \( C_2 = C/(1 - 2^{-\min\{\delta_1, \delta_2\}/N})^N \) with \( C, \delta_1, \delta_2, N \) in (6.5),

\[
\sum_{J \in \mathbb{Z}^n} \left| \hat{H}_J(\xi) \right| \leq C_2
\]

which implies that for \( A_J \) of the form defined as in (6.2) and for any \( Z \subset \mathbb{Z}^n \),

\[
\left\| \sum_{J \in Z} H_J * A_J * f \right\|_{L^2(\mathbb{R}^d)} \leq C_2 \|f\|_{L^2(\mathbb{R}^d)}.
\]

Let \( 1 < p < \infty \). If

\[
\left\| \sup_{J \in Z} \left| H_J * f \right| \right\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)} \text{ for } f \in L^p(\mathbb{R}^d),
\]

then,

\[
\left\| \sum_{J \in Z} H_J * A_J * f \right\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)} \text{ for } f \in L^p(\mathbb{R}^d).
\]

**Proof of (6.7).** Let

\[
(P_{-J q_i - \ell_i})^\wedge(\xi) = \eta \left( 2^{-J q_i - \ell_i} \xi_{\nu_i} \right) \text{ for } i = 1, \ldots, N
\]

with a view to restricting frequency variables as

\[
2^{-J q_i} |\xi_{\nu_i}| \approx 2^{\ell_i} \text{ for } i = 1, \ldots, N.
\]
We use for (6.5) and (6.10) to obtain that
\[
N \prod_{i=1}^{N} \eta \left( 2^{-J \cdot q_i - \ell_i} \xi_{\nu_i} \right) \left| \hat{H}_J(\xi) \right| \leq C 2^{-b |L|} \quad \text{for} \quad b = \frac{\min\{\delta_1, \delta_2\}}{N} \quad \text{and} \quad C \quad \text{in (6.5)},
\]
where $|L| = \sum_{i=1}^{N} |\ell_i|$. Then by using positivity of $\eta$ and $\sum_{\ell_i \in \mathbb{Z}} \eta \left( 2^{-J \cdot q_i - \ell_i} \xi_{\nu_i} \right) = 1$,
\[
\sum_{|J| \leq R} \left| \hat{H}_J(\xi) \right| = \sum_{L \in \mathbb{Z}^N} \prod_{i=1}^{N} \eta \left( 2^{-J \cdot q_i - \ell_i} \xi_{\nu_i} \right) \sum_{|J| \leq R} \left| \hat{H}_J(\xi) \right| \leq \sum_{L \in \mathbb{Z}^N} C 2^{-b |L|} \leq C \left( 1 - 2^{-\min\{\delta_1, \delta_2\}/N} \right)^N.
\]
where the first inequality follows from (6.11) and the observation that for each fixed $\xi$,
there exists finitely many $J$ such that $\prod_{i=1}^{N} \eta \left( 2^{-J \cdot q_i - \ell_i} \xi_{\nu_i} \right) \neq 0$. We proved (6.7).

\[\square\]

Proof of (6.9). Define
\[
\mathcal{P}^q_{J,L} = P_{-J \cdot q_1 - \ell_1} \ast \cdots \ast P_{-J \cdot q_N - \ell_N}, \quad \text{where} \quad L = (\ell_i)_{i=1}^{N} \in \mathbb{Z}^N.
\]
We use the Littlewood-Paley decomposition for each $J \in \mathbb{Z}$:
\[
\sum_{L \in \mathbb{Z}^N} \mathcal{P}^q_{J,L} \ast f = f.
\]
Define $\tilde{\mathcal{P}}^q_{J,L}$ by
\[
\tilde{\mathcal{P}}^q_{J,L} = \left( \sum_{r_1=-10}^{10} P_{-J \cdot q_1 - \ell_1 + r_1} \right) \ast \cdots \ast \left( \sum_{r_N=-10}^{10} P_{-J \cdot q_N - \ell_N + r_N} \right).
\]
Then $\tilde{\mathcal{P}}^q_{J,L} \ast \mathcal{P}^q_{J,L} = \mathcal{P}^q_{J,L}$. Thus
\[
\sum_{J \in \mathbb{Z}} H_J \ast A_J \ast \mathcal{P}^q_{J,L} \ast f = \sum_{J \in \mathbb{Z}} \tilde{\mathcal{P}}^q_{J,L} \ast H_J \ast A_J \ast \mathcal{P}^q_{J,L} \ast f.
\]
By applying the dual inequality of (6.3) in Lemma 6.1,
\[
\left\| \sum_{J \in \mathbb{Z}} \tilde{\mathcal{P}}^q_{J,L} \ast H_J \ast A_J \ast \mathcal{P}^q_{J,L} \ast f \right\|_{L^p(\mathbb{R}^d)} \lesssim \left\| \left( \sum_{J \in \mathbb{Z}} H_J \ast A_J \ast \mathcal{P}^q_{J,L} \ast f \right)^2 \right\|_{L^p(\mathbb{R}^d)}^{1/2}.
\]
Thus, it is sufficient to find a constant \( b > 0 \) independent of \( L \in \mathbb{Z}^N \) such that

\[
(6.12) \quad \left\| \sum_{J \in \mathbb{Z}} |H_J * A_J * P_{q}^{q}_{J,L} * f|^2 \right\|_{L^p(\mathbb{R}^d)}^{1/2} \leq \tilde{C} 2^{-b|L|} \|f\|_{L^p(\mathbb{R}^d)},
\]

where \( \tilde{C} \) is a multiple of \( C \) in (6.5). By the rank condition (6.6) and (6.4) in Lemma 6.1,

\[
(6.13) \quad \left\| \sum_{J \in \mathbb{Z}} |A_J^0 * P_{q}^{q}_{J,L} * f|^2 \right\|_{L^p(\mathbb{R}^d)}^{1/2} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.
\]

For \( p = 2 \), we use (6.7), (6.11) and (6.13) to obtain (6.12). Applying a standard bootstrap argument combined with (6.8), Lemmas 6.1 and 6.2, we obtain (6.12) for the other values of \( p \neq 2 \). The proof of (6.9) is now complete. \( \square \)

**Lemma 6.3.** Let \( \mathcal{G} = (\mathcal{G}_\nu) \in \mathcal{F}(\mathbb{N}(\Lambda, S)) \). Let \( H_J = \mathcal{H}^P_{\nu} \) as in (2.7). Then the \( L^p \) boundedness of maximal operator in (6.8) always holds. If \( \Lambda_\nu \)’s are pairwise disjoint, then the decay condition in (6.5) always holds:

\[
(6.14) \quad |\mathcal{I}_J(P_\mathcal{G}, \xi)| \leq C \min \left\{ |2^{-J \cdot m_\nu} \xi_\nu|^{-\delta}, 1 : m_\nu \in \mathcal{G}_\nu \cap \Lambda_\nu \text{ for } \nu = 1, \ldots, d \right\}
\]

where \( \delta \geq \frac{1}{\text{deg}(P_\mathcal{G})} \) with \( \text{deg}(P_\mathcal{G}) = \max\{ |m| : m \in \Lambda_\nu \cap \mathcal{G}_\nu \text{ for } \nu = 1, \ldots, d \} \).

**Proof.** Apply Theorem 3.2 to obtain (6.8). Next use the multi-dimensional Van der Corput lemma (Proposition 5 VIII in [18]) for obtaining (6.5). \( \square \)

### 7. Proof of Sufficiency

**7.1. Statement of Sufficiency Theorem.** We shall prove the sufficient part of Main Theorem 3.2 by showing Theorem 7.1 below. Theorem 7.1 will also applies to prove the sufficient part of Theorem 3.1 in Section 11. Let \( \Lambda = (\Lambda_\nu)_{\nu=1}^d \) with \( \Lambda_\nu \subset \mathbb{Z}_+^n \) and \( S \subset N_n \). To each \( F \in \mathcal{F}(\mathbb{N}(\Lambda, S)) \) and \( J \in \mathbb{Z}^n \), we recall (2.6) and (2.7):

\[
(7.1) \quad \mathcal{I}_J(P_\mathcal{F}, \xi) = \int e^{i \left( \xi_1 \sum_{m \in \mathbb{F}_1 \cap \Lambda_1} c_1^{m_1} 2^{-J \cdot m_1} + \cdots + \xi_d \sum_{m \in \mathbb{F}_d \cap \Lambda_d} c_d^{m_d} 2^{-J \cdot m_d} \right)} \prod h(t_\nu) dt_1 \cdots dt_n
\]

where \( \mathcal{I}_J(P_{\mathbb{N}(\Lambda,S)}, \xi) = \mathcal{I}_J(P_\Lambda, \xi) \) and \( \mathcal{I}_J(P_\mathcal{F}, \xi) \) is the Fourier multiplier of the operator

\[
 f \to H^P_J * f.
\]
Theorem 7.1 (Sufficiency Theorem). Let $\Lambda = (\Lambda_\nu)_{\nu=1}^d$ with $\Lambda_\nu \subset \mathbb{Z}_+^n$ and $S \subset \{1, \ldots, n\}$. Suppose that for $G = (G_\nu) \in \mathcal{F}(\tilde{N}(\Lambda, S))$,

$$|I_J(P_G, \xi)| \leq C \min \left\{ \left| 2^{-J} m_\nu \xi \right|^\delta : m_\nu \in G_\nu \cap \Lambda_\nu \text{ for } \nu = 1, \ldots, d \right\}. \quad (7.2)$$

Suppose that

$$\bigcup_{\nu=1}^d (F_\nu \cap \Lambda_\nu) \text{ is an even set for } F \in \mathcal{F}_{lo}(\tilde{N}(\Lambda, S)) \quad (7.3)$$

Then for any $F \in \mathcal{F}(\tilde{N}(\Lambda, S))$,

$$\sup_{\xi \in \mathbb{R}^d} \sum_{J \in \text{Cap}(F^*)} |I_J(P_F, \xi)| \leq C \quad \text{where } \text{Cap}(F^*) = \bigcap_{\nu=1}^d F^*_\nu, \quad (7.4)$$

and for $1 < p < \infty$,

$$\left\| \sum_{J \in \mathbb{Z} \subset \text{Cap}(F^*)} H_J^{P_F} \ast f \right\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}. \quad (7.5)$$

By (6.14), the condition (7.2) is satisfied if $\Lambda_\nu$’s are pairwise disjoint. So, Theorem 7.1 together with Proposition 4.3 immediately leads the sufficient part of Main Theorem 3.2.

Remark 7.1. In the above, $C_2$ in (7.4) is majorized by

$$C_R \prod_{\nu} \prod_{m \in \Lambda_\nu} (|c_m^\nu| + 1/|c_m^\nu|)^{1/R} \quad \text{for some large } R. \quad (7.6)$$

In order to prove Theorem 7.1, we first deal with the low rank case.

7.2. Main Estimate for Low Rank Case. We show that

Proposition 7.1. Suppose that $\text{rank} \left( \bigcup_{\nu=1}^d N(\Lambda_\nu, S) \right) \leq n - 1$ and that for $\Lambda = (\Lambda_\nu)_{\nu=1}^d$ with $\Lambda_\nu \subset \mathbb{Z}_+^n$ and $S \subset \{1, \ldots, n\}$. For each $G = (G_\nu) \in \mathcal{F}(\tilde{N}(\Lambda, S))$, we let $I_J(P_G, \xi)$ and $H_J^{P_G}$ be defined by (2.6) and (2.7). Suppose that

$$|I_J(P_G, \xi)| \leq C \min \left\{ \left| 2^{-J} m_\nu \xi \right|^\delta : m_\nu \in G_\nu \cap \Lambda_\nu \text{ for } \nu = 1, \ldots, d \right\}. \quad (7.7)$$
Suppose that

\[ \bigcup_{\nu=1}^{d} (\mathbb{F}_\nu \cap \Lambda_\nu) \]  

is an even set for \( \mathbb{F} \in \mathcal{F}_{lo}(\mathcal{N}(\Lambda, S)) \)

where

\[ \mathcal{F}_{lo}(\mathcal{N}(\Lambda, S)) = \left\{ (\mathbb{F}_\nu) \in \mathcal{F}(\mathcal{N}(\Lambda, S)) : \text{rank} \left( \bigcup_{\nu=1}^{d} \mathbb{F}_\nu \right) \leq n - 1 \right\}. \]

Let \( Z(S) \) be defined in Then

\[ \sup_{\xi} \sum_{J \in \mathcal{Z}} |I_{J}(P_\Lambda, \xi)| \lesssim C \]  

where \( I_{J}(P_\Lambda, \xi) = \left( H_{i}^{PA} \right)^{\wedge}(\xi) \).

For \( 1 < p < \infty \) and all subsets \( Z \subset Z(S) \),

\[ \left\| \sum_{J \in \mathcal{Z}} H_{i}^{PA} * f \right\|_{L^{p}(\mathbb{R}^d)} \lesssim \|f\|_{L^{p}(\mathbb{R}^d)}. \]

We now show Proposition 7.1 by using the projective cones in Lemmas 4.17 through 4.19. In view of (4.27),

\[ \sum_{J \in \mathcal{Z}(S)} |I_{J}(P_\Lambda, \xi)| \leq \sum_{F \in \mathcal{F}(\mathcal{N}(\Lambda, S))} \sum_{J / |J| \in \cap_{\nu=1}^{d} S_{\nu}[\mathcal{F}_\nu]} |I_{J}(P_\Lambda, \xi)|, \]

(7.9) \[ \left\| \sum_{J \in \mathcal{Z}} H_{i}^{PA} * f \right\|_{L^{p}(\mathbb{R}^d)} \leq \sum_{F \in \mathcal{F}(\mathcal{N}(\Lambda, S))} \left\| \sum_{J / |J| \in \cap_{\nu=1}^{d} S_{\nu}[\mathcal{F}_\nu]} H_{i}^{PA} * f \right\|_{L^{p}(\mathbb{R}^d)} \]

(7.10) \[ \]  

where \( J \) is taken over \( \mathbb{Z}^{d} \) as mentioned in Remark 4.5. Furthermore,

**Lemma 7.1.** Let \( \mathbb{P}_\nu = N(\Lambda_\nu, S) \) be a polyhedron in \( \mathbb{R}^{n} \) with \( \text{dim} (\mathbb{P}_\nu) = m_\nu \) and let \( \mathbb{F}_\nu \leq \mathbb{P}_\nu \) for each \( \nu = 1, \cdots, d \). Suppose that there exists \( \nu \in \{1, \cdots, d\} \) such that \( \mathbb{F}_\nu \in \mathcal{F}^{m_\nu - k_\nu}(\mathbb{P}_\nu) \) with \( k_\nu \geq 1 \), that is, \( \mathbb{F}_\nu \nsubseteq \mathbb{P}_\nu \). Let \( H_{i}^{PB} \) be given by (2.7). Then,

\[ \left\| (H_{i}^{PA} - H_{i}^{PB}) * f \right\|_{L^{p}(\mathbb{R}^d)} \leq 2^{-c|J|} \|f\|_{L^{p}(\mathbb{R}^d)} \]  

for \( J / |J| \in \cap_{\nu=1}^{d} S_{\nu}[\mathcal{F}_\nu] \)

(7.11) \[ \]  

**Proof.** Let

\[ B = \left\{ \nu : \mathbb{F}_\nu \nsubseteq N(\Lambda_\nu, S), \text{ that is, } \mathbb{F}_\nu \in \mathcal{F}^{m_\nu - k_\nu}(\mathbb{P}_\nu) \text{ where } k_\nu \geq 1 \right\}. \]
For each $\nu \in B$, choose $\tilde{m}_\nu \in F_\nu \cap \Lambda_\nu$ and $m \in \Lambda_\nu \setminus F_\nu$. By Lemma 4.19, observe that for there exists $\beta > 0$ such that

$$J/|J| \cdot (m - \tilde{m}_\nu) > \beta$$

for all $J/|J| \in \mathbb{S}_c[F_\nu^*]$ with $\nu \in B$. By (7.7), the Fourier multipliers of $H^P_{\Lambda_J} (= H^P_{\mathbf{N}(\Lambda,S)})$ and $H^P_{\tilde{F}}$ are given by

$$|I_J(P_\Lambda, \xi)|, |I_J(P_{\tilde{F}}, \xi)| \lesssim \min \left\{ \left| 2^{-J \cdot \tilde{m}_\nu} \xi_\nu c_\mu^\nu \right|^{-\delta} : \tilde{m}_\nu \in F_\nu, \ \nu = 1, \cdots, d \right\}.$$  

By the mean value theorem,

$$|I_J(P_\Lambda, \xi) - I_J(P_{\tilde{F}}, \xi)| = \left| \int \left( e^{i \sum_{\nu=1}^d \xi_\nu \sum_{m \in \Lambda_\nu} \omega_\mu^\nu 2^{-J \cdot m} m} - e^{i \sum_{\nu=1}^d \xi_\nu \sum_{m \in F_\nu \cup \Lambda_\nu} \omega_\mu^\nu 2^{-J \cdot m} m} \right) \prod h(t_\nu) dt \right| \lesssim \sum_{\nu \in B} \sum_{m \in \Lambda_\nu \setminus F_\nu} \left| 2^{-J \cdot m} \xi_\nu c_\mu^m \right|^\delta.$$ 

By (7.12)-(7.14),

$$\sup_{\xi} |I_J(P_\Lambda, \xi) - I_J(P_{\tilde{F}}, \xi)| \lesssim \sum_{m \in \Lambda_\nu \setminus F_\nu} \left| 2^{-J \cdot (m - \tilde{m}_\nu)} \right|^\delta \lesssim 2^{-\beta J/2}.$$ 

This implies that (7.11) holds for $p = 2$. Interpolation with $p = 1$ or $p = \infty$ yields the range $1 < p < \infty$. □

We sum up (7.11) of Lemma 7.1 together with (7.9) and (7.10) to obtain that

Lemma 7.2. For $Z \subset Z(S)$,

$$\sup_{\xi} \sum_{J \in Z(S)} |I_J(P_\Lambda, \xi)| \leq C + \sup_{\xi} \sum_{F_\nu \in F(\mathbf{N}(\Lambda,S))} \sum_{J/|J| \in \mathbb{S}_c[F_\nu]} |I_J(P_{\tilde{F}}, \xi)|,$$

$$\left\| \sum_{J \in Z} H^P_{\Lambda_J} * f \right\|_{L^p(\mathbb{R}^d)} \lesssim \left\| f \right\|_{L^p(\mathbb{R}^d)} + \sum_{F_\nu \in F(\mathbf{N}(\Lambda,S))} \left\| \sum_{J/|J| \in \mathbb{S}_c[F_\nu]} H^P_{\tilde{F}} * f \right\|_{L^p(\mathbb{R}^d)}.$$  

In proving Proposition 7.1, we combine (7.8) and Lemma 7.2 with the assumption:

$$\operatorname{rank} \left( \bigcup_{\nu=1}^d \mathbf{N}(\Lambda_\nu, S) \right) \leq n - 1.$$
Proof of Proposition 7.1. Since there are finitely many $F = (F_\nu)_{\nu=1}^d \in \mathcal{F}(\tilde{N}(\Lambda, S))$ in (7.16), it suffice to work with one fixed $F$ on the right hand side. By Lemmas 4.18 and 7.2, it suffices to show that

$$\sup_{\xi} \sum_{J/|J| \in \mathbb{N}(\Lambda, S)} |I_J(P_{\tilde{F}}, \xi)| \lesssim 1 \text{ under the assumption } \bigcap \mathcal{S}(\mathbb{F}_\nu^\circ) \neq \emptyset,$$

$$\left\| \sum_{J \in \mathbb{Z}, J/|J| \in \mathbb{N}(\Lambda, S)} H_{P_{\tilde{F}}} \ast f \right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \text{ under the assumption } \bigcap \mathcal{S}(\mathbb{F}_\nu^\circ) \neq \emptyset.$$

Note that

1. $\text{rank} \left( \bigcup_{\nu=1}^d \mathbb{F}_\nu \right) \leq \text{rank} \left( \bigcup_{\nu=1}^d \mathbb{N}(\Lambda_\nu, S) \right) \leq n - 1$,

2. $\bigcap_{\nu=1}^d (\mathbb{F}_\nu^\circ)^\circ \supset \bigcap_{\nu=1}^d \mathcal{S}(\mathbb{F}_\nu^\circ) \neq \emptyset$.

From this and (7.8), $I_J(P_{\tilde{F}}, \xi) \equiv 0$ for all $J$, which implies that the last term of each (7.15) and (7.16) vanishes. This completes the proof of Proposition 7.1. \qed

7.3. Proof of Theorem 7.1. Before proving Theorem 7.1, we show the following Van der Corput type lemma under the assumption of Theorem 7.1.

**Lemma 7.3.** Suppose that $F(s)$ for $s = 0, 1, \cdots, N$ is defined as in Definitions 5.2 and 5.3 satisfying Proposition 5.1. Define $I_J(P_{\tilde{F}(s)}, \xi)$ by (2.6). Then for $J \in \bigcap_{\nu=1}^d \mathbb{F}_\nu^s$,

$$|I_J(P_{\tilde{F}(s)}, \xi)| \leq CK \min \left\{ 2^{-J \cdot m_\nu} \xi_\nu |^{-\delta} : m_\nu \in P_\nu = \mathbb{N}(\Lambda_\nu, S) \text{ where } \nu = 1, \cdots, d \right\}$$

where $K = \prod_{\nu} \prod_{m \in \Lambda_\nu} \left( |c_{m_\nu}| + 1/|c_{m_\nu}| \right)^{1/\delta}$.

