A remark on the uniqueness of the Dirichlet series with a Riemann-type function equation

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Abstract

We show that if for a nonzero complex number $c$ the inverse images $L_1^{-1}(c)$ and $L_2^{-1}(c)$ of two functions in the extended Selberg class are the same, then $L_1(s)$ and $L_2(s)$ must be identical.

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1. Introduction

Automorphic $L$-functions which generalize the Riemann zeta function play a central role in investigating many arithmetic questions. Essential properties of these $L$-functions such as Euler products, functional equations and the Ramanujan conjecture can be axiomatized and this is what Selberg did in [6], specifying the following conditions.

(1) (Dirichlet series) For $\sigma > 1$, the $L$-function $L(s)$ is an absolutely convergent Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \quad (s = \sigma + it).$$

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(2) (Analytic continuation) For some integer $m \geq 0$, the function $(s - 1)^m L(s)$ is entire and of finite order.

(3) (Functional equation) The $L$-function $L(s)$ satisfies a functional equation of the form

$$
\Phi(s) = \omega \Phi(1 - \overline{s}),
$$

where

$$
\Phi(s) = Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \mu_j) L(s)
$$

with $Q > 0$, $\lambda_j > 0$, $\Re \mu_j \geq 0$ and $|\omega| = 1$.

(4) (Ramanujan hypothesis) For any $\epsilon > 0$, we have $a(n) \ll n^\epsilon$.

(5) (Euler product) For $\sigma$ sufficiently large,

$$
\log L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}, \quad s = \sigma + it,
$$

where $b(n) = 0$ unless $n$ is a positive power of a prime, and $b(n) \ll n^\theta$ for some $\theta < \frac{1}{2}$.

The set of $L$-functions which satisfy the conditions (1)–(5) is called the Selberg class and is denoted by $\mathcal{S}$. Note that the Riemann zeta function, the Dirichlet $L$-function with a primitive Dirichlet character, the Dedekind zeta function of an algebraic number field and the Hecke $L$-function with a primitive Hecke character all belong to the Selberg class $\mathcal{S}$. Kaczorowski and Perelli [4] introduced the extended Selberg class $\mathcal{S}^e$ of not identically vanishing functions $L(s)$ which satisfy the conditions (1)–(3) above. For a function $L(s)$ in the extended Selberg class, we define the degree $d$ as $d = 2 \sum_j \lambda_j$.

In the extended Selberg class $\mathcal{S}^e$, a natural question concerns the uniqueness of functions. On this topic, Steuding [7, p. 152] proved the following result.

**Theorem A.** If two functions $L_1(s)$ and $L_2(s)$ satisfy both the conditions (2) and (4) as well as the same functional equation (3) with $a(1) = 1$ and $L_1^{-1}(c_j) = L_2^{-1}(c_j)$ for two distinct complex numbers $c_1$ and $c_2$ such that

$$
\liminf_{T \to \infty} \frac{\tilde{N}_{L_1}^{c_1}(T) + \tilde{N}_{L_2}^{c_2}(T)}{N_{L_1}^{c_1}(T) + N_{L_2}^{c_2}(T)} > \frac{1}{2} + \epsilon
$$

for some positive $\epsilon$ with either $j = 1$ or $j = 2$, then $L_1 \equiv L_2$.

In Theorem A, the symbol $L^{-1}(c)$ denotes the preimage of $c$ under $L$, meaning $L^{-1}(c) = \{ s \in \mathbb{C} : L(s) = c \}$. Furthermore, the term $N_L^c(T)$ denotes the number of zeros of $L(s) - c$ in the region given by $0 \leq \Re s \leq T$ and $|t| \leq T$ counting multiplicities, while $\tilde{N}_L^c(T)$ stands for the number of zeros in the same region, but ignoring multiplicities.

Recently, Li [5] has substantially improved Theorem A as follows.

**Theorem B.** If two functions $L_1(s)$ and $L_2(s)$ satisfy both the conditions (2) and (4) as well as the same functional equation (3) with $a(1) = 1$ and $L_1^{-1}(c_j) = L_2^{-1}(c_j)$ for two distinct complex numbers $c_1$ and $c_2$, then $L_1 \equiv L_2$.

