On the zeros of sums of the Riemann zeta function

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Abstract

In this article, we study the zeros of $\zeta(\sigma_0+s)\pm\zeta(\sigma_0-s)$ for a fixed $\sigma_0\in\mathbb{R}$. We give a complete description where the zeros of the function are, except for $\frac{1}{2}\leqslant\sigma_0\leqslant\frac{3}{4}$. It turns out that the behavior of zeros of the function with $\sigma_0<\frac{1}{2}$ is very different from that of the function with $\sigma_0>\frac{3}{4}$. Roughly speaking, zeros of the function for $\sigma_0<\frac{1}{2}$ tend to be located on the imaginary axis or the real axis. On the other hand, almost all zeros of the functions for $\sigma_0>\frac{3}{4}$ are arbitrarily close to $\mathrm{Re}\,(s)=\pm\left(\sigma_0-\frac{1}{2}\right)$ and there are fewer zeros in any strip which does not contain these axes. We have the analogues for the function $\zeta(\sigma_0+s)+a\zeta(\sigma_0-s)$ ($\sigma_0>\frac{3}{4}$ and $|a|=1$; $\sigma_0>\frac{1}{2}$ and $|a|\neq 0,1$).

1 Introduction

We denote $\zeta(s)$ by the Riemann zeta function. J. Mozer [19] showed that for $\sigma_0>\frac{1}{2}$, there often exist zeros of $\mathrm{Re}\,\zeta(\sigma_0+it)-1$ and $\mathrm{Im}\,\zeta(\sigma_0+it)$ in the real line. We fix a real number σ_0. Throughout our paper, we denote $H(\sigma_0,s)$ by

$$H(\sigma_0,s)=\zeta(\sigma_0+s)+\zeta(\sigma_0-s)\ \text{ or }\ \zeta(\sigma_0+s)-\zeta(\sigma_0-s).$$

We have $H(\sigma_0,it)=2\mathrm{Re}\,\zeta(\sigma_0+it)$ or $2i\mathrm{Im}\,\zeta(\sigma_0+it)$. In this article, we further investigate the behavior of zeros of $H(\sigma_0,s)$ in the whole complex plane.

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Let $H(s)$ be a function. Let σ, σ_1, and σ_2 be in $(-\infty, \infty)$. For $H(s)$, we define the zero counting functions $N(T)$, $N_0(T)$, $N(\sigma, T)$ and $N(\sigma_1, \sigma_2, T)$:

$$\begin{aligned}
&N(T) = \text{ the number of zeros of } H(s) \text{ with } 0 < \operatorname{Im} s < T,\\
&N_0(T) = \text{ the number of zeros of } H(s) \text{ with } 0 < \operatorname{Im} s < T \text{ and } \operatorname{Re} s = 0,\\
&N(\sigma, T) = \text{ the number of zeros of } H(s) \text{ with } 0 < \operatorname{Im} s < T \text{ and } \sigma < \operatorname{Re} s,\\
&N_0^*(T) = N(0,T) + N_0(T)/2,\\
&N(\sigma_1, \sigma_2, T) = \text{ the number of zeros of } H(s) \text{ with } 0 < \operatorname{Im} s < T \text{ and}\\
&\qquad\qquad \sigma_1 < \operatorname{Re} s < \sigma_2.
\end{aligned}$$

Here, zeros are counted with multiplicities.

M. Z. Garaev [7] investigated the number of zeros of $H(\sigma_0, s)$ in a given region, due to the suggestion of A. A. Karatsuba. In [7], he showed the following theorems.

Theorem A (Garaev)**.** *For any $\sigma_0 < \frac{1}{2}$, there exists $c = c(\sigma_0) > 0$ such that for $H(\sigma_0, s)$, we have*

$$N(T) = \frac{T}{\pi} \log \frac{T}{2\pi e} + O(\log T)$$

and

$$N_0(T) > \frac{T}{\pi} \log \frac{T}{2\pi e} - \widetilde{N}(1 - \sigma_0, T) - c \log T$$

as $T \to \infty$, where $\widetilde{N}(1 - \sigma_0, T)$ is the number of zeros of $\zeta(s)$ with $0 < \operatorname{Im} s < T$ and $1 - \sigma_0 < \operatorname{Re} s$. In particular, there is a constant $c_1 < 1$ such that as $T \to \infty$, we have

$$N_0(T) = \frac{T}{\pi} \log \frac{T}{2\pi e} + O(T^{c_1}).$$

Theorem B (Garaev)**.** *Let $\frac{1}{2} < \sigma_0 < 1$. For $H(\sigma_0, s)$, as $T \to \infty$, we have*

$$T \ll N_0(T) \ll T \log T.$$

Let $\sigma_0 > 1$. As $T \to \infty$, we have

$$T \ll N_0(T) \ll T \log T$$

for $\zeta(\sigma_0 + s) - \zeta(\sigma_0 - s)$. For σ_0 close to 1, the same holds for $\zeta(\sigma_0 + s) + \zeta(\sigma_0 - s)$.

J. B. Conrey showed (subject to a conjecture) in his unpublished paper [5] that the zeros of $\zeta(s)$ and the "Gram-points", well-known from the literature on calculations related to the zeros of $\zeta(s)$, are connected via the curves $\operatorname{Re} \zeta(s) = 0$ and $\operatorname{Im} \zeta(s) = 0$. In particular, concerning Theorem A, J. B. Conrey [5] demonstrated that for $H(\sigma_0, s)$ with a fixed $\sigma_0 < -3$, we have

$$N_0(T) = \frac{T}{\pi} \log \frac{T}{2\pi e} + O_{\sigma_0}(1)$$

as $T \to \infty$ and the zeros of $\operatorname{Re} \zeta(\sigma_0 + it)$ are interlaced with the zeros of $\operatorname{Im} \zeta(\sigma_0 + it)$.

The Riemann zeta function has trivial zeros at $s = -2, -4, -6, \ldots$. Similarly, we can expect that there are trivial zeros for the function $H(\sigma_0, s)$. In fact, M. Z. Garaev [7] proved

Theorem C (Garaev)**.** *Let σ_0 be a real number. There exists $M_1 = M_1(\sigma_0) > 0$ such that all zeros of $H(\sigma_0, s)$ in $|\mathrm{Re}\, s| > M_1$ are real and exactly one in each interval $(2n-1+\sigma_0, 2n+1+\sigma_0)$.*

In Theorem A, M. Z. Garaev obtained the full quota of zeros of $H(\sigma_0, s)$ in $0 < \mathrm{Im}\, s < T$. Is it true that all complex zeros of $H(\sigma_0, s)$ with $\sigma_0 < \frac{1}{2}$ are on $\mathrm{Re}\, s = 0$?

Theorem C says that we know the behavior of zeros of $H(\sigma_0, s)$ as $|\mathrm{Re}\, s| \to \infty$. Immediate questions for this matter are:

Can we have a precise information of real zeros of the function in a given strip $\alpha < \mathrm{Re}\, s < \beta$ and are there any exceptional complex zeros in the region?

Concerning Theorem B, it is interesting to know more precise information of zeros of $H(\sigma_0, s)$ for the case $\sigma_0 > \frac{1}{2}$. For instance, does the function $H(\sigma_0, s)$ ($\sigma_0 > \frac{1}{2}$) have the similar behavior of zeros as the function $H(\sigma_0, s)$ ($\sigma_0 < \frac{1}{2}$)? In fact, M. Z. Garaev [7] proposed a problem as follows.

Problem . *Is it true that for any σ_0, $\frac{1}{2} < \sigma_0 < 1$, the inequality for $H(\sigma_0, s)$*

$$N_0(T) > T\phi(T) \qquad (T \to \infty)$$

holds for some real-valued function $\phi(t)$ with $\phi(t) \to \infty$ as $t \to \infty$?

In this article, we disprove this problem for the case, $\sigma_0 > \frac{3}{4}$. Furthermore, we shall discuss the behavior of zeros of $H(\sigma_0, s)$ near $\mathrm{Re}\, s = \sigma_0 - \frac{1}{2}$, for the case $\sigma_0 > \frac{1}{2}$. It turns out that for $\sigma_0 > \frac{3}{4}$, almost all complex zeros of $H(\sigma_0, s)$ are arbitrarily close to $\mathrm{Re}\, s = \pm\left(\sigma_0 - \frac{1}{2}\right)$. We also complement and strengthen M. Z. Garaev's result for the case $\sigma_0 < \frac{1}{2}$. One of interesting results is that there are one pair of exceptional conjugate complex zeros near the real axis in the left half-plane and by symmetry of zeros around the imaginary axis, one pair of exceptional conjugate complex zeros in the right half-plane, as $\sigma_0 \to -\infty$.

First, we state our results for the case $\sigma_0 < \frac{1}{2}$.

Theorem 1. *We have the following.*

(1) *If $\sigma_0 \leqslant \frac{1}{2}$, then all but finitely many complex zeros of $H(\sigma_0, s)$ are on $\mathrm{Re}\, s = 0$ provided that $\zeta(s)$ has only finitely many complex zeros in $\mathrm{Re}\, s < \sigma_0$.*

(2) *If $\sigma_0 \leqslant 0$, then all but finitely many complex zeros of $H(\sigma_0, s)$ are on $\mathrm{Re}\, s = 0$.*

(3) *If $0 < \sigma_0 < \frac{1}{2}$, then $H(\sigma_0, s)$ has only finitely many complex zeros in $|\mathrm{Re}\, s| \geqslant \frac{1}{2}$.*

Theorem 1 roughly describes the behavior of zeros of $H(\sigma_0, s)$ when $\sigma_0 < \frac{1}{2}$. With careful considerations of $H(\sigma_0, s)$, we are able to prove finer results about the distribution of zeros of the function. In order to introduce them, we need the following.

For each positive integer n, we define $\mathcal{S}_1(n)$, $\mathcal{S}_2(n)$, $\mathcal{S}_3(n)$ and $\mathcal{S}_4(n)$ by

$$\begin{aligned}\mathcal{S}_1(n) =&\{0<\eta<1/2 : \zeta(-4n-x)+\zeta(x-2\eta)<0 \text{ for some } 2\eta+1<x<2\},\\ \mathcal{S}_2(n) =&\{1/2<\eta<1 : \zeta(-4n-x)+\zeta(x-2\eta)<0 \text{ for some } 2<x<2\eta+1\},\\ \mathcal{S}_3(n) =&\{1<\eta<3/2 : \zeta(-4n-x)-\zeta(x-2\eta)>0 \text{ for some } 2\eta+1<x<4\},\\ \mathcal{S}_4(n) =&\{3/2<\eta<2 : \zeta(-4n-x)-\zeta(x-2\eta)>0 \text{ for some } 4<x<2\eta+1\}.\end{aligned}$$

Theorem 2. *Let $\sigma_0 \geqslant 8.5$. Let η_0 and n_0 be such that n_0 is a positive integer, $\sigma_0 = 2n_0 + \eta_0$ and $0 \leqslant \eta_0 < 2$. We set*

$$\eta_1 = \sup \mathcal{S}_1(n_0), \quad \eta_2 = \inf \mathcal{S}_2(n_0), \quad \eta_3 = \sup \mathcal{S}_3(n_0) \quad \textit{and} \quad \eta_4 = \inf \mathcal{S}_4(n_0).$$

Then we have the following;

(1) $0 < \eta_1 < \frac{1}{2} < \eta_2 < 1 < \eta_3 < \frac{3}{2} < \eta_4 < 2$ *and more precisely* $|h - \eta_k| < \sqrt{\pi}(2\pi)^{2n_0+2}\Gamma(4n_0+2.8)^{-\frac{1}{2}}$ *for* $k = 1, 2, 3, 4$, *where* $h = \frac{1}{2}$ *for* $k = 1, 2$ *and otherwise* $h = \frac{3}{2}$;

(2) *all zeros of* $\zeta(-\sigma_0+s)+\zeta(-\sigma_0-s)$ *are either simple and on the real line or on* $\operatorname{Re} s = 0$ *for any* η_0 *satisfying* $0 \leqslant \eta_0 < \eta_1$ *or* $\eta_2 < \eta_0 < 2$, *and the same holds for* $\zeta(-\sigma_0+s)-\zeta(-\sigma_0-s)$ *with* $\zeta'(-\sigma_0) \neq 0$ *and any* η_0 *satisfying* $0 \leqslant \eta_0 < \eta_3$ *or* $\eta_4 < \eta_0 < 2$;

(3) *as in (2), the same statement holds for* $\zeta(-\sigma_0+s)+\zeta(-\sigma_0-s)$ *with* $\eta_0 = \eta_1, \eta_2$, *except one double zero in* $\operatorname{Re} s < 0$ *(by symmetry of zeros around the imaginary axis, one double zero in* $\operatorname{Re} s > 0$*), and the same holds for* $\zeta(-\sigma_0+s)-\zeta(-\sigma_0-s)$ *with* $\eta_0 = \eta_3, \eta_4$;

(4) *as in (2), the same statement holds for* $\zeta(-\sigma+s)+\zeta(-\sigma-s)$ *with* η_0 *satisfying* $\eta_1 < \eta_0 < \eta_2$, *except for one pair of conjugate complex zeros near* $\operatorname{Re} s = -\sigma_0 - 1$ *in* $\operatorname{Re} s < 0$ *(by symmetry of zeros around the imaginary axis, one pair of conjugate complex zeros in* $\operatorname{Re} s > 0$*), and the same holds for* $\zeta(-\sigma_0+s)-\zeta(-\sigma_0-s)$ *with* η_0 *satisfying* $\eta_3 < \eta_0 < \eta_4$.

In Theorem 2 (4), we have exceptional zeros, i.e., one pair of conjugate complex zeros in $\operatorname{Re} s < 0$ (by symmetry of zeros around the imaginary axis, also in $\operatorname{Re} s > 0$). For particular σ_0, these two exceptional zeros occur because the Riemann zeta function has a simple pole at $s = 1$. We note that these exceptional zeros are extremely close to the real axis and in fact, they go to the real axis very quickly as those particular $-\sigma_0$'s in (4) tend to $-\infty$.

Theorem 3. *We have the following.*

(1) *Let* $\sigma_0 < \frac{1}{2}$. *Then, all zeros of* $H(\sigma_0, s)$ *in* $|\operatorname{Im} s| \geqslant 100$ *are on* $\operatorname{Re} s = 0$ *provided that* $\zeta(s)$ *in* $\operatorname{Re} s < \sigma_0$ *and* $\operatorname{Im} s \neq 0$ *has no zeros.*

(2) *The Riemann hypothesis holds if and only if for any* $\sigma_0 < \frac{1}{2}$, *all zeros of* $H(\sigma_0, s)$ *in* $|\operatorname{Im} s| \geqslant 100$ *are on* $\operatorname{Re} s = 0$.

The author developed some methods in proving similar results as Theorems 1–3. For these methods and the results, we refer to [11]–[14]. Together with some other necessary facts, by virtue of the methods as in the papers, we will be enable to prove Theorems 1–3.

Remark. Theorem 1 or 3 implies that under the Riemann hypothesis, all but finitely many complex zeros of $H(\sigma_0, s)$ are on $\mathrm{Re}\, s = 0$ for any $0 < \sigma_0 \leqslant \frac{1}{2}$. On the other hand, we numerically observe the location of zeros of the function for $-8.5 < \sigma_0 < \frac{1}{2}$. According to our numerical computations, the function $H(\sigma_0, s)$ for some $0 < \sigma_0 < \frac{1}{2}$ in $0 < \mathrm{Im}\, s < 10$ and $0 < \mathrm{Re}\, s < 10$ has complex zeros that are different from the exceptional zeros in Theorem 2 (4). The same phenomenon occurs for the function $H(\sigma_0, s)$ with some $-8.5 < \sigma_0 \leqslant 0$.

For $\sigma_0 < \frac{1}{2}$, we have seen that the complex zeros of $H(\sigma_0, s)$ tend to lie on $\mathrm{Re}\, s = 0$. For instance, all complex zeros of $\zeta(s) + \zeta(-s)$ in $|\mathrm{Im}\, s| \geqslant 100$ are on $\mathrm{Re}\, s = 0$. However, this phenomenon for the function $H(\sigma_0, s)$ with $\sigma_0 > \frac{1}{2}$ does not appear anymore. We expect that the complex zeros of the function for $\sigma_0 > \frac{1}{2}$ tend to lie on $\mathrm{Re}\, s = \pm\left(\sigma_0 - \frac{1}{2}\right)$, because we have

$$H(\sigma_0, s) = \zeta(2\sigma_0 - 1/2 + it) \pm \zeta(1/2 - it) \qquad (s = \sigma_0 - 1/2 + it,\ 2\sigma_0 - 1/2 > 1/2)$$

so that the function $H(\sigma_0, s)$ on $s = \sigma_0 - 1/2 + it$ is dominated by the function $\zeta(1/2 - it)$, and by symmetry of zeros around the imaginary axis, we have the same situation at $-\sigma_0 + \frac{1}{2}$. It turns out that the behavior of complex zeros of $H(\sigma_0, s)$ is closely related to that of zeros of $\zeta(s) - a$ ($a \neq 0$). We introduce results for the distribution of zeros of $\zeta(s) - a$. For the behavior of zeros of the function in vertical strips, Borchsenius and Jessen [4, Theorem 14] showed the following.

Theorem D (Borchsenius and Jessen)**.** *Let a be any nonzero constant. Let σ_1 and σ_2 be such that $\frac{1}{2} < \sigma_1 < \sigma_2$. For $\zeta(s) - a$, we have*

$$N(\sigma_1, \sigma_2, T) = h(\sigma_1, \sigma_2)T + o(T)$$

as $T \to \infty$, where $h(\sigma_1, \sigma_2)$ is a constant depending on a, σ_1 and σ_2.

Concerning Theorem D, we refer to [2], in which Bohr and Jessen investigated the behavior of zeros of the function $\log \zeta(s) - a$ in vertical strips and proved interesting theorems including the analogue of Theorem D. We note that Borchsenius and Jessen [4] rounded off the study on zeros in vertical strips of the functions $\zeta(s) - a$ and $\log \zeta(s) - a$ in the half-plane $\sigma > \frac{1}{2}$.

In the region $0 < \mathrm{Im}\, s < T$, the number of zeros of $\zeta(s) - a$ is

$$\frac{T}{2\pi} \log T + O_a(T).$$

Thus, by Theorem D, we know that almost all zeros of $\zeta(s) - a$ are arbitrarily close to $\mathrm{Re}\, s = \frac{1}{2}$. A basic problem for the behavior of zeros of this function related to the Riemann hypothesis is how close to $\mathrm{Re}\, s = \frac{1}{2}$ the zeros of $\zeta(s) - a$ lie. The best known result about this matter was done by Selberg [20]. Also, see [10]. Namely, he obtained the behavior of zeros of $\zeta(s) - a$ arbitrarily close to $\mathrm{Re}\, s = \frac{1}{2}$ as follows.

Theorem E (Selberg)**.** *Let a be any nonzero constant. Let $\phi(t)$ be a real valued function such that*

$$\phi(t) \to \infty, \qquad \psi(t) \to 0 \qquad (\textit{as } t \to \infty),$$

where

$$\psi(t) = \frac{\phi(t)\sqrt{\log\log t}}{\log t}.$$

Let $\beta_a + i\gamma_a$ denote zeros of $\zeta(s) - a$. As $T \to \infty$, we have

$$\begin{aligned}
\sum_{\substack{0<\gamma_a<T \\ \beta_a>\frac{1}{2}}} \left(\beta_a - \frac{1}{2}\right) =& \frac{1}{4\pi^{\frac{3}{2}}} T\sqrt{\log\log T} + O(T);\\
\sum_{\substack{0<\gamma_a<T \\ \left|\beta_a-\frac{1}{2}\right|>\psi(T)}} 1 =& O\left(\frac{T\log T}{\phi(T)}\right);\\
\sum_{\substack{0<\gamma_a<T \\ \left|\beta_a-\frac{1}{2}\right|<\psi(T)}} 1 =& \frac{T}{2\pi}\log T + O\left(\frac{T\log T}{\phi(T)}\right).
\end{aligned}$$

Originally, under the Riemann hypothesis, a kind of Theorem E was demonstrated by Bohr and Landau [3]. Also, Levinson [16] proved a weaker version of Theorem E.

Based on Theorems D and E, we naturally define $H(s;\sigma_0,a)$ by

$$\zeta(\sigma_0 + s) + a\zeta(\sigma_0 - s) \qquad (\sigma_0 \in \mathbb{R};\ 0 \neq a \in \mathbb{C}).$$

Then, the function $H(\sigma_0, s)$ is $H(s;\sigma_0,1)$ or $H(s;\sigma_0,-1)$.

We have some precise results for the function $H(\sigma_0, s)$ with $\sigma_0 < \frac{1}{2}$. However, the behavior of zeros of $H(s;\sigma_0,a)$ with $\sigma_0 < \frac{1}{2}$ and $a \neq -1, 1$ is less interesting, because in this case, we do not have precise results anymore as the function $H(\sigma_0, s)$. On the other hand, applying the same methods as in the proof of Theorem 1 and the method in [21, p. 230], we can demonstrate similar results as in Theorem 1.

Theorem 4. *Let a be a nonzero complex number and let $\delta > 0$. Then, we have the following.*

(1) *If $\sigma_0 \leqslant \frac{1}{2}$, then all but finitely many complex zeros of $H(s;\sigma_0,a)$ are in $|\mathrm{Re}\, s| < \delta$, provided that $\zeta(s)$ has only finitely many complex zeros in* $\mathrm{Re}\, s < \sigma_0$.

(2) *If $\sigma_0 \leqslant 0$, then all but finitely many complex zeros of $H(\sigma_0, s)$ are on $|\mathrm{Re}\, s| < \delta$.*

(3) *If $0 < \sigma_0 < \frac{1}{2}$, then $H(\sigma_0, s)$ has only finitely many complex zeros in $|\mathrm{Re}\, s| \geqslant \frac{1}{2}$.*

(4) *If $\sigma_0 \leqslant \frac{1}{2}$, then the number of zeros of $H(s;\sigma_0,a)$ in $|\mathrm{Re}\, s| > \delta$ and $0 < \mathrm{Im}\, s < T$ is $O_\delta(T)$.*

We omit the proof of Theorem 4.