**Proof.** By the hypothesis (7.2) in Theorem 7.1 and the fact that $F_\nu(s) \supset F_\nu(N) = F_\nu$ for all $s = 0, 1, \cdots, N$,

$$|I_J(P_{\tilde{F}(s)}, \xi)| \leq C \min \left\{ 2^{-J \cdot m_\nu} \xi_\nu |^{-\delta} : m_\nu \in \mathbb{F}_\nu \cap \Lambda_\nu \right\}$$

$$\leq C \min \left\{ 2^{-J \cdot \tilde{m}_\nu} \xi_\nu |^{-\delta} : \tilde{m}_\nu \in \mathbb{F}_\nu \right\}$$

(7.19)

where the second inequality follows from (4.35) in Lemma 4.20. On the other hand, we have for $J \in \mathbb{F}_\nu^s$ with each $\nu = 1, \cdots, d$,

$$2^{-J \cdot m_\nu} \leq 2^{-J \cdot \tilde{m}_\nu} \text{ where } m_\nu \in \mathbb{F}_\nu(0) = P_\nu \text{ and } \tilde{m}_\nu \in \mathbb{F}_\nu(N) = F_\nu,$$
This together with (7.19) yields (7.18).

Choose \( d \) vectors

\[ \hat{m}_\nu \in \mathbb{F}_\nu(N) \cap \Lambda_\nu = \mathbb{F}_\nu \cap \Lambda_\nu \quad \text{for each } \nu = 1, \cdots, d. \]  

(7.21)

Then, we see in view of \( \mathbb{F}_\nu(N) = \mathbb{F}_\nu \) that for every \( J \in \text{Cap}(\mathbb{F}^*) \cap \text{Cap}(\mathbb{F}^*) = \bigcap_{\nu} \mathbb{F}_\nu^* \),

\[ \hat{m}_\nu' \cdot J = \hat{m}_\nu \cdot J \quad \text{for all } \hat{m}_\nu' \in \mathbb{F}_\nu(N) \cap \Lambda_\nu. \]  

(7.22)

According to (6.1), define for each \( \{ \alpha, \beta \} \subset \{ 1, \cdots, d \} \) with \( \alpha > \beta \),

\[ A_{j}^{(\alpha, \beta)}(\xi) = \psi \left( \frac{2^{-j \hat{m}_\alpha} \xi_\alpha}{2^{-j \hat{m}_\beta} \xi_\beta} \right) \quad \text{and} \quad A_{j}^{(\beta, \alpha)}(\xi) = 1 - A_{j}^{(\alpha, \beta)}(\xi). \]

By (7.22), we observe that the function \( A_J \) for \( J \in \text{Cap}(\mathbb{F}^*) \) is independent of choices of \( \hat{m}_\alpha \in \mathbb{F}_\alpha(N) \cap \Lambda_\alpha \) and \( \hat{m}_\beta \in \mathbb{F}_\beta(N) \cap \Lambda_\beta \). Note that there are \( M = \binom{d}{2} \) collections of \( (\alpha, \beta) \) with \( \alpha > \beta \) in \( \{ 1, \cdots, d \} \). Then

\[ 1 = \prod_{(\alpha, \beta) \subset \{ 1, \cdots, d \}, \alpha < \beta} \left( A_{j}^{(\alpha, \beta)}(\xi) + A_{j}^{(\beta, \alpha)}(\xi) \right) = \sum_{\gamma} A_{j}^{\gamma}(\xi) \]

where \( A_{j}^{\gamma}(\xi) = \prod_{k=1}^{M} A_{j}^{(\alpha_k, \beta_k)}(\xi) \) with \( \gamma = ((\alpha_k, \beta_k))_{k=1}^{M} \) and the summation above is over all possible \( 2^M \) choices of \( \gamma \) having \( \alpha_k < \beta_k \) or \( \alpha_k > \beta_k \) for each \( k \in \{ 1, \cdots, M \} \). In order to show (7.4) and (7.5), we prove that for each \( \gamma = ((\alpha_j, \beta_j))_{j=1}^{M} \) and an arbitrary subset \( Z \) of \( \text{Cap}(\mathbb{F}^*) \cap \text{Cap}(\mathbb{F}^*) \),

\[ \sup_{\xi \in \mathbb{R}^d} \sum_{J \in Z} \left| \mathcal{I}_J(P, \xi) \hat{A}_{j}^{\gamma}(\xi) \right| \lesssim 1, \]

(7.23)

and

\[ \left\| \sum_{J \in Z} H_{j}^{P_{\gamma}(\Lambda, S)} * A_{j}^{\gamma} * f \right\|_{L^p(\mathbb{R}^d)} \lesssim \| f \|_{L^p(\mathbb{R}^d)} \]

where \( J \) is taken over \( J \in \mathbb{Z}^n \). Note that there exists an \( n \)-permutation \( \sigma \) such that

\[ \text{supp} \left( \hat{A}_{j}^{\gamma} \right) \subset \left\{ \xi \in \mathbb{R}^d : |2^{-j \hat{m}_{\sigma(1)}(1)} \xi_{\sigma(1)}| \lesssim \cdots \lesssim |2^{-j \hat{m}_{\sigma(d)}(d)} \xi_{\sigma(d)}| \right\}. \]

Without loss of generality,

\[ \text{supp} \left( \hat{A}_{j}^{\gamma} \right) \subset \left\{ \xi \in \mathbb{R}^d : |2^{-j \hat{m}_1|} \lesssim \cdots \lesssim |2^{-j \hat{m}_d} \xi_d| \right\}. \]  

(7.24)
In proving (7.23), it suffices to show that for $A_J = A^*_J$ satisfying (7.24) and for an arbitrary subset $Z$ of $\text{Cap}(\mathbb{R}^*)(\text{id})$,

$$\sup_{\xi \in \mathbb{R}^d} \sum_{J \in Z} \left| \mathcal{I}_J(P_\Lambda, \xi) \widehat{A}_J(\xi) \right| \lesssim 1,$$

(7.25)

and

$$\left\| \sum_{J \in Z} H^P(0) \ast A_J \ast f \right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)},$$

where $F(0) = (F_\nu(0))_{\nu=1}^d = (N(\Lambda_\nu, S)) = N(\Lambda, S)$.

**Remark 7.2.** From now on, we write $A \lesssim B$ when $A \leq CB$ where $C$ is a constant multiple of the constant $K$ in Lemma 7.3.

By Proposition 7.1 and $N(\Lambda_\nu, S) = F_\nu(0)$, it suffices to assume that

$$\text{rank} \left( \bigcup_{\nu=1}^d F_\nu(0) \right) = n.$$  

(7.26)

We will show in a moment that (7.25) follows from the next lemma.

**Lemma 7.4.** Let $Z$ be an arbitrary subset of $\text{Cap}(\mathbb{R}^*)(\text{id})$. Suppose that $F(s)$ is defined as in Definitions 5.2 and 5.3 satisfying Proposition 5.1. Then we have

If $\text{rank} \left( \bigcup_{\nu=1}^d F_\nu(s-1) \right) = n,$ then

$$\sup_{\xi \in \mathbb{R}^d} \sum_{J \in Z} \left| \mathcal{I}_J(P_{F(s-1)}, \xi) - \mathcal{I}_J(P_{F(s)}, \xi) \right| \widehat{A}_J(\xi) \lesssim 1,$$

and

$$\left\| \sum_{J \in Z} \left( H^P_{\nu(s-1)} - H^P_{\nu(s)} \right) \ast A_J \ast f \right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$  

Also, we have

$$\sup_{\xi \in \mathbb{R}^d} \sum_{J \in Z} \left| \mathcal{I}_J(P_{F(N)}, \xi) \widehat{A}_J(\xi) \right| \lesssim 1,$$

(7.28)

and

$$\left\| \sum_{J \in Z} H^P_{\nu(N)} \ast A_J \ast f \right\|_{L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$
Indeed assume that (7.27) and (7.28) are true. Let \( \text{rank} \left( \bigcup_{\nu=1}^{d} F_\nu(s-1) \right) = n \) for all \( s = 1, \ldots, N \). Then (7.27) and (7.28) yield that
\[
\sup_{\xi \in \mathbb{R}^d} \sum_{J \in \mathbb{Z}} | \mathcal{I}_J(P_{\mathbb{F}(0)}, \xi) \hat{A}_J(\xi) | \lesssim 1 \quad \text{and} \quad \left\| \sum_{J \in \mathbb{Z}} H^P_{J(0)} \ast A_J \ast f \right\|_{L^p(\mathbb{R}^d)} \lesssim \| f \|_{L^p(\mathbb{R}^d)}.
\]

Let \( \text{rank} \left( \bigcup_{\nu=1}^{d} F_\nu(s-1) \right) = n \) for \( s = 1, \ldots, r \), and \( \text{rank} \left( \bigcup_{\nu=1}^{d} F_\nu(r) \right) \leq n - 1 \). Then (7.27) yields that
\[
\sup_{\xi \in \mathbb{R}^d} \sum_{J \in \mathbb{Z}} | \mathcal{I}_J(P_{\mathbb{F}(0)}, \xi) \hat{A}_J(\xi) | \lesssim 1 + \sup_{\xi \in \mathbb{R}^d} \sum_{J \in \mathbb{Z}} | \mathcal{I}_J(P_{\mathbb{F}(r)}, \xi) \hat{A}_J(\xi) | ,
\]

and
\[
\left\| \sum_{J \in \mathbb{Z}} H^P_{J(0)} \ast A_J \ast f \right\|_{L^p(\mathbb{R}^d)} \lesssim \left\| f \right\|_{L^p(\mathbb{R}^d)} + \left\| \sum_{J \in \mathbb{Z}} H^P_{J(r)} \ast A_J \ast f \right\|_{L^p(\mathbb{R}^d)}.
\]

By (7.26), \( r \geq 1 \). Thus, by Lemma 5.5, we have an overlapping condition
\[
\bigcup_{\nu=1}^{d} (F'_\nu)^{\circ} \neq \emptyset.
\]

From the hypothesis of Theorem 7.1 and the rank condition
\[
\text{rank} \left( \bigcup_{\nu=1}^{d} F_\nu(r) \right) \leq n - 1,
\]

it follows that \( \bigcup_{\nu=1}^{d} (F_\nu(r) \cap \Lambda_\nu) \) is an even set. Thus, \( \mathcal{I}_J(P_{\mathbb{F}(r)}, \xi) \equiv 0 \) and \( H^P_{J(r)} \) also vanishes in (7.29). This implies (7.4) and (7.5). We now prove Lemma 7.4.

**Proof of (7.27) in Lemma 7.4.** Let \( s \in \{1, \ldots, N\} \) fixed. Choose \( \mu \in \{1, \ldots, d\} \) such that
\[
\text{rank} \left( \bigcup_{\nu=1}^{d} F_\nu(s-1) \right) = n, \quad \text{rank} \left( \bigcup_{\nu=1}^{d} F_\nu(s-1) \right) \leq n - 1
\]

where \( \bigcup_{\nu=1}^{d} F(s-1, \nu) \cap \Lambda_\nu = \emptyset \) for the case \( \mu = d \). For each \( s \), set
\[
F'(s-1) = (\emptyset, \ldots, \emptyset, F_{\nu=\mu+1}(s-1), \ldots, F_d(s-1)) \quad F'(s) = (\emptyset, \ldots, \emptyset, F_{\nu=\mu+1}(s), \ldots, F_d(s)).
\]
Therefore, we next consider the case $s \in J$. Thus, by Lemma 5.5, for
\[ (7.34) \]
\[ \sum_{J \in Z} \left( H_j^{p_\nu(s-1)} - H_j^{p_\nu'(s)} - H_j^{p_\nu(s)} + H_j^{p_\nu'(s)} \right) * A_J * f \right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}. \]

In order to show (7.27), we shall prove
\[ (7.33) \]
\[ \sup_{\xi \in \mathbb{R}^d} \sum_{J \in Z} \left| (I_J(P_{\mathbb{F}}(s-1), \xi) - I_J(P_{\mathbb{F}}(s), \xi)) \hat{A}_J(\xi) \right| \lesssim 1, \]
\[ \left\| \sum_{J \in Z} H_j^{p_\nu(s-1)} * A_J * f \right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}. \]

Proof of (7.33). Note that from (7.31) and $\mathbb{F}_\mu(s) \subseteq \mathbb{F}_\mu(s-1)$,
\[ \text{rank} \left( \bigcup_{\nu=\mu+1}^{d} \mathbb{F}_\nu(s-1) \right) \leq n - 1, \]
and
\[ \text{rank} \left( \bigcup_{\nu=\mu+1}^{d} \mathbb{F}_\nu(s) \right) \leq n - 1. \]

By Lemma 5.5, for $s = 2, \ldots, N$,
\[ \bigcap_{\nu=\mu+1}^{d} \mathbb{F}_\nu(s-1) \cap \Lambda_\nu \neq \emptyset \text{ and } \bigcap_{\nu=\mu+1}^{d} \mathbb{F}_\nu(s) \cap \Lambda_\nu \neq \emptyset. \]

Thus,
\[ \bigcup_{\nu=\mu+1}^{d} (\mathbb{F}_\nu(s-1) \cap \Lambda_\nu) \text{ and } \bigcup_{\nu=\mu+1}^{d} (\mathbb{F}_\nu(s) \cap \Lambda_\nu) \text{ are even sets.} \]

Therefore,
\[ I_J(P_{\mathbb{F}}(s-1), \xi) = I_J(P_{\mathbb{F}}(s), \xi) \equiv 0. \]

We next consider the case $s = 1$, that is,
\[ \sum_{J \in Z} \left| (I_J(P_{\mathbb{F}}(0), \xi) - I_J(P_{\mathbb{F}}(1), \xi)) \hat{A}_J(\xi) \right| \text{ and } \left\| \sum_{J \in Z} \left( H_j^{p_\nu(0)} - H_j^{p_\nu(1)} \right) * A_J * f \right\|_{L^p(\mathbb{R}^d)}. \]
where \( \mathcal{I}_J(F'(1), \xi) \equiv 0 \) by the previous argument. By applying the Proposition 7.1 with \[
\text{rank} \left( \bigcup_{\nu=\mu+1}^{d} F_\nu(0) \right) \leq n - 1 \quad \text{and} \quad F_\nu(0) = N(\Lambda_\nu, S),
\]
we obtain
\[
\sum_{J \in \mathbb{Z}} \left| I_J(P_{F'}(0), \xi) \right| \lesssim 1 \quad \text{and} \quad \left\| \sum_{J \in \mathbb{Z}} H^F_J(0) * A_J * f \right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.
\]
Therefore we proved (7.33). \( \square \)

**Proof of (7.34).** We shall show that for all \( J \in \mathbb{Z} \subset \text{Cap}(F^*) \)(id),
\[
|I_J(P_{F^{(s-1)}}, P_{F(s)}, \xi)| \lesssim \min \left\{ |2^{-J-n_\nu} \xi_\nu|^{\pm \epsilon} : n_\nu \in F_\nu(s-1) \right\}^d_{\nu=\mu}.
\]
This and the rank condition (7.30) combined with Proposition 6.1 and Lemma 6.3 yield (7.34). To show (7.35), by Lemma 7.3, it suffices to show that
\[
|I_J(P_{F^{(s-1)}}, P_{F(s)}, \xi)| \lesssim \min \left\{ |2^{-J-n_\nu} \xi_\nu|^{\pm \epsilon} : n_\nu \in F_\nu(s-1) \text{ for } \nu=\mu \right\}^d.
\]
Thus, the proof of (7.34) is finished if (7.36) is proved. \( \square \)

**Proof of (7.36).** By using mean value theorem for the exponential functions in (7.32),
\[
|I_J(P_{F^{(s-1)}}, P_{F(s)}, \xi)| \lesssim \left| \langle \xi, P_{F^{(s-1)}}(2^{-j_1}t_1, \ldots, 2^{-j_n}t_n) - P_{F(s)}(2^{-j_1}t_1, \ldots, 2^{-j_n}t_n) \rangle \right|
\]
+ \[
\sum_{\nu=1}^{d} \left| \sum_{m_\nu \in (F_\nu(s-1) \cap \Lambda_\nu) \setminus (F_\nu(s) \cap \Lambda_\nu)} |c_\nu 2^{-J-m_\nu} \xi_\nu| \right|.
\]
where \( |t_j| \leq 1 \) above. By (5.39) of Proposition 5.2, for any \( m_\nu \in F_\nu(s-1) \setminus F_\nu(s) \) and \( \tilde{m}_\nu \in F_\nu(N) \) in (7.21), there exists a constant \( b > 0 \) independent of \( J \) and \( \alpha_s \) such that
\[
J \cdot (m_\nu - \tilde{m}_\nu) \geq b \alpha_s \quad \text{where} \quad J = \sum_{j=1}^{N} \alpha_j p_j \in \text{Cap}(F^*) \)(id).
\]
Thus, we have in (7.37)
\[
|2^{-J-n_\nu} \xi_\nu| \lesssim 2^{-b \alpha_s} |2^{-J-m_\nu} \xi_\nu|.
\]
This together with Lemma 7.3 yields that in (7.37),

\[
(7.38) \quad |I_j(P_{\mathbb{F}(s-1)}, P_{\mathbb{F}(s)}, \xi)| \lesssim \sum_{\nu=1}^{d} 2^{-b_\alpha s} |2^{-J \tilde{m}_\nu} \xi_\nu| \lesssim 2^{-c_1 \alpha s}.
\]

By using the mean value theorem in (7.32) together with (7.20) and the support condition of \(\hat{A}_J\) in (7.24),

\[
|I_j(P_{\mathbb{F}(s-1)}, P_{\mathbb{F}(s)}, \xi)| \lesssim \left| \langle \xi, P_{\mathbb{F}(s-1)}(2^{-j_1} t_1, \ldots, 2^{-j_n} t_n) - P_{\mathbb{F}(s-1)}(2^{-j_1} t_1, \ldots, 2^{-j_n} t_n) \rangle \right|
\]

\[
+ \left| \langle \xi, P_{\mathbb{F}(s)}(2^{-j_1} t_1, \ldots, 2^{-j_n} t_n) - P_{\mathbb{F}(s)}(2^{-j_1} t_1, \ldots, 2^{-j_n} t_n) \rangle \right|
\]

\[
\lesssim \sum_{\nu=1}^{\mu} \sum_{\mu_\nu \in \{1, 2, \ldots, \mu \}} |\xi_\nu 2^{-j \tilde{m}_\nu}|.
\]

\[
(7.39) \quad \lesssim |\xi_\nu 2^{-J \tilde{m}_\nu}| \text{ for any } \tilde{m}_\nu \in \mathbb{F}_\nu(N).
\]

We note from (7.22) that the above \(\tilde{m}_\mu\) can be any vector in \(\mathbb{F}_\nu(N)\). By (7.38),(7.39) and (7.24),

\[
|I_j(P_{\mathbb{F}(s-1)}, P_{\mathbb{F}(s)}, \xi)| \lesssim \min \{|\xi_\nu 2^{-J \tilde{m}_\nu}|, 2^{-c_1 \alpha s} : \tilde{m}_\nu \in \mathbb{F}_\mu(N)\}
\]

\[
(7.40) \quad \lesssim \min \{|\xi_\nu 2^{-J \tilde{m}_\nu}|, 2^{-c_1 \alpha s} : \tilde{m}_\nu \in \mathbb{F}_\nu(N)\}^d_{\nu=\mu}.
\]

By (5.40) of Proposition 5.2,

\[
J : (n_\nu - \tilde{m}_\nu) \lesssim \alpha_s \text{ where } n_\nu \in \mathbb{F}_\nu(s-1) \text{ and } \tilde{m}_\nu \in \mathbb{F}_\nu(N).
\]

Hence for any \(n_\nu \in \mathbb{F}_\nu(s-1)\) and \(\tilde{m}_\nu \in \mathbb{F}_\nu(N)\) with \(\nu = \mu, \mu + 1, \ldots, d\) in (7.40),

\[
|2^{-J \tilde{m}_\nu} \xi_\nu| \lesssim 2^{c_2 \alpha s} |2^{-J \tilde{m}_\nu} \xi_\nu|.
\]

Then by (7.40) and (7.41),

\[
|I_j(P_{\mathbb{F}(s-1)}, P_{\mathbb{F}(s)}, \xi)| \lesssim \min \left\{ 2^{c_2 \alpha s} |\xi_\nu 2^{-J \tilde{m}_\nu}|, 2^{-c_1 \alpha s} : n_\nu \in \mathbb{F}_\nu(s-1) \right\}^d_{\nu=\mu}
\]

\[
\lesssim \min \left\{ |\xi_\nu 2^{-J \tilde{m}_\nu}| : n_\nu \in \mathbb{F}_\nu(s-1) \right\}^d_{\nu=\mu}.
\]

This yields (7.36).