The proof of Theorem B is based on Nevanlinna’s theory, in particular on Nevanlinna’s uniqueness theorem: two nonconstant meromorphic functions $f, g : \mathbb{C} \to \mathbb{C}$ must be identically equal if $f^{-1}(c_j) = g^{-1}(c_j)$ for five distinct values $c_j \in \mathbb{C} \cup \{ \infty \}$ (see [3] or [7]).
It is natural to ask whether Theorem B still holds if \( c_1 = c_2 \). In this note, we answer this question for functions in the extended Selberg class by proving the following result.

**Theorem 1.** If two functions \( L_1(s) \) and \( L_2(s) \) in the extended Selberg class \( S^\sharp \) satisfy the same functional equation with positive degree, if \( a(1) = 1 \) and \( L_1^{-1}(c) = L_2^{-1}(c) \) for a nonzero complex number \( c \), then \( L_1 \equiv L_2 \). The conclusion need not hold for \( c = 0 \) or if the functional equation is of degree zero.

The key point of the proof of Theorem 1 is the following. For a nonzero complex number \( c \), positive degrees of the \( L \)-functions and \( L_1^{-1}(c) = L_2^{-1}(c) \), we observe that for a sufficiently large \( \kappa > 0 \), the zeros of \( \prod_{j=1}^{K} \Gamma(\lambda_j s + \mu_j)^{-1} \) and common zeros of \( L_1(s) - c \) and \( L_2(s) - c \) in \( \text{Re} \ s < -\kappa \) should be zeros of \( L_2(s) - L_1(s) \). However, we see that in the region \( \text{Re} \ s < -\kappa \), the zeros of \( L_2(s) - L_1(s) \) are the same as the zeros of \( \prod_{j=1}^{K} \Gamma(\lambda_j s + \mu_j)^{-1} \). Based on these observations we are able to prove the first part of our theorem. For degree \( d = 0 \), we can readily construct counterexamples. These imply trivial counterexamples for any degree \( d \) and \( c = 0 \). We shall also give nontrivial counterexamples for \( d \geq 5 \) and \( c = 0 \).

**2. Proof of Theorem 1**

We divide the proof into two parts. Recall that \( d \) denotes the degree of \( L_1(s) \) and \( L_2(s) \).

2.1. Part I: \( c \neq 0 \) and \( d > 0 \)

Let us assume the contrary, namely that \( L_1 \neq L_2 \). Then there exists a smallest integer \( n_0 > 1 \) such that \( a_1(n_0) = a_2(n_0) \). Condition (1) now implies

\[
L_1(s) = 1 + O \left( 2^{-\sigma} \right), \quad L_2(s) = 1 + O \left( 2^{-\sigma} \right),
\]

\[
L_2(s) - L_1(s) = \frac{a_2(n_0) - a_1(n_0)}{n_0^s} \left[ 1 + O \left( \left( \frac{n_0}{n_0 + 1} \right)^\sigma \right) \right], \quad s = \sigma + it, \sigma \to \infty.
\]

Therefore we can choose a constant \( \kappa_0 > 0 \) such that neither of the three functions \( L_1(s) \), \( L_2(s) \) or \( L_2(s) - L_1(s) \) vanishes in the region \( \text{Re} \ s \geq \kappa_0 \).

For any meromorphic function \( f \) and for \( T > 0, \kappa > 0 \), we define \( N_f(T), N_f(T, \kappa) \) as follows.

\[
N_f(T) = \text{the number of zeros of } f, \quad \text{counting multiplicities, in } -T < \text{Re} \ s < -\kappa_0;
\]

\[
N_f(T, \kappa) = \text{the number of zeros of } f, \quad \text{counting multiplicities, in } -T < \text{Re} \ s < -\kappa_0, |\text{Im} \ s| < \kappa.
\]

Note that \( N_f(T) \) can be infinite.

**Lemma.** (a) There exists a constant \( \kappa > 0 \) such that if \( L(s) \) stands for one of the functions \( L_1(s) \), \( L_2(s) \) or \( L_2(s) - L_1(s) \), we have

\[
N_L(T) = N_L(T, \kappa) = \left( \sum_{j=1}^{K} \lambda_j \right) T + O(1), \quad T > 0.
\]

(b) There exists a constant \( \kappa_1 > 0 \) such that if \( L(s) \) stands for either \( L_1(s) \) or \( L_2(s) \), we have

\[
N_{L-c}(T, \kappa_1) = \left( \sum_{j=1}^{K} \lambda_j \right) T + O(1), \quad T > 0.
\]
Proof of Lemma. (a) Using the functional equation, we can write
\[ L_j(s) = \chi(s)\Gamma(1-s) \quad j = 1, 2, \]
where
\[ \chi(s) = \omega Q^{1-s} \prod_{j=1}^{K} \Gamma(\lambda_j (1-s) + \mu_j) \prod_{j=1}^{K} \Gamma(\lambda_j s + \mu_j). \]