For $\sigma_0 > \frac{1}{2}$, we will have analogues of Theorems D and E for the function $H(s;\sigma_0,a)$ with $a \neq 0$. In order to investigate the behavior of zeros of $H(s;\sigma_0,a)$ for the case, we state one basic theorem concerning the sum of distances from a given axis $\mathrm{Re}\, s = \alpha$ to zeros in $\mathrm{Re}\, s > \alpha$ and $0 < \mathrm{Im}\, s < T$. We note that the theorem follows from Littlewood's lemma. For convenience, we define $\mathcal{L}_{\sigma_0,a}(T,\sigma)$ by

$$\mathcal{L}_{\sigma_0,a}(T,\sigma) = \int_0^T \log |a^{-1} H(\sigma + it;\sigma_0,a)| dt$$

for $\sigma_0, \sigma \in \mathbb{R}$ and $0 \neq a \in \mathbb{C}$.

Theorem 5. *Let $a \neq 0$ and $\sigma_0 > \frac{1}{2}$. Let $\beta + i\gamma$ denote zeros of $H(s;\sigma_0,a)$. As $T \to \infty$, we have*

$$N(T) = \frac{T}{\pi} \log \frac{T}{2\pi e} + O(\log T); \tag{1}$$

$$2\pi \sum_{\substack{0<\gamma<T \\ \beta>\sigma}} (\beta - \sigma) = \left(\sigma_0 - \frac{1}{2} - \sigma\right) T \log \frac{T}{2\pi e} + O(\log T) + \mathcal{L}_{\sigma_0,a}(T,\sigma); \tag{2}$$

$$N_0^*(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T) \qquad (|a| = 1); \tag{3}$$

$$N_0^*(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T) \pm \widetilde{N}(\sigma_0, T) \qquad (|a| \neq 0, 1), \tag{4}$$

where the implied constant in 'O' does not depend on T, $\widetilde{N}(\sigma_0,T)$ is the number of zeros of $\zeta(s)$ in $\mathrm{Re}\, s > \sigma_0$ and $0 < \mathrm{Im}\, s < T$, and the sign in (4) is $+$ if $0 < |a| < 1$, and $-$ if $|a| > 1$.

Theorem 5 (2) is useful to understand the behavior of zeros of $H(s;\sigma_0,a)$, although the proof of this theorem follows immediately from Littlewood's lemma. We will apply this theorem to investigate the distribution of zeros of $H(s;\sigma_0,a)$. It turns out that Theorem 5 works well in knowing the location of zeros of $H(s;\sigma_0,a)$ for the case $\sigma_0 > \frac{1}{2}$ and $a \neq 0$.

The crucial point for the proof of Theorem E using Littlewood's lemma is

$$\int_0^T \log |\zeta(1/2 + it) - a|\, dt = \frac{1}{2\sqrt{\pi}} T \sqrt{\log \log T} + O(T)$$

as $T \to \infty$. For the detailed proof of this, see [20] or [10]. In order to investigate zeros of $H(s;\sigma_0,a)$ for the case $\sigma_0 > \frac{1}{2}$, an analogue of the above formula will be required.

Theorem 6. *Let $\sigma > \frac{1}{2}$ and let a be any nonzero complex number. Define $f(s)$ by*

$$f(s) = \zeta(s) + a\zeta(\sigma - 1/2 + s) \ \ \textit{or} \ \ \zeta(s) + a\zeta(1/2 + \sigma - s).$$

As $T \to \infty$, we have

$$\int_0^T \log |f(1/2 + it)|\, dt = \frac{1}{2\sqrt{\pi}} T \sqrt{\log \log T} + O(T).$$

For the proof of Theorem 6, we will adjust the proof of Selberg's theorem in [20] or [10]. In particular, see the argument in [10, pp. 155–157] that was done by H. L. Montgomery. We note that Theorem 6 is valid unconditionally. In fact, we need a careful treatment of zeros of $\zeta(\sigma + s)$ and $\zeta(1 + \sigma - s)$ in justifying Theorem 6 without the Riemann hypothesis.

Theorem 7. *We assume that $\sigma_0 > \frac{1}{2}$.*

(1) *Let σ_1 and σ_2 be such that $-\sigma_0 + \frac{1}{2} < \sigma_1 \leqslant \sigma_2 < \sigma_0 - \frac{1}{2}$. If $\sigma_0 > \frac{3}{4}$ and $|a| = 1$, then for $H(s; \sigma_0, a)$, we have*

$$N(\sigma_1, \sigma_2, T) = O(T)$$

as $T \to \infty$, where the implied constant in 'O' does not depend on T.

(2) *If $\sigma_0 > \frac{1}{2}$ and $|a| \neq 0, 1$, then the same holds for $H(s; \sigma_0, a)$.*

(3) *If $\sigma_0 > 1$ and $a \neq 0$, then for $H(s; \sigma_0, a)$, we have*

$$N(\sigma_1, \sigma_2, T) = \widetilde{h}(\sigma_1, \sigma_2) T + o(T)$$

as $T \to \infty$, where $\widetilde{h}(\sigma_1, \sigma_2)$ is a constant depending on a, σ_1 and σ_2.

(4) *Let $\sigma_1 > \sigma_0 - \frac{1}{2}$. Then, there exists a constant $\widetilde{\theta} < 1$ such that for $H(s; \sigma_0, a)$ with $a \neq 0$, as $T \to \infty$, we have*

$$\begin{aligned} &N(\sigma_1, T) = O\left(T^{\tilde{\theta}}\right), \ N(-\infty, -\sigma_1, T) = O\left(T^{\tilde{\theta}}\right), \\ &N(\sigma_1, T) = O(1), \ N(-\infty, -\sigma_1, T) = O(1) \qquad (\sigma_0 - \sigma_1 \leqslant 0). \end{aligned}$$

Thus, Theorem 7 disproves Garaev's problem for the function $H(\sigma_0, s)$ with $\frac{3}{4} < \sigma_0 < 1$.

Assuming the Riemann hypothesis, we immediately have '$O(1)$' in place of '$O(T^{\kappa})$' in Theorem 7 (4). On the other hand, our unconditional result requires a more complicated argument. Namely, for the proof of Theorem 7 (4), we mimic some parts of the proof of Theorem 6 together with necessary properties of $\zeta(s)$. In order to show Theorem 7 (1)–(3), we adopted Borchsenius and Jessen's argument [4]. In section 2, we shall use their methods and theorems for our purpose.

Theorem 8. *Let $\sigma_0 > \frac{1}{2}$ and $a \in \mathbb{C}$ with $a \neq 0$. Let $\phi(t)$ be a real valued function such that*

$$\phi(t) \to \infty, \qquad \psi(t) \to 0 \qquad (\text{as } t \to \infty),$$

where

$$\psi(t) = \frac{\phi(t)\sqrt{\log\log t}}{\log t}.$$

Let $\beta + i\gamma$ denote zeros of $H(s;\sigma_0,a)$. As $T \to \infty$, we have

$$\sum_{\substack{0<\gamma<T \\ \beta>\sigma_0-\frac{1}{2}}} \left(\beta - \left(\sigma_0 - \frac{1}{2}\right)\right) = \frac{1}{4\pi^{\frac{3}{2}}} T\sqrt{\log\log T} + O(T);$$

$$\sum_{\substack{0<\gamma<T \\ \beta<-\sigma_0+\frac{1}{2}}} \left(\beta + \left(\sigma_0 - \frac{1}{2}\right)\right) = -\frac{1}{4\pi^{\frac{3}{2}}} T\sqrt{\log\log T} + O(T);$$

$$\sum_{\substack{0<\gamma<T \\ |\beta|>\sigma_0-\frac{1}{2}+\psi(T)}} 1 = O\left(\frac{T\log T}{\phi(T)}\right).$$

Theorem 8 says that for any $\sigma_0 > \frac{1}{2}$, there exist a chunk of zeros in $\mathrm{Re}\, s > \sigma_0 - \frac{1}{2}$, provably the number of zeros of $H(s;\sigma_0,a)$ in $\mathrm{Re}\, s > \sigma_0 - \frac{1}{2}$ and $0 < \mathrm{Im}\, s < T$ is not less than $\frac{T}{4\pi}\log T$.

Theorem 9. *Let $\sigma_0 > \frac{3}{4}$, $a \in \mathbb{C}$ with $|a| = 1$ and let $\psi(t)$ be as in Theorem 8. In addition, we suppose*

$$\frac{\phi(t)}{\sqrt{\log\log t}} \to \infty$$

as $t \to \infty$. As $T \to \infty$, we have

$$\sum_{\substack{0<\gamma<T \\ \left|\beta-\left(\sigma_0-\frac{1}{2}\right)\right|<\psi(T)}} 1 = \frac{T}{2\pi}\log T + O\left(\frac{T\log T\sqrt{\log\log T}}{\phi(T)}\right).$$

The same formula with '$\left|\beta + \left(\sigma_0 - \frac{1}{2}\right)\right| < \psi(T)$' in place of '$\left|\beta - \left(\sigma_0 - \frac{1}{2}\right)\right| < \psi(T)$' holds. In particular, almost all complex zeros of $H(s;\sigma_0,a)$ are arbitrarily close to $\pm\left(\sigma_0 - \frac{1}{2}\right)$. If $\sigma_0 > \frac{1}{2}$ and $|a| \neq 0, 1$, the same formulas hold for $H(s;\sigma_0,a)$.

According to Theorems 7, 8 and 9, we now see that the behavior of zeros of $H(s;\sigma_0,a)$ for $\sigma_0 > \frac{1}{2}$ is much the same as that for $\zeta(s) - a$ with $a \neq 0$.

We remark that for $H(s;\sigma_0,a)$ with $\frac{1}{2} < \sigma_0 \leqslant \frac{3}{4}$ and $|a| = 1$, we expect the similar results as in Theorems 7 and 9. For instance, we speculate that for $H(s;\sigma_0,a)$ with $\frac{1}{2} < \sigma \leqslant \frac{3}{4}$ and $|a| = 1$,

$$N(\sigma_1,\sigma_2,T) = O(T) \qquad (-\sigma_0 + 1/2 < \sigma_1 \leqslant \sigma_2 < \sigma_0 - 1/2)$$

as $T \to \infty$. If this is valid, then we can prove the same result as in Theorem 9 for the case $\frac{1}{2} < \sigma_0 \leqslant \frac{3}{4}$ and $|a| = 1$. Furthermore, as Theorem E, we expect that for $H(s; \sigma_0, a)$ with $\frac{1}{2} < \sigma_0 \leqslant 1$ and $a \neq 0$,

$$N(\sigma_1, \sigma_2, T) = \widetilde{h}(\sigma_1, \sigma_2)T + o(T) \qquad (-\sigma_0 + 1/2 < \sigma_1 \leqslant \sigma_2 < \sigma_0 - 1/2)$$

as $T \to \infty$, where $\widetilde{h}(\sigma_1, \sigma_2)$ is a constant depending on a, σ_1 and σ_2.

This paper is organized as follows. In section 2, we will introduce some known results and demonstrate necessary facts in proving our theorems. In section 3, we shall prove Theorems 1–3. In section 4, we shall prove Theorems 5 and 6 by virtue of Littlewood's lemma in [21, p. 220] and a modified argument of the proof of Selberg's theorem in [20] or [10]. In section 5, applying Theorems 5 and 6, we will demonstrate Theorems 7–9.

2 Preliminaries

In this section, we state some known results for convenience and also we prove necessary facts for our purposes.

Proposition 2.1. *We denote the Riemann ξ-function $\xi(s)$ by*

$$\xi(s) = \frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s).$$

We have the following:

(1) *All zeros of $\xi(s)$ are on $0 < \mathrm{Re}\, s < 1$.*

(2) $\xi(s) = \xi(1-s)$.

(3) $\zeta(s) = \chi(s)\zeta(1-s)$, *where* $\chi(s) = 2^s\pi^{s-1}\sin\frac{\pi}{2}s\Gamma(1-s)$.

(4)
$$\zeta(s) = \frac{e^{bs}}{2(s-1)\Gamma(\frac{s}{2}+1)}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}$$

or

$$\xi(s) = \frac{1}{2}e^{b_0 s}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}.$$

Here, ρ runs through all zeros of $\zeta(s)$ in $0 < \mathrm{Re}\, s < 1$, $b = \log 2\pi - 1 - \frac{C_0}{2}$, $b_0 = b - \frac{1}{2}\log\pi$, $\sum_\rho \mathrm{Re}\,\frac{1}{\rho} = \frac{C_0}{2} + 1 - \frac{1}{2}\log 4\pi$, and $\sum_\rho \frac{1}{|\rho|^2} < \infty$, where C_0 is Euler's constant.

For Proposition 2.1 (1)–(3), see [21, p. 30, p. 45, (2.1.13) and Theorem 2.1] or [18, Theorem 6.6, Corollary 10.3, Corollary 10.4 and Theorem 10.12]. For Proposition 2.1 (4), see [21, 30–31] and [18, pp. 348–349].

Proposition 2.2. *Let* $\varepsilon > 0$. *Let* $s = \sigma + it$. *Then, we have*

$$\zeta(s) = O\left(|t|^{\frac{1}{2}-\frac{\sigma}{2}+\varepsilon}\right) \quad (0 \leqslant \sigma < 1); \qquad \zeta(s) = O\left(|t|^{\frac{1}{2}-\sigma+\varepsilon}\right) \quad (\sigma < 0);$$

$$\frac{1}{\zeta(s)} = O\left(|t|^{\sigma-\frac{1}{2}}\right) \qquad (\sigma < \alpha < 0); \qquad \chi(\sigma + it) = \left(\frac{2\pi}{t}\right)^{\sigma+it-\frac{1}{2}} e^{i(t+\frac{\pi}{4})}\left(1 + O\left(\frac{1}{t}\right)\right)$$

in any fixed strip $\alpha \leqslant \sigma \leqslant \beta$ *as* $t \to \infty$.

For Proposition 2.2, we refer to [21, pp. 95–96 and p. 78].

Proposition 2.3. *We have*

$$|\zeta(\sigma + it)| \leqslant 2t \qquad (\sigma \geqslant 1/2,\ t > 4).$$

Proof of Proposition 2.3. In [21, p. 49], we recall

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + s\int_N^\infty \frac{[x] - x + \frac{1}{2}}{x^{s+1}} dx + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} \qquad (\mathrm{Re}\, s > 0,\ \mathrm{Im}\, s > 1).$$

Taking $N = [t]$, we readily show Proposition 2.3. □

Proposition 2.4. *For* s *with* $|\arg s| \leqslant \pi - \epsilon$ *and* $0 < \epsilon < \pi$, *we have*

$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi - \int_0^\infty \frac{x - [x] - \frac{1}{2}}{s + x} dx. \tag{1}$$

For s *with* $|\arg s| < \pi$, *we have*

$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + \frac{1}{2}\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)}\zeta(n+1, s+1), \tag{2}$$

where

$$\zeta(z, \nu) = \sum_{0}^{\infty} (\nu + n)^{-z}.$$

For any $s \in \mathbb{C}$, *we have*

$$\frac{1}{\Gamma(s)} = s e^{C_0 s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}}, \tag{3}$$

where C_0 *is Euler's constant. We have*

$$\left|\frac{\Gamma(\sigma_1 + it_2)}{\Gamma(\sigma_2 + it_2)}\right| \leqslant \left|\frac{\Gamma(\sigma_1 + it_1)}{\Gamma(\sigma_2 + it_1)}\right| \quad (\sigma_1 \leqslant \sigma_2,\ \sigma_2 \geqslant 1,\ |t_1| \leqslant |t_2|,\ t_1, t_2 \in \mathbb{R}). \tag{4}$$

Similarly, we have

$$\left|\frac{\Gamma(\sigma_1 + it_2)}{\Gamma(\sigma_2 + it_2)}\right| \leqslant \left|\frac{\Gamma(\sigma_1 + it_1)}{\Gamma(\sigma_2 + it_1)}\right| \tag{5}$$

for $n \in \mathbb{Z}$, $n - \frac{1}{2} = \sigma_1 \leqslant \sigma_2$, $\frac{1}{2} \leqslant \sigma_2$, $|t_1| \leqslant |t_2|$, $t_1, t_2 \in \mathbb{R}$.

Proof of Proposition 2.4. For Proposition 2.4 (1), see [15, p. 405]. For Proposition 2.4 (2), see [6, p. 48]. For Proposition 2.4 (3), see [15, p. 396] or [18, p. 520].

For the proof of Proposition 2.4 (4), we use (3). We may assume $0 \leqslant t_1 \leqslant t_2$. Choose a nonnegative integer n such that $0 < \sigma_1 + n \leqslant \sigma_2$. Then we have

$$\begin{aligned}\left|\frac{\Gamma(\sigma_1+it_2)}{\Gamma(\sigma_2+it_2)}\right| &= \left|\frac{1}{\prod_{0\leqslant k<n}(\sigma_1+k+it_2)}\frac{\Gamma(\sigma_1+n+it_2)}{\Gamma(\sigma_2+it_2)}\right| \\ &\leqslant \left|\frac{1}{\prod_{0\leqslant k<n}(\sigma_1+k+it_1)}\frac{\Gamma(\sigma_1+n+it_2)}{\Gamma(\sigma_2+it_2)}\right|\end{aligned}$$

for $0 \leqslant t_1 \leqslant t_2$. Thus it suffices to show (4) for $0 < \sigma_1 \leqslant \sigma_2$ and $\sigma_2 > 1$. By (3) and the fact that

$$\frac{b+t_2^2}{a+t_2^2} \leqslant \frac{b+t_1^2}{a+t_1^2} \qquad (0 < a \leqslant b,\ 0 \leqslant t_1 \leqslant t_2),$$

we have

$$\begin{aligned}\left|\frac{\Gamma(\sigma_1+it_2)}{\Gamma(\sigma_2+it_2)}\right|^2 &= \frac{\sigma_2^2+t_2^2}{\sigma_1^2+t_2^2}e^{2C_0(\sigma_2-\sigma_1)}\prod_{n=1}^{\infty}\frac{(\sigma_2+n)^2+t_2^2}{(\sigma_1+n)^2+t_2^2}e^{-2\frac{\sigma_2-\sigma_1}{n}} \\ &\leqslant \frac{\sigma_2^2+t_1^2}{\sigma_1^2+t_1^2}e^{2C_0(\sigma_2-\sigma_1)}\prod_{n=1}^{\infty}\frac{(\sigma_2+n)^2+t_1^2}{(\sigma_1+n)^2+t_1^2}e^{-2\frac{\sigma_2-\sigma_1}{n}} \\ &= \left|\frac{\Gamma(\sigma_1+it_1)}{\Gamma(\sigma_2+it_1)}\right|^2.\end{aligned}$$

This proves (4). Similarly, we prove (5). We are done. □

Proposition 2.5. *We have the following.*

(1) $\zeta(s)$ *has simple trivial zeros* $-2, -4, -6, \ldots$ *only in* $\mathrm{Re}\, s < 0$*;*

(2) $$\zeta(1-2n) = (-1)^n\frac{B_n}{2n} \qquad (n = 1, 2, 3, \ldots)$$

where B_1*,* B_2*,* B_3*... are Bernoulli's numbers;*

(3) $$|\zeta(1-2n+it)| > |\zeta(-\sigma+it)| \qquad (-2n+2 \leqslant -\sigma \leqslant -1,\ n = 8, 9, \ldots);$$

(4) $$|\zeta(1-2n+it)| > |\zeta(-\sigma+it)| \qquad (-2n+2 \leqslant -\sigma \leqslant 0.5,\ n = 8, 9, \ldots,\ t \geqslant 10);$$

(5) $$|\zeta(-2n-1/2)| > |\zeta(-\sigma)| \qquad (-14 \leqslant -\sigma \leqslant 0.4,\ n = 8, 9 \ldots);$$

(6) $$|\zeta(-2n-1)| > |\zeta(-\sigma)| \qquad (-14 \leqslant -\sigma \leqslant 0.5,\ n = 8, 9 \ldots);$$

(7) $$|\zeta(-2n-x)| > |\zeta(-2n+2-x)| \qquad (n = 4, 5, \ldots,\ 0 < x < 2).$$

Proof of Proposition 2.5. For Proposition 2.5 (1), (2), see [21, p. 19 and p. 30].

We justify Proposition 2.5 (3). By virtue of Proposition 2.1 (3), Proposition 2.4 (1), (4) and the fact that

$$\left|\frac{\sin\frac{\pi}{2}(-\sigma+it)}{\sin\frac{\pi}{2}(1-2n+it)}\right| \leqslant 1,$$

we have

$$\begin{aligned}\log\left|\frac{\zeta(-\sigma+it)}{\zeta(1-2n+it)}\right| \leqslant&(2n-\sigma-1)\log 2\pi+\log\left|\frac{\Gamma(1+\sigma-it)}{\Gamma(2n-it)}\right|+\log\left|\frac{\zeta(1+\sigma-it)}{\zeta(2n-it)}\right|\\ \leqslant&(2n-\sigma-1)\log 2\pi+\log\left|\frac{\Gamma(1+\sigma)}{\Gamma(2n)}\right|+\log\frac{\zeta(2)}{2-\zeta(16)}\\ <&-(2n-\sigma-1)\log\frac{n}{\pi e}+\sigma\log\frac{\sigma+1}{2n}+\frac{1}{2}.\end{aligned}$$

Define $h(\sigma,n)$ by

$$h(\sigma,n)=-(2n-\sigma-1)\log\frac{n}{\pi e}+\sigma\log\frac{\sigma+1}{2n}+\frac{1}{2}.$$

We will show that $h(\sigma,n)<0$ for $n\geqslant 8$ and $1\leqslant\sigma\leqslant 2n-2$. For $n\geqslant 9$, we have

$$h(\sigma,n)<\sigma\log\frac{\sigma+1}{2n}+\frac{1}{2}\leqslant(2n-2)\log\frac{2n-1}{2n}+\frac{1}{2}<-\frac{n-1}{n}+\frac{1}{2}<0.$$

For $n=8$, we have

$$(h(\sigma,n))''=\frac{1}{\sigma+1}+\frac{1}{(\sigma+1)^2}>0.$$

We check the values of $h(\sigma,8)$ at the endpoints $1,14$:

$$h(1,8)=-14\log\frac{8}{\pi e}+\log\frac{1}{8}+\frac{1}{2}<0;\quad h(14,8)=-\log\frac{8}{\pi e}+14\log\frac{15}{16}+\frac{1}{2}<0.$$

Namely, we get $h(\sigma,8)<0$ for $1\leqslant\sigma\leqslant 14$. So, we conclude that $h(\sigma,n)<0$ for $n\geqslant 8$ and $1\leqslant\sigma\leqslant 2n-2$. Thus we obtain

$$\log\left|\frac{\zeta(-\sigma+it)}{\zeta(1-2n+it)}\right|<0.$$

We prove (4). By (3), it suffice to prove (4) for $-1\leqslant-\sigma\leqslant 0.5$. By Proposition 2.3 and Proposition 2.4 (4), we have

$$\begin{aligned}\left|\frac{\zeta(-\sigma+it)}{\zeta(1-2n+it)}\right| \leqslant&(2\pi)^{15-\sigma}\left|\frac{\Gamma(\sigma+1+it)}{\Gamma(16+it)}\right|\left|\frac{\zeta(1+\sigma+it)}{\zeta(16+it)}\right|\\ \leqslant&\frac{(2\pi)^{15.5}}{0.9999}\frac{2t}{|(15+it)\ldots(2+it)|}\left|\frac{\Gamma(\sigma+1+it)}{\Gamma(2+it)}\right|\\ \leqslant&\frac{\sqrt{2}(2\pi)^{16}}{0.9999}\frac{1}{15!}\frac{5!}{t^4}t\\ <&1\end{aligned}$$

for $t \geqslant 10$, $n \geqslant 8$ and $-1 \leqslant -\sigma \leqslant 0.5$. This justifies (4).