Therefore the proof of (7.34) is finished. □
Proof of (7.28) in Lemma 7.4. Assume that rank \( \bigcup_{\nu=1}^{d} F_\nu(N) \) \( \leq n - 1 \). By this and Lemma 5.5, \( \bigcup_{\nu=1}^{d} F_\nu(N) \cap \Lambda_\nu \) is an even set so that \( \mathcal{I}_J(F(N), \xi) \equiv 0 \). Thus, suppose that 
\[
\text{rank} \left( \bigcup_{\nu=1}^{d} F_\nu(N) \right) = n.
\]
As in (7.30) and (7.31), we choose \( \mu \in \{1, \cdots, d\} \) such that
\[
\text{rank} \left( \bigcup_{\nu=\mu}^{d} F_\nu(N) \right) = n \quad \text{and} \quad \text{rank} \left( \bigcup_{\nu=\mu+1}^{d} F_\nu(N) \right) \leq n - 1.
\]
(7.42)

Set
\[
F'(N) = (\emptyset, \cdots, \emptyset, F_{\mu+1}(N), \cdots, F_d(N)) \quad \text{for} \quad \mu \leq d - 1,
\]
and \( F'(N) = (\emptyset, \cdots, \emptyset) \) for \( \mu = d \). By Lemma 5.5
\[
\bigcap_{\nu=\mu+1}^{d} (F^*_\nu(N))^c \neq \emptyset.
\]
By this and (7.42), we see that \( \bigcup_{\nu=\mu+1}^{d} (F_\nu(N) \cap \Lambda_\nu) \) is an even set. So, \( \mathcal{I}_J(F'(N), \xi) \equiv 0 \). Thus it suffices to show that
\[
\sup_{\xi \in \mathbb{R}^d} \sum_{J \in \mathcal{Z}} \left| (\mathcal{I}_J(P_{F(N)}, \xi) - \mathcal{I}_J(P_{F'(N)}, \xi)) \hat{A}_J(\xi) \right| \lesssim 1,
\]
and
\[
\left\| \sum_{J \in \mathcal{Z}} \left( H^P_{\mu}(N) - H^P_{\mu}(N) \right) * A_J * f \right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.
\]
(7.44)
Let \( J \in \text{Cap}((\mathcal{F}^*))(\text{id}) \subset \bigcap F^*_\nu \subset F^*_\mu = F^*_\mu(N) \). Then for every \( n_\nu \in F_\nu(N) = F_\nu \),
\[
J \cdot (n_\nu - \tilde{m}_\nu) = 0 \quad \text{where} \quad \tilde{m}_\nu \in F_\nu(N) \quad \text{in} \quad (7.21).
\]
By this and the support condition (7.24) for \( \hat{A}_J(\xi) \) such that \( |2^{-J} \tilde{m}_1 \xi_1| \lesssim \cdots \lesssim |2^{-J} \tilde{m}_d \xi_d| \),
\[
\left| (\mathcal{I}_J(P_{F(N)}, \xi) - \mathcal{I}_J(P_{F'(N)}, \xi)) \hat{A}_J(\xi) \right| \lesssim \sum_{\nu=1}^{\mu} \sum_{m_\nu \in F_\nu(N) \cap \Lambda_\nu} \sum_{\nu=\mu}^{d} |2^{-J} m_\nu \xi_\nu| \approx |2^{-J} \tilde{m}_\nu \xi_\nu| \
\lesssim \min \left\{ \left| 2^{-J} \tilde{m}_\nu \xi_\nu \right| : \nu = \mu, \cdots, d \right\}
\]
(7.45)
\[
\lesssim \min \left\{ \left| 2^{-J} n_\nu \xi_\nu \right| : n_\nu \in F_\nu(N) \right\} \lesssim 1.
\]
Hence, Lemma 7.3 together with \( \text{rank} \left( \bigcup_{\nu=\mu}^{d} F_{\nu}(N) \right) = n \) and Proposition 6.1 yields (7.44). This completes the proof of (7.28). Consequently, Lemma 7.4 is proved. \( \square \)

8. Necessity Theorem

To prove the necessity part of Main Theorem, we need more properties of dual faces.

8.1. Transitivity Rule for Dual Faces.

**Proposition 8.1.** Let \( \mathbb{P} \subset \mathbb{R}^{n} \) be a polyhedron and \( F, G \in F(\mathbb{P}) \) such that \( G \preceq F \). Suppose that \( q \in (F^{*})^{\circ} \mid \mathbb{P} \). Suppose that \( p \in (G^{*})^{\circ} \mid F \). Then there exists \( \epsilon > 0 \) such that

\[
0 < \epsilon < \epsilon_{0} \quad \text{implies that} \quad q + \epsilon p \in (G^{*})^{\circ} \mid \mathbb{P}.
\]

See Figure 8.1 that visualizes Proposition 8.1 and Lemma 8.1.

**Definition 8.1.** Let \( V \) be a subspace of \( \mathbb{R}^{n} \). Denote a projection from \( \mathbb{R}^{n} \) to \( V \) by \( P_{V} \).

**Lemma 8.1.** Let \( \mathbb{P} \) be a polyhedron in \( \mathbb{R}^{n} \) and \( G \preceq \mathbb{P} \). Given \( q \in (G^{*})^{\circ} \mid \mathbb{P} \), there exists \( r > 0 \) depending only \( q \) such that for any \( n \in \mathbb{P} \setminus G \) and \( m \in G \),

\[
q \cdot \frac{P_{V^{\perp}(G)}(n - m)}{|P_{V^{\perp}(G)}(n - m)|} \geq r > 0.
\]
Proof. We start with the case that $G = \{m_0\}$ is a vertex of $P$. Let

$$S(P-m_0) = \left\{ \frac{n-m_0}{|n-m_0|} : n \in P \setminus \{m_0\} \right\}.$$  \hfill (8.2)

Since $P$ has finitely many vertices,

$$\left\{ \frac{n-m_0}{|n-m_0|} : n \in P \setminus \{m_0\} \right\} = \left\{ \frac{n-m_0}{|n-m_0|} : n \in P \setminus \{m_0\}, |n-m_0| = \epsilon \right\}.$$

This set is compact because it is the intersection of a compact set with a closed one. Since $q \in (G^*)^\circ \|P$, we have for all $n \in P \setminus \{m_0\}$,

$$q \cdot \frac{n-m_0}{|n-m_0|} > 0,$$

which with (8.2) implies that $q \cdot s > 0$ for $s \in S(P-m_0)$.

A map $s \to q \cdot s$ is continuous on the compact set $S(P-m_0)$. So it has a minimum $r > 0$:

$$q \cdot s \geq r \text{ for all } s \in S(P-m_0).$$ \hfill (8.3)

From $V(G) = \{0\}$ and $V^\perp(G) = \mathbb{R}^n$,

$$P_{V^\perp(G)}(n-m_0) = n-m_0.$$

From this together with (8.2),

$$S(P-m_0) = \left\{ \frac{P_{V^\perp(G)}(n-m_0)}{|P_{V^\perp(G)}(n-m_0)|} : n \in P \setminus \{m_0\} \right\}.$$

By this and (8.3),

$$q \cdot \frac{P_{V^\perp(G)}(n-m_0)}{|P_{V^\perp(G)}(n-m_0)|} \geq r \text{ for all } n \in P \setminus \{m_0\}.$$

We next consider the general case that $G$ is a $k$-dimensional face of $P$. We shall use the following two properties:

$$\left\{ x : P_{V^\perp(G)}(x) = 0 \right\} = V(G)$$ \hfill (8.4)

and

$$\text{If } n \in P \setminus G \text{ and } m \in G, \text{ then } n-m \not\in V(G).$$ \hfill (8.5)

Choose any $m_0 \in G \subseteq P$. Since an image of polyhedron under any linear transform is also a polyhedron, we see that $P_{V^\perp(G)}(P-m_0)$ is a polyhedron in $V^\perp(G)$. Moreover,

$$0 \text{ is a vertex of } P_{V^\perp(G)}(P-m_0).$$ \hfill (8.6)
Proof of (8.6). By Definition 2.11, we see that for \( q \in (G^*)^0|P \) and for \( m \in G \) and \( n \in P \setminus G \),
\begin{equation}
0 < q \cdot (n - m).
\end{equation}

Let \( P_{V^\perp(G)}(n - m_0) \in P_{V^\perp(G)}(P - m_0) \setminus \{0\} \), that is, \( P_{V^\perp(G)}(n - m_0) \neq 0 \). Then we have \( n - m_0 \notin V(G) \) by (8.4), that is, \( n \notin G \). Thus \( n \in P \setminus G \). By (8.7),
\[ q \cdot 0 = 0 < q \cdot (n - m) = q \cdot (n - m_0) = q \cdot P_{V^\perp(G)}(n - m_0) \]
where \( q \perp V(G) \) in the last equality. Thus the condition (2.3) of Definition 2.7 holds. \( \Box \)

In view of (8.2) and (8.6), we set a compact set
\begin{equation}
K = S(P_{V^\perp(G)}(P - m_0) - 0) = \left\{ \frac{n - 0}{|n - 0|} : n \in P_{V^\perp(G)}(P - m_0) \setminus \{0\} \right\}.
\end{equation}

In the above,
\begin{equation}
P_{V^\perp(G)}(P - m_0) \setminus \{0\} = P_{V^\perp(G)}((P \setminus G) - G),
\end{equation}

Proof of (8.9). Let \( z \in P_{V^\perp(G)}(P - m_0) \setminus \{0\} \). Then \( z = P_{V^\perp(G)}(n - m_0) \neq 0 \) with \( n \in P \).
From (8.4) \( n - m_0 \notin V(G) \), which also implies that \( n \notin G \). Thus \( z \in P_{V^\perp(G)}((P \setminus G) - G) \).
Let \( z \in P_{V^\perp(G)}((P \setminus G) - G) \). Then \( z = P_{V^\perp(G)}(n - m) \) with \( n \in P \setminus G \) and \( m \in G \). Thus
\[ z = P_{V^\perp(G)}(n - m_0 + m_0 - m) = P_{V^\perp(G)}(n - m_0) + P_{V^\perp(G)}(m_0 - m) \]
where the last inequality follows from (8.4) and (8.5). Hence \( z \in P_{V^\perp(G)}(P - m_0) \setminus \{0\} \). \( \Box \)

By (8.9), we rewrite the compact set in (8.8) as
\begin{equation}
K = \left\{ \frac{P_{V^\perp(G)}(n - m)}{|P_{V^\perp(G)}(n - m)|} : n \in P \setminus G, m \in G \right\}.
\end{equation}

By \( q \in (G^*)^0|P \), for all \( n \in P \setminus G \) and \( m \in G \),
\[ q \cdot P_{V^\perp(G)}(n - m) = q \cdot (n - m) > 0 \]
because of \( q \perp V(G) \) and Definition 2.11. Therefore
\[ q \cdot s > 0 \text{ for } s \in K. \]
From this combined with the compactness of $K$, there exists $r > 0$ such that

$$q \cdot s \geq r \quad \text{for all } s \in K.$$ 

By (8.10), for any $n \in P \setminus G$ and $m \in G$,

$$q \cdot \frac{P_{V^\perp(G)}(n - m)}{|P_{V^\perp(G)}(n - m)|} \geq r > 0.$$

This completes the proof of Lemma 8.1. \hfill \square

**Proof of Proposition 8.1.** It suffices to show that $0 < \epsilon < \epsilon_0$ implies that for all $n \in P \setminus G$ and $m \in G$,

$$q \cdot (n - m) > 0.$$  \hfill (8.11)

We first observe from $q \in (F^*)^G|P$,

$$q \cdot (n - m) \geq 0 \quad \text{for all } n \in P \setminus G \text{ and } m \in G.$$ \hfill (8.12)

Moreover, combined with $p \in (G^*)^F|F$,

$$q \cdot (n - m) > 0 \quad \text{for all } n \in P \setminus F \text{ and } m \in G,$$

$$p \cdot (n - m) > 0 \quad \text{for all } n \in F \setminus G \text{ and } m \in G.$$ \hfill (8.13) \hfill (8.14)

Let $\pi_p$ be a supporting plane of $G \preceq F$,

$$G \subseteq \pi_p \quad \text{and} \quad F \setminus G \subseteq (\pi^+_p)^G.$$ 

Split $P = (P \cap \pi^-_p) \cup (P \cap \pi^+_p) = \mathbb{P}^- \cup \mathbb{P}^+$ that are visualized in Figure 8.1.

**Case 1.** Suppose $n \in P^+ \setminus G$. Then in view that $n \in P^+ \subseteq \pi^+_p$,

$$p \cdot (n - m) \geq 0 \quad \text{for all } n \in P \setminus G \text{ and } m \in G.$$ \hfill (8.15)

By (8.12) and (8.15), we have $\geq$ in (8.11). Thus either (8.13) or (8.14) yields $> \text{ in (8.11)}$.

**Case 2.** Suppose that $n \in P^- \setminus G$. Note that $G \preceq \mathbb{P}^-$. Consider a hyperplane $\pi_q(F)$ containing $F$ and whose normal vector is $q$. Then $\pi_q(F)$ is a supporting plane of $G \preceq \mathbb{P}^-$. Thus

$$q \in (G^*)^G|\mathbb{P}^- \cup \mathbb{P}^+.$$
Hence by Lemma 8.1, for all $n \in \mathbb{P}^{-} \setminus \mathbb{G}$ and $m \in \mathbb{G}$,

\begin{equation}
q \cdot \frac{P_{V^\perp(G)}(n-m)}{|P_{V^\perp(G)}(n-m)|} \geq r > 0 \quad \text{and} \quad P_{V^\perp(G)}(n-m) \neq 0
\end{equation}

where the last follows from (8.4) and (8.5). Split

\[ n - m = P_{V(G)}(n-m) + P_{V^\perp(G)}(n-m). \]

Since $q \in (\mathbb{G}^*)^0((\mathbb{P}^-, \mathbb{R}^n)$ and $p \in (\mathbb{G}^*)^0\mathbb{F}$, that is $q, p \perp V(G)$,

\[ q \cdot P_{V(G)}(n-m) = p \cdot P_{V(G)}(n-m) = 0. \]

So

\[ (q + \epsilon p) \cdot (n - m) = (q + \epsilon p) \cdot P_{V^\perp(G)}(n - m). \]

Choose $\epsilon_0 = \frac{r}{100|p|}$ and $0 < \epsilon < \epsilon_0$. Then, by (8.16),

\[ (q + \epsilon p) \cdot (n - m) = (q + \epsilon p) \cdot P_{V^\perp(G)}(n - m) \geq r |P_{V^\perp(G)}(n-m)| - \epsilon |p| |P_{V^\perp(G)}(n-m)| \geq \frac{99}{100} r |P_{V^\perp(G)}(n-m)| > 0. \]

This completes the proof of Proposition 8.1. \qed

**Lemma 8.2.** Let $\mathbb{P}_\nu \subset \mathbb{R}^n$ be a polyhedron and $\mathbb{F}_\nu, \mathbb{G}_\nu \in \mathcal{F}(\mathbb{P}_\nu)$ such that $\mathbb{G}_\nu \preceq \mathbb{F}_\nu$. Suppose that $q \in \bigcap_\nu (\mathbb{F}_\nu^*)^0((\mathbb{P}_\nu, \mathbb{R}^n)$. Suppose $p \in \bigcap_\nu (\mathbb{G}_\nu^*)^0((\mathbb{F}_\nu, \mathbb{R}^n)$. Then there exists a vector $w \in \bigcap_\nu (\mathbb{G}_\nu^*)^0((\mathbb{P}_\nu, \mathbb{R}^n)$.\]

**Proof.** By Proposition 8.1, there exists $\epsilon_\nu > 0$ such that for $0 < \epsilon < \epsilon_\nu$, $q + \epsilon p \in (\mathbb{G}_\nu^*)^0((\mathbb{P}_\nu, \mathbb{R}^n)$. Choose $\epsilon_0 = \min\{\epsilon_\nu : \nu = 1, \cdots, d\}$. Then for $0 < \epsilon < \epsilon_0$, $q + \epsilon p \in \bigcap_{\nu=1}^d (\mathbb{G}_\nu^*)^0((\mathbb{P}_\nu, \mathbb{R}^n)$. \qed

**8.2. Lemma for Necessity.**
Lemma 8.3. Let $\Lambda = (\Lambda_\nu)$ with $\Lambda_\nu \subset \mathbb{Z}_+^n$ and let $P_\Lambda \in \mathcal{P}_\Lambda$. Fix $S \subset \{1, \cdots, n\}$. Given $F = (F_\nu) \in \mathcal{F}(\mathcal{N}(\Lambda, S))$, define

$$I(P_\Lambda, \xi, r) = \int_{\prod(-r_j, r_j)} e^{i\xi_1 \sum_{m \in F_1 \cap \Lambda_1} c_1^m t^m + \cdots + \xi_d \sum_{m \in F_d \cap \Lambda_d} c_d^m t^m} \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n},$$

$$I(P_\Lambda, \xi, a, b) = \int_{\prod\{a_j < |t_j| < b_j\}} e^{i\xi_1 \sum_{m \in F_1 \cap \Lambda_1} c_1^m t^m + \cdots + \xi_d \sum_{m \in F_d \cap \Lambda_d} c_d^m t^m} \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}.$$

Suppose that

$$\sup_{r \in I(S), \xi \in \mathbb{R}^d} |I(P_\Lambda, \xi, r)| < \infty$$

where $I(S)$ is as defined in (1.3). Suppose also that

$$u = (u_1, \cdots, u_n) \in \bigcap_{\nu=1}^d (\mathbb{F}_\nu^*)^C \neq \emptyset$$

where

(8.17) $u_j > 0$ for $j \in S \setminus S_0$ and $u_j = 0$ for $j \in S_0 \subset S$ by Lemma 4.14.

Then

(8.18) $\sup_{\xi \in \mathbb{R}^d, a, b \in I(S_0)} |I(P_\Lambda, \xi, a, b)| < \infty.$

Proof of (8.18). By the definition of $(\mathbb{F}_\nu^*)^C$, there exists $\rho_\nu$ such that

(8.19) $u \cdot m = \rho_\nu < u \cdot n$ for all $m \in F_\nu$ and $n \in \mathcal{N}(\Lambda_\nu, S) \setminus F_\nu.$

Let $a = (a_j), b = (b_j) \in I(S_0)$ where $I(S_0) = \prod_{j=1}^n I_j$ where

(8.20) $I_j = (0, \infty)$ for $j \in \{1, \cdots, n\} \setminus S_0$ and $I_j = (0, 1)$ for $j \in S_0$ as in (1.3).

Let $\rho = (\rho_\nu)$ with $\rho_\nu$ in (8.19). Set

$$I(a(\delta), b(\delta)) = \prod_{j=1}^n \{a_j \delta^{u_j} < |t_j| < b_j \delta^{u_j}\} \text{ and } \delta^{-\rho} \xi = (\delta^{-\rho_1} \xi_1, \cdots, \delta^{-\rho_d} \xi_d).$$

Then

$$I(P_\Lambda, \delta^{-\rho} \xi, a(\delta), b(\delta)) = \int_{I(a(\delta), b(\delta))} e^{i(\delta^{-\rho_1} \xi_1 \sum_{m \in \Lambda_1} c_1^m t^m + \cdots + \delta^{-\rho_d} \xi_d \sum_{m \in \Lambda_d} c_d^m t^m)} \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}.$$
By (8.17) and (8.20), we find a sufficiently small $\delta$ such that
\[
\delta^{u_j} a_j, \delta^{v_j} b_j < 1 \quad \text{for } j \in S \setminus S_0 \quad \text{and} \quad \delta^{u_j} a_j = a_j, \delta^{v_j} b_j = b_j < 1 \quad \text{for } j \in S_0 \subset S.
\]

Thus $a(\delta), b(\delta) \in I(S)$. Hence, by our hypothesis
\[
(8.21) \quad \left| I(P_{\Lambda}, \delta^{-\rho} \xi, a(\delta), b(\delta)) \right| \leq C \quad \text{uniformly in } \xi \text{ and } a, b, \delta.
\]

Consider the difference of two multipliers given by
\[
M(\xi, \delta, a, b) = I(P_{\Lambda}, \delta^{-\rho} \xi, a(\delta), b(\delta)) - I(P_{\mathbb{R}}, \delta^{-\rho} \xi, a(\delta), b(\delta))
\]
\[
= \int_{I(a(\delta), \delta, b(\delta))} e^{i(\xi \delta^{-\rho_1} \sum_{m \in \Lambda_1} c_1^m t_m + \cdots + \xi_d \delta^{-\rho_d} \sum_{m \in \Lambda_d} c_d^m t_m)} \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}
\]
\[
- \int_{I(a(\delta), \delta, b(\delta))} e^{i(\xi \delta^{-\rho_1} \sum_{m \in \mathbb{F}_1 \cap \Lambda_1} c_1^m t_m + \cdots + \xi_d \delta^{-\rho_d} \sum_{m \in \mathbb{F}_d \cap \Lambda_d} c_d^m t_m)} \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}.
\]

By the mean value theorem and change of variable $t'_j = \delta^{-u_j} t_j$ in the above two integrals,
\[
|M(\xi, \delta, a, b)|
\leq \int_{I(a, b)} \left| \sum_{n \in \Lambda_1 \setminus \mathbb{F}_1} \delta^{u_1 - \rho_1} c_1^n t^n + \cdots + \sum_{n \in \Lambda_d \setminus \mathbb{F}_d} \delta^{u_d - \rho_d} c_d^n t^n \right| \frac{dt_1}{|t_1|} \cdots \frac{dt_n}{|t_n|}
\]
\[
\leq |\xi_1| \sum_{n \in \Lambda_1 \setminus \mathbb{F}_1} \delta^{u_1 - \rho_1} |c_1^n| C^1_n(a, b) + \cdots + |\xi_d| \sum_{n \in \Lambda_d \setminus \mathbb{F}_d} \delta^{u_d - \rho_d} |c_d^n| C^d_n(a, b).
\]

The constants $C^1_m(a, b), \ldots, C^d_m(a, b)$ above are absolute value for the integral of \( \frac{t^m}{|t_1| \cdots |t_n|} \) on the region $I(a, b)$. From $u \cdot n - \rho_\nu > 0$ in (8.19), we can choose $\delta > 0$ so that $\delta^{u \cdot n - \rho_\nu}$ above is small enough to satisfy
\[
|M(\xi, \delta, a, b)| \leq 1.
\]

By this and (8.21),
\[
(8.22) \quad \left| I(P_{\mathbb{R}}, \delta^{-\rho} \xi, a(\delta), b(\delta)) \right| \leq C.
\]
By (8.19) and the change of variables $\delta^{-u_j} t_j = t'_j$ for all $j = 1, \ldots, n$,

$$I(P_F, \delta^{-\rho} \xi, a(\delta), b(\delta))$$

$$= \int_{I(a(b), b(\delta))} e^{i\sum_{m \in \mathbb{F}_1} c_1^m t_1^m + \cdots + \sum_{m \in \mathbb{F}_d} c_d^m t_d^m} \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}$$

Hence this identity combined with (8.22) yields (8.18). □

8.3. Necessity Theorem.

Definition 8.2. We define a set of rank $m$-subsets of $\mathbb{R}^n$ by

$$\mathcal{M}_{m,n} = \{ M \subset \mathbb{R}^n : \dim(\text{span}(M)) = m, \text{ and span}(P_{\mathbb{R}^m \times \{0\}^{n-m}}(M)) = \mathbb{R}^m \times \{0\}^{n-m}\}.$$