From this equality and the fact that \( L_1(s), L_2(s) \) and \( L_2(s) - L_1(s) \) have no zeros in \( \text{Re} s \geq \kappa_0 \), that \( \Gamma(s) \) is analytic except for (simple) poles at \( s = 0, -1, -2, \ldots \) and that \( \chi(s) \) has no poles in \( |\text{Im} s| \geq \kappa \) for \( \kappa > 0 \) sufficiently large, we readily see that the zeros of \( L_1(s), L_2(s) \) and \( L_2(s) - L_1(s) \) are the same as zeros of \( \prod_{j=1}^{K} \Gamma(\lambda_j s + \mu_j)^{-1} \) in the region defined by \( \text{Re} s < -\kappa_0 \) and \( |\text{Im} s| < \kappa \). In fact it suffices to find the number of poles of \( \prod_{j=1}^{K} \Gamma(\lambda_j s + \mu_j) \) in \( -T < \text{Re} s < -\kappa_0 \). The observation that \( \Gamma(s) \) has (simple) poles only at \( s = -n \) \((n = 0, 1, 2, \ldots)\) then completes the proof of part (a) of the Lemma.

(b) We need the following claim.

Claim. Let \( r > 0 \). Then there exist constants \( T_r > 0 \) and \( \eta > 0 \) such that for any \( T > T_r \), we have
\[ |\chi(s)| > r \quad \text{for} \quad \text{Re} s = -T^* \quad \text{and} \quad |\text{Im} s| < \eta \quad \text{or} \quad \text{Re} s < -T_r \quad \text{and} \quad |\text{Im} s| = \eta, \]
where \( T^* \) is a real number in \((T, T + 1)\).

Proof of Claim. Fix a constant \( \eta \) larger than \( 1 + \left| \frac{\text{Im} \mu_j}{\lambda_j} \right| \) for \( 1 \leq j \leq K \). From [2, pp. 47 and 3], we recall that
\[ \Gamma(s) = e^{(s-\frac{1}{2})\log(s)-s} (2\pi)^{\frac{1}{2}} \left[ 1 + O \left( |s|^{-1} \right) \right] (\arg s < \pi), \]
\[ \Gamma(s) = \frac{\pi}{\Gamma(1-s) \sin \pi s}. \]

This implies
\[ \chi(s) = \omega Q^{1-2s} \prod_{j=1}^{K} \left[ \Gamma(\lambda_j (1-s) + \mu_j) \Gamma(1-\lambda_j s - \mu_j) \frac{\sin \pi(\lambda_j s + \mu_j)}{\pi} \right], \]
\[ \left| \chi(s) \right| \prod_{j=1}^{K} \sin \pi(\lambda_j s + \mu_j) \rightarrow \infty \quad (\text{Re} s \rightarrow -\infty \text{ and } |\text{Im} s| \leq \eta). \]

It therefore suffices to show that there exists a fixed constant \( \delta > 0 \) such that for any \( T > 0 \),
\[ \prod_{j=1}^{K} \sin \pi(\lambda_j s + \mu_j) > \delta, \quad |\text{Im} s| = \eta \quad \text{or} \quad \text{Re} s = -T^* \quad \text{and} \quad |\text{Im} s| < \eta, \quad (*) \]
where \( T^* \) is a real number in the interval \((T, T + 1)\). This can be seen as follows. Since the function \( \prod_{j=1}^{K} \sin \pi(\lambda_j s + \mu_j) \) has zeros only at \( s = -\frac{n+\mu_j}{\lambda_j}, j = 1, \ldots, K, n = 0, \pm 1, \pm 2, \ldots \), we see that, counting multiplicities, the number of zeros of the function in the region \(-T - 1 <
\[ \Re s < -T \] is less than \( \lambda^* := 1 + \sum_{j=1}^{K} \frac{1}{\kappa_j} \). Therefore there exist \( a_T, b_T \) in \((T, T + 1)\) such that
\[ b_T - a_T = \frac{1}{\kappa^*} \] and \( \prod_{j=1}^{K} \sin \pi (\lambda_j s + \mu_j) \) has no zeros in \(-b_T \leq \Re s \leq -a_T\). Setting \( T^* = \frac{ar + bT}{2} \), the inequality (\( \star \)) follows for a fixed constant \( \delta > 0 \) and the claim is proved. \( \square \)