We prove (5). Using Proposition 2.1 (3), we have

$$\begin{aligned}\left|\frac{\zeta(-2m-1/2)}{\zeta(-2n-1/2)}\right| =&(2\pi)^{2(n-m)}\frac{\Gamma(2m+3/2)}{\Gamma(2n+3/2)}\frac{\zeta(2m+3/2)}{\zeta(2n+3/2)}\\ \leqslant&\left(\frac{2\pi}{2m+3/2}\right)^{2(n-m)}\zeta(2m+3/2)\\ <&1\end{aligned}$$

for positive integers m, n with $n > m \geqslant 3$. Thus, (5) follows from (3), because we have

$$|\zeta(-16.5)| > \zeta(-15) \ \text{ and } \ |\zeta(0.4)/\zeta(-16.5)| < 1 \qquad (-1 \leqslant -\sigma \leqslant 1/2).$$

Similarly, we prove (6) and (7).

We complete the proof of Proposition 2.5. □

Proposition 2.6. *Let $0 \leqslant \alpha < \beta < \sigma_1$. Let $f(s)$ be an analytic function, real for real s, regular for $\sigma \geqslant \alpha$ except at $s = \sigma_0 + 2n\pi$ ($n = 1, 2, 3, \ldots$); let*

$$|\mathrm{Re}\, f(\sigma_1 + it)| \geqslant m > 0 \text{ and } |f(\sigma' + it')| \leqslant M_{\sigma,t} \ (\sigma' \geqslant \sigma,\ 1 \leqslant t' \leqslant t).$$

Then if T is not the ordinate of a zero of $f(s)$,

$$|\arg f(\sigma + iT)| \leqslant \frac{\pi}{\log[(\sigma_1-\alpha)/(\sigma_1-\beta)]}\left(\log M_{\alpha,T+2} + \log\frac{1}{m}\right) + \frac{3}{2}\pi$$

for $\sigma \geqslant \beta$.

As in the proof of the lemma in [21, p. 213], we can justify Proposition 2.6.

Proposition 2.7. *Let $\sigma > \frac{1}{2}$. For $\zeta(s)$, as $T \to \infty$, we have*

$$N(T,\sigma) = O\left(T^{\theta}\right),$$

where θ is an absolute constant strictly less than 1.

For Proposition 2.7, see [21, pp. 232–245, 252–253].

Proposition 2.8. *As $T \to \infty$, we have*

$$\int_1^T |\zeta(\sigma+it)|^2 dt = O\left(T \min\left(\log T, \frac{1}{\sigma - 1/2}\right)\right) \qquad (1/2 \leqslant \sigma \leqslant 2),$$
$$\int_0^T \log|\zeta(\sigma+it)| dt = o(T) \qquad (\sigma > 1/2).$$

For Proposition 2.8, we refer to [21, Theorem 7.2 (A)] for the first statement and we can immediately prove the second one using Proposition 2.7 and Theorem 9.15 in [21].

Proposition 2.9. *We let* $\log^+ x = \max(0, \log x)$ *and* $\log^- x = \min(0, \log x)$ *for* $x > 0$. *As* $T \to \infty$, *we have*

$$\int_0^T \log^+ |\zeta(1/2+it)|\, dt = \frac{1}{2\sqrt{\pi}} T\sqrt{\log\log T} + O(T). \tag{2.1}$$

For Proposition 2.9, see [20, p. 55] or [10, p. 155].

Proposition 2.10. *Without any hypothesis, we have*

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n<x^2} \frac{\Lambda_x(n)}{n^s} + \frac{x^{2(1-s)} - x^{1-s}}{(1-s)^2 \log x} + \frac{1}{\log x}\sum_{q=1}^{\infty} \frac{x^{-2q-s} - x^{-2(2q+s)}}{(2q+s)^2} + \frac{1}{\log x}\sum_{\rho} \frac{x^{\rho-s} - x^{2(\rho-s)}}{(s-\rho)^2},$$

where the ρ *runs through all zeros of* $\zeta(s)$ *in* $\operatorname{Re} s \geqslant 0$ *and*

$$\Lambda_x(n) = \Lambda(n) \qquad (1 \leqslant n \leqslant x), \qquad \frac{\Lambda(n)\log(\frac{x^2}{n})}{\log x} \qquad (x \leqslant n \leqslant x^2).$$

For Proposition 2.10, see [21, Theorem 14.20] or [18, p. 433].

We introduce results of mean motions in [4]. Suppose that $p > 0$ and that $f(s)$, $f_1(s)$, $f_2(s), \ldots$ are functions defined in the half strip $\alpha < \sigma < \beta$, $t \geqslant 0$. Then, we say that $f_n(s)$ converges in the mean with index p, towards $f(s)$ in $[\alpha, \beta]$ if

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \int_{\alpha_1}^{\beta_1} |f(\sigma+it) - f_n(\sigma+it)|^p d\sigma dt \to 0 \qquad (n \to \infty),$$

for any reduced strip $(\alpha <)\alpha_1 < \sigma < \beta_1 (< \beta)$.

V. Borchsenius and B. Jessen [4, Theorem 1] proved

Proposition 2.11. *Let* $-\infty \leqslant \alpha < \alpha_0 < \beta_0 < \beta \leqslant \infty$, $t \geqslant 0$ *and let* $f_1(s), f_2(s), \ldots$ *be a sequence of functions almost periodic in* $[\alpha, \beta]$ *converging uniformly in* $[\alpha_0, \beta_0]$ *towards a function* $f(s)$, *which is then almost periodic in* $[\alpha_0, \beta_0]$. *Suppose that none of the functions is identically zero. Suppose further that* $f(s)$ *may be continued as a regular function in the half-strip* $\alpha < \sigma < \beta$, $t \geqslant 0$, *and that* $f_n(s)$ *converges in mean with an index* $p > 0$ *towards* $f(s)$ *in* $[\alpha, \beta]$. *Then, the Jensen function*

$$\varphi(\sigma) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \log|f(\sigma+it)| dt$$

exists uniformly in $[\alpha, \beta]$*, i.e. the function*

$$\varphi(\sigma; T) = \frac{1}{T} \int_0^T \log |f(\sigma + it)| dt$$

converges for $T \to \infty$ *uniformly in* $[\alpha, \beta]$ *towards a limit function* $\varphi(\sigma)$*. The Jensen function* $\varphi_n(\sigma)$ *of* $f_n(s)$ *converges for* $n \to \infty$ *uniformly in* $[\alpha, \beta]$ *towards* $\varphi(\sigma)$*. For* $f(s)$*, we have*

$$\liminf_{T\to\infty} \frac{N(\sigma_1, \sigma_2, T)}{T} \geqslant \frac{\varphi'(\sigma_2 - 0) - \varphi'(\sigma_1 + 0)}{2\pi};$$
$$\limsup_{T\to\infty} \frac{N(\sigma_1, \sigma_2, T)}{T} \leqslant \frac{\varphi'(\sigma_2 + 0) - \varphi'(\sigma_1 - 0)}{2\pi}$$

as $T \to \infty$*, where* $\varphi'(\sigma - 0)$ *and* $\varphi'(\sigma + 0)$ *denote the left and right derivatives of* $\varphi(\sigma)$*.*

For the earlier version of Proposition 2.11, we refer to [8].

$\mathbb{T}$ denotes $\mathbb{R}/\mathbb{Z}$. Let p_j be the jth prime number ($j = 1, 2, 3, \ldots$). For σ_1, $\sigma_2 > \frac{1}{2}$, we define $\zeta_m^*(\sigma_1; \theta_1^*, \ldots, \theta_m^*)$, $\zeta_n(\sigma_2; \theta_1, \ldots, \theta_n)$ and $\zeta_{m,n}(\sigma_1, \sigma_2; \theta_1^*, \ldots, \theta_m^*, \theta_1, \ldots, \theta_n)$ from $\mathbb{T}^m$, $\mathbb{T}^n$, $\mathbb{T}^{m+n}$ to $\mathbb{C}$ by

$$\zeta_m^*(\sigma_1; \theta_1^*, \ldots, \theta_m^*) = \prod_{j=1}^{m} \left(1 - p_j^{-\sigma_1} e^{2\pi i \theta_j^*}\right),$$
$$\zeta_n(\sigma_2; \theta_1, \ldots, \theta_n) = \prod_{k=1}^{n} \left(1 - p_k^{-\sigma_2} e^{2\pi i \theta_k}\right)^{-1},$$
$$\zeta_{m,n}(\sigma_1, \sigma_2; \theta_1^*, \ldots, \theta_m^*, \theta_1, \ldots, \theta_n) = \zeta_m^*(\sigma_1, \theta_1^*, \ldots, \theta_m^*) \cdot \zeta_n(\sigma_2, \theta_1, \ldots, \theta_n).$$

Let $\widetilde{\mu}_{m,\sigma_1}^*$, $\widetilde{\mu}_{n,\sigma_2}$ and $\widetilde{\mu}_{m,n,\sigma_1,\sigma_2}$ denote distribution functions of $\zeta_m^*(\sigma_1; \theta_1^*, \ldots, \theta_m^*)$, $\zeta_n(\sigma_2; \theta_1, \ldots, \theta_n)$ and $\zeta_{m,n}(\sigma_1, \sigma_2; \theta_1^*, \ldots, \theta_m^*, \theta_1, \ldots, \theta_n)$ defined by

$$\widetilde{\mu}_{m,\sigma_1}^*(E) = |\Omega_1(E)|, \;\; \widetilde{\mu}_{n,\sigma_2}(E) = |\Omega_2(E)|, \;\; \widetilde{\mu}_{m,n,\sigma_1,\sigma_2}(E) = |\Omega_3(E)|,$$

where '$|A|$' means the Lebesgue measure of a subset of $\mathbb{T}^m$, $\mathbb{T}^n$ or $\mathbb{T}^{m+n}$ and for an arbitrary Borel set E in $\mathbb{C}$,

$$\Omega_1(E) = \{(\theta_1^*, \ldots, \theta_m^*) \in \mathbb{T}^m : \zeta_m^*(\sigma_1; \theta_1^*, \ldots, \theta_m^*) \in E\},$$
$$\Omega_2(E) = \{(\theta_1, \ldots, \theta_n) \in \mathbb{T}^n : \zeta_n(\sigma_2; \theta_1, \ldots, \theta_n) \in E\},$$
$$\Omega_3(E) = \{(\theta_1^*, \ldots, \theta_m^*, \theta_1, \ldots, \theta_n) \in \mathbb{T}^{m+n} : \zeta_{m,n}(\sigma_1, \sigma_2; \theta_1^*, \ldots, \theta_m^*, \theta_1, \ldots, \theta_n) \in E\}.$$

We call $F(x)$ the density of a distribution function μ on $\mathbb{C}$ if

$$\mu(E) = \int_E F(x) dx$$

for any Borel set E in $\mathbb{C}$.

We define $\zeta_n(s)$ $(n = 1, 2, 3, \ldots)$ by

$$\zeta_n(s) = \prod_{p \leqslant p_n} (1 - p^{-s})^{-1},$$

where p runs through all primes $\leqslant p_n$.

Proposition 2.12. *Let $\sigma_1,\ \sigma_2 > \frac{1}{2}$. The distribution functions $\widetilde{\mu}^*_{m,\sigma_1}$, $\widetilde{\mu}_{n,\sigma_2}$ and $\widetilde{\mu}_{m,n,\sigma_1,\sigma_2}$ are absolutely continuous with continuous densities $\widetilde{F}^*_{m,\sigma_1}(x)$, $\widetilde{F}_{n,\sigma_2}(x)$ and $\widetilde{F}_{m,n,\sigma_1,\sigma_2}(x)$. The distribution functions $\widetilde{\mu}^*_{m,\sigma_1}$, $\widetilde{\mu}_{n,\sigma_2}$ and $\widetilde{\mu}_{m,n,\sigma_1,\sigma_2}$ converge for $m \to \infty$ and $n \to \infty$ towards distributions $\widetilde{\mu}^*_{\sigma_1}$, $\widetilde{\mu}_{\sigma_2}$ and $\widetilde{\mu}_{m,\sigma_1,\sigma_2}$ which are absolutely continuous with continuous densities $\widetilde{F}^*_{\sigma_1}(x)$, $\widetilde{F}_{\sigma_2}(x)$ and $\widetilde{F}_{m,\sigma_1,\sigma_2}(x)$ which are zero for $x = 0$. For absolute constants $\widetilde{K}_0,\ \widetilde{\lambda} > 0$, we have*

$$\left|\widetilde{F}^*_{m,\sigma_1}(x)\right|, \left|\widetilde{F}^*_{\sigma_1}(x)\right|, \left|\widetilde{F}_{n,\sigma_2}(x)\right|, \left|\widetilde{F}_{\sigma_2}(x)\right|, \left|\widetilde{F}_{m,n,\sigma_1,\sigma_2}(x)\right|, \left|\widetilde{F}_{m,\sigma_1,\sigma_2}(x)\right| \leqslant \widetilde{K}_0 e^{-\widetilde{\lambda}(\log|x|)^2}.$$

For $a \in \mathbb{C}$, the Jensen function

$$\varphi_m(\sigma_1, \sigma_2) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \log\left|\frac{\zeta(\sigma_2 + it)}{\zeta_m(\sigma_1 - it)} + a\right| dt$$

exists and

$$\varphi_m(\sigma_1, \sigma_2) = O(1),$$

where the implied constant in 'O' depends only on σ_1 and σ_2.

We remark that Proposition 2.12 does not follow from theorems in [4] directly, because we have the extra function $\zeta^*_m(\sigma_1; \theta^*_1, \ldots, \theta^*_m)$ in $\zeta_{m,n}(\sigma_1, \sigma_2; \theta^*_1, \ldots, \theta^*_m, \theta_1, \ldots, \theta_n)$. Fortunately, all techniques in [4] work well for Proposition 2.12. Concerning this proposition, we also refer to [9].

Proof of Proposition 2.12. We follow the methods as in Theorems 5, 6, 11 and 12 [4]. Let μ^*_{m,σ_1}, μ_{n,σ_2} and $\mu_{m,n,\sigma_1,\sigma_2}$ denote distribution functions of $\log\zeta^*_m(\sigma_1; \theta^*_1, \ldots, \theta^*_m)$, $\log\zeta_n(\sigma_2; \theta_1, \ldots, \theta_n)$ and $\log\zeta_{m,n}(\sigma_1, \sigma_2; \theta^*_1, \ldots, \theta^*_m, \theta_1, \ldots, \theta_n)$. By Theorems 5, 6 in [4], we have the following.

The distribution functions μ^*_{m,σ_1}, μ_{n,σ_2} and $\mu_{m,n,\sigma_1,\sigma_2}$ are for $m, n \geqslant 11$ absolutely continuous with continuous densities $F^*_{m,\sigma_1}(x)$, $F_{n,\sigma_2}(x)$ and $F_{m,n,\sigma_1,\sigma_2}(x)$. Also, the distribution functions μ^*_{m,σ_1}, μ_{n,σ_2} and $\mu_{m,n,\sigma_1,\sigma_2}$ converge for $m \to \infty$ and $n \to \infty$ towards distribution functions $\mu^*_{\sigma_1}$, μ_{σ_2} and $\mu_{m,\sigma_1,\sigma_2}$ $(m \geqslant 11)$ which are absolutely continuous with continuous densities $F^*_{\sigma_1}(x)$, $F_{\sigma_2}(x)$ and $F_{\sigma_1,\sigma_2}(x)$. For absolute constants $K_0,\ \lambda > 0$, we have

$$\left|F^*_{m,\sigma_1}(x)\right|, \left|F^*_{\sigma_1}(x)\right|, \left|F_{n,\sigma_2}(x)\right|, \left|F_{\sigma_2}(x)\right|, \left|F_{m,n,\sigma_1,\sigma_2}(x)\right| \leqslant K_0 e^{-\lambda|x|^2}. \tag{2.2}$$

One crucial fact is that by the equation (65) in [4, p. 145], we have

$$F_{m,n,\sigma_1,\sigma_2}(x) = \int_{\mathbb{C}} F^*_{m,\sigma_1}(x-u)d\mu_{n,\sigma_2}(u)$$

for sufficiently large m, n. Using this and (2.2), we have

$$|F_{m,\sigma_1,\sigma_2}(x)| \leqslant K_0' e^{-\lambda'|x|^2} \tag{2.3}$$

for absolute constants K_0', $\lambda' > 0$. From this, as in Theorem 12 in [4], for absolute constants $\widetilde{K}_0$, $\widetilde{\lambda} > 0$, we get

$$\left|\widetilde{F}_{m,\sigma_1,\sigma_2}(x)\right| \leqslant \widetilde{K}_0 e^{-\widetilde{\lambda}(\log|x|)^2}.$$

As in the proof of Theorems 12 and 14 in [4], we have all requirements of Proposition 2.12, except for

$$\varphi(\sigma_1,\sigma_2) = O(1).$$

As in the equation (88) in [4], we write

$$\varphi(\sigma_1,\sigma_2) = \int_{\mathbb{C}} \log|u-a|\widetilde{F}_{m,\sigma_1,\sigma_2}(u)du.$$

Then, we have

$$|\varphi(\sigma_1,\sigma_2)| \leqslant \widetilde{K_0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \log|x_1+ix_2-a| e^{-\widetilde{\lambda}\left(\log(x_1^2+x_2^2)\right)^2} dx_1dx_2 = O(1).$$

Hence, we complete the proof of Proposition 2.12. □

Finally, we need the following.

Proposition 2.13. *Let $\sigma_0 > 1$ and $0 \neq a \in \mathbb{C}$. Then, the Jensen function*

$$\varphi(\sigma) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \log|H(\sigma+it;\sigma_0,a)|\,dt$$

exists and is differentiable for $-\sigma_0 + \frac{1}{2} < \sigma < \sigma_0 - \frac{1}{2}$.

Proof of Proposition 2.13. We fix $\sigma_0 > 1$ and $a \neq 0$. It suffices to show the proposition for $-\sigma_0 + \frac{1}{2} < \sigma \leqslant 0$. It is known that $\zeta_n(s)$ converges in mean with the index 2 towards $\zeta(s)$ in $[\frac{1}{2}, \infty]$. For the proof of this, apply a result in [1, pp. 163–169]. We put

$$f(s) = \frac{\zeta(\sigma_0+s)}{\zeta(\sigma_0-s)} + a \quad \text{and} \quad f_n(s) = \frac{\zeta_n(\sigma_0+s)}{\zeta_n(\sigma_0-s)} + a \quad (n = 1,2,3,\ldots).$$

Choose $\sigma_* > 0$ such that $\sigma_0 - \sigma_* > 1$. Then, $f_n(s)$ converges uniformly to $f(s)$ in $[-\sigma_*,\sigma_*]$ and $f_n(s)$ converges in mean with the index 2 towards $f(s)$ in $\left[-\sigma_0+\frac{1}{2},\sigma_*\right]$. Thus, by Proposition 2.11, we see that $\varphi_n^*(\sigma)$ converges uniformly to $\varphi^*(\sigma)$ in $\left[-\sigma_0+\frac{1}{2},\sigma_*\right]$, where

$$\varphi_n^*(\sigma) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \log|f_n(\sigma+it)|\,dt \qquad (n = 1,2,\ldots)$$

and

$$\varphi^*(\sigma) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \log |f(\sigma + it)|\, dt.$$

Applying Proposition 2.8, we have

$$\varphi(\sigma) = \varphi^*(\sigma) \qquad (-\sigma_0 + 1/2 < \sigma < \sigma_*).$$

Now, we work on

$$\zeta_n^*(\sigma_1; \theta_1^*, \ldots, \theta_n^*) = \prod_{j=1}^{n} \left(1 - p_j^{-\sigma_1} e^{2\pi i \theta_j^*}\right),$$

$$\zeta_n(\sigma_2; \theta_1, \ldots, \theta_n) = \prod_{j=1}^{n} \left(1 - p_j^{-\sigma_2} e^{2\pi i \theta_j}\right)^{-1},$$

$$\zeta_{n,n}(\sigma_1, \sigma_2; \theta_1^*, \ldots, \theta_n^*, \theta_1, \ldots, \theta_n) = \zeta_n^*(\sigma_1, \theta_1^*, \ldots, \theta_n^*) \cdot \zeta_n(\sigma_2, \theta_1, \ldots, \theta_n)$$

as in the proof of Proposition 2.12, where $\sigma_1 = \sigma_0 - \sigma$ and $\sigma_2 = \sigma_0 + \sigma$. Then, we have an analogue of Theorem 12 in [4]. By virtue of this, the differentiability of $\varphi^*(\sigma)$ in $\left(-\sigma_0 + \frac{1}{2}, \sigma_1\right)$ follows from the same argument as in the proof of Theorem 14 in [4]. Thus, $\varphi(\sigma)$ is differentiable in $(-\sigma_0 + 1/2, \sigma_*)$. □

We remark that for $\frac{1}{2} < \sigma_0 \leqslant 1$, we are not able to prove the same conclusions as in Proposition 2.13, using our methods. The reason is that Proposition 2.11 doesn't work for the function $H(s; \sigma_0, a)$ with $\frac{1}{2} < \sigma_0 \leqslant 1$. Nevertheless, we expect that Proposition 2.13 holds for $\frac{1}{2} < \sigma_0 \leqslant 1$. We may need a different way in demonstrating it.