Theorem 8.1 (Necessity Part of Main Theorems 3.2). Let $\Lambda = (\Lambda_\nu)$ where $\Lambda_\nu \subset \mathbb{Z}_+^n$ is a finite set for $\nu = 1, \ldots, d$ and let $S \subset \{1, \ldots, n\}$. Suppose that there exist a face $\mathbb{F} = (\mathbb{F}_\nu) \in \mathcal{F}_0(\mathbb{N}(\Lambda, S))$ such that

$$\bigcup_{\nu=1}^{d} (\mathbb{F}_\nu \cap \Lambda_\nu) \text{ is not an even set.} \quad (8.23)$$

Here we remind $(\mathbb{F}_\nu) \in \mathcal{F}_0(\mathbb{N}(\Lambda, S))$ means that

$$\text{rank} \left( \bigcup_{\nu} \mathbb{F}_\nu \right) \leq n - 1 \quad \text{and} \quad \bigcap_{\nu} (\mathbb{F}_\nu^\circ)^0 \mid \mathbb{N}(\Lambda_\nu, S) \neq \emptyset. \quad (8.24)$$

Then there exist a vector polynomial $P_\Lambda \in \mathcal{P}_\Lambda$ so that

$$\sup_{\xi \in \mathbb{R}^d, r \in I(S)} |I(P_\Lambda, \xi, r)| = \infty.$$ 

Proof of Theorem 8.1. Choose the integer $m$ such that

$$m = \min \left\{ \text{rank} \left( \bigcup_{\nu} \mathbb{F}_\nu \right) : \exists \mathbb{F} \in \mathcal{F}(\mathbb{N}(\Lambda, S)) \text{ satisfying } (8.23), (8.24) \right\}. \quad (8.25)$$

Then we have $\mathbb{F} = (\mathbb{F}_\nu)$ with $\mathbb{F}_\nu \in \mathcal{F}(\mathbb{N}(\Lambda_\nu, S))$ such that

$$\bigcup_{\nu=1}^{d} (\mathbb{F}_\nu \cap \Lambda_\nu) \text{ is not an even set.} \quad (8.26)$$
and
\begin{equation}
\text{rank} \left( \bigcup_{\nu} F_{\nu} \right) = m \leq n - 1 \quad \text{and} \quad \bigcap_{\nu} (F_{\nu}^*)^o|N(\Lambda_{\nu}, S) \neq \emptyset.
\end{equation}

By this rank condition and Definition 8.2, we can assume that without loss of generality,
\begin{equation}
\text{Sp} \left( \bigcup_{\nu} F_{\nu} \right) \in M_{m,n}
\end{equation}
where $M_{m,n}$ is in Definition 8.2. By (8.27) and (8.17), we have for some $S_0 \subset S \subset N_n$
\begin{equation}
(u_j) \in \bigcap_{\nu} (F_{\nu}^*)^o|N(\Lambda_{\nu}, S) \quad \text{and} \quad u_j = 0 \quad \text{for} \quad j \in S_0 \quad \text{and} \quad u_j > 0 \quad \text{for} \quad j \in S \setminus S_0.
\end{equation}

By Lemma 4.20,
\begin{equation}
\{ e_j : j \in S_0 \} \subset \text{Sp} \left( \bigcup_{\nu} F_{\nu} \right).
\end{equation}
Moreover, we see from (8.25) that for any $G_{\nu} \preceq F_{\nu}$,
\begin{equation}
\bigcup_{\nu} G_{\nu} \cap \Lambda_{\nu} \quad \text{is even whenever} \quad \text{rank} \left( \bigcup_{\nu} G_{\nu} \right) \leq m - 1 \quad \text{and} \quad \bigcap_{\nu} (G_{\nu}^*)^o|N(\Lambda_{\nu}, S) \neq \emptyset.
\end{equation}

In order to show Theorem 8.1, by Lemma 8.3, it suffices to find, under the assumption (8.26)-(8.31), a polynomial $P_{\Lambda}(t) = \left( \sum_{q \in \Lambda} c_q^o t^q \right) \in \mathcal{P}_{\Lambda}$ with an appropriate $c_q^o$ such that
\begin{equation}
\sup_{\xi \in \mathbb{R}^d, a, b \in I(S_0)} |I(P_{\xi}, \xi, a, b)| = \infty
\end{equation}
where
\begin{equation}
I(P_{\xi}, \xi, a, b) = \int_{\Pi_{\{a_j < |t_j| < b_j\}}} e^{i \left( \sum_{m \in \mathbb{F} \cap \Lambda_{\nu}} c_{m}^o t^m + \cdots + \xi d \sum_{m \in \mathbb{F} \cap \Lambda_{\nu}} c_{m}^d t^m \right) \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}.
\end{equation}

9. Preliminary Lemmas for Necessity Proof


**Proposition 9.1.** In proving (8.32) under the assumptions (8.26)-(8.31), we can impose another assumption that
\begin{equation}
\{ q \in \Sigma \left( \bigcup_{\nu} (F_{\nu} \cap \Lambda_{\nu}) \right) : q = (\underbrace{odd, \cdots, odd}_m, *, \cdots, *) \}
\end{equation}
\begin{equation}
\subset \{ q \in \Sigma \left( \bigcup_{\nu} (F_{\nu} \cap \Lambda_{\nu}) \right) : q = (odd, \cdots, odd) \}
\end{equation}
where \( \Sigma \) was defined in (3.2) as \( \Sigma (\bigcup (F_{\nu} \cap \Lambda_{\nu})) = \{ \alpha_1 m_1 + \cdots + \alpha_N m_N : \alpha_j = 0 \text{ or } 1 \} \) for \( \bigcup (F_{\nu} \cap \Lambda_{\nu}) = \{ m_1, \cdots, m_N \} \).

**Definition 9.1.** Given a permutation \( \sigma : \{1, \cdots, n\} \rightarrow \{1, \cdots, n\} \), we define the permutation matrix \( T_{\sigma} \) by

\[
T_{\sigma}(m) = (m_{\sigma(1)}, \cdots, m_{\sigma(n)}) \quad \text{for} \quad m = (m_1, \cdots, m_n).
\]

In proving Proposition 9.1, we need the following lemma.

**Lemma 9.1.** Let \( E \subset \mathbb{Z}^n \) be a finite set and let \( E \in \mathcal{M}_{m,n} \) where \( \mathcal{M}_{m,n} \) is in Definition 8.2. Suppose there exists a vector \((\text{odd}, \cdots, \text{odd}) \in E\)

(9.2)

Then, there exists a permutation \( \sigma \) satisfying the following two properties:

(9.3) \( \{ q \in T_{\sigma}(E) : q = (\text{odd}, \cdots, \text{odd}, *, \cdots, *) \} \subset \{ q \in T_{\sigma}(E) : q = (\text{odd}, \cdots, \text{odd}) \} \)

and

(9.4) \( T_{\sigma}(E) \in \mathcal{M}_{m,n} \).

Our proof of Lemma 9.1 is based on iterated applications of the following observation.

**Observation 9.1.** Let \( E \in \mathcal{M}_{m,n} \) with \( E \subset \mathbb{Z}^n \) be a finite set containing a vector \((\text{odd}, \cdots, \text{odd}) \). For each \( \mu \in \{m+1, \cdots, n\} \), we denote

(9.5) \( Z(m|\mu) = \{ q \in \mathbb{Z}^n : q = (\text{odd}, \cdots, \text{odd}, *, \cdots, *) \}, \)

equivalently \( q = (q_j) \in Z(m|\mu) \) if and only if \( q_j \) is odd numbers for \( j = 1, \cdots, m \) and \( q_{\mu} \) is even number. Then we observe that for each \( \mu \in \{m+1, \cdots, n\} \), there exist \( k \in \{1, \cdots, m\} \) such that

(9.6) \( Z(m|\mu) \cap T_{\sigma_{k,\mu}}(E) = \emptyset \) and \( T_{\sigma_{k,\mu}}(E) \in \mathcal{M}_{m,n} \).

where \( \sigma_{k,\mu} \) is a transposition switching \( k \) and \( \mu \).
Proof of Observation 9.1. To each subset \( M = \{q_1, \cdots, q_N\} \subset \mathbb{R}^n \), we associate a matrix:

\[
M_{tr}(M) = \begin{pmatrix}
q_1 \\
\vdots \\
q_N
\end{pmatrix}
\]

whose rows are vectors in \( M \). Let \( E = \{q_1, \cdots, q_N\} \). Since \( E \in \mathcal{M}_{m,n} \), by reordering if necessary, \( M_{tr}(E) \sim \begin{pmatrix} U' \\ 0 \end{pmatrix} \) where 0 is the zero matrix of size \((N - m) \times n\) and \( U \) is the \( m \times n \) matrix given by

\[
U = \begin{pmatrix}
1 & 0 & \cdots & 0 & c_{1,m+1} & \cdots & c_{1,\mu} & \cdots & c_{1,n} \\
0 & 1 & 0 & \vdots & c_{2,m+1} & \cdots & c_{2,\mu} & \cdots & c_{2,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & c_{m,m+1} & \cdots & c_{m,\mu} & \cdots & c_{m,n}
\end{pmatrix}.
\]

Here \( \sim \) means row equivalence and \((c_{ij})_{1 \leq i \leq m, m+1 \leq j \leq n}\) a real \( m \times (n - m) \) matrix. We add \((\text{odd}, \cdots, \text{odd})\) to the last row of the above matrix. Then, \( M_{tr}(E) \sim \begin{pmatrix} R \\ 0 \end{pmatrix} \)

\[
R = \begin{pmatrix}
1 & 0 & \cdots & 0 & c_{1,m+1} & \cdots & c_{1,\mu} & \cdots & c_{1,n} \\
0 & 1 & 0 & \vdots & c_{2,m+1} & \cdots & c_{2,\mu} & \cdots & c_{2,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & c_{m,m+1} & \cdots & c_{m,\mu} & \cdots & c_{m,n}
\end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \\ r_{m+1}
\end{pmatrix}.
\]

In order to prove (9.6), it suffices to prove that there exists \( k \in \{1, \cdots, m\} \) such that

\[
T_{\sigma_{k,\mu}}(\mathbb{Z}(m|\mu)) \cap E = \emptyset \quad \text{and} \quad c_{k,\mu} \neq 0.
\]

Indeed, the first equation in (9.9) implies

\[
\mathbb{Z}(m|\mu) \cap T_{\sigma_{k,\mu}}(E) = T_{\sigma_{k,\mu}}(T_{\sigma_{k,\mu}}(\mathbb{Z}(m|\mu)) \cap E) = \emptyset.
\]

which is the first part of (9.6). The second property \( c_{k,\mu} \neq 0 \) in (9.9) yields \( T_{\sigma_{k,\mu}}(E) \in \mathcal{M}_{m,n} \), which is the second part in (9.6). This can be checked by switching the \( k^{th} \) column.
\( e_k \) and the \( \mu^{th} \) column \((c_{\ell,\mu})_\ell\) with \( c_{k,\mu} \neq 0 \) of the matrix in (9.7) and adding some multiple of \( k^{th} \) row to another row so as to remove nonzero terms of \( k^{th} \) columns.

**Proof of (9.9).** If \( c_{\ell,\mu} = 0 \) for all \( \ell \in \{1, \cdots, m\} \), then the submatrix \( R|_{1,\cdots,m,\mu} \) consisting of 1 through \( m \) columns and \( \mu^{th} \) column of the matrix \( R \) in (9.8) given by

\[
R|_{1,\cdots,m,\mu} = \begin{pmatrix}
1 & 0 & \cdots & 0 & c_{1,\mu} \\
0 & 1 & 0 & \vdots & c_{2,\mu} \\
\vdots & 0 & 1 & 0 & \vdots \\
0 & \cdots & 0 & 1 & c_{m,\mu} \\
\mathrm{odd} & \cdots & \mathrm{odd} & \mathrm{odd} & \mathrm{odd}
\end{pmatrix}
\]

with \( c_{1,\mu} = \cdots = c_{m,\mu} = 0 \) has rank \( \geq m + 1 \), which is impossible. Thus, at least one of \( c_{\ell,\mu} \neq 0 \). So, we set

\( A = \{ \ell : c_{\ell,\mu} \neq 0 \} \neq \emptyset \) and \( B = \{ \ell : c_{\ell,\mu} = 0 \} \) with \( A \cup B = \{1, \cdots, m\} \). Note that the set \( T_{\sigma_{\ell,\mu}}(\mathbb{Z}(m|\mu)) \cap E \) is expressed as

\[
\{ q \in E : q = (\underbrace{\mathrm{odd}, \cdots, \mathrm{odd}, \mathrm{even}, \cdots, odd}_{\ell \ \mathrm{components}}, \underbrace{odd, \cdots, odd}_{m \ \mathrm{components}}, \underbrace{*, \cdots, *, odd}_{\mu \ \mathrm{components}}, \cdots, *) \}\.
\]

Assume that

\[
T_{\sigma_{\ell,\mu}}(\mathbb{Z}(m|\mu)) \cap E \neq \emptyset \quad \text{for every} \quad \ell \in A = \{ \ell : c_{\ell,\mu} \neq 0 \}.
\]

Choose \( p_\ell \in T_{\sigma_{\ell,\mu}}(\mathbb{Z}(m|\mu)) \cap E \) for each \( \ell \in A \) and define a matrix \( \tilde{R} \) by

\[
\tilde{R} = \begin{pmatrix}
\tilde{r}_1 \\
\tilde{r}_2 \\
\vdots \\
\tilde{r}_m \\
\tilde{r}_{m+1}
\end{pmatrix}
\]

where \( \tilde{r}_\ell = p_\ell \) for \( \ell \in A \) and \( \tilde{r}_\ell = r_\ell \) for \( \ell \in B \cup \{m + 1\} \).
where $r_\ell$ is in (9.8). We express the above rows $\tilde{r}_\ell$ of $\tilde{R}$ more precisely as

$$
\tilde{r}_\ell = \begin{cases} 
(\text{odd}, \cdots, \text{odd}, \text{even}, \text{odd}, \cdots, \text{odd}, *, \cdots, *, \text{odd}, *, \cdots, *) = p_\ell & \text{if } \ell \in A = \{c_{\ell,\mu} \neq 0\} \\
(0, \cdots, 0, 1, 0, \cdots, 0, c_{\ell,m+1}, \cdots, c_{\ell,\mu}, c_{\ell,\mu+1}, \cdots, c_{\ell,n}) = r_\ell & \text{if } \ell \in B = \{c_{\ell,\mu} = 0\} \\
(\text{odd}, \cdots, \text{odd}) = r_{m+1} & \text{if } \ell = m + 1.
\end{cases}
$$

Since $\tilde{R}$ is obtained by replacing some rows in $R$ with vectors in $E$,

$$(9.14) \quad \tilde{R} \sim R.
$$

Next, consider the restriction of $\tilde{R}$ to the matrix consisting of $1, \cdots, m$ and $\mu$th columns:

$$\tilde{R}|_{1,\cdots,m,\mu} = \begin{pmatrix} 
\tilde{r}_1|_{1,\cdots,m,\mu} \\
\tilde{r}_2|_{1,\cdots,m,\mu} \\
\vdots \\
\tilde{r}_m|_{1,\cdots,m,\mu} \\
\tilde{r}_{m+1}|_{1,\cdots,m,\mu}
\end{pmatrix}
$$

where

$$\tilde{r}_\ell|_{1,\cdots,m,\mu} = \begin{cases} 
(\text{odd}, \cdots, \text{odd}, \text{even}, \text{odd}, \cdots, \text{odd}, \text{odd}) & \text{if } \ell \in A = \{c_{\ell,\mu} \neq 0\} \\
(0, \cdots, 0, 1, 0, \cdots, 0, 0) & \text{if } \ell \in B = \{c_{\ell,\mu} = 0\} \\
(\text{odd}, \cdots, \text{odd}) & \text{if } \ell = m + 1.
\end{cases}
$$
Thus the determinant of \((m + 1) \times (m + 1)\) matrix \(\tilde{R}_{1,\ldots,m,\mu}\) is computed as

\[
\begin{align*}
\det \left( \tilde{R}_{1,\ldots,m,\mu} \right) & = \det \begin{pmatrix}
\text{even} & \text{odd} & \text{odd} & \text{odd} \\
\text{odd} & \text{even} & \text{odd} & \text{odd} \\
\text{odd} & \text{odd} & \text{even} & \text{odd} \\
\text{odd} & \text{odd} & \text{odd} & \text{odd}
\end{pmatrix} \\
& = \det \begin{pmatrix}
\text{odd} & \text{even} & \text{even} & \text{even} \\
\text{even} & \text{odd} & \text{even} & \text{even} \\
\text{even} & \text{even} & \text{odd} & \text{even} \\
\text{odd} & \text{odd} & \text{odd} & \text{odd}
\end{pmatrix} = \text{odd} \neq 0.
\end{align*}
\]

The second equality above follows by adding the last row to the other rows. This with (9.14) implies that \(\text{rank}(R) = \text{rank}(\tilde{R}) \geq \text{rank}(\tilde{R}_{1,\ldots,m,\mu}) \geq m+1\), which is a contradiction to \(E \in \mathcal{M}_{mn}\). The assumption (9.13) does not always hold, which means that there exists \(k \in \{1, \ldots, m\}\) such that

\[
T_{\sigma_{k,\mu}}(Z(m|\mu)) \cap E = \emptyset \quad \text{and} \quad c_{k,\mu} \neq 0. \tag{9.15}
\]

Hence (9.9) is proved. \(\square\)

Therefore we have finished the proof of Observation 9.1 as we have already justified by (9.10) and the consecutive remarks. \(\square\)

**Proof of Lemma 9.1.** We are given a finite set \(E \in \mathcal{M}_{m,n}\) with \(E \cap \{(\text{odd}, \cdots, \text{odd})\} \neq \emptyset\). So, we are able to apply Observation 9.1 to the set \(E\) for \(\mu = m + 1\). Then we have a permutation \(\sigma_1 = \sigma_{k_1,m+1}\) with \(k_1 \in \{1, \cdots, m\}\) so that

\[
T_{\sigma_1}(E) \in \mathcal{M}_{m,n}
\]

and

\[
\{q \in T_{\sigma_1}(E) : q = (\underbrace{\text{odd}, \cdots, odd, even, \ast, \cdots, \ast}_{m \text{ components}})\} = \emptyset. \tag{9.16}
\]

Obviously, for any permutation \(\sigma\),

\[
\{(\text{odd}, \cdots, \text{odd})\} \cap E \neq \emptyset \implies \{(\text{odd}, \cdots, \text{odd})\} \cap T_\sigma(E) \neq \emptyset.
\]
We thus apply Observation 9.1 to $T_{\sigma_1}(E) \in \mathcal{M}_{m,n}$ with $T_{\sigma_1}(E) \cap \{(odd, \cdots, odd)\} \neq \emptyset$ for $\mu = m + 2$. We then have a permutation $\sigma_2 = \sigma_{k_2,m+2}$ with $k_2 \in \{1, \cdots, m\}$ where

$$T_{\sigma_2}T_{\sigma_1}(E) \in \mathcal{M}_{m,n}$$

and

(9.17) $\{q \in T_{\sigma_2}T_{\sigma_1}(E) : q = (odd, \cdots, odd, *, even, *, \cdots, *)\} = \emptyset.$

By (9.16) and $\sigma_2 = \sigma_{k_2,m+2},$ (9.18) $\{q \in T_{\sigma_2}T_{\sigma_1}(E) : q = (odd, \cdots, odd, even, odd, *, \cdots, *)\} = \emptyset.$

Apply Observation 9.1 to $T_{\sigma_2}T_{\sigma_1}(E) \in \mathcal{M}_{m,n}$ with $T_{\sigma_2}T_{\sigma_1}(E) \cap \{(odd, \cdots, odd)\} \neq \emptyset$ for $\mu = m + 3$. We then have a permutation $\sigma_3 = \sigma_{k_3,m+3}$ with $k_3 \in \{1, \cdots, m\}$ where

$$T_{\sigma_3}T_{\sigma_2}T_{\sigma_1}(E) \in \mathcal{M}_{m,n}$$

and

(9.19) $\{q \in T_{\sigma_3}T_{\sigma_2}T_{\sigma_1}(E) : q = (odd, \cdots, odd, *, *, even, *, \cdots, *)\} = \emptyset.$

By (9.17),(9.18) and $\sigma_3 = \sigma_{k_3,m+3},$ (9.20) $\{q \in T_{\sigma_3}T_{\sigma_2}T_{\sigma_1}(E) : q = (odd, \cdots, odd, *, even, odd, *, \cdots, *)\} = \emptyset.$

(9.21) $\{q \in T_{\sigma_3}T_{\sigma_2}T_{\sigma_1}(E) : q = (odd, \cdots, odd, even, odd, odd, *, \cdots, *)\} = \emptyset.$

We repeat this process we have $\sigma = \sigma_{n-m} \cdots \sigma_1$ such that

$$T_{\sigma}(E) = T_{\sigma_{n-m}} \cdots T_{\sigma_1}(E) \in \mathcal{M}_{m,n}$$
and

\[
\{ q \in T_{\sigma_{n-m}} \cdots T_{\sigma_1}(E) : q = (odd, \ldots, odd, even, odd, odd, odd) \} = \emptyset,
\]

\[
\{ q \in T_{\sigma_{n-m}} \cdots T_{\sigma_1}(E) : q = (odd, \ldots, odd, * \text{ even}, odd, odd) \} = \emptyset,
\]

\[
\{ q \in T_{\sigma_{n-m}} \cdots T_{\sigma_1}(E) : q = (odd, \ldots, odd, * \text{ even}, odd) \} = \emptyset,
\]

\[
\{ q \in T_{\sigma_{n-m}} \cdots T_{\sigma_1}(E) : q = (odd, \ldots, odd, * \text{ even}, odd) \} = \emptyset.
\]

Thus \( q = (odd, \ldots, odd, *, \ldots, *) \in T_{\sigma}(E) \) is always of the form \((odd, \ldots, odd)\). Therefore the proof of Lemma 9.1 is finished.