3 Location of zeros when $\sigma_0 < \frac{1}{2}$

In this section, we shall justify the theorems 1, 2 and 3.

First, we investigate a rough behavior of the function $H(\sigma_0, s)$. Namely, we prove Theorem 1. We shall use author's argument in [11]–[14]. In fact, the proof of Theorem 1 is rather short and simple compare to the ones in the papers.

Proof of Theorem 1. In Theorem 1, (2) follows from (1) and Proposition 2.1 (1).

We prove (1). So, we assume that $\zeta(s)$ has only finitely many complex zeros in Re $s < \sigma_0$. By Theorem C and the facts that $H(\sigma_0, s) = H(\sigma_0, -s)$ or $H(\sigma_0, s) = -H(\sigma_0, -s)$ and $H(\sigma_0, s) = \overline{H(\sigma_0, \overline{s})}$, it suffices to show that $H(\sigma_0, s)$ has only finitely many zeros in $-M_1 \leqslant$ Re $s < 0$ and Im $s > 0$. We compute

$$\left|\frac{\zeta(\sigma_0 - s)}{\zeta(\sigma_0 + s)}\right|$$

for $-M_1 \leqslant \operatorname{Re} s < 0$ and $\operatorname{Im} s > 0$. Let $s = -\sigma + it$ where $0 < \sigma \leqslant M_1$ and $t > 0$. Using Proposition 2.1 (4), we have

$$\frac{\zeta(\sigma_0 - s)}{\zeta(\sigma_0 + s)} = e^{-2bs} \frac{(\sigma_0 + s - 1)}{(\sigma_0 - s - 1)} \frac{\Gamma\left(\frac{\sigma_0+s}{2} + 1\right)}{\Gamma\left(\frac{\sigma_0-s}{2} + 1\right)} \prod_{\rho} \frac{\rho - \sigma_0 + s}{\rho - \sigma_0 - s} e^{-\frac{2s}{\rho}}.$$

Thus, by Proposition 2.1 (4) and Proposition 2.4 (3), we get

$$\begin{aligned}\left|\frac{\zeta(\sigma_0 + \sigma - it)}{\zeta(\sigma_0 - \sigma + it)}\right|^2 =& e^{4b\sigma} \frac{(\sigma_0 - 1 - \sigma)^2 + t^2}{(\sigma_0 - 1 + \sigma)^2 + t^2} e^{2C_0\sigma} \prod_{n=1}^{\infty} \frac{(\sigma_0 + \sigma + 2n)^2 + t^2}{(\sigma_0 - \sigma + 2n)^2 + t^2} e^{-\frac{2\sigma}{n}} \\ & \prod_{\rho} \frac{(\beta - \sigma - \sigma_0)^2 + (\gamma - t)^2}{(\beta + \sigma - \sigma_0)^2 + (\gamma - t)^2} e^{\frac{4\sigma\beta}{\beta^2+\gamma^2}},\end{aligned}$$

where $\rho = \beta + i\gamma$ runs through all zeros of $\xi(s)$. By this equation, Proposition 2.1 (4) and the fact that

$$1 + a < e^a, \qquad \frac{(b-a)^2 + t^2}{(b+a)^2 + t^2} < 1 \qquad (a, b > 0,\ t > 0),$$

we obtain

$$(*) \qquad \log\left|\frac{\zeta(\sigma_0 + \sigma - it)}{\zeta(\sigma_0 - \sigma + it)}\right|^2 \leqslant -2\sigma\left(-c(t) + \sum_{n=1}^{\infty} \frac{1}{n} - \frac{2\sigma_0 + 4n}{(\sigma_0 - \sigma + 2n)^2 + t^2}\right)$$

where $t > 0$ and

$$c(t) = \log \pi + C_0 + \frac{2(1 - \sigma_0)}{(\sigma_0 - 1 + \sigma)^2 + t^2} + \frac{1}{2\sigma} \sum_{\beta < \sigma_0} \log \frac{(\beta - \sigma - \sigma_0)^2 + (\gamma - t)^2}{(\beta + \sigma - \sigma_0)^2 + (\gamma - t)^2}.$$

Since σ_0 is fixed and $0 < \sigma \leqslant M_1$, it is not hard to see that we have

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{2\sigma_0 + 4n}{(\sigma_0 - \sigma + 2n)^2 + t^2} \geqslant c_1 + \sum_{n=1}^{\infty} \frac{1}{n} - \frac{4n}{4n^2 + t^2} \geqslant c_2 + \log t$$

for sufficiently large t and absolute constants c_1, c_2. By our assumption, there are only finitely many β's satisfying $\beta < \sigma_0$. Hence, for sufficiently large t, we have

$$\frac{1}{2\sigma} \sum_{\beta < \sigma_0} \log \frac{(\beta - \sigma - \sigma_0)^2 + (\gamma - t)^2}{(\beta + \sigma - \sigma_0)^2 + (\gamma - t)^2} < c_3,$$

where $c_3 > 0$ is an absolute constant. Thus, it is not hard to see that we get

$$\log\left|\frac{\zeta(\sigma_0 - s)}{\zeta(\sigma_0 + s)}\right|^2 < 0 \qquad \text{or} \qquad \left|\frac{\zeta(\sigma_0 - s)}{\zeta(\sigma_0 + s)}\right| < 1$$

uniformly in $-M_1 \leqslant \operatorname{Re} s < 0$ and $\operatorname{Im} s > t_0$ for a sufficiently large $t_0 > 0$. Then we readily conclude that in $-M_1 \leqslant \operatorname{Re} s < 0$ and $\operatorname{Im} s > t_0$, $H(\sigma_0, s)$ has no zeros. Thus, (1) follows.

Similarly, by virtue of the previous arguments, we can justify (3). We omit the proof of it.

We complete the proof of Theorem 1. □

We shall give a precise information about the zeros of $H(-\sigma_0, s)$ for $\sigma_0 \geqslant 8.5$. The proof will require fine estimations of ratios of the Riemann zeta function in the region $\operatorname{Re} s < 0$. By virtue of these and the sign change method, we are enable to prove Theorem 2.

Proof of Theorem 2. We assume that $\sigma_0 \geqslant 8.5$. Let n_0 and η_0 be such that $\sigma_0 = 2n_0 + \eta_0$, n_0 is a positive integer and $0 \leqslant \eta_0 < 2$.

We need the following lemma in proving Theorem 2.

Lemma 3.1. *Let* $0 < \widetilde{\sigma} \leqslant 1$. *Then there exist a sufficiently large integer* $m_0 > 0$ *with* $2m_0 - 1 > 2\sigma_0 + 10$ *and a positive number* t_1 *such that*

$$\left|\frac{\zeta(-\sigma_0 - s)}{\zeta(-\sigma_0 + s)}\right| < 1$$

holds in $\mathcal{R}_1 \cup \ldots \cup \mathcal{R}_{10}$, *where*

$$\begin{aligned}
\mathcal{R}_1 &= \{(s, \eta_0) : -1 \leqslant \operatorname{Re} s < 0,\ 0 \leqslant \eta_0 \leqslant 1\},\\
\mathcal{R}_2 &= \{(s, \eta_0) : \operatorname{Re} s = -1,\ 1 < \eta_0 < 2\},\\
\mathcal{R}_3 &= \{(s, \eta_0) : -1 \leqslant \operatorname{Re} s < 0,\ |\operatorname{Im} s| \geqslant 1\},\\
\mathcal{R}_4 &= \{(s, \eta_0) : \operatorname{Re} s = \sigma_0 - (2m - 1),\ m \geqslant m_0\},\\
\mathcal{R}_5 &= \{(s, \eta_0) : \operatorname{Re} s < -\widetilde{\sigma},\ |\operatorname{Im} s| \geqslant t_1\},\\
\mathcal{R}_6 &= \{(s, \eta_0) : \operatorname{Re} s = -2n_0 - (5 - \eta_0) - 1/2\},\\
\mathcal{R}_7 &= \{(s, \eta_0) : \operatorname{Re} s = -2n_0 - (3 - \eta_0) - 1/2,\ 0 \leqslant \eta_0 \leqslant 1\},\\
\mathcal{R}_8 &= \{(s, \eta_0) : \operatorname{Re} s = -2n_0 - (-1 - \eta_0)\},\\
\mathcal{R}_9 &= \{(s, \eta_0) : \operatorname{Re} s = -2n_0 - (1 - \eta_0),\ 0.3 \leqslant \eta_0 < 2\},\\
\mathcal{R}_{10} &= \{(s, \eta_0) : \operatorname{Re} s = -2n_0 - (3 - \eta_0),\ 1.3 \leqslant \eta_0 < 2\}.
\end{aligned}$$

Intuitively, one can observe this technical lemma without a difficulty. However, the proof of this is rather elaborate. By virtue of this lemma, we can use Rouché's theorem in knowing the exact number of zeros of $H(-\sigma_0, s)$ with $\sigma_0 \geqslant 8.5$ in a given region. Then, using the sign change method, we will prove Theorem 2.

Proof of Lemma 3.1. Using Proposition 2.1 (3), we have

$$\frac{\zeta(-\sigma_0 - s)}{\zeta(-\sigma_0 + s)} = (2\pi)^{-2s} \frac{\sin\frac{\pi}{2}(\sigma_0 + s)}{\sin\frac{\pi}{2}(\sigma_0 - s)} \frac{\Gamma(1 + \sigma_0 + s)}{\Gamma(1 + \sigma_0 - s)} \frac{\zeta(1 + \sigma_0 + s)}{\zeta(1 + \sigma_0 - s)}. \tag{3.1}$$

We gather properties of the right equation of this. We set

$$s = -\sigma + it,$$

where $\sigma > 0$ and $t \geqslant 0$. Using Euler's product

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1} \quad \text{and} \quad \left|\frac{1 - p^{-\sigma_2 - it}}{1 - p^{-\sigma_1 - it}}\right| \leqslant \frac{1 - p^{-\sigma_2}}{1 - p^{-\sigma_1}} \quad (1 < \sigma_1 < \sigma_2,\, t \in \mathbb{R}),$$

we have

$$\left|\frac{\zeta(1+\sigma_0+s)}{\zeta(1+\sigma_0-s)}\right| \leqslant \frac{\zeta(1+\sigma_0-\sigma)}{\zeta(1+\sigma_0+\sigma)} \qquad (\sigma_0 - \sigma > 0).$$

Thus, we get

$$\begin{aligned}
\log\left|\frac{\zeta(1+\sigma_0+s)}{\zeta(1+\sigma_0-s)}\right| &\leqslant \log\frac{\zeta(1+\sigma_0-\sigma)}{\zeta(1+\sigma_0+\sigma)} \\
&= \sum_{n=1}^{\infty} \sum_{p\text{: prime}} \frac{1}{np^{n+n\sigma_0}}(p^{n\sigma} - p^{-n\sigma}) \\
&\leqslant \sum_n \sum_p \frac{2\sigma \log p}{p^{n(\sigma_0+1-\sigma)}} \\
&< 2\sigma \sum_p \frac{\log p}{p^{\sigma_0+1-\sigma} - 1}
\end{aligned} \tag{3.2}$$

for $\sigma_0 - \sigma > 0$. Since

$$\frac{2^a - 1}{x^a - 1} < \frac{2^6}{x^6} \qquad (a \geqslant 6,\, x > 2),$$

we obtain

$$\sum_p \frac{\log p}{p^{\sigma_0+1-\sigma} - 1} \leqslant \frac{1}{2^{\sigma_0+1-\sigma} - 1} \cdot 2^6 \sum_p \frac{\log p}{p^6} < \frac{1}{2^{\sigma_0+1-\sigma} - 1} \qquad (\sigma_0 - \sigma \geqslant 5).$$

Using this with (3.2), we obtain

$$\log\left|\frac{\zeta(1+\sigma_0+s)}{\zeta(1+\sigma_0-s)}\right| \leqslant \frac{2\sigma}{2^{\sigma_0+1-\sigma} - 1} \qquad (\sigma_0 \geqslant 6,\, 0 \leqslant \sigma \leqslant 1,\, t \geqslant 0). \tag{3.3}$$

We see that

$$\frac{1}{2}\log\left(1 + \frac{2\sigma}{\sigma'}\right) \leqslant \frac{\sigma}{\sigma'},$$

$$\frac{d}{d\sigma}\left((\sigma' + 2\sigma)\log\left(1 + \frac{2\sigma}{\sigma'}\right) - 2\sigma\right) = 2\log\left(1 + \frac{2\sigma}{\sigma'}\right) \geqslant 0$$

for $\sigma' > 0$ and $\sigma \geqslant 0$. Using these, we can prove

$$\begin{aligned}
-\left(\sigma'' - \frac{1}{2}\right)\log\frac{\sigma''}{\sigma'} + 2\sigma - \frac{\sigma}{\sigma'} &= -\sigma''\log\frac{\sigma''}{\sigma'} + 2\sigma + \frac{1}{2}\log\frac{\sigma''}{\sigma'} - \frac{\sigma}{\sigma'} \\
&\leqslant -\sigma''\log\frac{\sigma''}{\sigma'} + 2\sigma \\
&\leqslant 0
\end{aligned}$$

for $\sigma'' > \sigma' > 0$ with $\sigma'' - \sigma' = 2\sigma \geqslant 0$. By this inequality and Proposition 2.4 (1), (4), we have

$$\begin{aligned}\log\left|\frac{\Gamma(\sigma'+it)}{\Gamma(\sigma''+it)}\right| \leqslant& \log\left|\frac{\Gamma(\sigma')}{\Gamma(\sigma'')}\right| \\ \leqslant& \left(\sigma'-\frac{1}{2}\right)\log\sigma' - \left(\sigma''-\frac{1}{2}\right)\log\sigma'' + 2\sigma + \sigma\int_0^\infty \frac{dx}{(\sigma'+x)^2} \\ \leqslant& 2\sigma\left(-\log\sigma' + \frac{1}{\sigma'}\right) - \left(\sigma''-\frac{1}{2}\right)\log\frac{\sigma''}{\sigma'} + 2\sigma - \frac{\sigma}{\sigma'} \\ \leqslant& 2\sigma\left(-\log\sigma' + \frac{1}{\sigma'}\right)\end{aligned} \tag{3.4}$$

for $t \in \mathbb{R}$ and $\sigma'' > \sigma' > 0$ with $\sigma'' - \sigma' = 2\sigma \geqslant 0$. We recall $\sigma_0 = 2n_0 + \eta_0$, where $\sigma_0 \geqslant 8.5$ and $0 \leqslant \eta_0 < 2$. Set

$$G(s) = \left|\frac{\sin\frac{\pi}{2}(\sigma_0+s)}{\sin\frac{\pi}{2}(\sigma_0-s)}\right|^2.$$

We get

$$G(-\sigma+it) = \left|\frac{e^{\pi t} - e^{i\pi(\eta_0-\sigma)}}{e^{\pi t} - e^{i\pi(\eta_0+\sigma)}}\right|^2 = 1 + \frac{-4e^{\pi t}\sin\pi\eta_0\sin\pi\sigma}{e^{2\pi t} + 1 - 2e^{\pi t}\cos\pi(\eta_0+\sigma)}.$$

From this, we have

$$G(-\sigma+it) \leqslant 1 \qquad (0 \leqslant \eta_0 \leqslant 1,\ 0 < \sigma \leqslant 1); \tag{3.5}$$

$$G(-\sigma+it) = 1 \qquad (1 < \eta_0 < 2,\ \sigma = 1); \tag{3.6}$$

$$G(-\sigma+it) \leqslant 1 + \frac{4\sigma\pi e^{\pi t_0}}{(e^{\pi t_0}-1)^2} \qquad (0 \leqslant \eta_0 < 2,\ t \geqslant t_0 > 0,\ 0 < \sigma \leqslant 1). \tag{3.7}$$

Here we mean that (3.5) and (3.6) hold for arbitrary real number t. By (3.1), (3.3)–(3.6), we obtain

$$\log\left|\frac{\zeta(-\sigma_0-s)}{\zeta(-\sigma_0+s)}\right|^2 \leqslant 4\sigma\left(\log 2\pi - \log(1+\sigma_0-\sigma) + \frac{1}{\sigma_0+1-\sigma} + \frac{1}{2^{\sigma_0+1-\sigma}-1}\right)$$

for any $0 < \sigma \leqslant 1$ with $0 \leqslant \eta_0 \leqslant 1$, or $\sigma = 1$ with $1 < \eta_0 < 2$. From this inequality, for $\sigma_0 \geqslant 8.5$, we have

$$\left|\frac{\zeta(-\sigma_0-s)}{\zeta(-\sigma_0+s)}\right| < 1 \qquad (0 < \sigma \leqslant 1,\ 0 \leqslant \eta_0 \leqslant 1,\ \text{or}\ \sigma = 1,\ 1 < \eta_0 < 2), \tag{3.8}$$

because

$$\log 2\pi - \log\sigma' + \frac{1}{\sigma'} + \frac{1}{2^{\sigma'}-1}$$

decreases for $\sigma' \geqslant 8.5$ and its maximum at $\sigma' = 8.5$ is less than 0. By (3.1), (3.3) and (3.7), for any $0 \leqslant \eta < 2$, we obtain

$$\begin{aligned}\log\left|\frac{\zeta(-\sigma_0-s)}{\zeta(-\sigma_0+s)}\right|^2 \leqslant& 4\sigma\left(\log 2\pi + \frac{\pi e^{\pi t_0}}{(e^{\pi t_0}-1)^2} - \log(1+\sigma_0-\sigma)\right) + \\ & 4\sigma\left(\frac{1}{\sigma_0+1-\sigma} + \frac{1}{2^{\sigma_0+1-\sigma}-1}\right)\end{aligned}$$

for any $t \geqslant t_0$ and any $0 < \sigma \leqslant 1$. From this, we have

$$\left|\frac{\zeta(-\sigma_0 - s)}{\zeta(-\sigma_0 + s)}\right| < 1 \qquad (t \geqslant 1,\, 0 < \sigma \leqslant 1), \tag{3.9}$$

because

$$\log 2\pi + \frac{\pi e^{\pi t}}{(e^{\pi t} - 1)^2} - \log \sigma' + \frac{1}{\sigma'} + \frac{1}{2^{\sigma'} - 1}$$

decreases for both $t \geqslant 1$ and $\sigma' \geqslant 8.5$, and its maximum at $t = 1$ and $\sigma' = 8.5$ is less than 0.

Applying Proposition 2.1 (3) and Proposition 2.4 (1), there exists a sufficiently large integer $m_0 > 0$ with $2m_0 - 1 > 2\sigma_0 + 10$ such that we have

$$\left|\frac{\zeta(-\sigma_0 - s)}{\zeta(-\sigma_0 + s)}\right| < 1 \qquad (\mathrm{Re}\, s = \sigma_0 - (2m - 1),\, m \geqslant m_0). \tag{3.10}$$

Using Proposition 2.2, for any positive number ε, we obtain

$$\left|\frac{\zeta(-\sigma_0 - s)}{\zeta(-\sigma_0 + s)}\right| = \begin{cases} O\left(t^{-2\sigma+\varepsilon}\right), & s = -\sigma + it,\, 0 < \sigma \leqslant \sigma_0; \\ O\left(t^{-\frac{3\sigma}{2} - \frac{\sigma_0}{2} + \varepsilon}\right), & s = -\sigma + it,\, \sigma_0 < \sigma \leqslant \sigma_0 + 1 \\ O\left(t^{-\sigma - \sigma_0 - \frac{1}{2}}\right), & s = -\sigma + it,\, \sigma > \sigma_0 + 1. \end{cases}$$

From this, we see that for a fixed $\widetilde{\sigma}$ with $0 < \widetilde{\sigma} \leqslant 1$, we can choose a positive number t_1 such that for $s = -\sigma + it$,

$$\left|\frac{\zeta(-\sigma_0 - s)}{\zeta(-\sigma_0 + s)}\right| < 1 \qquad (t \geqslant t_1,\, \sigma > \widetilde{\sigma}). \tag{3.11}$$

We set $s = -2n_0 - (5 - \eta_0) - \frac{1}{2} + it$ with $t \geqslant 0$. We have

$$\log\left|\frac{\zeta(5 - 2\eta_0 + \frac{1}{2} + it)}{\zeta(4n_0 + 6 + \frac{1}{2} + it)}\right| \leqslant \log \frac{\zeta(\frac{3}{2})}{2 - \zeta(22)} < 1; \quad \left|2 \sin \frac{\pi}{2}\left(-4n_0 - 5 - \frac{1}{2} + it\right)\right| \geqslant e^{\frac{\pi}{2}t}.$$

Using these and Proposition 2.1 (3), we have

$$\log\left|\frac{\zeta(-\sigma_0 - s)}{\zeta(-\sigma_0 + s)}\right| \leqslant 1 + (4n_0 + 6 + \frac{1}{2}) \log 2\pi - \frac{\pi}{2}t - \log\left|\Gamma(4n_0 + 6 + \frac{1}{2} + it)\right| \tag{3.12}$$

on $s = -2n_0 - (5 - \eta_0) - \frac{1}{2} + it$ with $t \geqslant 0$. By Proposition 2.4 (2), we obtain

$$\begin{aligned} \log|\Gamma(\sigma' + it)| &\geqslant \left(\sigma' - \frac{1}{2}\right) \log \sigma' - \frac{\pi}{2}t - \sigma' - \frac{1}{2}\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)} \zeta(n+1, \sigma' + 1) \\ &\geqslant \left(\sigma' - \frac{1}{2}\right) \log \sigma' - \frac{\pi}{2}t - \sigma' - \frac{1}{12(\sigma' - 1)} \end{aligned} \tag{3.13}$$

Thus, we have

$$\left|\frac{\zeta(-\sigma_0 - s)}{\zeta(-\sigma_0 + s)}\right| < 1 \qquad (t \geqslant 0,\ \mathrm{Re}\, s = -2n_0 - (5 - \eta_0) - 1/2), \tag{3.14}$$

because by (3.12) and (3.13), we obtain

$$\begin{aligned}\log\left|\frac{\zeta(-\sigma_0 - s)}{\zeta(-\sigma_0 + s)}\right| &< 4n_0 + 8 + (4n_0 + 6)\log 2\pi - (4n_0 + 6)\log(4n_0 + 6)\\ &< 24 + 22\log 2\pi - 22\log 22\\ &< 0.\end{aligned}$$

For $0 \leqslant \eta_0 \leqslant 1$, we similarly prove

$$\begin{aligned}\log\left|\frac{\zeta(-\sigma_0 - s)}{\zeta(-\sigma_0 + s)}\right| &< 4n_0 + 6 + (4n_0 + 4)\log 2\pi - (4n_0 + 4)\log(4n_0 + 4)\\ &< 22 + 20\log 2\pi - 20\log 20\\ &< 0\end{aligned}$$

on $s = -2n_0 - (3 - \eta_0) - \frac{1}{2} + it$ with $t \geqslant 0$. Thus, we have

$$\left|\frac{\zeta(-\sigma_0 - s)}{\zeta(-\sigma_0 + s)}\right| < 1 \qquad (t \geqslant 0,\ \mathrm{Re}\, s = -2n_0 - (3 - \eta_0) - 1/2,\ 0 \leqslant \eta_0 \leqslant 1), \tag{3.15}$$

Let $s = -2n_0 - (-1 - \eta_0) + it$. We note that $n_0 \geqslant 4$. Thus, by Proposition 2.5, we get

$$\left|\frac{\zeta(-\sigma_0 - s)}{\zeta(-\sigma_0 + s)}\right| < 1 \qquad (t \geqslant 0,\ \mathrm{Re}\, s = -2n_0 - (-1 - \eta_0))\,, \tag{3.16}$$

because Re $(-\sigma_0 + s) = -4n_0 + 1$, Re $(-\sigma_0 - s) = -1 - 2\eta_0$ and $-2n_0 + 2 \leqslant -1 - 2\eta_0$.