**Proof of Proposition 9.1.** Let \( E = \Sigma \left( \bigcup_{\nu=1}^{d} (F_{\nu} \cap \Lambda_{\nu}) \right) \). Then by (8.26) and (8.28),

\[
E \in \mathcal{M}_{m,n} \quad \text{and} \quad \exists (odd, \ldots, odd) \in E.
\]

We apply Lemma 9.1 for this set \( E \) to find a permutation \( \sigma \) satisfying (9.3) and (9.4) such that

\[
\{ q \in \Sigma \left( \bigcup_{\nu=1}^{d} T_{\sigma} (F_{\nu} \cap \Lambda_{\nu}) \right) : q = (odd, \ldots, odd, *, \ldots, *) \}
\]

\[
\subset \left\{ q \in \Sigma \left( \bigcup_{\nu=1}^{d} T_{\sigma} (F_{\nu} \cap \Lambda_{\nu}) \right) : q = (odd, \ldots, odd) \right\}.
\]

(9.22)

and

\[
\Sigma \left( \bigcup_{\nu=1}^{d} T_{\sigma} (F_{\nu} \cap \Lambda_{\nu}) \right) \in \mathcal{M}_{m,n}.
\]

(9.23)

We rewrite

\[
I(P_{\xi}, \xi, a, b) = \int_{\Pi \{ a_{\sigma(j)} < [t_{\sigma(j)}] < b_{\sigma(j)} \}} e^{i \left( \sum_{m \in F_{1} \cap \Lambda_{1}} c_{1}^{m} t_{1}^{m} + \cdots + \sum_{m \in F_{d} \cap \Lambda_{d}} c_{d}^{m} t_{d}^{m} \right)} \frac{dt_{\sigma(1)}}{t_{\sigma(1)}} \cdots \frac{dt_{\sigma(n)}}{t_{\sigma(n)}}.
\]
We work with \( \mathbf{N}(\tilde{\Lambda}_\nu, \tilde{S}) \) where \( \tilde{\Lambda}_\nu = T_\sigma(\Lambda_\nu) \) and \( \tilde{S} = \sigma(S) \) instead of \( \mathbf{N}(\Lambda_\nu, S) \). By the invariance properties of Lemma 4.24,

\[
(9.24) \quad \mathbf{N}(\tilde{\Lambda}_\nu, \tilde{S}) = (T_\sigma(\mathbf{N}(\Lambda_\nu, S))) \quad \text{and} \quad \tilde{\mathcal{F}}_\nu = T_\sigma(\mathcal{F}_\nu) \in \mathcal{F}(\mathbf{N}(\tilde{\Lambda}_\nu, \tilde{S}))
\]

Using the change of variables \( T_\sigma(t) = u \) \((t_\sigma(j) = u_j)\) with \( t^m = \prod_{j=1}^m t_\sigma(j) = \prod_{j=1}^m u_j \),

\[
\mathcal{I}(P_{\tilde{\mathcal{F}}}, \xi, a, b)
\]

\[
(9.25) \quad e^{i(\xi_1 \sum_{n \in T_\sigma(\mathcal{F}) \cap T_\sigma(\Lambda_\nu)} c_n^\nu u^n + \cdots + \xi_d \sum_{n \in T_\sigma(\mathcal{F}) \cap T_\sigma(\Lambda_\nu)} c_n^\nu u^n)} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}
\]

where \( n = T_\sigma(m) \) and \( c_n^\nu = c_n^{\nu \cdot T_\sigma(n)} \). If we denote the integral (9.25) by \( \mathcal{I}(P_P, \xi, \tilde{a}, \tilde{b}) \) with \( P_\tilde{\Lambda}(u) = (\sum_{n \in \tilde{\Lambda}_\nu} c_n^\nu u^n) \in \mathcal{P}_\tilde{\Lambda} \) and \( \tilde{a} = (a_\sigma(j)), \tilde{b} = (b_\sigma(j)) \), then we see that (8.32) is equivalent to

\[
(9.26) \quad \sup_{\xi \in \mathbb{R}^d, \tilde{a}, \tilde{b} \in I(\sigma(S_0))} \left| \mathcal{I}(P_{\tilde{\mathcal{F}}}, \xi, \tilde{a}, \tilde{b}) \right| = \infty.
\]

We need to check (8.26)-(8.31). By (2) and (3) of Lemma 4.24 and \( T_\sigma^{-1} = T_\sigma^t \), for any \( \mathcal{G}_\nu \subseteq \mathcal{F}_\nu \)

\[
(9.27) \quad \tilde{\mathcal{G}}_\nu = T_\sigma(\mathcal{G}_\nu) \in \mathcal{F}^k(\mathbf{N}(\tilde{\Lambda}_\nu, \tilde{S})) \iff \mathcal{G}_\nu \in \mathcal{F}^k(\mathbf{N}(\Lambda_\nu, S)),
\]

\[
(9.28) \quad ((\tilde{\mathcal{G}}_\nu)^\circ)(\mathbf{N}(\tilde{\Lambda}_\nu, \tilde{S}), \mathbb{R}^n) = T_\sigma((\mathcal{G}_\nu)^\circ)(\mathbf{N}(\Lambda_\nu, S), \mathbb{R}^n)).
\]

Using (9.27) with \( \mathcal{G}_\nu = \mathcal{F}_\nu \) and the obvious observation that \( \Omega \subseteq \mathbb{Z}^n \) is an even (odd) set if and only if \( T_\sigma(\Omega) \) is an even (odd) set, we replace (8.26) by

\[
(9.29) \quad \bigcup_{\nu=1}^d (\tilde{\mathcal{F}}_\nu \cap \tilde{\Lambda}_\nu) = T_\sigma \left( \bigcup_{\nu=1}^d (\mathcal{F}_\nu \cap \Lambda_\nu) \right) \quad \text{is not an even set.}
\]

Using (9.28) with \( \mathcal{G}_\nu = \mathcal{F}_\nu \), we replace (8.27) by

\[
(9.30) \quad \text{rank} \left( \bigcup_{\nu=1}^d (\tilde{\mathcal{F}}_\nu \cap \tilde{\Lambda}_\nu) \right) = \text{rank} \left( T_\sigma \left( \bigcup_{\nu=1}^d (\mathcal{F}_\nu \cap \Lambda_\nu) \right) \right) = m,
\]

\[
\bigcap_{\nu=1}^d (\tilde{\mathcal{F}}_\nu)^\circ \mathbf{N}(\tilde{\Lambda}_\nu, \tilde{S}) = T_\sigma \left( \bigcap_{\nu=1}^d (\mathcal{F}_\nu)^\circ \mathbf{N}(\Lambda_\nu, S) \right) \neq \emptyset.
\]
Using (9.23), we replace (8.28) by
\[ \Sigma(\bigcup_{\nu=1}^{d} \tilde{F}_\nu \cap \tilde{\Lambda}_\nu) \in \mathcal{M}_{m,n}. \]

By (8.31) together with (9.27) and (9.28), we replace (8.31) with
\[ \bigcup_{\nu=1}^{d} \tilde{G}_\nu \cap \tilde{\Lambda}_\nu \] is an even set whenever
\[ \text{rank} \left( \bigcup_{\nu=1}^{d} \tilde{G}_\nu \cap \tilde{\Lambda}_\nu \right) \leq m - 1 \] and \[ \bigcap_{\nu=1}^{d} (\tilde{G}_\nu^* N(\tilde{\Lambda}_\nu, \tilde{S})) \neq \emptyset. \]
where \( \tilde{G}_\nu \preceq \tilde{F}_\nu \). Using (9.28) and (8.29),
\[ (u_j) \in \bigcap_{\nu} (\tilde{G}_\nu^* N(\tilde{\Lambda}_\nu, \tilde{S})) \] and \( u_j = 0 \) for \( j \in \tilde{S}_0 = \sigma(S_0) \) and \( u_j > 0 \) for \( j \in \tilde{S} \setminus \tilde{S}_0 \).

By this and Lemma 4.20,
\[ \{ e_j : j \in \tilde{S}_0 \} \subset \text{Sp} \left( \bigcup_{\nu} \tilde{F}_\nu \right). \]
where we see that \( \tilde{S}_0 \subset \{1, \ldots, m\} \cap \tilde{S} \) from (9.31). Thus proving (8.32) under the assumption (8.26)-(8.31) can be replaced by proving (9.26) under the assumption (9.29)-(9.33). Hence we are able to apply Lemma 9.1 with \( E = \Sigma (\bigcup (\tilde{F}_\nu \cap \Lambda_\nu)) \) to conclude that it suffices to prove (9.26) under the assumptions (9.22) with \( \tilde{F}_\nu \cap \tilde{\Lambda}_\nu = T_\nu(\tilde{F}_\nu \cap \Lambda_\nu) \) as well as (9.29)-(9.33). We therefore finish the proof of Proposition 9.1 by identifying \( \tilde{F}_\nu \) and \( \Lambda_\nu \) to \( \tilde{F}_\nu \) and \( \tilde{\Lambda}_\nu \) respectively.

\[ \square \]

9.2. Scheme of Necessity Proof.

**Definition 9.2.** Let \( 1 \leq m < n \). Using \( \mathbb{R}^n = X \oplus Y \) with \( X = \mathbb{R}^m \times \{0\} \) and \( Y = \{0\} \times \mathbb{R}^{n-m} \), we write \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \) as \( a = (a_X, a_Y) \) so that
\[ a_X = (a_1, \ldots, a_m) \in \mathbb{R}^m \] and \( a_Y = (a_{m+1}, \ldots, a_n) \in \mathbb{R}^{n-m} \).

Define
\[ 1_X(S_0) = (r_j)_{j=1}^m \] with \( r_j = 1 \) for \( j \in S_0 \) and \( r_j = \infty \) for \( j \in N_m \setminus S_0 \),
and

\[ I_X(S_0) = \prod_{j=1}^{m} I_j \] where \( I_j = (0, 1) \) for \( j \in S_0 \) and \( I_j = (0, \infty) \) for \( j \in N_m \setminus S_0 \).

To show (8.32), we prove that there exists \( P_{\Lambda}(t) = \left( \sum_{q \in \Lambda_{\nu}} c_q^{a} \right) \in \mathcal{P}_{\Lambda} \) with an appropriate \( c_q^{a} \) and \( \xi \in \mathbb{R}^d \) so that

\[
\lim_{a_X \to 0, b_X \to 1_X(S_0)} |I(P_{\xi}, a, b)| = \infty \quad \text{as} \quad a_Y \to 0.
\]

Indeed, we show that for some \( C(\xi) > 0 \),

\[
\lim_{a_X \to 0, b_X \to 1_X(S_0)} \left| I(P_{\xi}, a, b) \right| \geq C(\xi) \prod_{j=m+1}^{n} \log(b_j/a_j) \to \infty \quad \text{as} \quad a_Y \to 0.
\]

We record

\[
I(P_{\xi}, a, b) = \int_{\prod_{j=m+1}^{n} \{a_j < |t_j| < b_j\}} \left[ \int_{\prod_{j=1}^{m} \{a_j < |t_j| < b_j\}} e^{i \xi \cdot P_{\xi}(t)} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} \right] \frac{dt_{m+1}}{t_{m+1}} \cdots \frac{dt_n}{t_n}.
\]

As a first step to show (9.35), we write the integral in (9.35) as

\[
\lim_{a_X \to 0, b_X \to 1_X(S_0)} \left[ \lim_{a_X \to 0, b_X \to 1_X(S_0)} \int_{\prod_{j=m+1}^{n} \{a_j < |t_j| < b_j\}} e^{i \xi \cdot P_{\xi}(t)} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} \right] \frac{dt_{m+1}}{t_{m+1}} \cdots \frac{dt_n}{t_n}.
\]

This follows from the dominated convergence theorem. To apply the convergence theorem, we shall prove that the following lemma:

**Lemma 9.2.** For each \( t_Y = (t_{m+1}, \ldots, t_n) \in \prod_{j=m+1}^{n} \{a_j < |t_j| < b_j\} \),

\[
\sup_{\xi \in \mathbb{R}^d, a_X, b_X \in I_X(S_0)} \left| \int_{\prod_{j=1}^{m} \{a_j < |t_j| < b_j\}} e^{i \xi \cdot P_{\xi}(t)} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} \right| \leq C((a_j)_{j=m+1}^{n}, (b_j)_{j=m+1}^{n}),
\]

and that

\[
\lim_{a_X \to 0, b_X \to 1_X(S_0)} \int_{\prod_{j=m+1}^{n} \{a_j < |t_j| < b_j\}} e^{i \xi \cdot P_{\xi}(t)} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} \text{ exists}.
\]

Assume that (9.37) and (9.38) are proved in the next section. Then, by writing each integral in the \( n \)-tuple integral (9.36) as the sum of 2 integrals by separating the variables into the positive and negative parts, we decompose the above \( n \)-tuple integral into the sum
of $2^n$ pieces. For this purpose, we denote $O = \{\sigma = (\sigma_j) = (\pm 1, \cdots, \pm 1)\}$, the sign-index set of $2^n$ elements. Next by using the change of variables $t_1 = \sigma_1 t'_1, \cdots, t'_n = \sigma_n t_n$ in each integration, we write the integral (9.36) as

$$\int_{\prod_{j=m+1}^{n}\{a_j < t_j < b_j\}} J(P_{\tilde{F}}, \xi, (t_{m+1}, \cdots, t_n)) \frac{dt_{m+1}}{t_{m+1}} \cdots \frac{dt_n}{t_n}. \tag{9.39}$$

Here the integrand above is

$$J(P_{\tilde{F}}, \xi, (t_{m+1}, \cdots, t_n)) = \lim_{a_X \to 0, b_X \to 1} \int_{\prod_{j=m+1}^{n}\{a_j < t_j < b_j\}} \sum_{\sigma \in O} (-1)^{|\sigma|} \exp(i P_{\tilde{F}}(\xi, \sigma t)) \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m}. \tag{9.40}$$

where $|\sigma|$ is a number of $-1$’s in the components of $\sigma \in O$ and $P_{\tilde{F}}(\xi, \sigma t)$ is

$$P_{\tilde{F}}(\xi, \sigma t_1, \cdots, \sigma t_n) = \sum_{\nu=1}^{d} \left( \sum_{q \in F_{\nu}} \begin{pmatrix} \nu' \nu q \nu' \nu \end{pmatrix} \right) \xi_{\nu}.$$ 

The limit in (9.40) exists by (9.38). Moreover, $J(P_{\tilde{F}}, \xi, (t_{m+1}, \cdots, t_n))$ is finite by (9.37). We shall in the next section prove that it is independent of $t_Y = (t_{m+1}, \cdots, t_n)$. Indeed, we show that

**Lemma 9.3.** The integral value $J(P_{\tilde{F}}, \xi, (t_{m+1}, \cdots, t_n))$ in (9.40) does not depend on $t_Y = (t_{m+1}, \cdots, t_n)$ to be written as

$$J(P_{\tilde{F}}, \xi, (t_{m+1}, \cdots, t_n)) = J(P_{\tilde{F}}, \xi) \text{ for every } t_Y = (t_{m+1}, \cdots, t_n). \tag{9.41}$$

Moreover, $J(P_{\tilde{F}}, \xi)$ is non-vanishing in the sense that

$$\exists \ P_{\Lambda} \in P_{\Lambda} \text{ and } \xi \in \mathbb{R}^d \text{ such that } J(P_{\tilde{F}}, \xi) \neq 0. \tag{9.42}$$

Therefore (9.35) follows from Lemma 9.3 in (9.39). We shall prove Lemma 9.2 in Section 10.1, and show Lemma 9.3 in Section 10.2.

10. **Proof of Necessity**

10.1. **Proof of Lemma 9.2.** Let $\Omega = (\Omega_{\nu})$ where $\Omega_{\nu} \subset \mathbb{R}^m$ is given by

$$\Omega_{\nu} = (\mathbb{F}_{\nu} \cap \Lambda_{\nu}) \chi = \{(q_1, \cdots, q_m) : (q_1, \cdots, q_n) \in \mathbb{F}_{\nu} \cap \Lambda_{\nu}\}. \tag{10.1}$$
For each \( t_Y = (t_{m+1}, \ldots, t_n) \in \prod_{i=m+1}^{n} \{ a_j < |t_j| < b_j \} \), define
\[
\mathcal{I}(P_\Omega, \xi, aX, bX, t_Y) = \int_{\prod_{i=m+1}^{n} \{ a_j < |t_j| < b_j \}} e^{i\xi \cdot P_\Omega(t_{1}, \ldots, t_{m}, t_Y)} \frac{dt_1 \ldots dt_m}{t_1 \ldots t_m}.
\]

In view of (2.6), we let
\[
\mathcal{I}_f(P_\Omega, \xi, t_Y) = \int e^{i\xi \cdot P_\Omega(t_{1}, \ldots, t_{m}, t_Y)} \prod_{\ell=1}^{m} h \left( \frac{t_\ell}{2^{j_\ell}} \right) dt.
\]

We regard \( P_\Omega(t_{1}, \ldots, t_{m}, t_Y) \) as the polynomial of variables \( t_{1}, \ldots, t_{m} \) with coefficients depending on the fixed numbers in \( t_Y = (t_{m+1}, \ldots, t_n) \). In view of \( \text{Sp}(\bigcup_{\nu} F_\nu) \in M_{m,n} \) in (8.28), there is a one to one correspondence between \( \text{Sp}(\bigcup_{\nu} F_\nu) \) and \( \mathbb{R}^m \times \{ 0 \}^{n-m} \).

Thus, we observe that for \( q = (q_1, \cdots, q_n) \in F_\nu \cap \Lambda_\nu \), the coefficient \( c_{qX}^\nu \) of the monomial \( t_X^q = \prod_{j=1}^{m} t_j^{q_j} \) in \( P_\Omega(t_{1}, \ldots, t_{m}, t_Y) \) is \( c_{qY}^\nu \). More precisely,
\[
P_\Omega(t_{1}, \ldots, t_{m}, t_Y) = \left( \sum_{q \in F_\nu \cap \Lambda_\nu} (c_{qY}^\nu) t_X^q \right) \text{ where } t_X^q = \prod_{j=1}^{m} t_j^{q_j} \text{ and } t_Y^q = \prod_{j=m+1}^{n} t_j^{q_j}.
\]

Let \( D_q = \max\{ \prod_{j=m+1}^{n} |t_j^{q_j}| : a_j < |t_j| < b_j \} \) and \( d_q = \min\{ \prod_{j=m+1}^{n} |t_j^{q_j}| : a_j < |t_j| < b_j \} \).

Then the coefficient \( c_{qX}^\nu \) of the monomial \( t_X^q \) is between \( |d_q c_q^\nu| \) and \( |D_q c_q^\nu| \). To show (9.37) and (9.38) of Lemma 9.2, it suffices to show that for all \( P_\Omega \in \mathcal{P}_\Omega \),

\[
(10.2) \quad \sup_{\xi \in \mathbb{R}^d} \sum_{J \in Z(S_0) \cap Z^m} |\mathcal{I}_f(P_\Omega, \xi, t_Y)| \leq C_R \prod_{\nu} \prod_{q \in \Lambda_\nu} (|D_q c_q^\nu| + 1/|d_q c_q^\nu|)^{1/R}
\]

where \( Z(S_0) = \prod_{i=1}^{m} Z_i \) with \( Z_i = \mathbb{R}_+ \) for \( i \in S_0 \) and \( Z_i = \mathbb{R} \) as in Proposition 4.3. By Theorem 7.1 with Remark 7.1, we obtain (10.2) with the similar bound in (7.6) because the hypotheses of Theorem 7.1 are satisfied as it is checked in the following proposition:

**Proposition 10.1.** Suppose (8.26)-(8.31) hold. Then
\[
\bigcup_{\nu=1}^{d} (K_\nu \cap \Omega_\nu) \text{ is an even set}
\]

whenever \( K = (K_\nu) \in \mathcal{F}_{lo}(\mathbf{N}(\Omega, S_0)) \) where \( \mathcal{F}_{lo}(\mathbf{N}(\Omega, S_0)) \) is
\[
\left\{ K \in \mathcal{F}(\mathbf{N}(\Omega, S_0)) : \text{rank} \left( \bigcup_{\nu=1}^{d} K_\nu \right) \leq m - 1 \text{ and } \bigcap_{\nu=1}^{d} (K_\nu)^* \mathbf{N}(\Omega_\nu, S_0) \neq \emptyset \right\}.
\]
**Proof of Proposition 10.1.** We start with defining an isomorphism between two vector spaces $\text{Sp} \left( \bigcup_{d \nu = 1}^d \mathbb{F}_\nu \right) \in \mathcal{M}_{m,n}$ and $\mathbb{R}^m$.

**Definition 10.1.** Let $\left\{ \mathbb{F}_\nu \right\}_{d \nu = 1}^d$ be the collection of faces in Proposition 10.1 having the properties (8.26)-(8.31). Denote the space $\text{Sp} \left( \bigcup_{d \nu = 1}^d \mathbb{F}_\nu \right)$ by $V$. Then by (8.28),

$$V = \text{Sp} \left( \bigcup_{d \nu = 1}^d \mathbb{F}_\nu \right) \in \mathcal{M}_{m,n}$$

where

$$\mathcal{M}_{m,n} = \{ M \subset \mathbb{R}^n : \dim(\text{span}(M)) = m, \text{ and } \text{span}(P_{\mathbb{R}^m \times \{0\}^{n-m}}(M)) = \mathbb{R}^m \times \{0\}^{n-m} \}.$$ 

We consider a map $T_X : V = \text{Sp} \left( \bigcup \mathbb{F}_\nu \right) \rightarrow X = \mathbb{R}^m$ defined by

$$T_X(q_1, \cdots, q_m, q_{m+1}, \cdots, q_n) = (q_1, \cdots, q_m).$$

Since rank($V$) = rank($X$) = $m$ and $T_X$ is an onto map, the linear map $T_X : V \rightarrow X$ is an isomorphism. We shall denote $T_X(q) = q_X$.