From (3.8)–(3.11) and (3.14)–(3.16), the inequality of $|\zeta(-\sigma_0 - s)/\zeta(-\sigma_0 + s)|$ in Lemma 3.1 holds for the regions $\mathcal{R}_1, \ldots, \mathcal{R}_8$.

We shall show that the inequality in Lemma 3.1 holds for $\mathcal{R}_9$. For the case $1 \leqslant \eta_0 < 2$, by virtue of Proposition 2.5 (3), we can show that the inequality in Lemma 3.1 holds. So, we suppose $0.3 \leqslant \eta_0 < 1$. We need the following.

Claim 3.2. *We have*

$$\frac{a + (r - t)^2}{b + (r - t)^2} \cdot \frac{a + (r + t)^2}{b + (r + t)^2} \leqslant \left(\frac{a + r^2}{b + r^2}\right)^2 \quad (0 \leqslant a \leqslant 4,\ a < b,\ 0 \leqslant t \leqslant r,\ 3 \leqslant r).$$

Proof of Claim 3.2. We put

$$f(t) = (a + r^2)^2(b^2 + 2b(r^2 + t^2) + (r^2 - t^2)^2) - (b + r^2)^2(a^2 + 2a(r^2 + t^2) + (r^2 - t^2)^2).$$

It suffices to show that $f(t) \geqslant 0$ for $0 \leqslant t \leqslant r$. Clearly, $f(0) = 0$. Set

$$p(t) = (a+r^2)^2(b+t^2-r^2) - (b+r^2)^2(a+t^2-r^2).$$

Since $f'(t) = 4tp(t)$, the sign of $f'(t)$ in $[0, r]$ depends on the quadratic polynomial $p(t)$. Using the facts that the leading coefficient of $p(t)$ is $(a+r^2)^2 - (b+r^2)^2 < 0$ and $p(0) > 0$, we can see that $f(t)$ in $[0, r]$ attains its minimum at 0 provided that $f(r)$ is positive. We get

$$f(r) = r^2(b-a)[r^2(a+b) - 2ab + 4r^4] > 0,$$

because $r^2(a+b) - 2ab \geqslant 9b - 8b > 0$. Thus, we complete the proof of Claim 3.2. □

By Proposition 2.1 (4), we have

$$\left|\frac{\zeta(-\sigma_0 - s)}{\zeta(-\sigma_0 + s)}\right|^2 = e^{4b(2n_0+1-\eta_0)} \frac{(4n_0+2)^2 + t^2}{(2\eta_0)^2 + t^2} \left|\frac{\Gamma(\frac{-4n_0+1+it}{2})}{\Gamma(\frac{3-2\eta_0+it}{2})}\right|^2 \prod_\rho \left|\frac{\rho - 1 + 2\eta_0 - it}{\rho + 4n_0 + 1 - it} e^{-\frac{4n_0+2-2\eta_0}{\rho}}\right|^2. \tag{3.17}$$

We note that the smallest imaginary part of ρ's in $\operatorname{Im} s > 0$ is between 14 and 15. Thus, by Claim 3.2, we have

$$\left|\frac{\overline{\rho} - 1 + 2\eta_0 - it}{\overline{\rho} + 4n_0 + 1 - it}\right|^2 \cdot \left|\frac{\rho - 1 + 2\eta_0 - it}{\rho + 4n_0 + 1 - it}\right|^2 \leqslant \left|\frac{\overline{\rho} - 1 + 2\eta_0}{\overline{\rho} + 4n_0 + 1}\right|^2 \cdot \left|\frac{\rho - 1 + 2\eta_0}{\rho + 4n_0 + 1}\right|^2$$

for $0 \leqslant t \leqslant 14$. Using this, (3.17), Proposition 2.4 (5), Proposition 2.5 (3), (6) and the facts that $(3 - 2\eta_0)/2 \geqslant 1/2$ and

$$\frac{a+t}{b+t} \leqslant \frac{a}{b} \qquad (0 < b \leqslant a,\ 0 \leqslant t),$$

we get

$$\left|\frac{\zeta(-\sigma_0 - s)}{\zeta(-\sigma_0 + s)}\right|^2 \leqslant \left|\frac{\zeta(1 - 2\eta_0)}{\zeta(-4n_0 - 1)}\right|^2 \leqslant \left|\frac{\zeta(1 - 2\eta_0)}{\zeta(-17)}\right|^2 < 1$$

for $0 \leqslant t \leqslant 14$ and $\operatorname{Re} s = -2n_0 - (1 - \eta_0)$. By Proposition 2.5 (4), we have

$$\left|\frac{\zeta(-\sigma_0 - s)}{\zeta(-\sigma_0 + s)}\right|^2 \leqslant \left|\frac{\zeta(1 - 2\eta_0 + it)}{\zeta(-4n_0 - 1 + it)}\right|^2 < 1,$$

for $0.3 \leqslant \eta_0 \leqslant 1$, $t \geqslant 14$ and $\operatorname{Re} s = -2n_0 - (1 - \eta_0)$. This shows that the inequality in Lemma 3.1 holds for $\mathcal{R}_9$. Similarly, we prove Lemma 3.1 for $\mathcal{R}_{10}$.

We complete the proof of Lemma 3.1. □

Using Lemma 3.1, we demonstrate the proof of Theorem 2. For convenience, we divide our proof of the theorem into two parts.

Proof of Theorem 2 for $\zeta(-\sigma_0+s)+\zeta(-\sigma_0-s)$. It is not hard to see that at $s=0$, $\zeta(-\sigma_0+s)+\zeta(-\sigma_0-s)$ has a simple zero if $\sigma_0=2n_0$ and no zero otherwise.

We recall that $\sigma_0\geqslant 8.5$ and $\sigma_0=2n_0+\eta_0$, where n_0 is a positive integer and $0\leqslant\eta_0<2$. We define $f_{\eta_0}(s)$ by

$$f_{\eta_0}(s)=(s+\sigma_0+1)(\zeta(-\sigma_0+s)+\zeta(-\sigma_0-s)).$$

We divide our proof Theorem 2 into two cases.

Case 1. $0\leqslant\eta_0\leqslant 1$.

We observe that $f_{\eta_0}(s)$ has no zeros in $|\mathrm{Im}\, s|\geqslant t_1$, $\mathrm{Re}\, s<0$, or in $-1\leqslant\mathrm{Re}\, s<0$ by $\mathcal{R}_1$ and $\mathcal{R}_5$ of Lemma 3.1. So, it suffices to consider the zeros of $f_{\eta_0}(s)$ in $|\mathrm{Im}\, s|<t_1$ and $\mathrm{Re}\, s<-1$ for the proof of Theorem 2 when $0\leqslant\eta_0\leqslant 1$.

Lemma 3.3. *We have the following:*

(1) $f_{\eta_0}(s)$ *has real and simple zeros only in* $\mathrm{Re}\, s<-2n_0-(3-\eta_0)-\frac{1}{2}$ *or* $-2n_0-(-1-\eta_0)<\mathrm{Re}\, s<0$*;*

(2) $f_{\eta_0}(s)$ *has three zeros in* $-2n_0-(3-\eta_0)-\frac{1}{2}\leqslant\mathrm{Re}\, s\leqslant -2n_0-(-1-\eta_0)$ $(0\leqslant\eta_0\leqslant 1)$*;*

(3) $f_{\eta_0}(s)$ *has two zeros in* $-2n_0-(3-\eta_0)-\frac{1}{2}\leqslant\mathrm{Re}\, s\leqslant -2n_0-(1-\eta_0)$ $(0.3<\eta_0\leqslant 1)$.

Proof of Lemma 3.3. We fix $m\geqslant m_0$. By Lemma 3.1,

$$\left|\frac{\zeta(-\sigma_0-s)}{\zeta(-\sigma_0+s)}\right|<1 \quad (\mathrm{Re}\, s=\sigma_0-(2m-1)\text{ or }-2n_0-(3-\eta_0)-1/2).$$

Thus, by Rouché's theorem, $\zeta(-\sigma_0+s)$ and $\zeta(-\sigma_0+s)+\zeta(-\sigma_0-s)$ have the same number of zeros in $|\mathrm{Im}\, s|<t_1$ and $\sigma_0-(2m-1)<\mathrm{Re}\, s<-2n_0-(3-\eta_0)-\frac{1}{2}$. By Proposition 2.5 (1), (2), we see that for a positive integer m satisfying $\sigma_0-(2m-1)<-2n_0-(3-\eta_0)-1/2$, the sign of $\zeta(-\sigma_0+s)$ at $s=\sigma_0-(2m-1)$ is $(-1)^m$, and $\zeta(-\sigma_0+s)$ has simple and real zeros only at each σ_0-2m in the region $\mathrm{Re}\, s<-2n_0-(3-\eta_0)-1/2$. On the other hand, it is easy to see that for $\sigma_0-(2m-1)<-2n_0-(3-\eta_0)-1/2$ with a positive integer m, we have

$$|\zeta(-(2m-1))|>\zeta(3/2)\geqslant\zeta(3-\eta_0+1/2)>\zeta(2m-1-2\sigma_0).$$

Thus, the sign of $\zeta(-\sigma_0+s)+\zeta(-\sigma_0-s)$ is the same as $\zeta(-\sigma_0+s)$ for each $s=\sigma_0-(2m-1)<-2n_0-(3-\eta_0)-1/2$. Thus, we conclude that by the sign changes and the fact that $\zeta(-\sigma_0+s)$ and $\zeta(-\sigma_0+s)+\zeta(-\sigma_0-s)$ have the same number of zeros in $\mathrm{Re}\, s<-2n_0-(3-\eta_0)-1/2$, $\zeta(-\sigma_0+s)+\zeta(-\sigma_0-s)$ has real and simple zeros only in the region.

Similarly, we can show that $\zeta(-\sigma_0+s)+\zeta(-\sigma_0-s)$ has real zeros only in $|\mathrm{Im}\, s|<t_1$ and $-2n_0-(-1-\eta_0)<\mathrm{Re}\, s<-1$. Here we note that in order to know the sign of

$\zeta(-\sigma_0+s)+\zeta(-\sigma_0-s)$ at each $s=\sigma_0-(2m-1)$ in the region, we can use Proposition 2.5 (3), because the rightmost possible case of $\zeta(-\sigma_0+s)+\zeta(-\sigma_0-s)$ to $\operatorname{Re} s=-1$ is $\zeta(-4n_0+1)+\zeta(-1)$ with $2n_0 \geqslant 8$. (1) follows.

We now investigate the zeros of $f_{\eta_0}(s)$ in $-2n_0-(3-\eta_0)-\frac{1}{2} \leqslant \operatorname{Re} s \leqslant -2n_0-(-1-\eta_0)$. Using $\mathcal{R}_7$, $\mathcal{R}_8$ of Lemma 3.1, we see that $(s+\sigma_0+1)\zeta(-\sigma_0+s)$ and $f_{\eta_0}(s)$ have the same number of zeros in $-2n_0-(3-\eta_0)-\frac{1}{2} \leqslant \operatorname{Re} s \leqslant -2n_0-(-1-\eta_0)$. Thus, $f_{\eta_0}(s)$ has three zeros in $-2n_0-(3-\eta_0)-\frac{1}{2} \leqslant \operatorname{Re} s \leqslant -2n_0-(-1-\eta_0)$, because $(s+\sigma_0+1)\zeta(-\sigma_0+s)$ has three real zeros only in the region. (2) follows.

Similarly, for $0.3<\eta_0 \leqslant 1$, $f_{\eta_0}(s)$ has two zeros in $-2n_0-(3-\eta_0)-\frac{1}{2} \leqslant \operatorname{Re} s \leqslant -2n_0-(1-\eta_0)$, because $(s+\sigma_0+1)\zeta(-\sigma_0+s)$ has two real zeros only in the region. (3) follows.

We complete the proof of Lemma 3.3. □

By Proposition 2.2 (3), we recall

$$\zeta(s)=2(2\pi)^{s-1}\sin\frac{\pi}{2}s\Gamma(1-s)\zeta(1-s).$$

For proofs of the following claims, we shall use this repeatedly (without mentioning) in such a way that for $s<0$, the sign of $\zeta(s)$ is the one of $\sin\frac{\pi}{2}s$.

Claim 3.4. *Suppose* $0 \leqslant \eta_0 \leqslant \frac{1}{2}$. $f_{\eta_0}(s)$ *is negative in* $[-2n_0-(3-\eta_0)-\frac{1}{2}, -2n_0-(2-\eta_0)]$ *and* $[-2n_0-(1+\eta_0), -2n_0-(1-\eta_0)]$.

Proof of Claim 3.4. Let $s=-2n_0-(x-\eta_0)$ with $2 \leqslant x \leqslant 3.5$. Then

$$f_{\eta_0}(s)=(1+2\eta_0-x)(\zeta(-4n_0-x)+\zeta(x-2\eta_0))<0,$$

since $1+2\eta_0-x<0$, $\sin(-x\pi/2) \geqslant 0$ and $\zeta(x-2\eta_0)>0$ where $2 \leqslant x \leqslant 3.5$ and $0 \leqslant \eta_0<\frac{1}{2}$ or $2<x \leqslant 3.5$ and $0 \leqslant \eta_0 \leqslant \frac{1}{2}$. We also have $f_{\eta_0}(s)=-1$ at $x=2$ and $\eta_0=\frac{1}{2}$.

Let $s=-2n_0-1-y$ with $-\eta_0 \leqslant y \leqslant \eta_0$. Then

$$f_{\eta_0}(s)=(\eta_0-y)(\zeta(-4n_0-\eta_0-1-y)+\zeta(1+y-\eta_0))<0,$$

since $f_{\eta_0}(-2n_0-1-\eta_0)=-1$, and $\sin\frac{\pi}{2}(-\eta_0-1-y) \leqslant 0$ and $\zeta(1+y-\eta_0)<0$ with $-\eta_0 \leqslant y<\eta_0$. Thus Claim 3.4 follows. □

Claim 3.5. *Suppose* $\frac{1}{2}<\eta_0<1$. $f_{\eta_0}(s)$ *is negative in* $[-2n_0-(3-\eta_0)-\frac{1}{2}, -2n_0-(1+\eta_0)]$ *and* $[-2n_0-(2-\eta_0), -2n_0-(1-\eta_0)]$.

Proof of Claim 3.5. Let $s=-2n_0-(x-\eta_0)$ with $1 \leqslant x \leqslant 2$. Then

$$f_{\eta_0}(s)=(1+2\eta_0-x)(\zeta(-4n_0-x)+\zeta(x-2\eta_0))<0,$$

since $\sin(-x\pi/2) \leqslant 0$ and $\zeta(x-2\eta_0) < 0$.

Let $s = -2n_0 - 2 - x$ with $-1+\eta_0 \leqslant x \leqslant 1.5-\eta_0$. Then

$$f_{\eta_0}(s) = (\eta_0 - 1 - x)(\zeta(-4n_0 - \eta_0 - 2 - x) + \zeta(-\eta_0 + 2 + x)) < 0,$$

since $\sin\frac{\pi}{2}(-\eta_0 - 2 - x) > 0$ and $\zeta(-\eta_0 + 2 + x) > 0$ with $-1+\eta_0 < x \leqslant 1.5-\eta_0$, and $f_{\eta_0}(-2n_0 - 1 - \eta_0) = -1$. So, Claim 3.5 follows. □

Lemma 3.6. *$f_{\eta_0}(s)$ has at least one real zero in $-2n_0-(1-\eta_0) < \mathrm{Re}\, s < -2n_0-(-1-\eta_0)$.*

Proof of Lemma 3.6. From Claim 3.4 and 3.5, we see that $f_{\eta_0}(s)$ is negative at $s = -2n_0 - (1-\eta_0)$. Since by Lemma 3.1, $|\zeta(-\sigma_0 - s)/\zeta(-\sigma_0 + s)| < 1$ and by Proposition 2.5 (2), $\zeta(-\sigma_0 + s) = \zeta(-4n_0 + 1) > 0$ at $s = -2n_0 - (-1-\eta_0)$, $f_{\eta_0}(s)$ is positive at $s = -2n_0 - (1-\eta_0)$. This proves Lemma 3.6 by virtue of the sign change method. □

Hence we have at most two zeros of $f_{\eta_0}(s)$ in $-2n_0-(3-\eta_0) < \mathrm{Re}\, s < -2n_0-(1-\eta_0)$. In fact, we shall show that $f_{\eta_0}(s)$ has two zeros in the region.

We put

$$I_{\eta_0} = [-2n_0 - (3-\eta_0) - 1/2, -2n_0 - (1-\eta_0)].$$

Lemma 3.7. *Let η_1 and η_2 be as in Theorem 2. Then, $f_{\eta_0}(s)$ has one pair of complex zeros in $\mathrm{Re}\, s \in I_{\eta_0}$ for any $\eta_1 < \eta_0 \leqslant \frac{1}{2}$ (any $\frac{1}{2} < \eta_0 < \eta_2$), two simple real zeros for any $0 \leqslant \eta_0 < \eta_1$ (any $\eta_2 < \eta_0 \leqslant 1$) and one double real zero for $\eta_0 = \eta_1$ ($\eta_0 = \eta_2$). η_1 and η_2 satisfy*

$$0.3 < \eta_1 < 1/2,$$

$$|\eta_k - 1/2| < \sqrt{\pi}(2\pi)^{2n_0+2}\Gamma(4n_0+2.8)^{-\frac{1}{2}} \qquad (k = 1,\, 2).$$

Proof of Lemma 3.7. We start with the following.

Claim 3.8. *Let $0 \leqslant \eta_1' < \eta_0' < \frac{1}{2}$ ($\frac{1}{2} < \eta_0' < \eta_1' \leqslant 1$). If $f_{\eta_1'}(s) \leqslant 0$ for $s \in I_{\eta_1'}$, then $f_{\eta_0'}(s) < 0$ for $s \in I_{\eta_0'}$.*

Proof of Claim 3.8. Suppose $0 \leqslant \eta_1' < \eta_0' < \frac{1}{2}$ and $f_{\eta_1'}(s) \leqslant 0$ for $s \in I_{\eta_1'}$. By assumption, we have

$$f_{\eta_1'}(s) \leqslant 0 \qquad (-2n_0 - (2-\eta_1') \leqslant s \leqslant -2n_0 - (1+\eta_1'))\,.$$

Namely, we have

$$\zeta(-4n_0 - x) + \zeta(x - 2\eta_1') \geqslant 0 \qquad (2\eta_1' + 1 \leqslant x \leqslant 2).$$

We get

$$(\zeta(-4n_0 - x) + \zeta(x-2\eta_1')) - (\zeta(-4n_0 - x) + \zeta(x - 2\eta_0')) = \zeta(x-2\eta_1') - \zeta(x-2\eta_0') < 0$$

for any $2\eta_0' + 1 < x \leqslant 2$. Hence, it is easy to see that

$$\zeta(-4n_0 - x) + \zeta(x - 2\eta_0') > 0 \qquad (2\eta_0' + 1 < x \leqslant 2).$$

Thus, we have

$$f_{\eta_0'}(s) < 0 \qquad (-2n_0 - (2 - \eta_0') \leqslant s \leqslant -2n_0 - (1 + \eta_0')).$$

By this and Claim 3.4, we have $f_{\eta_0'}(s) < 0$ for $s \in I_{\eta_0'}$.

For the case $\frac{1}{2} < \eta_0' < \eta_1'$ with the assumption that $f_{\eta_1'}(s) \leqslant 0$ for $s \in I_{\eta_1'}$, by virtue of Claim 3.5, we can similarly prove $f_{\eta_0'}(s) < 0$ for $s \in I_{\eta_0'}$.

We complete the proof of Claim 3.8. □

By Claim 3.8, it is easy to see that we have

$$\eta_1 = \inf\{0 \leqslant \eta_0 \leqslant 1/2 : f_{\eta_0}(s) < 0 \text{ in } I_{\eta_0}\}$$

and

$$\eta_2 = \sup\{1/2 \leqslant \eta_0 \leqslant 1 : f_{\eta_0}(s) < 0 \text{ in } I_{\eta_0}\}$$

If $\eta_1 = \frac{1}{2}$, then $-2n_0 - (2 - \eta_0) = -2n_0 - (1 + \eta_0) = -2n_0 - \frac{3}{2}$ and $f_{\frac{1}{2}}(-2n_0 - \frac{3}{2}) = -1$. By Claim 3.4, $f_{\eta_1}(s) < 0$ in I_{η_1}. Thus, by continuity, we can find η_1' such that $0 \leqslant \eta_1' < 1/2$ and $f_{\eta_1'}(s) < 0$ in $I_{\eta_1'}$. This is absurd. Thus, we conclude that $0 \leqslant \eta_1 < 1/2$. Similarly, we can show $1/2 < \eta_2 \leqslant 1$.

We note that $n_0 \geqslant 5$ if $\sigma_0 = 2n_0 + 0.3$, because $\sigma_0 \geqslant 8.5$. By Proposition 2.5 (7), we have

$$\left|\frac{\zeta(1.2)}{\zeta(-4n_0 - 1.8)}\right| \leqslant \left|\frac{\zeta(1.2)}{\zeta(-21.8)}\right| < 1.$$

So, $f_{0.3}(-21.8) > 0$. By this and Claim 3.4, $f_{0.3}(s)$ has two simple zeros in $I_{0.3}$. Hence, we know that $0.3 < \eta_1 < \frac{1}{2}$.

Before we prove Lemma 3.7, we compute η_1 and η_2. Since for $\eta_1 < \eta < \frac{1}{2}$, $\zeta(-4n_0 - x) + \zeta(x - 2\eta) > 0$ for all $2\eta + 1 \leqslant x \leqslant 2$, we get

$$2(2\pi)^{-4n_0-x-1} \sin\frac{\pi}{2}(-x)\Gamma(4n_0 + 1 + x)\zeta(4n_0 + 1 + x) + \zeta(x - 2\eta) > 0$$

for all $2\eta + 1 \leqslant x \leqslant 2$. We set $\eta = \frac{1}{2} - \epsilon$. Then $0 < \epsilon < \frac{1}{5}$, because $0.3 < \eta_1 < \frac{1}{2}$. We have

$$\zeta(x - 1 + 2\epsilon) > 2(2\pi)^{-4n_0-x-1} \sin\frac{\pi}{2}x\Gamma(4n_0 + 1 + x)\zeta(4n_0 + 1 + x)$$

for all $2 - 2\epsilon \leqslant x \leqslant 2$. In particular, we obtain

$$\zeta(1 + \epsilon) > 2(2\pi)^{-4n_0-3} \sin\frac{\pi}{2}\epsilon\Gamma(4n_0 + 2.8)$$

for $0 < \epsilon < \frac{1}{5}$. We have

$$\frac{\epsilon}{\sin\frac{\pi}{2}\epsilon}\frac{\zeta(1+\epsilon)}{\epsilon} < \frac{7}{10}\frac{\zeta(1+\epsilon)}{\epsilon} < \frac{1}{\epsilon^2},$$

because

$$\frac{\sin x}{x} > 1 - \frac{x^2}{6} > \frac{20}{7\pi} \qquad (0 < x < \pi/10).$$

Thus, we obtain

$$\epsilon < \sqrt{\pi}(2\pi)^{2n_0+2}\Gamma(4n_0+2.8)^{-\frac{1}{2}}.$$

Therefore, we have (1) for η_1. We similarly approximate η_2.

By Lemma 3.3 (3), we know that $f_{\eta_0}(s)$ has two zeros in $-2n_0-(3-\eta_0)-\frac{1}{2} < \mathrm{Re}\, s < -2n_0-(1-\eta_0)$ for $0.3 < \eta_0 \leqslant 1$.

Suppose that for $0 \leqslant \eta < \eta_1$ ($\eta_2 < \eta \leqslant 1$), $f_\eta(s)$ has one pair of complex zeros in $-2n_0-(3-\eta)-\frac{1}{2} < \mathrm{Re}\, s < -2n_0-(1-\eta)$. Then, $f_\eta(s)$ should be negative in I_η by the claims 3.4 and 3.5. Thus $\eta_1 \leqslant \eta < \frac{1}{2}$ or $\frac{1}{2} < \eta \leqslant \eta_2$. This is absurd. Hence, for any $0 \leqslant \eta < \eta_1$ ($\eta_2 < \eta \leqslant 1$), $f_\eta(s)$ has two real zeros and by Claim 3.8 these two real zeros are simple. It is not hard to observe that for $\eta = \eta_1$ or $\eta = \eta_2$, $f_\eta(s)$ has one double real zero. Let η be such that $\eta_1 < \eta < \eta_2$. Then, we have $0.3 < \eta < 1$. Thus, by Claim 3.8, we conclude that $f_\eta(s)$ has one pair of complex zeros in $-2n_0-(3-\eta)-\frac{1}{2} \leqslant \mathrm{Re}\, s \leqslant -2n_0-(1-\eta)$. Hence, we complete the proof of Lemma 3.7. □

Thus, Theorem 2 for Case 1 follows from the lemmas 3.3, 3.6 and 3.7.

Case 2. $1 < \eta_0 < 2$.

We need two lemmas to justify Theorem 2 for this case.

Lemma 3.9. *$f_{\eta_0}(s)$ has real and simple zeros only for the region* $\mathrm{Re}\, s < -1$.

Proof of Lemma 3.9. We recall from Lemma 3.1 that we have $|\zeta(-\sigma_0-s)/\zeta(-\sigma_0+s)| < 1$ in $\mathcal{R}_2, \mathcal{R}_4, \mathcal{R}_5, \mathcal{R}_6$ and $\mathcal{R}_8$. Using these facts together with Proposition 2.5 (3), as in the proof of Case 1, we can demonstrate that $f_{\eta_0}(s)$ has real and simple zeros in $-2n_0+1+\eta_0 \leqslant \mathrm{Re}\, s < -1$ and $\mathrm{Re}\, s \leqslant -2n_0-5-1/2+\eta_0$, and $f_{\eta_0}(s)$ and $(s+\sigma_0+1)\zeta(-\sigma_0+s)$ have the same number of zeros in $-2n_0-5-1/2+\eta_0 < \mathrm{Re}\, s < -2n_0+1+\eta_0$. We note that $(s+\sigma_0+1)\zeta(-\sigma_0+s)$ has four zeros at $-2n_0+\eta_0$, $-2n_0-2+\eta_0$, $-2n_0-1-\eta_0$and $-2n_0-4+\eta_0$ in the region $-2n_0-5-1/2+\eta_0 < \mathrm{Re}\, s < -2n_0+1+\eta_0$. Thus, it suffices to show that $f_{\eta_0}(s)$ has four simple zeros in the interval $(-2n_0-5-1/2+\eta_0, -2n_0+1+\eta_0)$. Clearly, we have

$$f_{\eta_0}(-2n_0-5-1/2+\eta_0) > 0,\ f_{\eta_0}(-2n_0-1-\eta_0) < 0,\ f_{\eta_0}(-2n_0+1+\eta_0) > 0. \quad (3.18)$$

We have

$$\zeta(-4n_0-2-1/2) > 0 \qquad \text{and} \qquad \zeta(-4n_0-1/2) < 0. \quad (3.19)$$

By Proposition 2.5 (5), we have

$$|\zeta(-4n_0 - 2 - 1/2)| > |\zeta(2 + 1/2 - 2\eta_0)| \ \text{ and } \ |\zeta(-4n_0 - 1/2)| > |\zeta(1/2 - 2\eta_0)|$$

for $n_0 \geqslant 4$ and $1 < \eta_0 < 2$. By this and (3.19), we get

$$f_{\eta_0}(-2n_0 - 2 - 1/2 + \eta_0) > 0 \qquad \text{and} \qquad f_{\eta_0}(-2n_0 - 1/2 + \eta_0) < 0.$$

Hence, by this and (3.18), $f_{\eta_0}(s)$ has four simple zeros in $(-2n_0-5-1/2+\eta_0, -2n_0+1+\eta_0)$, because for $1 \leqslant \eta_0 < 2$,

$$-2n_0-5-1/2+\eta_0 < -2n_0-1-\eta_0 < -2n_0-2-1/2+\eta_0 < -2n_0-1/2+\eta_0 < -2n_0+1+\eta_0.$$

Thus, we complete the proof of Lemma 3.9. □

Lemma 3.10. *Let* $\sigma_0 \geqslant 8.5$*. For* $1 < \eta_0 < 2$*,* $H(-\sigma_0, s)$ *has at most one real zeros only in* $-1 < \operatorname{Re} s < 0$ *and* $|\operatorname{Im} s| < 1$*.*

Proof of Lemma 3.10. We may assume that $H(-\sigma_0, s) \neq 0$ at $s = \pm i$. We will work only on $\zeta(-\sigma_0 + s) + \zeta(-\sigma_0 - s)$, because the same method works for the function $\zeta(-\sigma_0 + s) - \zeta(-\sigma_0 - s)$. Define $f(s)$ and $h(s)$ by

$$f(s) = \zeta(-\sigma_0 + s) + \zeta(-\sigma_0 - s) \qquad \textit{and} \qquad h(s) = 1 + \frac{\zeta(-\sigma_0 + s)}{\zeta(-\sigma_0 - s)}.$$

Put

$N_0 =$ the number of zeros of $H(-\sigma_0, s)$ on $\operatorname{Re} s = 0$ and $|\operatorname{Im} s| < 1$,
$N_1 =$ the number of zeros of $H(-\sigma_0, s)$ in the interior of C.

Define curves C and C_1 by

$C =$ the rectangle with vertices at $1 + i$, $-1 + i$, $-1 - i$, $1 - i$,
$C_1 =$ the straight line joining 1, $1 + i$, i,

where we take the orientation of curves counterclockwise. By Lemma 3.1 and the assumption that $H(-\sigma_0, s) \neq 0$ at $s = \pm i$, we have $f(s) \neq 0$ on the curve C. Then, by the facts that $f(s) = f(-s)$ and $f(s) = \overline{f(\overline{s})}$, we have

$$N_1 = \frac{1}{2\pi i}\int_C \frac{f'(s)}{f(s)} ds = \frac{2}{\pi i}\int_{C_1} \frac{f'(s)}{f(s)} ds = \frac{2}{\pi}\Delta_1 \arg f(s),$$

where $\Delta_1 \arg f(s)$ is the argument change of $f(s)$ along the curve C_1. Recall that $\zeta(s) = \chi(s)\zeta(1-s)$ and $\chi(s) = 2(2\pi)^{s-1}\sin\frac{\pi}{2}s\Gamma(1-s)$. Thus, from the facts that $(2\pi)^s$, $\Gamma(1+\sigma_0+s)$, $\zeta(1 + \sigma_0 + s)$ are real on the real axis and have no zeros in the interior of C and on C, and $\sin\frac{\pi}{2}(-\sigma_0 - s)$ is real on the real axis and has one zero at $2 - \eta_0$ in $0 \leqslant \operatorname{Re} s \leqslant 1$, we get

$$\Delta_1 \arg f(s) = \Delta_0 \arg \zeta(-\sigma_0 - s) + \pi + \Delta_1 \arg h(s),$$

where $\Delta_0 \arg \zeta(-\sigma_0 - s)$ is the argument change of $\zeta(-\sigma_0 - s)$ from 0 to i and $\Delta_1 \arg h(s)$ is the argument change of $h(s)$ along the curve C_1. We note that

$$N_0 \geqslant \frac{2}{\pi}\Delta_0 \arg \zeta(-\sigma_0 - s) - 1.$$

Hence, we get

$$0 \leqslant N_1 - N_0 \leqslant 3 + \frac{2}{\pi}\Delta_1 \arg h(s) < 4,$$

because $h(s) = 1 + \zeta(-\sigma_0 + s)/\zeta(-\sigma_0 - s)$ and $|\zeta(-\sigma_0 + s)/\zeta(-\sigma_0 - s)| < 1$ on C_1 by $\mathcal{R}_2$, $\mathcal{R}_3$ of Lemma 3.1. Suppose that we have an extra complex zero s_0 in the interior of C with $\operatorname{Re} s_0 \neq 0$. Then, by symmetry of zeros around the imaginary and real axes, $-s_0$, $\overline{s_0}$ and $-\overline{s_0}$ are also zeros of $f(s)$. However, it is impossible, since $0 \leqslant N_1 - N_0 \leqslant 3$. Thus, extra zeros of $f(s)$ off the imaginary axis in the interior of C are real. By symmetry of zeros around the imaginary axis, we have $N_1 - N_0 = 0$ or $N_1 - N_0 = 2$. In any case, we see that the possible real zeros in the region $-1 < \operatorname{Re} s < 0$ and $|\operatorname{Im} s| < 1$ are simple. We complete the proof of the lemma. □

Thus, from the lemmas 3.9 and 3.10, we complete the proof of Theorem 2 for Case 2.

From Case 1 and Case 2, Theorem 2 for $\zeta(-\sigma_0 + s) + \zeta(-\sigma_0 - s)$ follows. □

Proof of Theorem 2 for $\zeta(-\sigma_0 + s) - \zeta(-\sigma_0 - s)$. It is not hard to see that at $s = 0$, $\zeta(-\sigma_0 + s) - \zeta(-\sigma_0 - s)$ has a simple zero if $\zeta'(-\sigma_0) \neq 0$ and a double zero otherwise.

We will follow the same methods as in the proof of Theorem 2 for the function $\zeta(-\sigma_0 + s) + \zeta(-\sigma_0 - s)$. We define $\widetilde{f}_{\eta_0}(s)$ by

$$\widetilde{f}_{\eta_0}(s) = (s + \sigma_0 + 1)(\zeta(-\sigma_0 + s) - \zeta(-\sigma_0 - s)).$$

First, we note that

$$\widetilde{f}_{\eta_0}(-\sigma_0 - 1) = 1,$$

while we have $f_{\eta_0}(-\sigma_0 - 1) = -1$. Thus, naturally, we may expect that we have exceptional zeros for $H(-\sigma, s)$ with some $1 < \eta_0 < 2$.

We divide our proof into two cases.

Case 1. $0 \leqslant \eta_0 \leqslant 1$.

As in Lemma 3.3, we obtain that $\widetilde{f}_{\eta_0}(s)$ has real and simple zeros only in $\operatorname{Re} s < -2n_0 - (3-\eta_0) - \frac{1}{2}$ or $-2n_0 - (-1-\eta_0) < \operatorname{Re} s < 0$, and $f_{\eta_0}(s)$ has three zeros in $-2n_0 - (3-\eta_0) - \frac{1}{2} \leqslant \operatorname{Re} s \leqslant -2n_0 - (-1-\eta_0)$. We note that we have

$$\widetilde{f}_{\eta_0}(-2n_0 - 3 - 1/2 + \eta_0) < 0,\ \ \widetilde{f}_{\eta_0}(-2n_0 - 1 - \eta_0) > 0,\ \ \widetilde{f}_{\eta_0}(-2n_0 + 1 + \eta_0) > 0.$$

By Proposition 2.5 (5), we have

$$|\zeta(-4n_0 - 1/2)| > |\zeta(1/2 - 2\eta_0)|,$$

because $n_0 \geqslant 5$ if $0 \leqslant \eta_0 \leqslant 0.1$ and $\frac{1}{2} - 2\eta_0 \leqslant 0.3$ if $0.1 < \eta_0 \leqslant 1$. Hence, we have $\widetilde{f}_{\eta_0}(-2n_0 - 1/2 + \eta_0) < 0$, for $\zeta(-4n_0 - 1/2) < 0$ and $-2n_0 - 1/2 + \eta_0 > -\sigma_0 - 1 = -2n_0 - \eta_0 - 1$. Thus, $\widetilde{f}_{\eta_0}(s)$ has three simple and real zeros in $-2n_0 - (3 - \eta_0) - \frac{1}{2} \leqslant \mathrm{Re}\, s \leqslant -2n_0 - (-1 - \eta_0)$. Therefore, $\widetilde{f}_{\eta_0}(s)$ has simple and real zeros only in $\mathrm{Re}\, s < 0$.

Case 2. $1 < \eta_0 < 2$.

We follow the same methods as in Case 2 for $\zeta(-\sigma_0 + s) + \zeta(-\sigma_0 - s)$. Then, we can demonstrate that $\zeta(-\sigma_0 + s) - \zeta(-\sigma_0 - s)$ does not have any zero in the region $-1 < \mathrm{Re}\, s < 0$. The rest part of our proof is identical to the proof for Case 1 with $\zeta(-\sigma_0 + s) + \zeta(-\sigma_0 - s)$. So, we omit the proof. Here, we use the fact that the inequality holds for $\mathcal{R}_{10}$ together with $\mathcal{R}_2, \mathcal{R}_4, \mathcal{R}_5, \mathcal{R}_6, \mathcal{R}_9$.

From Case 1 and Case 2, Theorem 2 for $\zeta(-\sigma_0 + s) - \zeta(-\sigma_0 - s)$ follows. □

We have completed the proof of Theorem 2. □

Proof of Theorem 3. (1). Let $\sigma_0 < \frac{1}{2}$. Suppose that $\zeta(s)$ in $\mathrm{Re}\, s < \sigma_0$ and $\mathrm{Im}\, s \neq 0$ has no zeros. By Theorem 2, we can assume that $-8.5 \leqslant -\sigma_0 < \frac{1}{2}$. For this case, we use the same method as in the proof of Lemma 3.1.

First, we suppose that $\sigma \geqslant \sigma_0 + 2$. By Proposition 2.4 (1), we readily get

$$\log|\Gamma(1 + \sigma + \sigma_0 + it)| \geqslant \left(\sigma_0 + \sigma + \frac{1}{2}\right)\log t - (\sigma_0 + \sigma + 1) - \frac{\pi}{2}t - \frac{1}{2}.$$

Using this and Proposition 2.1 (3), we obtain

$$\left|\frac{\zeta(-\sigma_0 - s)}{\zeta(-\sigma_0 + s)}\right| \leqslant \frac{e^{\frac{1}{2}}\zeta(2)}{2 - \zeta(8)} \frac{(2\pi e)^{\sigma_0+\sigma}}{1 - e^{-\pi t}} \frac{1}{t^{\sigma_0+\sigma+\frac{1}{2}}} < 1 \tag{3.20}$$

for $s = -\sigma + it$, $\sigma \geqslant 2 + 2\pi$, $t \geqslant 100$. We suppose that $0 < \sigma \leqslant \sigma_0 + 2 \leqslant 11$. We recall the formula $(*)$ in the proof of Theorem 1:

$$\log\left|\frac{\zeta(-\sigma_0 - s)}{\zeta(-\sigma_0 + s)}\right|^2 \leqslant -2\sigma\kappa(\sigma, t)$$

where $s = -\sigma + it$ with σ, $t > 0$ and

$$\kappa(\sigma, t) = -\log\pi - C_0 - \frac{2(1 + \sigma_0)}{(-\sigma_0 - 1 + \sigma)^2 + t^2} + \sum_{n=1}^{\infty}\frac{1}{n} - \frac{-2\sigma_0 + 4n}{(-\sigma_0 - \sigma + 2n)^2 + t^2}.$$

Here, we do not have the term '$\sum_{\beta<\sigma_0}$', because we are assuming that $\zeta(s)$ has no zeros in $\operatorname{Re} s < \sigma_0$ and $\operatorname{Im} s \neq 0$. Since $0 < \sigma < 11$, $-\frac{1}{2} < \sigma_0 \leqslant 9$ and $t \geqslant 100$, it is easy to see that

$$-\log\pi - C_0 - \frac{2(1+\sigma_0)}{(-\sigma_0-1+\sigma)^2+t^2} > -2. \tag{3.21}$$

We easily get

$$\sum_{n=1}^{\infty}\frac{1}{n} - \frac{-2\sigma_0+4n}{(-\sigma_0-\sigma+2n)^2+t^2} = \sum_{n=1}^{\infty}\frac{1}{n} - \frac{n-\sigma_1}{(n-\sigma_1)^2+t_1^2} - \frac{\frac{\sigma}{2}}{(n-\sigma_1)^2+t_1^2},$$

where $\sigma_1 = \frac{\sigma_0+\sigma}{2}$ and $t_1 = \frac{t}{2}$. By the partial summation formula [21, p.13] or [18, p.489], we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n} - \frac{n-\sigma_1}{(n-\sigma_1)^2+t_1^2} &= \int_{\frac{1}{2}}^{\infty}\frac{1}{x} - \frac{x-\sigma_1}{(x-\sigma_1)^2+t_1^2}dx - \\ &\int_{\frac{1}{2}}^{\infty}(x-[x]-\frac{1}{2})\left(\frac{1}{x^2} + \frac{t_1^2-(x-\sigma_1)^2}{((x-\sigma_1)^2+t_1^2)^2}\right)dx \\ &\geqslant \log 2 + \log t_1 - \frac{1}{2}\int_{\frac{1}{2}}^{\infty}\frac{1}{x^2} + \frac{1}{(x-\sigma_1)^2+t_1^2}dx \\ &> \log t_1 - 1.\end{aligned}$$

Similarly, we have

$$\sum_{n=1}^{\infty}\frac{\sigma}{(n-\sigma_1)^2+t_1^2} < 1.$$

By these and (3.21), we have

$$\kappa(\sigma,t) > -2 + \log 50 - 1 - 1/2 > 0.$$

From this and (3.20), we have

$$\left|\frac{\zeta(-\sigma_0-s)}{\zeta(-\sigma_0+s)}\right| < 1 \qquad (s = -\sigma+it, \sigma > 0, |t| \geqslant 100).$$

Thus, all zeros of $H(-\sigma_0, s)$ in $\operatorname{Im} s \geqslant 100$ are on $\operatorname{Re} s = 0$.

(2). $\Rightarrow$) This follows from (1).

$\Leftarrow$) We suppose that for any $\sigma_0 < \frac{1}{2}$, all zeros of $H(\sigma_0, s)$ in $\operatorname{Im} s \geqslant 100$ are on $\operatorname{Re} s = 0$. We may assume that $H(\sigma_0, s) = \zeta(\sigma_0+s) + \zeta(\sigma_0-s)$, because the same proof works for $H(\sigma_0, s) = \zeta(\sigma_0+s) - \zeta(\sigma_0-s)$. Since $\zeta(\sigma_0+s) + \zeta(\sigma_0-s)$ converges uniformly to $\zeta(1/2+s) + \zeta(1/2-s)$ as $\sigma_0 \to \frac{1}{2}$ in any compact subset of $\{s : \operatorname{Im} s > 0\}$, all zeros of $\zeta(1/2+s) + \zeta(1/2-s)$ in $\operatorname{Im} s \geqslant 100$ are on $\operatorname{Re} s = 0$. Using Proposition 2.1 (3), we have

$$\zeta(1/2+s) + \zeta(1/2-s) = (1+\chi(s))\zeta(s+1/2),$$

where $\chi(s) = 2(2\pi)^{s-1}\sin\frac{\pi}{2}s\Gamma(1-s)$. Therefore, all zeros of $\zeta(s+\frac{1}{2})$ in $|\mathrm{Im}\, s| \geqslant 100$ are on $\mathrm{Re}\, s = 0$. Namely, all zeros of $\zeta(s)$ in $|\mathrm{Im}\, s| \geqslant 100$ are on $\mathrm{Re}\, s = \frac{1}{2}$. For $|\mathrm{Im}\, s| \leqslant 100$, all complex zeros of $\zeta(s)$ are on $\mathrm{Re}\, s = \frac{1}{2}$. We refer to [17] for this. Thus, the Riemann hypothesis follows.