To show Proposition 10.1, we use the invariance properties:

**Lemma 10.1.** Let $\left\{ \mathbb{F}_\nu \right\}_{d \nu = 1}^d$ be the collection of faces in Proposition 10.1 having the properties (8.26)-(8.31). Suppose that $T_X : V = \text{Sp} \left( \bigcup \mathbb{F}_\nu \right) \rightarrow X = \mathbb{R}^m$ as in Definition 10.1. Let $S_0 \subset \{1, \cdots, m\}$ be the set in (8.30) so that $\{e_j\}_{j \in S_0} \subset \text{Sp}(\bigcup_{d \nu = 1}^d \mathbb{F}_\nu) = V$. Then,

$$T_X(F_\nu \cap \Lambda_\nu, S_0) = T_X(N(F_\nu \cap \Lambda_\nu, S_0)),$$

$$K_\nu \in F(N(T_X(F_\nu \cap \Lambda_\nu, S_0))) \text{ if and only if } T_X^{-1}(K_\nu) \in F(N(F_\nu \cap \Lambda_\nu, S_0)),$$

$$(T_X^{-1}(K_\nu))^{\circ} | (N(F_\nu \cap \Lambda_\nu, S_0), V) = T_X^T((K_\nu)^{\circ} | (N(T_X(F_\nu \cap \Lambda_\nu, S_0)), X)).$$

where $T_X^T : X \rightarrow V$ is a transpose of $T_X : V \rightarrow X$.

**Proof.** Since $\{e_j\}_{j \in S_0} \subset \text{Sp}(\bigcup_{d \nu = 1}^d \mathbb{F}_\nu)$ in (8.30), the polyhedron $N(F_\nu \cap \Lambda_\nu, S_0)$ defined by $\text{Ch} \left( (F_\nu \cap \Lambda_\nu) + \mathbb{R}^S_+ \right)$ is contained in $V$

$$N(F_\nu \cap \Lambda_\nu, S_0) \subset V = \text{Sp} \left( \bigcup \mathbb{F}_\nu \right).$$
In view of (10.6), we are able to use Lemma 4.24 for proving Lemma 10.1 by setting
\( V = \text{Sp}(\bigcup_{\nu=1}^{d} F_{\nu}) \), \( W = X = \mathbb{R}^{m} \) and \( P = N(F_{\nu} \cap \Lambda_{\nu}, S_{0}) \). By (4) of Lemma 4.24 with Definition 2.12 and the fact \( T_{X}(B + \mathbb{R}_{+}^{S_{0}}) = T_{X}(B) + \mathbb{R}_{+}^{S_{0}} \),

\[
T_{X}(N(F_{\nu} \cap \Lambda_{\nu}, S_{0})) = T_{X}(\text{Ch}\left((F_{\nu} \cap \Lambda_{\nu}) + \mathbb{R}_{+}^{S_{0}}\right)) = \text{Ch}\left(T_{X}(F_{\nu} \cap \Lambda_{\nu}) + \mathbb{R}_{+}^{S_{0}}\right) = N(T_{X}(F_{\nu} \cap \Lambda_{\nu}), S_{0})
\]

which yields (10.3). Next (10.4) follows from (10.3) and (2) of Lemma 4.24. Lastly, by (10.3) and (3) of Lemma 4.24,

\[
(T_{X}^{-1}(K_{\nu})^{*})^{\circ}\mid(N(F_{\nu} \cap \Lambda_{\nu}, S_{0}), V) = (T_{X}^{-1}(K_{\nu})^{*})^{\circ}\mid(T_{X}^{-1}(N(T_{X}(F_{\nu} \cap \Lambda_{\nu}), S_{0})), V)
\]

\[
= [(T_{X}^{-1})^{-1}]^{\dagger}\left(\text{Ch}_{\nu}^{*}\mid(N(T_{X}(F_{\nu} \cap \Lambda_{\nu}), S_{0}), X)\right)
\]

\[
= [T_{X}]^{\dagger}\left(\text{Ch}_{\nu}^{*}\mid(N(T_{X}(F_{\nu} \cap \Lambda_{\nu}), S_{0}), X)\right),
\]

which proves (10.5). \( \square \)

We continue the proof of Proposition 10.1. Let \( K_{\nu} \in F(N(T_{X}(F_{\nu} \cap \Lambda_{\nu}), S_{0})) \), where \( \Omega_{\nu} = T_{X}(F_{\nu} \cap \Lambda_{\nu}) \) as in (10.1). By (10.4) of Lemma 10.1, there exists

\[
G_{\nu} = T_{X}^{-1}(K_{\nu}) \in F(N(F_{\nu} \cap \Lambda_{\nu}, S_{0})).
\]

From \( \bigcup_{\nu=1}^{d} G_{\nu} = T_{X}^{-1}(\bigcup_{\nu=1}^{d} K_{\nu}) \),

\[
\text{rank}\left(\bigcup_{\nu=1}^{d} G_{\nu}\right) = \text{rank}\left(\bigcup_{\nu=1}^{d} K_{\nu}\right) \leq m - 1
\]
because $\mathcal{T}_X$ is an isomorphism. By $\mathbb{F}_\nu = \mathbb{N}(\Lambda_\nu \cap \mathbb{F}_\nu, S_0)$ in (4.35) and $\mathbb{R}^n \supset V$,

$$\bigcap_{\nu=1}^{d} (G^*_\nu)^{\circ} | (\mathbb{F}_\nu, \mathbb{R}^n) \supset \bigcap_{\nu=1}^{d} (G^*_\nu)^{\circ} | (\mathbb{N}(\Lambda_\nu \cap \mathbb{F}_\nu, S_0), V)$$

$$= \bigcap_{\nu=1}^{d} (T^{-1}_X(K^*_\nu)^{\circ} | (\mathbb{N}(\mathbb{F}_\nu \cap \Lambda_\nu, S_0), V))$$

$$= \bigcap_{\nu=1}^{d} [T_X] (\bigcap_{\nu=1}^{d} ((K^*_\nu)^{\circ} | (\mathbb{N}(T_X(\mathbb{F}_\nu \cap \Lambda_\nu), S_0), \mathbb{R}^m}))$$

$$= [T_X] \left( \bigcap_{\nu=1}^{d} ((K^*_\nu)^{\circ} | (\mathbb{N}(\Omega_\nu, S_0)) \right) \neq \emptyset.$$ 

The second equality above follows from (10.5) and the last line follows from the second condition defining $\mathcal{F}_{10}(\mathbb{N}(\Omega, S_0))$ in Proposition 10.1. Thus

$$d \bigcap_{\nu=1}^{d} (G^*_\nu)^{\circ} | \mathbb{F}_\nu = \bigcap_{\nu=1}^{d} (G^*_\nu)^{\circ} | (\mathbb{F}_\nu, \mathbb{R}^n) \neq \emptyset.$$

By (8.27),

$$d \bigcap_{\nu=1}^{d} (F^*_\nu)^{\circ} | \mathbb{N}(\Lambda_\nu, S) \neq \emptyset.$$ 

By applying Lemma 8.2 together with (10.8) and (10.9),

$$d \bigcap_{\nu=1}^{d} (G^*_\nu)^{\circ} | \mathbb{N}(\Lambda_\nu, S) \neq \emptyset.$$

By (10.7),(10.10) and (8.31),

$$\bigcup_{\nu=1}^{d} G_\nu \cap \Lambda_\nu \text{ is an even set having no point of \(odd, \cdots, odd\) in } \Sigma \left( \bigcup_{\nu=1}^{d} G_\nu \cap \Lambda_\nu \right).$$
By this together with Proposition 9.1,
\[
\Sigma \left( \bigcup_{\nu=1}^{d} G_{\nu} \cap \Lambda_{\nu} \right) \cap \left\{ q \in \Sigma \left( \bigcup_{\nu=1}^{d} (F_{\nu} \cap \Lambda_{\nu}) \right) : q = (\text{odd, \ldots, odd, } \ast, \ldots, \ast) \right\}
\]
\[
\subset \Sigma \left( \bigcup_{\nu=1}^{d} G_{\nu} \cap \Lambda_{\nu} \right) \cap \left\{ q \in \Sigma \left( \bigcup_{\nu=1}^{d} (F_{\nu} \cap \Lambda_{\nu}) \right) : q = (\text{odd, \ldots, odd}) \right\} = \emptyset.
\]
Therefore
\[
\Sigma \left( \bigcup_{\nu=1}^{d} K_{\nu} \cap \Omega_{\nu} \right) = T_{X} \left( \Sigma \left( \bigcup_{\nu=1}^{d} G_{\nu} \cap \Lambda_{\nu} \right) \right)
\]
contains no point of \((\text{odd, \ldots, odd})\).

Hence \(\bigcup_{\nu=1}^{d} (K_{\nu} \cap \Omega_{\nu})\) is an even set. So, the proof of Proposition 10.1 is finished. \(\square\)

10.2. Proof of Lemma 9.3. We prove the independence (9.41) and the non-vanishing property (9.42) to finish the necessity proof for Theorems 3.2.

Proof of (9.41). Recall (9.40)
\[
J(P_{\xi}, t_{Y}) = \lim_{a_{X} \rightarrow 0, b_{X} \rightarrow 1} \mathcal{J}(P_{\xi}, \xi, t_{Y}, a_{X}, b_{X}),
\]
with
\[
(10.11) \mathcal{J}(P_{\xi}, \xi, t_{Y}, a_{X}, b_{X}) = \int_{\prod_{j=1}^{m} \{a_{j} < t_{j} < b_{j}\}} \sum_{\sigma \in \mathcal{O}} (-1)^{|\sigma|} \exp(iP_{\xi}(\xi, \sigma t)) \frac{dt_{1}}{t_{1}} \cdots \frac{dt_{m}}{t_{m}}
\]
where
\[
P_{\xi}(\xi, \sigma_{1}t_{1}, \ldots, \sigma_{n}t_{n}) = \sum_{\nu=1}^{d} \left( \sum_{q \in F_{\nu}} c_{q}^{\nu} \sigma^{q} t^{q} \right) \xi_{\nu}.
\]
By (8.30), we let \(S_{0} = \{1, \ldots, k\} \subset \{1, \ldots, m\}\). In view of (8.28) and (8.30), choose \(\{q_{1}, \ldots, q_{m}\} \subset \mathbb{R}^{n}\) with \(\text{Sp} (q_{1}, \ldots, q_{m}) = \text{Sp} (\bigcup F_{\nu})\) such that

(i) For \(i = 1, \ldots, m\), \(q_{i} = (q_{i})_{j=1}^{m} = (q_{i})_{X} = e_{i} \in \mathbb{R}^{m}\),

(ii) For \(i = 1, \ldots, k\), \(q_{i} = e_{i} \in \mathbb{R}^{n}\).

Fix \(t_{Y} = (t_{m+1}, \ldots, t_{n})\) and use the change of variables:
\[
(10.12) \quad x_{i} = \left( \prod_{j=1}^{m} t_{j}^{q_{ij}} \right) \left( \prod_{j=m+1}^{n} t_{j}^{q_{ij}} \right) \quad \text{so that} \quad x_{1} = t^{q_{1}}, \ldots, x_{m} = t^{q_{m}}
\]
where for $0 \leq i, j \leq m$, $q_{ij} = \delta_{ij}$ and $\delta_{ij}$ equals 1 if $i = j$ and 0 otherwise. As $q \in \bigcup_{\nu}(F_\nu \cap \Lambda_\nu) \subset \text{Sp}(\bigcup F_\nu)$ is expressed as a linear combination of $q_1, \ldots, q_m$, there exists a vector $b(q) = (b_1, \ldots, b_m) \in \mathbb{R}^m$ such that

$$t^q = t^b_1 q_1 + \cdots + t^b_m q_m = x_1^{b_1} \cdots x_m^{b_m} = x^{b(q)}.$$  

This implies that the phase function $P_\xi(\xi, \sigma t)$ is written as

$$P_\xi(\xi, \sigma t) = \sum_{\nu=1}^d \left( \sum_{q \in F_\nu \cap \Lambda_\nu} c^\nu_q \sigma^q t^q \right) \xi_\nu$$

$$= \sum_{\nu=1}^d \left( \sum_{q \in F_\nu \cap \Lambda_\nu} c^\nu_q \sigma^q x^{b(q)} \right) \xi_\nu = Q_\xi(\xi, x).$$

Then we compute in (10.12) with fixed $t_Y = (t_{m+1}, \ldots, t_n)$,

$$\frac{\partial (x_1, \ldots, x_m)}{\partial (t_1, \ldots, t_m)} = \det \begin{pmatrix} \frac{\delta_{11} t_1^q}{t_1} & \cdots & \frac{\delta_{1m} t_1^q}{t_m} \\ \vdots & \ddots & \vdots \\ \frac{\delta_{m1} t_m^q}{t_1} & \cdots & \frac{\delta_{mm} t_m^q}{t_m} \end{pmatrix} = \det(\delta_{ij}) \frac{x_1 \cdots x_m}{t_1 \cdots t_m}.$$  

Solve $(t_1, \ldots, t_m)$ in (10.12) in terms of $(x_1, \ldots, x_m)$ and $t_Y = (t_{m+1}, \ldots, t_n)$,

$$t_i = \frac{x_i}{\frac{q_{i,m+1}}{t_{m+1}} \cdots \frac{q_{i,n}}{t_n}} \text{ for } i = 1, \ldots, m.$$  

Note from this together with $q_1 = e_1, \ldots, q_k = e_k$ in (ii) above,

$$t_1 = x_1, \ldots, t_k = x_k, t_i = x_i/\left(\frac{q_{i,m+1}}{t_{m+1}} \cdots \frac{q_{i,n}}{t_n}\right) \text{ for } i = k+1, \ldots, m.$$  

So the region $\prod_{j=1}^m \{ a_j < t_j < b_j \}$ in (9.40) is transformed to the region

$$U(a_X, b_X, t_Y) = \prod_{i=1}^k \left\{ a_i < x_i < b_i \right\} \prod_{i=k+1}^m \left\{ a_i < \frac{x_i}{\frac{q_{i,m+1}}{t_{m+1}} \cdots \frac{q_{i,n}}{t_n}} < b_i \right\}$$

Thus as $a_X = (a_i)_{i=1}^m \to 0_X$ and $b_X = (b_i)_{i=1}^m \to 1_X(S_0) = \left( 1, \ldots, 1, \infty, \ldots, \infty \right)$, $k$ components $m-k$ components

$$J(P_\xi, (t_{m+1}, \ldots, t_n))$$

$$= \lim_{a_X \to 0, b_X \to 1_X(S_0)} \int_{U(a_X, b_X, t_Y)} \sum_{\sigma \in O} (-1)^{\left| \sigma \right|} \exp(iQ_\xi(\xi, x)) \frac{dx_1}{x_1} \cdots \frac{dx_m}{x_m},$$
is independent of $t_Y \in \prod_{j=m+1}^{n} \{a_j < t_j < b_j\}$ since $t_{m+1}^{q_{m+1}} \cdots t_n^{q_n}$ is absorbed in the limit of $a_X \to 0_X$ and $b_X \to 1_X(S_0)$ in (10.14). \[\square\]

**Proof of (9.42).** Since $\mathcal{J}(P_F, \xi, (t_{m+1}, \cdots, t_n))$ is independent of $t_Y = (t_{m+1}, \cdots, t_n)$, it suffices to work with $t_Y = 1_Y$ and find $P_\Lambda \in \mathcal{P}_\Lambda$ that for some choices of $\xi$ and coefficients in $P_F$,

\begin{equation}
\mathcal{J}(P_F, \xi) = \mathcal{J}(P_F, \xi, 1_Y) \neq 0 \quad \text{where} \quad 1_Y = \left(1, \cdots, 1\right). \tag{10.15}
\end{equation}

Let $\mathbb{Z}_2 = \{0, 1\}$ be the additive group and let $\mathbb{Z}_2^n = \{(v_1, \cdots, v_n) : v_i \in \mathbb{Z}_2\}$. Define a function $\Gamma : \mathbb{Z}^n \to \mathbb{Z}_2^n$ by

\begin{equation}
\Gamma(q_1, \cdots, q_n) = (\gamma(q_1), \cdots, \gamma(q_n)) \tag{10.16}
\end{equation}

where

\[\gamma(q_i) = \begin{cases} 
0 & \text{if } q_i \text{ is an even number}, \\
1 & \text{if } q_i \text{ is an odd number}.
\end{cases}\]

We put

\begin{equation}
\Gamma\left(\bigcup_{\nu=1}^{d} (F_\nu \cap \Lambda_\nu)\right) = \{z_1, \cdots, z_L\} \subset \mathbb{Z}_2^n. \tag{10.17}
\end{equation}

For $\ell = 1, \cdots, L$, let

\[\Gamma^{-1}\{z_\ell\} = \left\{q \in \bigcup_{\nu=1}^{d} (F_\nu \cap \Lambda_\nu) : \Gamma(q) = z_\ell\right\}.
\]

We then have

\begin{equation}
\bigcup_{\nu=1}^{d} (F_\nu \cap \Lambda_\nu) = \bigcup_{\ell=1}^{L} \Gamma^{-1}\{z_\ell\}. \tag{10.17}
\end{equation}

Since $\bigcup_{\nu=1}^{d} (F_\nu \cap \Lambda_\nu)$ is an odd set, there exist $z_1, \cdots, z_s \in \Gamma\left(\bigcup_{\nu=1}^{d} (F_\nu \cap \Lambda_\nu)\right)$ such that

\begin{equation}
z_1 \oplus \cdots \oplus z_s = (1, \cdots, 1) \quad \text{in } \mathbb{Z}_2^n. \tag{10.18}
\end{equation}
Assume the contrary to (10.15). Then from (9.40) and (9.41), for all \( \xi_1, \cdots, \xi_d \) and all choices of coefficients \( c'_q \in \mathbb{R} \setminus \{0\} \), we have in (10.11),

\[
\mathcal{J}(P_F, \xi, 1_Y) = \lim_{a_X \to 0, b_X \to 1_X(S_0)} \mathcal{J}(P_F, \xi, 1_Y, a_X, b_X)
\]

(10.19) \[
= \lim_{a_X \to 0, b_X \to 1_X(S_0)} \int_{\prod_{j=1}^m \{a_j < t_j < b_j\}} (-1)^{|\sigma|} \exp(iP_F(\xi, \sigma t)) \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m}
\]

\[
= 0 \quad \text{where } t = (t_1, \cdots, t_m, 1_Y) \text{ in the above integral.}
\]

In view of (10.13),

(10.20) \[
P_F(\xi, \sigma t) = \sum_{\nu=1}^d \left( \sum_{q \in F \cap \Lambda_\nu} c'_{\nu} \sigma^q t^q \right) \xi_\nu
\]

\[
= \sum_{q \in \bigcup_{\nu=1}^d (F \cap \Lambda_\nu)} \xi_{\nu(q)} c'_{\nu(q)} \sigma^q t^q.
\]

Rearrange monomials \( t^q \) in (10.20) by using (10.17) and reset their coefficients so that

1. \( \xi_{\nu(q)} c'_{\nu(q)} = \zeta_\ell \) if \( q \in \Gamma^{-1}\{z_\ell\} \) for each \( \ell = 1, \cdots, s \),
2. \( \xi_{\nu(q)} c'_{\nu(q)} = \zeta_{s+1} \) if \( q \in E = \bigcup_{\nu=1}^d (F \cap \Lambda_\nu) \setminus \bigcup_{\ell=1}^s \Gamma^{-1}\{z_\ell\} \),

which is possible because (10.19) holds for all \( \xi \) and all coefficients \( c'_q \in \mathbb{R} \setminus \{0\} \). Then,

(10.21) \[
P_F(\xi, \sigma t) = \sum_{\ell=1}^s \sum_{q \in \Gamma^{-1}\{z_\ell\}} \xi_{\nu(q)} c'_{\nu(q)} \sigma^q t^q
\]

\[
= \zeta_1 \sum_{q \in \Gamma^{-1}\{z_1\}} \sigma^q t^q + \cdots + \zeta_s \sum_{q \in \Gamma^{-1}\{z_s\}} \sigma^q t^q + \zeta_{s+1} \sum_{q \in E} \sigma^q t^q
\]

\[
= \sum_{\ell=1}^{s+1} Q_\ell(\sigma, t) \zeta_\ell.
\]

Rewrite \( \mathcal{J}(P_F, \xi, 1_Y, a_X, b_X) \) in (10.19) as

\[
\mathcal{L}(\zeta, a_X, b_X) = \int_{\prod_{j=1}^m \{a_j < t_j < b_j\}} \sum_{\sigma \in O} (-1)^{|\sigma|} \exp \left( i \sum_{\ell=1}^{s+1} Q_\ell(\sigma, t) \zeta_\ell \right) \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m}.
\]

Then we see in view of (10.19) that for almost every \( \zeta = (\zeta_1, \cdots, \zeta_{s+1}) \in \mathbb{R}^{s+1} \)

(10.22) \[
\mathcal{L}(\zeta) = \lim_{a_X \to 0, b_X \to 1_X(S_0)} \mathcal{L}(\zeta, a_X, b_X) = 0.
\]
On the other hand by Lemma 10.1 and Theorem 7.1 with Remark 7.1,

$$\sup_{\xi, a_X, b_X \in I_X(S_0)} |\mathcal{F}(P_{\xi}, \xi, 1_Y, a_X, b_X)| \leq C_R \prod_{\nu \in \Lambda_\nu} \prod_{q \in \Lambda_\nu} (|c^\nu_q| + 1/|c^\nu_q|)^{1/R}.$$  

Thus, by simply plug $\xi \nu(q) = 1$ where $\xi \nu(q) c^{\nu(q)} = \zeta_\ell$ in (10.21),

$$(10.23) \quad \sup_{a_X, b_X \in I_X(S_0)} |\mathcal{L}(\zeta, a_X, b_X)| \leq C_M \prod_{\ell=1}^{s+1} (|\zeta_\ell| + 1/|\zeta_\ell|)^{1/M} \text{ for some large } M > 0.$$  