We complete the proof of Theorem 3. □

4 Proofs of Theorems 5 and 6

Proof of Theorem 5. We let $\sigma_0 > \frac{1}{2}$ and $a \in \mathbb{C}$ with $a \neq 0$. We recall

$$H(s;\sigma_0,a) = \zeta(\sigma_0+s) + a\zeta(\sigma_0-s).$$

By Proposition 2.2, we can choose a sufficiently large $\sigma_1 > 0$ such that

$$\sigma_1 - \sigma_0 > 3 \text{ and } \left| a\frac{\zeta(\sigma_0+\sigma_1+it)}{\chi(\sigma_0-\sigma_1-it)} \right| < \frac{1}{4}. \tag{4.1}$$

We set

$$g(s) = \zeta(1-\sigma_0+s) + a^{-1}\frac{\zeta(\sigma_0+s)}{\chi(\sigma_0-s)}.$$

Then, we have

$$H(s;\sigma_0,a) = ag(s)\chi(\sigma_0-s).$$

By (4.1), Proposition 2.2 and Proposition 2.6,

$$\arg g(\tau+iT) = O(\log T) \qquad (-\sigma_1 \leqslant \tau \leqslant \sigma_1). \tag{4.2}$$

We start with the proof of (1).

(1). Let $T > 1$. We may assume that $H(s;\sigma_0,a) \neq 0$ on $\mathrm{Im}\, s = T$. Choose $0 < t_0 < 1$ such that $H(s;\sigma_0,a) \neq 0$ on $\mathrm{Im}\, s = t_0$. Then, for $H(s;\sigma_0,a)$, we have

$$N(T) = \frac{1}{2\pi i}\int_{C_T} \frac{H'(s;\sigma_0,a)}{H(s;\sigma_0,a)}ds + O(1),$$

where C_T is the rectangle with vertices σ_1+it_0, σ_1+iT, $-\sigma_1+iT$ and $-\sigma_1+it_0$. Using (4.1) and Proposition 2.1 (3), it is not hard to see that

$$N(T) = \frac{1}{\pi}\Delta_1 \arg\chi(\sigma_0-s) + \frac{1}{2\pi}\Delta_2 \arg g(s) + O(\log T),$$

where $\Delta_1 \arg\chi(\sigma_0-s)$ is the argument change from σ_1+it_0 to σ_1+iT and $\Delta_2 \arg g(s)$ is the argument change from σ_1+iT to $-\sigma_1+iT$. Using Proposition 2.2 for $\chi(s)$ and (4.2) with the previous formula, we obtain (1).

(2). For convenience, we set
$$H(s) = a^{-1}H(s;\sigma_0,a).$$
By Littlewood's Lemma (see [21, p. 220]), we obtain
$$\begin{aligned}2\pi\sum_{\substack{0<\gamma<T\\ \beta>\sigma}}(\beta-\sigma) = \int_0^T \log|H(\sigma+it)|dt - \int_0^T \log|H(\sigma_1+it)|dt+ \\ \int_\sigma^{\sigma_1}\arg(H(\tau+iT))d\tau + O(1).\end{aligned} \tag{4.3}$$
Using Proposition 2.2 for χ, we obtain
$$\int_0^T \log|\chi(\sigma_0-\sigma_1-it)|dt = \left(\frac{1}{2}-(\sigma_0-\sigma_1)\right)T\log\frac{T}{2\pi e}+O(\log T). \tag{4.4}$$
From Proposition 2.1 (3), we have
$$\zeta(\sigma_0-\sigma_1-it) = \chi(\sigma_0-\sigma_1-it)\zeta(1-\sigma_0+\sigma_1+it), \tag{4.5}$$
where $\chi(s) = 2^s\pi^{s-1}\sin\frac{\pi}{2}s\Gamma(1-s)$. Using the first inequality in (4.1) and Euler's product $\zeta(s) = \prod_p(1-p^{-s})^{-1}$, we can show
$$\int_0^T \log|\zeta(1-\sigma_0+\sigma_1+it)|dt = O(1). \tag{4.6}$$
Using Proposition 2.2, we have
$$\int_0^T \log\left|1+a^{-1}\frac{\zeta(\sigma_0+\sigma_1+it)}{\zeta(\sigma_0-\sigma_1-it)}\right|dt = O(1).$$
By this and (4.4)–(4.6), we have
$$\begin{aligned}\int_0^T \log|H(\sigma_1+it)|dt &= \int_0^T \log|\zeta(\sigma_0-\sigma_1-it)|dt + O(1)\\ &= \left(\frac{1}{2}-(\sigma_0-\sigma_1)\right)T\log\frac{T}{2\pi e}+O(\log T).\end{aligned} \tag{4.7}$$
By (4.2) and Proposition 2.2 for χ, we get
$$\begin{aligned}\int_\sigma^{\sigma_1}\arg(H(\tau+iT))d\tau &= \int_\sigma^{\sigma_1}\arg(\chi(\sigma_0-\tau-iT)) + \arg g(\tau+iT)d\tau\\ &=(\sigma_1-\sigma)T\log\frac{T}{2\pi e}+O(\log T).\end{aligned}$$
By this, (4.3) and (4.7), we prove (2).

(3). If $|a|=1$, then the zeros of $H(s;\sigma_0,a)$ are symmetric around the imaginary axis, because
$$a\overline{H(s;\sigma_0,a)} = H(-\overline{s};\sigma_0,a).$$

Thus, (3) follows from this fact and (1).

(4). We suppose that $|a| \neq 0, 1$. Let $T > 1$. We may suppose $\zeta(\sigma_0 - it) \neq 0$ for $0 < t \leqslant T$ and $H(s; \sigma_0, a) \neq 0$ on $\operatorname{Im} s = T$. Then, for $H(s; \sigma_0, a)$, as in (1), we have

$$N_0^*(T) = \frac{1}{2\pi}(\Delta_1 + \Delta_2 + \Delta_3) + O(1),$$

where Δ_1, Δ_2 and Δ_3 are the argument changes of $H(s; \sigma_0, a)$ from $\sigma_1 + it_0$ to $\sigma_1 + iT$, from $\sigma_1 + iT$ to iT and from iT to it_0, respectively. We write

$$H(\sigma_1 + it; \sigma_0, a) = \chi(\sigma_0 - \sigma_1 - it)\left(a\zeta(1 - \sigma_0 + \sigma_1 + it) + \frac{\zeta(\sigma_0 + \sigma_1 + it)}{\chi(\sigma_0 - \sigma_1 - it)}\right).$$

Then, using (4.2), Proposition 2.2 for χ and Proposition 2.6, we have

$$\Delta_1 = T \log \frac{T}{2\pi e} + O(1) \quad \text{and} \quad \Delta_2 = O(\log T).$$

We write

$$H(it; \sigma_0, a) = \begin{cases} \zeta(\sigma_0 - it)\,(a + \zeta(\sigma_0 + it)/\zeta(\sigma_0 - it)) & (|a| > 1), \\ \zeta(\sigma_0 + it)\,(1 + a\zeta(\sigma_0 - it)/\zeta(\sigma_0 + it)) & (|a| < 1). \end{cases}$$

Since $|a| \neq 0, 1$ and $|\zeta(\sigma_0 - it)/\zeta(\sigma_0 + it)| = 1$, we have

$$\Delta_3 = \Delta^* + O(1),$$

where Δ^* is the argument change of $\zeta(\sigma_0 - s)$ (or $\zeta(\sigma_0 + s)$) from iT to it_0. It is not hard to see that

$$\frac{1}{2\pi}\Delta^* = \pm \widetilde{N}(\sigma_0, T) + O(\log T),$$

where $\widetilde{N}(\sigma_0, T)$ is the number of zeros of $\zeta(s)$ in $\operatorname{Re} s > \sigma_0$ and $0 < \operatorname{Im} s < T$, and the sign in the formula is $+$ if $0 < |a| < 1$, $-$ if $|a| > 1$. Hence, we immediately get (4) from the formulas for Δ_1, Δ_2 and Δ_3.

We complete the proof of Theorem 5. □

The proof of Theorem 6 essentially follows from Selberg's argument. For this, we refer to [20, pp. 54–57] or [10, pp. 155–157]. However, while we follow Selberg's argument, we need to be careful with possible zeros of the Riemann zeta function in the right half-plane to $\operatorname{Re} s = \frac{1}{2}$. Namely, these exceptional zeros cause a problem in applying Selberg's argument for our purpose. Fortunately, the number of the possible zeros in $\operatorname{Re} s > \frac{1}{2} + \lambda$ $(\lambda > 0)$ and $0 < \operatorname{Im} s < T$ is very small, i.e., at most $O(T^\theta)$ $(\theta < 1)$ in $0 < \operatorname{Im} s < T$. We will use this fact crucially for our proof of Theorem 6.

Proof of Theorem 6. We fix $\sigma > \frac{1}{2}$ and $a \neq 0$. We work only on the function

$$f(s) = \zeta(s) + a\zeta(1/2 + \sigma - s),$$

because we can justify the other case using the same method. For convenience, we define $F(t)$ by

$$F(t) = a\zeta(\sigma + it) + \zeta(1/2 - it).$$

We will prove

$$\int_0^T \log|F(t)|\, dt = \frac{1}{2\sqrt{\pi}} T\sqrt{\log\log T} + O(T)$$

as $T \to \infty$.

Let ϵ be a positive number such that $\frac{1}{2} + \epsilon < \sigma$. Enumerate the zeros of $\zeta(s)$, $s_1, s_2, s_3, \ldots$ in $\mathrm{Re}\, s \geqslant \frac{1}{2} + \epsilon$ and $\mathrm{Im}\, s > 0$ with $\mathrm{Im}\, s_1 \leqslant \mathrm{Im}\, s_2 \leqslant \ldots$. If there exist only finitely many s_k's, then our proof of Theorem 6 is simpler. So, we suppose that there exist infinitely many s_k's. By Proposition 2.7, for some $\theta < 1$, we have

$$\#\{k : \mathrm{Im}\, s_k < T\} = O\left(T^{\theta}\right). \tag{4.8}$$

For $T > 0$, define $\mathcal{E}(T)$ by

$$\mathcal{E}(T) = \bigcup_{\mathrm{Im}\, s_k < T} [\mathrm{Im}\, s_k - \log T, \mathrm{Im}\, s_k + \log T].$$

We set

$$\mathcal{R}(T) = [0, T] \backslash \mathcal{E}(T).$$

It is easy to see that by (4.8)

$$|\mathcal{R}(T)| = T + O\left(T^{\theta}\log T\right) \qquad \text{and} \qquad |\mathcal{E}(T)| = O(T^{\theta}\log T), \tag{4.9}$$

where '$|A|$' means the Lebesgue measure of A $(\subseteq \mathbb{R})$.

Lemma 4.1. *As $T \to \infty$,we have*

$$\int_{\mathcal{R}(T)} \log^{+}|\zeta(1/2 + it)|\, dt = \frac{1}{2\sqrt{\pi}} T\sqrt{\log\log T} + O(T).$$

Proof. We note that by (4.9) with Proposition 2.2 or Proposition 2.3, we have

$$\int_{\mathcal{E}(T)} \log^{+}|\zeta(1/2 + it)|\, dt = O\left(\int_{\mathcal{E}(T)} \log T dt\right) = O(T^{\theta}\log^2 T).$$

From this and Proposition 2.9, Lemma 4.1 follows. □

We put
$$\widetilde{g}(s) = a\zeta(\sigma_0 + s) + \zeta(\sigma_0 - s),$$
where $\sigma_0 = \frac{\sigma}{2} + \frac{1}{4}$. We apply Theorem 5 to $\widetilde{g}(s)$ and then we have
$$2\pi \sum_{\substack{b<\gamma<c \\ \beta>\frac{\sigma}{2}-\frac{1}{4}}} \left(\beta - \left(\frac{\sigma}{2} - \frac{1}{4}\right)\right) = O(\log c) + \int_b^c \log|F(t)|\,dt,$$
where $0 < b < c$ and $\beta + i\gamma$ runs through all zeros of $\widetilde{g}(s)$ in $\beta > \frac{\sigma}{2} - \frac{1}{4}$ and $b < \gamma < c$. Thus, by this and (4.9), we get
$$\begin{aligned} 2\pi \sum_{\substack{\gamma\in\mathcal{E}(T) \\ \beta>\frac{\sigma}{2}-\frac{1}{4}}} \left(\beta - \left(\frac{\sigma}{2} - \frac{1}{4}\right)\right) =& O(\text{ the number of intervals } [b,c]\cdot\log T) + \int_{\mathcal{E}(T)} \log|F(t)|\,dt \\ =& O\left(T^\theta \log T\right) + \int_{\mathcal{E}(T)} \log|F(t)|\,dt. \end{aligned}$$
Using (4.8), this and the fact that by Theorem 5, the number of zeros of $\widetilde{g}(s)$ in $t < \text{Im}\, s < t+1$ $(t > 1)$ is $O(\log t)$, we see that
$$\int_{\mathcal{E}(T)} \log|F(t)|\,dt = \left(T^\theta \log^2 T\right). \tag{4.10}$$

Define $\mathcal{A}(T)$ and $\mathcal{B}(T)$ by
$$\begin{aligned} \mathcal{A}(T) =& \{t \in \mathcal{R}(T) : 1 < |F(t)| < 1 + 2|a\zeta(\sigma - it)|\}, \\ \mathcal{B}(T) =& \{t \in \mathcal{R}(T) : 1 + 2|a\zeta(\sigma - it)| < |F(t)|\}. \end{aligned}$$
Using the fact that for $t \in \mathcal{B}(T)$,
$$1 + |a\zeta(\sigma - it)| \leqslant |\zeta(1/2 + it)| \ \text{ and } \ \frac{|\zeta(1/2 + it)|}{2} \leqslant |F(t)| \leqslant 2|\zeta(1/2 + it)|,$$
we have
$$\begin{aligned} \int_{\mathcal{B}(T)} \log|F(t)|\,dt =& \int_{\mathcal{B}(T)} \log^+|\zeta(1/2 + it)|\,dt + O(T) \\ =& \int_{\mathcal{R}(T)} \log^+|\zeta(1/2 + it)|\,dt + O\left(\int_{\widetilde{\mathcal{B}}} \log|\zeta(1/2 + it)|\,dt\right) + O(T) \\ =& \int_{\mathcal{R}(T)} \log^+|\zeta(1/2 + it)|\,dt + O\left(\int_0^T \log(1 + 2|a\zeta(\sigma - it)|)dt\right) + \\ & O(T), \end{aligned}$$
where
$$\widetilde{\mathcal{B}} = \{t \in \mathcal{R}(T) : 1 \leqslant |\zeta(1/2 + it)| < 1 + 2|a\zeta(\sigma - it)|\}.$$

Thus we see that

$$\int_{\mathcal{R}(T)} \log^+ |F(t)| dt = \int_{\mathcal{A}(T)} \log |F(t)|\, dt + \int_{\mathcal{B}(T)} \log |F(t)|\, dt$$
$$= O\left(\int_0^T \log(1 + 2|a\zeta(\sigma - it)|) dt \right) + \tag{4.11}$$
$$\int_{\mathcal{R}(T)} \log^+ |\zeta(1/2 + it)| dt + O(T).$$

By Proposition 2.8, it is easy to see that

$$\int_0^T \log(1 + 2|a\zeta(\sigma - it)|) dt \leqslant T \log \left(\frac{1}{T} \int_0^T 1 + 2|a\zeta(\sigma - it)| dt \right) = O(T).$$

From this and Lemma 4.1 with (4.11), we get

$$\int_{\mathcal{R}(T)} \log^+ |F(t)| dt = \frac{1}{2\sqrt{\pi}} T \sqrt{\log \log T} + O(T). \tag{4.12}$$

Define $\mathcal{C}(T)$ by

$$\mathcal{C}(T) = \{ t \in \mathcal{R}(T) : \max \left(\frac{|a\zeta(\sigma - it)|}{2}, \frac{1}{2|a\zeta(\sigma - it)| + \frac{1}{2}} \right) < |F(t)| < 1 \}.$$

Since $\frac{1}{4} < |a\zeta(\sigma + it)| < 2$ for $t \in C(T)$, for some $c_1 > 0$, we have

$$\int_{\mathcal{C}(T)} \log |F(t)|\, dt \geqslant -c_1 T. \tag{4.13}$$

Now we suppose

$$|F(t)| \leqslant |a\zeta(\sigma - it)|/2.$$

Define $\mathcal{D}(T)$ by

$$\mathcal{D}(T) = \{ t \in \mathcal{R}(T) : |F(t)| \leqslant |a\zeta(\sigma - it)|/2 \}$$

Write

$$\mathcal{R}(T) = \bigcup_{k=1}^{n(T)} (a_k, b_k).$$

Let $a_k < t_1^{(k)} < t_2^{(k)} < t_3^{(k)} < \ldots < b_k$ be the solutions of

$$\arg \zeta(1/2 + it) \equiv \arg(a\zeta(\sigma - it)) \mod 2\pi.$$

Lemma 4.2. *Let* $t > 10$. *If* $\frac{1}{2} < \sigma < 1$ *and* $t \in \mathcal{R}(T)$, *we have*

$$\frac{\zeta'}{\zeta}(\sigma + it) = O\left((\log t)^{1+\epsilon-(\sigma-\frac{1}{2})} \right).$$

If $\sigma \geqslant 1$, *we have*

$$\frac{\zeta'}{\zeta}(\sigma + it) = O\left(\frac{\log t}{\log \log t} \right).$$

Proof. For the second statement of Lemma 4.2, see [21, Theorem 5.17].

Let $t \in \mathcal{R}(T)$ with $t > 10$. By Proposition 2.10, we obtain

$$\frac{\zeta'}{\zeta}(\sigma+it) = O\left(\sum_{n<x^2} \frac{\Lambda(n)}{n^\sigma} + \frac{x^{2(1-\sigma)}}{t^2 \log x} + \frac{1}{\log x}\left|\sum_\rho \frac{x^{\rho-s} - x^{2(\rho-s)}}{(s-\rho)^2}\right|\right).$$

We obtain

$$\sum_{n<x^2} \frac{\Lambda(n)}{n^\sigma} = O\left(\sum_{p<x^2} \frac{\log p}{p^\sigma}\right) = O\left(\sum_{n<x^2} \pi(n) \frac{\log n}{n^{\sigma+1}}\right) = O\left(x^{2(1-\sigma)}\right),$$

where $\pi(x)$ is the number of primes $\leqslant x$. We get

$$\begin{aligned}\left|\sum_\rho \frac{x^{\rho-s} - x^{2(\rho-s)}}{(s-\rho)^2}\right| &\leqslant 2x^{\frac{1}{2}+\epsilon-\sigma} \sum_{\gamma \in \mathcal{R}(T)} \frac{1}{|s-\rho|^2} + 2x^{1-\sigma} \sum_{\gamma \notin \mathcal{R}(T)} \frac{1}{|s-\rho|^2} \\ &= O\left(x^{\frac{1}{2}+\epsilon-\sigma} \sum_{k=1}^\infty \frac{\log(k+t)}{k^2} + x^{1-\sigma} \sum_{k=1}^\infty \frac{\log(k+t)}{\log^2 t + k^2}\right) \\ &= O\left(x^{\frac{1}{2}+\epsilon-\sigma} \log t + x^{1-\sigma} \log\log t\right).\end{aligned}$$

Thus, we obtain

$$\frac{\zeta'}{\zeta}(\sigma+it) = O\left(x^{2(1-\sigma)} + \frac{x^{\frac{1}{2}+\epsilon-\sigma} \log t}{\log\log t} + x^{1-\sigma} \log\log t\right)$$

for $t \in \mathcal{R}(T)$. Letting $x = \log t$, Lemma 4.2 follows. □

We recall that

$$\pi\widetilde{N}(T) = \Delta \arg s(s-1) + \Delta \arg \pi^{-s/2} + \Delta\Gamma(s/2) + \Delta \arg \zeta(s),$$

where $\widetilde{N}(T)$ is the number of zeros of $\zeta(s)$ in the region $0 < \operatorname{Im} s < T$ and Δ denotes the variation from 2 to $2+iT$, and then to $\frac{1}{2}+iT$, along straight lines. See [21, p. 212]. Using this, we have

$$\frac{d}{dt} \arg \zeta(1/2+it) \sim -\frac{1}{2} \log t.$$

By this and Lemma 4.2, we have

$$\frac{d}{dt}(\arg \zeta(1/2+it) - \arg \zeta(\sigma - it)) \sim -\frac{1}{2} \log t$$

for $t \in \mathcal{R}(T)$. Integrating from $t_{n-1}^{(k)}$ to $t_n^{(k)}$, there exists k_0 so that we obtain

$$\frac{4\pi - 1}{\log t_n^{(k)}} < t_n^{(k)} - t_{n-1}^{(k)} < \frac{4\pi + 1}{\log t_n^{(k)}}$$

for $k > k_0$ and $n = 1, 2, \ldots$. Thus, there exists a positive constant c_2 such that for $t_{n-1}^{(k)} < t < t_n^{(k)}$, $k > k_0$ and $n = 1, 2, \ldots$, we have

$$\begin{aligned}\log|F(t)| =& \log|a\zeta(\sigma - it)| + \log\left|\frac{|\zeta(1/2+it)|}{|a\zeta(\sigma-it)|}e^{i(\arg\zeta(1/2+it)-\arg(a\zeta(\sigma-it)))} - 1\right| \\ >& \log|a\zeta(\sigma-it)| - c_2\log|(t-t_n^{(k)})\log t_n^{(k)}| - 1.\end{aligned} \tag{4.14}$$

Define $\mathcal{F}(T)$ by

$$\mathcal{F}(T) = \{t \in \mathcal{R}(T) : |a\zeta(\sigma - it)| \geqslant 1\}.$$

Then, we have

$$\mathcal{F}(T) \asymp T.$$

For this, see [4, Theorem 13]. Then, by this and Proposition 2.8, we get

$$\int_0^T \log^+|a\zeta(\sigma-it)|dt \leqslant |\mathcal{F}(T)|\log\left(\frac{1}{|\mathcal{F}(T)|}\int_{\mathcal{F}(T)}|a\zeta(\sigma-it)|dt\right) = O(T).$$

We also note that by Proposition 2.8,

$$\int_0^T \log|a\zeta(\sigma-it)|dt = O(T).$$

Hence we conclude that

$$\int_0^T \log^-|a\zeta(\sigma-it)|dt = O(T). \tag{4.15}$$

From this and (4.14), there exist positive constants c_3, c_4 such that we obtain

$$\begin{aligned}\int_{\mathcal{D}(T)}\log|F(t)|\,dt \geqslant& \int_0^T \log^-|a\zeta(\sigma-it)|dt \\ &- c_2\sum_{0<t_n<T}\int_{t_n}^{t_{n+1}}\log|(t-t_n)\log t_n|dt - T \\ \geqslant& - c_3\left(T + \sum_{t_n^{(k)}\in\mathcal{R}(T)} t_{n+1}^{(k)} - t_n^{(k)}\right) \\ \geqslant& - c_4 T.\end{aligned} \tag{4.16}$$

Using (4.15), it is easy to see that for some $c_5 > 0$,

$$\int_{\mathcal{G}(T)}\log|F(t)|dt \geqslant -c_5 T,$$

where

$$\mathcal{G}(T) = \{t \in \mathcal{R}(T) : \frac{|a\zeta(\sigma-it)|}{2} \leqslant |F(t)| \leqslant \frac{1}{2|a\zeta(\sigma-it)| + \frac{1}{2}}\}.$$

From this, (4.10), (4.12), (4.13) and (4.16), Theorem 6 follows. □

5 Location of zeros when $\sigma_0 > \frac{1}{2}$

In this section, we show the theorems 7–9.