We now find a contradiction to (10.22). Let $f$ be a Schwartz function on $\mathbb{R}^{s+1}$ of the form $\hat{f}(\zeta) = \prod_{\ell=1}^{s+1} \hat{f}_\ell(\zeta_\ell)$ with $f_\ell$ a Schwartz function on $\mathbb{R}$. Then from (10.23),

$$\sup_{a_X, b_X \in I_X(S_0)} |\mathcal{L}(\zeta, a_X, b_X) \hat{f}(\zeta)| \leq C_M \prod_{\ell=1}^{s+1} (|\zeta_\ell| + 1/|\zeta_\ell|)^{1/M} |\hat{f}(\zeta)|$$

which is an integrable function on $\mathbb{R}^{s+1}$. This enables us to use the dominated convergence theorem for (10.22) multiplied by $\hat{f}(\zeta)$ to obtain that

$$(10.24) \quad 0 = \int_{\mathbb{R}^{s+1}} \mathcal{L}(\zeta, a_X, b_X) \hat{f}(\zeta) d\zeta = \lim_{a_X \to 0, b_X \to 1_X(S_0)} \int_{\mathbb{R}^{s+1}} \mathcal{L}(\zeta, a_X, b_X) \hat{f}(\zeta) d\zeta.$$

Rewrite the integral on the right-hand side as follows. Interchange, by Fubini’s theorem, the order of integration and apply the Fourier inversion formula for the Schwartz functions:

$$\int_{\mathbb{R}^{s+1}} \mathcal{L}(\zeta, a_X, b_X) \hat{f}(\zeta) d\zeta$$

$$= \int_{\mathbb{R}^{s+1}} \int_{\prod_{j=1}^{m} \{a_j, t_j < b_j\}} \sum_{\sigma \in \mathcal{O}} (-1)^{|\sigma|} \exp \left( i \sum_{\ell=1}^{s+1} Q_\ell(\sigma, t) \zeta_\ell \right) \hat{f}(\zeta) \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} d\zeta$$

$$= \int_{\mathbb{R}^{s+1}} \prod_{j=1}^{m} \int_{\{a_j, t_j < b_j\}} \sum_{\sigma \in \mathcal{O}} (-1)^{|\sigma|} \exp \left( i \sum_{\ell=1}^{s+1} Q_\ell(\sigma, t) \zeta_\ell \right) \hat{f}(\zeta) d\zeta \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m}$$

$$= \int_{\prod_{j=1}^{m} \{a_j, t_j < b_j\}} \sum_{\sigma \in \mathcal{O}} (-1)^{|\sigma|} \prod_{\ell=1}^{s+1} f_\ell(Q_\ell(\sigma, t)) \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m},$$

where $Q_\ell(\sigma, t)$ is defined in (10.21). Here we choose $f_1, \cdots, f_s$ to be odd Schwartz functions on $\mathbb{R}$ that are positive on $[0, \infty)$, and $f_{s+1}(x) = e^{-x^2}$. By using $\sigma^q = \sigma^{\nu(q)} = \sigma^{\zeta_\ell}$ for all
\[ q \in \Gamma^{-1}(z) \text{ and oddness of } f, \]
\[
\sum_{\sigma \in O} (-1)^{|\sigma|} \prod_{\ell=1}^{s+1} f_{\ell}(Q_{\ell}(\sigma, t)) = \sum_{\sigma \in O} (-1)^{|\sigma|} \prod_{\ell=1}^{s} f_{\ell}(\sum_{q \in \Gamma^{-1}(z)} t^q) \exp \left( -\left| \sum_{q \in E} \sigma q \right|^2 \right)
\]
(10.25)
\[
= \sum_{\sigma \in O} (-1)^{|\sigma|} \prod_{\ell=1}^{s} f_{\ell}(\sum_{q \in \Gamma^{-1}(z)} t^q) \exp \left( -\left| \sum_{q \in E} \sigma q \right|^2 \right)
\]
where the last equality follows from (10.18) and
\[
\sigma z_1 + \cdots + z_s = \sigma z_1 \oplus \cdots \oplus z_s = (-1)^{|\sigma|}.
\]
So, the limit in (10.24) is positive, which is a contradiction. Hence (10.15) is proved. \[ \square \]

11. Proofs of Corollary 3.1 and Main Theorem 3.1


Proof of Corollary 3.1. Sufficiency. By applying a linear transformation to \( P \), we may assume that \( e_{\nu} \notin \Lambda_{n+1} \) for \( 1 \leq \nu \leq n \) without loss of generality. Suppose that
\[
(F_{n+1} \cap \Lambda_{n+1}) \cup A \text{ is an even set whenever } \text{rank}(F_{n+1} \cup A) \leq n - 1
\]
(11.1)
where \( F_{n+1} \in \mathcal{F}(N(\Lambda_{n+1}, S)) \) and \( A \subset \{e_1, \cdots, e_n\} \). It suffices to deduce from (11.1) that the hypothesis of Main Theorem 3.2 holds for the case \( \Lambda = (\{e_1\}, \cdots, \{e_n\}, \Lambda_{n+1}) \), since we have already proved Main Theorem 3.2. Let \( \text{rank} \left( \bigcup_{\nu=1}^{n+1} F_{\nu} \right) \leq n - 1 \) and \( \bigcap_{\nu=1}^{n+1} (F_{\nu}^*) \neq \emptyset \).

We claim that \( \bigcup_{\nu=1}^{n+1} (F_{\nu} \cap \Lambda_{\nu}) \) is an even set. Observe that for every nonempty face \( F_{\nu} \in \mathcal{F}(N(\{e_\nu\}, S)) \), \( F_{\nu} \cap \Lambda_{\nu} = \{e_\nu\} \). Thus for \( A = \{e_\nu : F_{\nu} \neq \emptyset \ \text{for } \nu = 1, \cdots, n\} \), we write
\[
\bigcup_{\nu=1}^{n+1} F_{\nu} \cap \Lambda_{\nu} = (F_{n+1} \cap \Lambda_{n+1}) \cup A.
\]
By (11.1), \( \bigcup_{\nu=1}^{n+1} F_{\nu} \cap \Lambda_{\nu} \) is an even set.

Necessity. Suppose that (11.1) does not hold. Then there exists \( A \subset \{e_1, \cdots, e_n\} = N_n \) and \( F_{n+1} \) such that
\[
\text{rank} \left( A \cup F_{n+1} \right) \leq n - 1 \text{ and } A \cup (F_{n+1} \cap \Lambda_{n+1}) \text{ is odd.}
\]
To $A \subset N_n$, there corresponds the index set $I \subset \{1, \cdots , n\}$ such that $A = \{e_\nu : \nu \in I\}$. Let $\text{Sp}(F_{n+1}) \cap \{e_\nu : \nu \in S\} = \{e_\nu_1, \cdots , e_\nu_k\}$ where $\{\nu_1, \cdots , \nu_k\} = S_1 \subset S$. Choose

- For $\nu \in N_n \setminus I$, let $F_\nu = \emptyset$ with $(F_\nu^*)^c = Z(S) \setminus \{0\}$.
- For $\nu \in I$, let $F_\nu = \{e_\nu\} + \mathbb{R}^{S_1}$ with

$$(F_\nu^*)^c = \text{CoSp}^c(\{e_j : j \in S \setminus S_1\} \cup \{\pm e_j : j \in N_n \setminus S\}).$$

Then we can observe that $(F_{n+1}^*)^c \subset (F_\nu^*)^c$ for all $\nu = 1, \cdots , n$. Therefore,

$\text{rank} \left( \bigcup_{\nu=1}^{n+1} F_\nu \right) = \text{rank} \left( A \cup F_{n+1} \right) \leq n - 1$ and $\bigcap_{\nu=1}^{n+1} (F_\nu^*)^c \neq \emptyset$,

but $\bigcup_{\nu=1}^{n+1} F_\nu \cap \Lambda_\nu = A \cup (F_{n+1} \cap \Lambda_{n+1})$ is an odd set, which implies that the hypothesis of Main Theorem 3.2 breaks, which directly implies only

$$\sup_{r \in I(S)} \left\| \mathcal{H}_r^P \right\|_{L^2(\mathbb{R}^{n+1}) \rightarrow L^2(\mathbb{R}^{n+1})} = \infty.$$ 

In order to get

$$\left\| \mathcal{H}_r^P \right\|_{L^2(\mathbb{R}^{n+1}) \rightarrow L^2(\mathbb{R}^{n+1})} = \infty$$

we proceed as follows. As in the proof of Theorem 8.1, we have $F = (F_\nu)$ with $F_\nu \in \mathcal{F}(N(\Lambda_\nu, S))$ satisfying (8.26)-(8.31). We thus obtain (9.35) so that there exists $P_\Lambda \in \mathcal{P}_\Lambda$ such that

$$\left\| \mathcal{H}_r^P \right\|_{L^2(\mathbb{R}^{n+1}) \rightarrow L^2(\mathbb{R}^{n+1})} = \infty.$$ 

where $S_0$ is in (8.29) so that

$$(q_j) \in \bigcap_{\nu} (F_\nu^*)^c |N(\Lambda_\nu, S) \text{ and } q_j = 0 \text{ for } j \in S_0 \text{ and } q_j > 0 \text{ for } j \in S \setminus S_0.$$ 

This implies $\left\| \mathcal{H}_r^P \right\|_{L^2(\mathbb{R}^d)} = \infty$ by the following standard argument: For $\delta > 0$, define a dilation

$$f_\delta(x_1, \cdots , x_n, x_{n+1}) = f(\delta^{-q_1}x_1, \cdots , \delta^{-q_n}x_n, \delta^{-r}x_{n+1}) \text{ where } F_{n+1} \subset \pi_{q,r}$$

and a measure

$$\mu_\delta^S(\phi) = \int_{I(S)} \phi(\delta^{-q_1}t_1, \cdots , \delta^{-q_n}t_n, \delta^{-r}P(t_1, \cdots , t_n)) \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}$$
satisfying \( H_{1,\delta}^P(f) = [\mu^S_{\delta} * f_{\delta-1}]_{\delta} \). By using
\[
\lim_{\delta \to 0} \mu^S_{\delta} (\phi) = \int_{I(S_0)} \phi(t_1, \cdots, t_n, P_\delta(t_1, \cdots, t_n)) \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n},
\]
we conclude that the boundedness of \( \| H_{1,\delta}^P \|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \) implies the boundedness of \( \| H_{1,\delta}^P \|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \). □

11.2. Proof of Sufficient Part of Main Theorem 3.1. We now develop the argument of [16] for \( n \geq 3 \), and obtain Main Theorem 3.1.

Definition 11.1. Let \( P \in \mathcal{P}_\Lambda \) where \( \Lambda = (\Lambda_\nu) \) with \( \Lambda_\nu \subset \mathbb{Z}_+^n \) and \( S \subset \{1, \cdots, n\} \). Let \( A \in GL(d) \). We set the collection of all \( \vec{N}(AP, S) \) with \( A \in GL(d) \) in Definition 3.4,
\[
N(P, S) = \{ \vec{N}(AP, S) : A \in GL(d) \}.
\]
Consider the collection of equivalent classes \( A(P) = \{ [A] : A \in GL(d) \} \) with the equivalence relation for \( A, B \in GL(d) \),
\[
A \sim B \text{ if and only if } \Lambda(AP) = \Lambda(BP).
\]
We see that \( A(P) \) is finite and write \( A(P) \) as \( \{ [A_k] : k = 1, \cdots, N \} \). Thus we can regard \( N(P, S) \) as the ordered \( N \)-tuples of \( \vec{N}(A_k P, S) \) (indeed, \( Nd \) tuples of Newton Polyhedrons \( \vec{N}((A_k P)_\nu, S) \)):
\[
N(P, S) = \left( \vec{N}(AP, S) \right)_{[A] \in A(P)} = \left( \vec{N}(A_k P, S) \right)_{k=1}^N.
\]
So, we define the set of all combinations of \( Nd \)-tuples of faces by
\[
F(N(P, S)) = \left\{ (F_{[A]})_{[A] \in A(P)} : F_{[A]} \in F \left( \vec{N}(AP, S) \right) \right\} = \left\{ (F_{A_k})_{k=1}^N : F_{A_k} \in F \left( \vec{N}(A_k P, S) \right) \right\},
\]
where \( F_{A_k} = ((F_{A_k})_1, \cdots, (F_{A_k})_d) \) with \( (F_{A_k})_\nu \in F \left( \vec{N}((A_k P)_\nu, S) \right) \).

To prove Main Theorem 3.1, we apply the Proposition 4.3 for every \( \vec{N}(A_k P, S) \) with \( k = 1, \cdots, N \) to obtain the following general form of cone decomposition.
Lemma 11.1. Let \( P \in \mathcal{P}_\Lambda \) where \( \Lambda = (\Lambda_\nu) \) with \( \Lambda_\nu \subset \mathbb{Z}_+^n \) and \( S \subset \{1, \ldots, n\} \). Then,

\[
\bigcup_{(\mathcal{F}_A)_{[A]} \in \mathcal{A}(P)} \left( \bigcap_{[A] \in \mathcal{A}(P)} \text{Cap}(\mathcal{F}_{[A]}^*) \right) = Z(S).
\]

Given \( \Lambda \), there are finitely many Newton polyhedrons in \( \{\bar{\mathcal{N}}(AP, S) : A \in \text{GL}(d), P \in \mathcal{P}_\Lambda \} \).

Proof. By (11.2), the left hand side above is

\[
\bigcup_{(\mathcal{F}_A)_{[A]} \in \mathcal{F}(\mathcal{N}(P,S))} \left( \bigcap_{k=1}^{N} \bigcap_{\nu=1}^{d} (\mathcal{F}_{A_k})_{[A]}^* \right) = \bigcap_{k=1}^{N} \bigcup_{\mathcal{F}_{A_k} \in \mathcal{F}(\mathcal{N}(A_k P,S))} \bigcap_{\nu=1}^{d} (\mathcal{F}_{A_k})_{[A]}^* \bigcap_{\nu=1}^{d} (\mathcal{F}_{A_k})_{[A]}^* = Z(S),
\]

For each fixed \( A_k \), Proposition 4.3 yields that

\[
\bigcup_{\mathcal{F}_{A_k} \in \mathcal{F}(\mathcal{N}(A_k P,S))} \bigcap_{\nu=1}^{d} (\mathcal{F}_{A_k})_{[A]}^* = Z(S),
\]

which proves Lemma 11.1. \( \square \)

To each \( [A] \in \mathcal{A}(P) \), we first assign a \( d \)-tuple of faces \( \mathcal{F}_A \in \mathcal{F}(\bar{\mathcal{N}}(AP, S)) \). Next, fix

(11.3) \((\mathcal{F}_A)_{[A]} \in \mathcal{P}_\Lambda \).

To show Main Theorem 3.1, in view of Lemma 11.1, it suffices to show that

(11.4) \( \left\| \sum_{J \in Z} H_{P_A}^J \right\| \leq C \) where \( Z \subset \bigcap_{[A] \in \mathcal{A}(P)} \text{Cap}(\mathcal{F}_A^*) \) with \( \mathcal{F}_A \) chosen in (11.3).

To show (11.4), we can replace \( H_{P_A}^J \) by \( H_{UP_A}^J \) for some \( U \in \text{GL}(d) \) and prove that

(11.5) \( \left\| \sum_{J \in Z} H_{P_A}^J \right\| = \left\| \sum_{J \in Z} H_{UP_A}^J \right\| \leq C \)

where the equality follows from

\[
\int f(x - P(t)) \prod_{\nu=1}^{d} \frac{\chi(2^{j_\nu}t_\nu)}{t_\nu} dt = \int f(U^{-1}(Ux - UP(t)) \prod_{\nu=1}^{d} \frac{\chi(2^{j_\nu}t_\nu)}{t_\nu} dt.
\]

Without the disjointness of \( \Lambda_\nu \)'s, we are lack of the decay condition (6.14) in Lemma 6.3 and (7.2) of Theorem 7.1. In order to recover this, we shall modify the proof of [16] and
find an appropriate $U$ to satisfy the desirable decay estimate in Lemma 11.2. We work this process for $d = 3$. Let $[A_1] \in A(P)$ with $A_1 = I$. Then

$$(A_1P)(t) = P(t) = \left( \sum_{m \in \Lambda((A_1P)\nu) = \Lambda(P)\nu} c_m^\nu t^m \right)_{\nu=1}^3.$$ 

Take any vector $m(A_1, 1) \in (\mathbb{F}_A)_1 \cap \Lambda((A_1P)_1)$ where $\mathbb{F}_{A_1} \in \mathcal{F}(\hat{N}(A_1P, S))$ was chosen in (11.3) with $[A_1] \in A(P)$. Define

$$A_2 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{c^2m(A_1, 1)}{c^3m(A_1, 1)} & 1 & 0 \\ -\frac{c^2m(A_1, 1)}{c^3m(A_1, 1)} & 0 & 1 \end{pmatrix} \text{ so that } A_2A_1P(t) = \begin{pmatrix} (A_1P)_1(t) \\ (A_1P)_2(t) - \frac{c^2m(A_1, 1)}{c^3m(A_1, 1)} (A_1P)_1(t) \\ (A_1P)_3(t) - \frac{c^2m(A_1, 1)}{c^3m(A_1, 1)} (A_1P)_1(t) \end{pmatrix}$$

where

- $t^{m(A_1, 1)}$ does not appear in either of the $2^{nd}$ or the $3^{rd}$ component of $A_2A_1P(t)$.

Next choose $m(A_2, 2) \in (\mathbb{F}_{A_2A_1})_2 \cap \Lambda((A_2A_1P)_2)$ where $\mathbb{F}_{A_2A_1} \in \mathcal{F}(\hat{N}(A_2A_1P, S))$ was chosen in (11.3) with $[A_2A_1] \in A(P)$. Define a matrix

$$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{c^2m(A_2, 2)}{c^3m(A_2, 2)} & 1 \end{pmatrix} \text{ so that } A_3A_2A_1P(t) = \begin{pmatrix} (A_2A_1P)_1(t) = (A_1P)_1(t) \\ (A_2A_1P)_2(t) \\ (A_2A_1P)_3(t) - \frac{c^2m(A_2, 2)}{c^3m(A_2, 2)} (A_2A_1P)_2(t) \end{pmatrix}$$

where

(11.6) $t^{m(A_1, 1)}$ does not appear in either of the $2^{nd}$ or the $3^{rd}$ component of $A_3A_2A_1P(t)$,

(11.7) $t^{m(A_2, 2)}$ does not appear in the $3^{rd}$ component of $A_3A_2A_1P(t)$.

Choose $m(A_3, 3) \in (\mathbb{F}_{A_3A_2A_1})_3 \cap \Lambda((A_3A_2A_1P)_3)$ where $\mathbb{F}_{A_3A_2A_1} \in \mathcal{F}(\hat{N}(A_3A_2A_1P, S))$ was chosen in (11.3) with $[A_3A_2A_1] \in A(P)$. Since

$$m(A_k, k) \in (\mathbb{F}_{A_k \cdots A_1})_k \text{ and } J \in \bigcap_{[A] \in A(P)} \text{Cap}(\mathbb{F}_A)^c \subset (\mathbb{F}_{A_k \cdots A_1})_k^c \text{ for } k = 1, \cdots, 3,$$

we have for each $k = 1, 2, 3$,

$$2^{-Jm(A_k, k)} \geq 2^{-Jm} \text{ for } m \in \Lambda((A_k \cdots A_1P)_k)$$

where $\Lambda((A_k \cdots A_1P)_k) = \Lambda((A_3A_2A_1P)_k)$ for each $k$ by construction above.
Lemma 11.2. Let $U = A_3 A_2 A_1$ and $\mathcal{F}_U = \mathcal{F}_{A_3 A_2 A_1} \in \mathcal{F}(\mathbf{N}(U P, S))$ where $A_1, A_2, A_3$ and $\mathcal{F}_{A_3 A_2 A_1}$ were defined above. For $\mathcal{G} \in \mathcal{F}(\mathbf{N}(U P, S))$ such that $\mathcal{G} \geq \mathcal{F}_U$, let

$$I_J([U P]_{\mathcal{G}}, \xi) = \int e^{i(\sum_{\nu=1}^{3}(\sum_{m \in \mathcal{G} \cap \Lambda((U P)_{\nu})}2^{-J \cdot m}m)\xi_{\nu})} \prod_{\ell=1}^{n} h(t_{\ell})dt.$$ 

Then for $J \in Z \subset \bigcap_{A \in A(P)} \text{Cap}(\mathcal{F}_A^*) \subset \text{Cap}(\mathcal{F}_U^*)$, there exists $C > 0$ and $\delta$ that are independent of $J, \xi$ satisfying:

$$|I_J([U P]_{\mathcal{G}}, \xi)| \leq C \min \left\{ |2^{-J \cdot m} \xi_{\nu}|^{-\delta} : m \in \Lambda((U P)_{\nu}), \nu = 1, 2, 3 \right\}. \quad (11.9)$$

Proof of (11.9). By (11.8), it suffices to show that

$$|I_J([U P]_{\mathcal{G}}, \xi)| \leq C|2^{-J \cdot m(A_k,k)} \xi_k|^{-\delta} \quad \text{for} \quad k = 1, 2, 3. \quad (11.10)$$

The case $k = 1$ follows from (11.6). To show (11.10) for $k = 2$, it suffices to consider $|2^{-J \cdot m(A_2,2)} \xi_2| \gg |2^{-J \cdot m(A_1,1)} \xi_1|$, since we have already the decay $|2^{-J \cdot m(A_1,1)} \xi_1|^{-\delta}$ in (11.10). This and (11.7) yield the desired result for $k = 2$. Since (11.10) holds for $k = 1, 2$, we may assume that $|2^{-J \cdot m(A_3,3)} \xi_3| \gg |2^{-J \cdot m(A_k,k)} \xi_k|$ for $k = 1, 2$. So, the case $k = 3$ is obtained by the Van der Corput lemma. \hfill \Box

Proof of (11.4). The first hypothesis of Theorem 7.1 is satisfied by Lemma 11.2. The second hypothesis (7.3) of Theorem 7.1 is also satisfied by the hypothesis (3.6) of Main Theorem 3.1 such that

$$\bigcup_{\nu=1}^{d} [K_U]_{\nu} \cap [\Lambda(U P_{\Lambda})]_{\nu} \quad \text{is an even set}$$

whenever

$$K_U \in \left\{ K_U \in \mathcal{F}(\mathbf{N}(U P, S)) : \bigcap_{\nu=1}^{d} ([K_U]_{\nu})^o \neq \emptyset \quad \text{and} \quad \text{rank} \left( \bigcup_{\nu=1}^{d} [K_U]_{\nu} \right) \leq n - 1 \right\}.$$ 

Therefore by applying Theorem 7.1 for $Z \subset \bigcap_{A \in A(P)} \text{Cap}(\mathcal{F}_A^*) \subset \text{Cap}(\mathcal{F}_U^*)$ with $U = A_3 A_2 A_1$, we obtain (11.4). \hfill \Box
11.3. **Proof of Necessity Part of Main Theorem 3.1.** Finally, we prove the necessity part of Main Theorem 3.1.