Proof of Theorem 7. (1). We need the following lemma.

Lemma 5.1. *For any $a \in \mathbb{C}$, as $T \to \infty$, we have*

$$(1) \qquad \int_0^T \log|\zeta(\sigma_1+it)+a\zeta(\sigma_2-it)|dt < cT \qquad (\sigma_1 > 1/2,\ \sigma_2 > 1/2);$$

$$(2) \qquad \int_0^T \log|\zeta(\sigma_1+it)+a\zeta(\sigma_2-it)|dt = O(T) \qquad (\sigma_1 > 1,\ \sigma_2 > 1/2),$$

where the implied constant in 'O' and $c > 0$ does not depend on T.

Proof of Lemma 5.1. First, we prove (1). We fix $\sigma_1, \sigma_2 > \frac{1}{2}$. By Jensen's inequality and Proposition 2.8, we have

$$\int_0^T \log|\zeta(\sigma_1+it)+a\zeta(\sigma_2-it)|dt \leqslant T\log\left(\frac{1}{T}\int_0^T |\zeta(\sigma_1+it)+a\zeta(\sigma_2-it)|dt\right) = O(T).$$

This proves (1).

We prove (2). We see that

$$\int_0^T \log|\zeta_m(\sigma_1-it)|dt = \mathrm{Re}\sum_{p\leqslant p_m}\sum_{n=1}^{\infty}\frac{1}{np^{n\sigma_1}}\int_0^T p^{int}dt = O(1)$$

for $\sigma_1 > 1$. By this and Proposition 2.12, we have

$$\int_0^T \log|\zeta(\sigma_2+it)+a\zeta_m(\sigma_1-it)|dt = O(T), \tag{5.1}$$

where the implied constant in 'O' does not depend on m. We note that $\zeta_m(\sigma_1-it)$ converges uniformly to $\zeta(\sigma_1-it)$ in $[0,T]$ for fixed $\sigma_1 > 1$. Thus, by (5.1) and Lebesgue dominate theorem, we have

$$\int_0^T \log|\zeta(\sigma_2+it)+a\zeta(\sigma_1-it)|dt = O(T).$$

This proves Lemma 5.1 (2).

We complete the proof of Lemma 5.1. □

For the proof of (1), we fix $\sigma_0 > \frac{3}{4}$ and $|a| = 1$. In this case, a crucial point is that the zeros of $H(s;\sigma_0,a)$ are symmetric around the imaginary axis. Let σ be such that $0 < \sigma < \sigma_0 - \frac{1}{2}$ and $\sigma_0 + \sigma > 1$. Choose $\sigma_1 > 0$ such that $\sigma < \sigma_1$ and $\sigma_0 - \sigma_1 > \frac{1}{2}$. By Theorem 5 (2), for $H(s;\sigma_0,a)$, we have

$$2\pi \sum_{\substack{0<\gamma<T\\ \beta>\sigma}} (\beta-\sigma) = \left(\sigma_0 - \frac{1}{2} - \sigma\right) T \log \frac{T}{2\pi e} + O(\log T) + \mathcal{L}_{\sigma_0,a}(T,\sigma),$$

$$2\pi \sum_{\substack{0<\gamma<T\\ \beta>\sigma_1}} (\beta-\sigma_1) = \left(\sigma_0 - \frac{1}{2} - \sigma_1\right) T \log \frac{T}{2\pi e} + O(\log T) + \mathcal{L}_{\sigma_0,a}(T,\sigma_1).$$

Subtracting these two equations, we get

$$2\pi \sum_{\substack{0<\gamma<T\\ \sigma<\beta\leqslant\sigma_1}} (\beta-\sigma) + 2\pi \sum_{\substack{0<\gamma<T\\ \beta>\sigma_1}} (\sigma_1-\sigma) = (\sigma_1-\sigma) T \log \frac{T}{2\pi e} + O(\log T) + \mathcal{L}_{\sigma_0,a}(T,\sigma) - \mathcal{L}_{\sigma_0,a}(T,\sigma_1).$$

From this, we obtain

$$2\pi \sum_{\substack{0<\gamma<T\\ \beta>\sigma}} (\sigma_1-\sigma) \geqslant (\sigma_1-\sigma) T \log \frac{T}{2\pi e} + O(\log T) + \mathcal{L}_{\sigma_0,a}(T,\sigma) - \mathcal{L}_{\sigma_0,a}(T,\sigma_1).$$

By this and Lemma 5.1, we get

$$\sum_{\substack{0<\gamma<T\\ \beta>\sigma}} 1 \geqslant \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(T). \tag{5.2}$$

From Theorem 5 (3), we recall

$$N_0^*(T) = \frac{1}{2} \sum_{\substack{0<\gamma<T\\ \beta=0}} 1 + \sum_{\substack{0<\gamma<T\\ \beta>0}} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).$$

Thus, by this and (5.2), we conclude that

$$\frac{1}{2} \sum_{\substack{0<\gamma<T\\ \beta=0}} 1 + \sum_{\substack{0<\gamma<T\\ 0<\beta\leqslant\sigma}} 1 = O(T).$$

It is easy to see that (1) immediately follows from this equation.

(2). We fix $\sigma_0 > \frac{1}{2}$ and $|a| \neq 0, 1$. It is easy to see that

$$\begin{aligned} \mathcal{L}_{\sigma_0,a}(T,0) &\geqslant -c(a)T + \int_0^T \log|\zeta(\sigma_0+it)|dt = O(T); \\ \mathcal{L}_{\sigma_0,a}(T,\sigma) &\leqslant O(T) \qquad (-\sigma_0+1/2<\sigma<\sigma_0-1/2), \end{aligned} \tag{5.3}$$

where $c(a) = \max\{|\log ||a^{-1}| - 1||\,, \log ||a^{-1}| + 1|\}$. Thus, in particular, we have

$$\mathcal{L}_{\sigma_0,a}(T,0) = O(T). \tag{5.4}$$

We let $0 < \sigma < \sigma_0 - \frac{1}{2}$. As in the previous case, we get

$$2\pi \sum_{\substack{0<\gamma<T\\ 0<\beta\leqslant\sigma}} \beta + 2\pi \sum_{\substack{0<\gamma<T\\ \beta>\sigma}} \sigma = \sigma T \log \frac{T}{2\pi e} + O(\log T) + \mathcal{L}_{\sigma_0,a}(T,0) - \mathcal{L}_{\sigma_0,a}(T,\sigma). \tag{5.5}$$

Since $H(s;\sigma_0,a) \neq 0$ on $\operatorname{Re} s = 0$, we have

$$N_0^*(T) \geqslant \sum_{\substack{0<\gamma<T\\ \beta>0}} 1.$$

Using this, (5.4), (5.5) and Theorem 5 (4), we have

$$\mathcal{L}_{\sigma_0,a}(T,\sigma) \geqslant O(T) + O(\widetilde{N}(\sigma_0,T)) = O(T),$$

since $\widetilde{N}(\sigma_0,T) = O\left(T^{\theta}\right)$ $(\theta < 1)$. By this and (5.3), we have

$$\mathcal{L}_{\sigma_0,a}(T,\sigma) = O(T) \qquad (0 \leqslant \sigma < \sigma_0 - 1/2). \tag{5.6}$$

Let $0 < \sigma_2 < \sigma_1 < \sigma_0 - \frac{1}{2}$. We repeat the above argument for σ and σ_1. Then, we get

$$2\pi \sum_{\substack{0<\gamma<T\\ \beta>\sigma_2}} \sigma_1 - \sigma_2 \geqslant (\sigma_1 - \sigma_2) T \log \frac{T}{2\pi e} + O(\log T) + \mathcal{L}_{\sigma_0,a}(T,\sigma_2) - \mathcal{L}_{\sigma_0,a}(T,\sigma_1).$$

Using this, (5.6) and Theorem 5 (4), as in the previous case, we prove (2).

(3). We fix $\sigma_0 > 1$ and $a \neq 0$. Let σ_1 and σ_2 be such that $-\sigma_0 + \frac{1}{2} < \sigma_1 \leqslant \sigma_2 < \sigma_0 - \frac{1}{2}$. By Proposition 2.13, $\varphi(\sigma)$ for the function $H(s;\sigma_0,a)$ is differentiable in $\left(-\sigma_0 + \frac{1}{2}, \sigma_0 - \frac{1}{2}\right)$. Thus, we have

$$\varphi'(\sigma_2 + 0) - \varphi'(\sigma_1 - 0) = \varphi'(\sigma_2 - 0) - \varphi'(\sigma_1 + 0) = \varphi'(\sigma_2) - \varphi'(\sigma_1).$$

Thus, by Proposition 2.11, we have

$$N(\sigma_1,\sigma_2,T) = \frac{\varphi'(\sigma_2) - \varphi'(\sigma_1)}{2\pi} T + o(T) \qquad (T \to \infty)$$

for $H(s;\sigma_0,a)$. Thus, (3) follows.

(4). Let $\sigma_1 > \sigma_0 - \frac{1}{2}$. Using the functional equation for $\zeta(s)$, the fourth statement of Proposition 2.2 and Theorem 5.17 in [21], we have

$$\frac{1}{\zeta(it)} = O_\epsilon\left(t^{-\frac{1}{2}+\epsilon}\right) \qquad (t \to \infty)$$

for any $\epsilon > 0$. By this and the third statement in Proposition 2.2, we can see that $H(s;\sigma_0,a)$ does not vanish for sufficiently large $\operatorname{Im} s$ provided that $\sigma_1 - \sigma_0 \leqslant 0$.

We suppose $0 < \sigma_0 - \sigma_1 < \frac{1}{2}$. We set

$$\sigma = 2\sigma_0 - \frac{1}{2}.$$

Then, $\sigma > \frac{1}{2}$. We repeat some parts of the proof of Theorem 6. We recall $\mathcal{E}(T)$ and $\mathcal{R}(T)$ for σ. By Theorem 5 (2) and (5.3), we have

$$\sum_{\substack{\gamma\in\mathcal{E}(T)\\ \beta>\sigma_0-\frac{1}{2}}} \beta - \left(\sigma_0 - \frac{1}{2}\right) = O\left(T^{\theta}\log^2 T\right).$$

Thus, we get

$$\sum_{\substack{\gamma\in\mathcal{E}(T)\\ \beta>\sigma_1}} 1 = O\left(T^{\theta}\log^2 T\right). \tag{5.7}$$

Let $t \in \mathcal{R}(T)$. Then, by Lemma 4.2, we have

$$\log\zeta(\sigma^* + it) = \log\zeta(5+it) + \int_5^{\sigma^*} \frac{\zeta'}{\zeta}(\tau+it)d\tau = O\left((\log t)^{\sigma_0-\sigma_1+\epsilon+\frac{1}{2}}\right),$$

where $\sigma^* \geqslant 1-\sigma_0+\sigma_1$ and ϵ is a positive constant with $\sigma_0-\sigma_1+\epsilon < \frac{1}{2}$. Using this and the fourth statement in Proposition 2.2, we conclude that $\zeta(1-\sigma_0+s)+a^{-1}\chi(\sigma_0-s)^{-1}\zeta(\sigma_0+s)$ does not vanish for $\operatorname{Re} s \geqslant \sigma_1$ and sufficiently large $\operatorname{Im} s$ in $\mathcal{R}(T)$. Thus, we see that for $H(s;\sigma_0,a)$, we have

$$\sum_{\substack{\gamma\in\mathcal{R}(T)\\ \beta>\sigma_1}} 1 = O(1),$$

because by the functional equation for $\zeta(s)$, it is easy to see that $H(s;\sigma_0,a) = 0$ if and only if $\zeta(1-\sigma_0+s)+a^{-1}\chi(\sigma_0-s)^{-1}\zeta(\sigma_0+s) = 0$. Thus, by this and (5.7), (4) for $N(\sigma_1,T)$ follows. Similarly, we can demonstrate (4) for $N(-\infty,-\sigma_1,T)$, using $H(s;\sigma_0,\overline{a})$.

We complete the proof of Theorem 7. □

Proof of Theorem 8. We fix $\sigma_0 > \frac{1}{2}$ and $a \neq 0$. Let $\varphi(t)$ and $\psi(t)$ be as in Theorem 8. By the theorems 5 and 6, we obtain

$$2\pi \sum_{\substack{0<\gamma<T\\ \beta>\sigma_0-\frac{1}{2}}} \beta - \left(\sigma_0 - \frac{1}{2}\right) = \frac{1}{2\sqrt{\pi}}T\sqrt{\log\log T} + O(T)$$

as $T \to \infty$. Thus, the first formula follows. Using $H(s;\sigma_0,\overline{a})$, we obtain the second formula by virtue of the above argument. From the first and second formulas, we immediately derive that

$$2\pi \sum_{\substack{0<\gamma<T \\ |\beta|>\sigma_0-\frac{1}{2}+\psi(T)}} \psi(T) = O\left(T\sqrt{\log\log T}\right) \quad \text{or} \quad \sum_{\substack{0<\gamma<T \\ |\beta|>\sigma_0-\frac{1}{2}+\psi(T)}} 1 = O\left(\frac{T\log T}{\phi(T)}\right),$$

because $\psi(T) = \phi(T)\sqrt{\log\log T}/\log T$. This proves the third formula. We complete the proof of Theorem 8. □

Proof of Theorem 9. We fix $\sigma_0 > \frac{3}{4}$ and $|a| = 1$. Let κ be a positive real number. For $T > 10$, we set

$$\sigma(T,\kappa) = \sigma_0 - \frac{1}{2} - \kappa\psi(T) = \sigma_0 - \frac{1}{2} - \frac{\kappa\phi(T)\sqrt{\log\log T}}{\log T}.$$

Fix σ^* such that $0 \leqslant \sigma^* < \sigma(T,\kappa)$ and

$$\mathcal{L}_{\sigma_0,a}(\sigma^*, T) = O(T).$$

This follows from Lemma 5.1. By Theorem 5, we have

$$2\pi \sum_{\substack{0<\gamma<T \\ \beta>-\sigma(T,\kappa)}} \beta + \sigma(T,\kappa) = \left(\sigma_0 - \frac{1}{2} + \sigma(T,\kappa)\right) T\log\frac{T}{2\pi e} + O(\log T) +$$

$$\mathcal{L}_{\sigma_0,a}(T, -\sigma(T,\kappa)),$$

$$2\pi \sum_{\substack{0<\gamma<T \\ \beta>-\sigma^*}} \beta + \sigma^* = \left(\sigma_0 - \frac{1}{2} + \sigma^*\right) T\log\frac{T}{2\pi e} + O(\log T) + \mathcal{L}_{\sigma_0,a}(T, -\sigma^*).$$

We note that

$$\mathcal{L}_{\sigma_0}(T, -\sigma(T,\kappa)) = \mathcal{L}_{\sigma_0}(T, \sigma(T,\kappa)), \ \ \mathcal{L}_{\sigma_0}(T, -\sigma^*) = \mathcal{L}_{\sigma_0}(T, \sigma^*).$$

From the above equations, we obtain

$$2\pi \sum_{\substack{0<\gamma<T \\ -\sigma(T,\kappa)<\beta<-\sigma^*}} (\beta + \sigma(T,\kappa)) + 2\pi \sum_{\substack{0<\gamma<T \\ \beta>-\sigma^*}} (\sigma(T,\kappa) - \sigma^*) =$$

$$(\sigma(T,\kappa) - \sigma^*) T\log\frac{T}{2\pi e} + O(\log T) + \mathcal{L}_{\sigma_0,a}(T, \sigma(T,\kappa)) - \mathcal{L}_{\sigma_0,a}(T, \sigma^*).$$

Since we have

$$N_0^*(T) = \sum_{\substack{0<\gamma<T \\ \beta>0}} 1 + \frac{1}{2} \sum_{\substack{0<\gamma<T \\ \beta=0}} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T)$$

from Theorem 5 (3), we have

$$2\pi \sum_{\substack{0<\gamma<T \\ -\sigma(T,\kappa)<\beta<-\sigma^*}} (\beta+\sigma(T,\kappa)) + 2\pi(\sigma(T,\kappa)-\sigma^*) \left(\sum_{\substack{0<\gamma<T \\ -\sigma^*<\beta<0}} 1 + \frac{1}{2} \sum_{\substack{0<\gamma<T \\ \beta=0}} 1 \right) =$$
$$O(\log T) + \mathcal{L}_{\sigma_0,a}(T,\sigma(T,\kappa)) - \mathcal{L}_{\sigma_0,a}(T,\sigma^*).$$

Since the left side of the above equation is nonnegative, we can see that

$$\mathcal{L}_{\sigma_0,a}(T,\sigma(T,\kappa)) \geqslant \mathcal{L}_{\sigma_0,a}(T,\sigma^*) + O(\log T). \tag{5.8}$$

It is easy to observe that

$$\begin{aligned} \mathcal{L}_{\sigma_0,a}(T,\sigma(T,\kappa)) \leqslant & \frac{T}{2} \log \left(\frac{1}{T} \int_0^T |H(-\sigma(T,\kappa)+it;\sigma_0,a)|^2 dt \right) \\ =& O(T\log\log T), \end{aligned} \tag{5.9}$$

because by Proposition 2.8, we have

$$\int_0^T |\zeta(\sigma_0+\sigma(T,\kappa)-it)|^2 dt = O(T)$$

and

$$\int_0^T |\zeta(\sigma_0-\sigma(T,\kappa)+it)|^2 dt = \int_0^T |\zeta(1/2+\kappa\psi(T)+it)|^2 dt = O(T\log T).$$

Since $\mathcal{L}_{\sigma_0,a}(T,\sigma^*) = O(T)$, we derive

$$\mathcal{L}_{\sigma_0,a}(T,\sigma(T,\kappa)) = O(T\log\log T)$$

from (5.8) and (5.9). By this and Theorem 5, we have

$$2\pi \sum_{\substack{0<\gamma<T \\ \beta>\sigma(T,1)}} (\beta-\sigma(T,1)) = \left(\sigma_0 - \frac{1}{2} - \sigma(T,1)\right) T \log\frac{T}{2\pi e} + O(T\log\log T),$$

$$2\pi \sum_{\substack{0<\gamma<T \\ \beta>\sigma(T,1/2)}} (\beta-\sigma(T,1/2)) = \left(\sigma_0 - \frac{1}{2} - \sigma(T,1/2)\right) T \log\frac{T}{2\pi e} + O(T\log\log T),$$

Thus we get

$$2\pi \sum_{\substack{0<\gamma<T \\ \sigma(T,1)<\beta<\sigma(T,1/2)}} (\beta-\sigma(T,1)) + 2\pi \sum_{\substack{0<\gamma<T \\ \beta>\sigma(T,1/2)}} (\sigma(T,1/2)-\sigma(T,1)) =$$
$$(\sigma(T,1/2)-\sigma(T,1)) T \log\frac{T}{2\pi e} + O(T\log\log T).$$

Hence, we obtain

$$\sum_{\substack{0<\gamma<T\\ \beta>\sigma(T,1)}} 1 \geqslant \frac{T}{2\pi}\log\frac{T}{2\pi e} + O\left(\frac{T\log\log T}{\sigma(T,1/2)-\sigma(T,1)}\right).$$

Using this and the facts

$$\sigma(T,1/2)-\sigma(T,1) = \frac{\phi(T)\sqrt{\log\log T}}{2\log T}, \qquad \sum_{\substack{0<\gamma<T\\ \beta>0}} 1 \leqslant \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T)$$

we have

$$\sum_{\substack{0<\gamma<T\\ \beta>\sigma(T,1)}} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O\left(\frac{T\log T\sqrt{\log\log T}}{\phi(T)}\right).$$

Using this and Theorem 8, we prove the formula in Theorem 9 for the case $\sigma_0 > \frac{3}{4}$ and $|a| = 1$. Using $H(s;\sigma_0,\overline{a})$, we can similarly demonstrate the same formula with '$\left|\beta + \left(\sigma_0 - \frac{1}{2}\right)\right| < \psi(T)$' in place of '$\left|\beta - \left(\sigma_0 - \frac{1}{2}\right)\right| < \psi(T)$'.

Using Theorem 5, Theorem 6 and the fact that

$$\mathcal{L}_{\sigma_0,a}(T,\sigma) = O(T) \qquad (-\sigma_0 + 1/2 < \sigma < \sigma_1 - 1/2),$$

we can similarly justify Theorem 9 for the case $\sigma_0 > \frac{1}{2}$ and $|a| \neq 0, 1$. We note that the asymmetry of zeros of $H(s;\sigma_0,a)$ does not cause any problem for the argument. □

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