**Theorem 11.1** (Necessity of Main Theorem 3.1). Let \( \Lambda = (\Lambda_1, \ldots, \Lambda_d) \) with \( \Lambda_\nu \subset \mathbb{Z}_n^+ \) and \( S \subset N_n \). Suppose that there exist \( A \in GL(d) \) and \( P \in \mathcal{P}_\Lambda \),

\[
\bigcup_{\nu=1}^d (\mathcal{F}_A)_\nu \cap \Lambda((AP)_\nu) \text{ is an odd set and } \mathcal{F}_A = ((\mathcal{F}_A)_\nu) \in \mathcal{F}_{\text{lo}}(\tilde{N}(AP, S))
\]

where the set \( \mathcal{F}_{\text{lo}}(\tilde{N}(AP, S)) \) is defined as in Definition 3.3. Then there exists \( Q \in \mathcal{P}_\Lambda \) such that

\[
\sup_{\xi \in \mathbb{R}^d, r \in I(S)} \mathcal{I}(Q, \xi, r) = \infty.
\]

**Proof.** Suppose that there exists

\[
A = \begin{pmatrix} A_1 \\ \vdots \\ A_d \end{pmatrix} \in GL_d \text{ and } P(t) = \left( \sum_{m \in \Lambda_1} c_1^m t^m, \ldots, \sum_{m \in \Lambda_d} c_d^m t^m \right) \in \mathcal{P}_\Lambda,
\]

such that

\[
AP(t) = \left( \sum_{m \in \tilde{\Lambda}_1} \langle A_1, c_m \rangle t^m, \ldots, \sum_{m \in \tilde{\Lambda}_d} \langle A_d, c_m \rangle t^m \right) \text{ where } \tilde{\Lambda}_\nu = \Lambda((AP)_\nu)
\]

where

\[
c_m = \begin{pmatrix} c_1^m \\ \vdots \\ c_d^m \end{pmatrix} \quad \text{and} \quad \langle A_\nu, c_m \rangle \neq 0 \text{ for } m \in \tilde{\Lambda}_\nu.
\]

Suppose also that there exists

\[
\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_\nu) \in \mathcal{F}_{\text{lo}}(\tilde{N}(\tilde{\Lambda}, S)) \text{ where } \bigcup_{\nu=1}^d (\tilde{\mathcal{F}}_\nu \cap \tilde{\Lambda}_\nu) \text{ is an odd set.}
\]

We may assume that the rank of \( \cup_{\nu=1}^d \tilde{\mathcal{F}}_\nu \) is minimal among \( \tilde{\mathcal{F}} \) satisfying (11.12) as in (8.25) and (8.27). By the odd set condition and (9.34), there exists nonzero constants
(11.13) \[ \lim_{a_X \to 0, b_X \to 1} \int_{\prod_{j=1}^{n} \{a_j < |t_j| < b_j \}} e^{i \sum_{m \in \mathbb{F}_v \cap \tilde{\Lambda}_v} K_m^m \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}} = \infty \text{ as } a_Y \to 0. \]

**Proof of (11.13).** We adapt the same argument in Section 10. Assume that (11.13) does not hold. Then, in view of (10.19), for every \((K_m)_{m \in \cup_{\nu=1}^{d-1} \mathbb{F}_v \cap \tilde{\Lambda}_v} \) with \(K_m \neq 0, \)

\[ \lim_{a_X \to 0, b_X \to 1} \int_{\prod_{j=1}^{n} \{a_j < |t_j| < b_j \}} \sum_{\sigma \in O} (-1)^{s} e^{i \sum_{m \in \mathbb{F}_v \cap \tilde{\Lambda}_v} K_m^{m \sigma^m} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m}} = 0. \]

Let \( \Gamma \) and \( z_\ell \) are defined as in (10.16)-(10.18). Put \( K_m \) in (11.14) where

\[ K_m = \begin{cases} \zeta_\ell & \text{for all } m \in \Gamma^{-1}\{z_\ell\} \text{ where } \ell = 1, \ldots, s \\ \zeta_{s+1} & \text{for all } m \in E = \cup (\mathbb{F}_v \cap \tilde{\Lambda}_v) \setminus \bigcup_{\ell=1}^{s} \Gamma^{-1}\{z_\ell\}. \end{cases} \]

Then, as we have (10.22),

\[ \lim_{a_X \to 0, b_X \to 1} \mathcal{L}(\zeta, a_X, b_X) = 0 \text{ for all } \zeta_\ell \neq 0 \text{ with } \ell = 1, \ldots, s + 1 \text{ (a.e. } \zeta \in \mathbb{R}^{s+1}) \]

where \( \mathcal{L}(\zeta, a_X, b_X) \) is

\[ \int_{\prod_{j=1}^{n} \{a_j < |t_j| < b_j \}} \sum_{\sigma \in O} (-1)^{s} e^{i \sum_{m \in \mathbb{F}_v \cap \tilde{\Lambda}_v} K_m^{m \sigma^m} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m}}. \]

We have shown that (11.15) yields a contradiction as in (10.24) and (10.25). \qed

Since any finite union of \( d - 1 \) dimensional subspaces cannot be \( \mathbb{R}^d \), we can choose

\[ (11.16) \quad \eta \in \mathbb{R}^d \setminus \bigcup_{m \in \bigcup_{\nu=1}^{d-1} \mathbb{F}_v \cap \tilde{\Lambda}_v} \text{Span}^\perp \{c_m\} \neq \emptyset. \]

By (11.16), we have for each \( m \in \bigcup_{\nu=1}^{d-1} \mathbb{F}_v \cap \tilde{\Lambda}_v, \)

\[ \langle \eta, c_m \rangle \neq 0. \]

Thus for each \( m \in \bigcup_{\nu=1}^{d-1} \mathbb{F}_v \cap \tilde{\Lambda}_v, \) we can choose a nonzero number \( \alpha_m \in \mathbb{R} \) such that

\[ (11.17) \quad \langle \eta, \alpha_m c_m \rangle = K_m. \]
Define a vector

\begin{equation}
\mathbf{b}_m = \begin{cases} 
\alpha_m c_m & \text{if } m \in \bigcup_{\nu=1}^d \tilde{\mathcal{F}}_{\nu} \cap \tilde{\Lambda}_\nu \\
\mathbf{c}_m & \text{if } m \in \left( \bigcup_{\nu=1}^d \Lambda_\nu \right) \setminus \left( \bigcup_{\nu=1}^d \tilde{\mathcal{F}}_{\nu} \cap \tilde{\Lambda}_\nu \right) \end{cases}.
\end{equation}

Choose \( \xi \in \mathbb{R}^d \) such that \( A^* \xi = \eta \). Then, we have for \( m \in \bigcup_{\nu=1}^d \tilde{\mathcal{F}}_{\nu} \cap \tilde{\Lambda}_\nu \),

\begin{equation}
\langle \xi, A \mathbf{b}_m \rangle = \langle A^* \xi, \mathbf{b}_m \rangle = \langle \eta, \mathbf{b}_m \rangle = K_m.
\end{equation}

Define

\[ Q(t) = \left( \sum_{m \in \Lambda_1} b_1^m t^m, \ldots, \sum_{m \in \Lambda_d} b_d^m t^m \right) \in \mathcal{P}_\Lambda. \]

Then,

\[ AQ(t) = \left( \sum_{m \in \Lambda_1} \langle A_1, \mathbf{b}_m \rangle t^m, \ldots, \sum_{m \in \Lambda_d} \langle A_d, \mathbf{b}_m \rangle t^m \right) \]

where \( \langle A_\nu, \mathbf{b}_m \rangle \neq 0 \) for \( m \in \tilde{\Lambda}_\nu \) by (11.11) and (11.18). Note that

\[ \langle \xi, AQ(t) \rangle = \sum_{m \in \bigcup_{\nu=1}^d \tilde{\Lambda}_\nu} \langle \xi, \mathbf{b}_m \rangle t^m \]

and from (11.19),

\[ \langle \xi, [AQ]_{\tilde{\mathcal{F}}}(t) \rangle = \sum_{m \in \bigcup_{\nu=1}^d \tilde{\mathcal{F}}_{\nu} \cap \tilde{\Lambda}_\nu} \langle \xi, \mathbf{b}_m \rangle t^m = \sum_{m \in \bigcup_{\nu=1}^d \tilde{\mathcal{F}}_{\nu} \cap \tilde{\Lambda}_\nu} K_m t^m. \]

This together with (11.13) implies that

\[ \lim_{a_X \to 0, b_X \to 1_X(S_0)} \int_{\prod_{j=1}^n \{ |a_j| < |t_j| < b_j \}} e^{i \langle \xi, [AQ]_{\mathcal{F}}(t) \rangle} \prod_{j=1}^n \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} = \infty \text{ as } a_Y \to 0. \]

Thus

\[ \sup_{\xi \in \mathbb{R}^d, a, b \in I(S_0)} |\mathcal{I}([AQ]_{\mathcal{F}}, \xi, a, b)| = \infty. \]

Therefore by Lemma 8.3,

\[ \sup_{\xi \in \mathbb{R}^d, x \in I(S)} |\mathcal{I}([AQ], \xi, r)| = \infty, \] which implies that \( \sup_{\xi \in \mathbb{R}^d, x \in I(S)} |\mathcal{I}(Q, \xi, r)| = \infty. \)

\[ \square \]
12. Appendix

We give a proof of Lemma 4.10 by using the idea of [7]. Recall

**Lemma 4.10.** Let \( \mathcal{P} \) be a polyhedron in an inner product space \( V \) with

\[
\dim(\mathcal{P}) = \dim(V) = n.
\]

Let \( \mathcal{G} \in \mathcal{F}(\mathcal{P}) \) be a proper face of \( \mathcal{P} \) with a generator \( \Pi(\mathcal{G}) = \{q_j\}_{j=1}^M \), that is,

\[
\mathcal{G} = \bigcap_{j=1}^M \mathcal{F}_j
\]

where \( \mathcal{F}_j = \pi_{q_j,r_j} \cap \mathcal{P} \) so that \((\mathcal{F}_j^*)^\circ = \CoSp^\circ(\{q_j\})\) for \( j = 1, \ldots, M \). Then,

(12.1) \((\mathcal{G}^*)^\circ(\mathcal{P}, V) = \CoSp^\circ(\{q_j\}_{j=1}^M)\) and \( \mathcal{G}^\circ(\mathcal{P}, V) = \CoSp(\{q_j\}_{j=1}^M)\).

We first deal with the cone type case.

**Definition 12.1.** [Cone and its Dual Cone] Let \( V \) be an inner product space. We say that a set \( C \subset V \) is a cone in \( V \) (with apex 0 the origin) if

\[
C = \CoSp\{x_j\}_{j=1}^N = \CoSp\{x_1\} \oplus \cdots \oplus \CoSp\{x_N\}
\]

for some \( \{x_j\}_{j=1}^N \subset V \).

Next, we define \( C^\vee = \{u \in V : \langle u, x \rangle \geq 0 \text{ for all } x \in C\} \) and call it the dual cone of \( C \).

We can interpret \( C^\vee \) as the dual face of the apex 0 given by \( \{0\}^\circ(\mathcal{C}, V) \).

**Example 12.1.** Let \( V \) be a vector space and \( q \in V \). Then we see that the upper half space \((\pi_{q,0} \cap V)^+ = \{x \in V : \langle q, x \rangle \geq 0 \}\) is a cone in \( V \), which is expressed as \( \CoSp\{q, \pm n_j\}_{j=1}^M \)

where \( \{n_j\}_{j=1}^M \) is an orthonormal basis of \( \pi_{q,0} \cap V \). We can easily see that

\[
((\pi_{q,0} \cap V)^+)^\vee = \CoSp(q).
\]

Then we obtain the following two properties:

**Lemma 12.1.** Let \( V \) be a vector space. Then for each cone \( C, C_j \), where \( j = 1, \ldots, N \) in \( V \),

P1. \((C^\vee)^\vee = C\).

P2. \((\bigcap_{j=1}^N C_j)^\vee = \bigoplus_{j=1}^N C_j^\vee\).
Proof. We first show $P1$. The proof is based on the following fundamental observation coming from the theory of convex sets:

\[(12.2) \quad \text{for each } x_0 \notin C, \text{ there is some } u_0 \in C^\vee \text{ such that } \langle u_0, x_0 \rangle < 0.\]

Let $x \in C$. Then by Definition 12.1,

\[\langle u, x \rangle \geq 0 \text{ for all } u \in C^\vee.\]

This means that $x \in (C^\vee)^\vee$. Thus $C \subseteq (C^\vee)^\vee$. If $x \notin C$, then by (12.2), there is $u_0 \in C^\vee$ such that $\langle u_0, x \rangle < 0$, which is a contradiction to the fact that $x \in (C^\vee)^\vee$ implies $\langle u, x \rangle \geq 0$ for all $u \in C^\vee$.

Thus $x \in C$. This implies that $(C^\vee)^\vee \subseteq C$. We next prove $P2$. By using $P1$, it suffices to prove $(\bigoplus_{j=1}^N C_j)^\vee = \bigcap_{j=1}^N C_j^\vee$. This follows from the following relations: $u \in \left(\bigoplus_{j=1}^N C_j\right)^\vee$ if and only if $\langle u, \sum_{j=1}^N x_j \rangle \geq 0$ for all $x_j \in C_j$ if and only if $\langle u, x_\ell \rangle \geq 0$ for all $x_\ell \in C_\ell$ (choose $x_j = 0$ for $j \neq \ell$ in $\sum_{j=1}^N x_j$) if and only if $u \in \bigcap_{j=1}^N C_j^\vee$. \qed

Definition 12.2. Let $P$ be a polyhedron in $\mathbb{R}^n$ with $\dim(P) = n$ and $G \subseteq P$. Fix $m_0 \in G^\circ$ and define a polyhedron $P/G$:

\[P/G = (P - \{m_0\}) \cap V^\perp(G)\]

which looks like a cross section of $(P - \{m_0\})$ by $V^\perp(G)$. See Figure 8.1, where we can see the cross section $(P - \{m_0\}) \cap V^\perp(G)$ before the translation by $m_0$. In view of (8.6), we see that

\[(12.3) \quad 0 \text{ is a vertex of } P/G.\]

We thus observe that $P/G \cap B(0, \epsilon)$ with some small $\epsilon > 0$ is a portion of the cone around the apex 0 in the vector space $V^\perp(G)$.

Definition 12.3. To each $P/G$, we assign this cone $C(P/G)$ in the vector space $V^\perp(G)$ by

\[C(P/G) = \{rn : n \in P/G \text{ and } r \geq 1\}\]

which is the same set as $\{rn : n \in P/G \cap B(0, \epsilon), r \geq 1\}$ for any $\epsilon > 0$. 

We can observe that $C(P/G)$ is independent of choices of $m_0 \in G^\circ$, and $C(P/G)$ is the smallest cone containing $P/G$.

**Lemma 12.2.** Let $P$ be a polyhedron in $\mathbb{R}^n$ with $\text{dim}(P) = n$ and $G \subseteq P$.

\begin{align*}
(12.4) & \quad \mathcal{G}^*(P, \mathbb{R}^n) = \mathcal{G}^*(P, V^\perp(G)) \\
(12.5) & \quad \mathcal{G}^*(P, V^\perp(G)) = C(P/G)^\vee
\end{align*}

**Proof.** The first equality (12.4) follows directly from (2.4) in Definition 2.11 that for $q \in \mathcal{G}^*(P, \mathbb{R}^n)$, $\langle q, u \rangle = r$ for all $u \in G$. We now show (12.5). Let $q \in \mathcal{G}^*(P, V^\perp(G))$. Observe that

\begin{equation}
(12.6) \quad \mathcal{G}^*(P, V^\perp(G)) \subset \{ p \in V^\perp(G) : \langle p, (n - m) \rangle \geq 0 \text{ for all } n \in P \text{ and } m \in G \}.
\end{equation}

Fix $m_0 \in G^\circ$ as in Definition 12.2 and let

$$u \in C(P/G) = \{rn : n \in (P - \{m_0\}) \cap V^\perp(G) \text{ and } r \geq 1\}.$$ 

Then $u = r(n_0 - m_0)$ with $n_0 \in P$ and $r \geq 1$. Thus by (12.6),

$$\langle q, u \rangle = r\langle q, n_0 - m_0 \rangle \geq 0,$$

which implies $q \in C(P/G)^\vee$.

Hence

$$\mathcal{G}^*(P, V^\perp(G)) \subset C(P/G)^\vee.$$

To prove the other direction, let $q \in C(P/G)^\vee$. Notice that the underlying vector space of $C(P/G)$ is $V^\perp(G)$. By this together with Definition 12.1,

\begin{equation}
(12.7) \quad q \in V^\perp(G).
\end{equation}

Let $n \in P \setminus G, m \in G$ and $m_0 \in G^\circ$ as in Definition 12.2. Then we can observe that for some small positive $\delta$,

$$P_{V^\perp(G)}((P - \{m_0\}) \cap B(0, \delta)) \subset (P - \{m_0\}) \cap V^\perp(G).$$

This implies that for $\eta > 0$ such that $|\eta(n - m_0)| < \delta$,

\begin{equation}
(12.8) \quad P_{V^\perp(G)}(\eta(n - m_0)) \in (P - \{m_0\}) \cap V^\perp(G) = P/G \subset C(P/G).
\end{equation}
We write
\[ n - m = (n - m_0) + (m_0 - m) = \frac{1}{\eta} \eta(n - m_0) + (m_0 - m) = \frac{1}{\eta} P V_\perp(\eta(n - m_0)) + \frac{1}{\eta} P V(\eta(n - m_0)) + (m_0 - m). \]

By (12.8) and \( q \in C(\mathbb{P}/\mathbb{G})^\vee \),
\[ \left\langle q, \frac{1}{\eta} P V_\perp(\eta(n - m_0)) \right\rangle \geq 0. \]

From (12.7),
\[ \left\langle q, \frac{1}{\eta} P V(\eta(n - m_0)) \right\rangle = 0 \text{ and } \left\langle q, m_0 - m \right\rangle = 0. \]

Hence \( \left\langle q, n - m \right\rangle \geq 0 \). So, \( q \in \mathbb{G}^\ast(\mathbb{P}, V_\perp(\mathbb{G})) \). Therefore, \( C(\mathbb{P}/\mathbb{G})^\vee \subset \mathbb{G}^\ast(\mathbb{P}, V_\perp(\mathbb{G})) \). □

**Proof of (12.1).** Let \( \Pi(\mathbb{G}) = \{ q_j \}_{j=1}^M \) and let \( \pi_{q_j,0} \) be the translation of \( \pi_{q_j,r_j} \) by \( m_0 \in \mathbb{G}^0 \) which is chosen as in Definition 12.2. From \( V(\mathbb{G}) = \text{Span}(\mathbb{G} - \{ m_0 \}) \subset \pi_{q_j,0} \),
\[ \tag{12.9} q_j \in V_\perp(\mathbb{G}). \]

From this combined with \( \mathbb{P} \cap B(m_0, \epsilon) = \bigcap_{\pi \in \Pi(\mathbb{G})} \pi^+ \cap B(m_0, \epsilon) \) for small \( \epsilon > 0 \),
\[ (\mathbb{P} - \{ m_0 \}) \cap V_\perp(\mathbb{G}) \cap B(0, \epsilon) = \left( \bigcap_{j=1}^M \pi_{q_j,0}^+ \right) \cap V_\perp(\mathbb{G}) \cap B(0, \epsilon) \]
\[ \tag{12.10} = \bigcap_{j=1}^M \left( \pi_{q_j,0} \cap V_\perp(\mathbb{G}) \right)^+ \cap B(0, \epsilon). \]

where \( \pi_{q_j,0} \cap V_\perp(\mathbb{G}) = \{ x \in V_\perp(\mathbb{G}) : \left\langle q_j, x \right\rangle = 0 \} \) is a hyperplane in \( V_\perp(\mathbb{G}) \). By (12.10) and Definition 12.3,
\[ C(\mathbb{P}/\mathbb{G}) = \left\{ rn : n \in (\mathbb{P} - \{ m_0 \}) \cap V_\perp(\mathbb{G}) \cap B(0, \epsilon) \text{ and } r \geq 1 \right\} \]
\[ \tag{12.11} = \left\{ rn : n \in \bigcap_{j=1}^M \left( \pi_{q_j,0} \cap V_\perp(\mathbb{G}) \right)^+ \cap B(0, \epsilon) \text{ and } r \geq 1 \right\} \]
\[ = \bigcap_{j=1}^M (\pi_{q_j,0} \cap V_\perp(\mathbb{G}))^+. \]
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Here \((\pi_{q_j,0} \cap V^\perp(G))^+\) is a cone in \(V^\perp(G)\) where \(q_j \in V^\perp(G)\) in (12.9). By Example 12.1,

\[
\left( (\pi_{q_j,0} \cap V^\perp(G))^+ \right)^\vee = \text{CoSp}(q_j).
\]

This and (12.11) together with the property \(P2\) in Lemma 12.1 yield,

\[
\mathcal{C}(\mathbb{P}/G)^\vee = \bigoplus_{j=1}^M \left( (\pi_{q_j,0} \cap V^\perp(G))^+ \right)^\vee = \text{CoSp}\{q_j\}_{j=1}^M.
\]

This combined with (12.4) and (12.5) in Lemma 12.2 proves (12.1). \(\square\)

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