In addition to the positive real number \( \kappa \) chosen in the proof of part (a) of the lemma, we set \( r = 2|c| \). By the Claim and the fact that \( L_1(s), L_2(s) \sim 1 \) for \( \Re s \to \infty \), we can choose a constant \( \kappa_1 > \kappa \) and sequences \( \langle a_n \rangle \) and \( \langle b_n \rangle \) with \( \kappa_0 < a_n < b_n < a_n + 1(n = 1, 2, 3, \ldots) \) and \( a_n \to \infty \) such that we get
\[ |L_j(s) - (L_j(s) - c)| = |c| < |\chi(s)L_j(1 - \overline{s})| = |L_j(s)|, \quad j = 1, 2 \]
for \(-b_n \leq \Re s \leq -a_n(n = 1, 2, 3, \ldots) \) and \( |\Im s| \leq \kappa_1 \) or \( \Re s < -\kappa_1 \) and \( |\Im s| = \kappa_1 \). Together with Rouché’s theorem, this estimate and part (a) now imply part (b) of the lemma. \( \square \)

We now prove part I of Theorem 1. Observe that \( L_j(s) \) and \( L_j(s) - c, j = 1, 2 \) do not have zeros in common; furthermore, in the region \( \Re s < -\kappa \) for \( \kappa > 0 \) sufficiently large, zeros of \( \prod_{j=1}^{K} \Gamma(\lambda_j s + \mu_j)^{-1} \) (which are zeros of \( L_j(s), j = 1, 2 \)) and common zeros of \( L_1(s) - c \) and \( L_2(s) - c \) should be zeros of \( L_2(s) - L_1(s) \). Thus it is easy to see that
\[ N_{L_2-L_1}(T) \geq N_{L_1}(T) + N_{L_1-c}(T) + O(1), \]
where
\[ N_{L_1-c}(T) = \text{the number of zeros (ignoring multiplicities) of } L_1(s) - c \]
in the region \(-T < \Re s < -\kappa_0 \) and \( |\Im s| < \kappa_1 \).

By this inequality and part (a) of the Lemma, we have
\[ N_{L_1-c}(T) = O(1). \]
On the other hand, part (b) of the lemma implies
\[ N_{L_1-c}(T) \to \infty, \quad T \to \infty. \]
From this contradiction we conclude that we must have \( L_1 \equiv L_2 \).

2.2. Part II: \( c = 0 \) or \( d = 0 \)

We shall give counterexamples for each case.

Case (i) \( c = 0 \) and \( d = 0 \).

Let \( a_1 \) and \( a_2 \) be distinct complex numbers. We set
\[
l_1(s) = 1 + \frac{a_1}{2^s} + \frac{3a_1/\sqrt{6}}{3^s} + \frac{\sqrt{6}}{6^s},
\]
\[
l_2(s) = 1 + \frac{a_2}{2^s} + \frac{3a_2/\sqrt{6}}{3^s} + \frac{\sqrt{6}}{6^s}.
\]
It is easy to see that
\[ (\sqrt{6})^s l_j(s) = (\sqrt{6})^{1-s} \overline{l_j(1 - \overline{s})}, \quad j = 1, 2. \]

We set
\[ L_1(s) = l_1^2(s) l_2(s) \quad \text{and} \quad L_2(s) = l_1(s) l_2^2(s). \]
Clearly \( L_1(s) \) and \( L_2(s) \) are in the extended Selberg class \( \mathcal{S}^\sharp \) and we have \( L^{-1}(0) = L_2^{-1}(0) \), but \( L_1 \neq L_2 \).

**Case (ii) \( c = 0 \) and \( d > 0 \).**

Starting with the examples in Case (i), we can find an obvious counterexample consisting of \( L_1(s)L(s) \) and \( L_2(s)L(s) \) for any \( L(s) \) in \( \mathcal{S}^\sharp \). On top of that, we shall now provide a nontrivial counterexample for each case \( d \geq 5 \) and \( c = 0 \).

We let \( \chi_1 \) be the primitive character modulo 5 such that \( \chi_1(2) = i \) and we set \( \chi_2 = \bar{\chi}_1 \) and

\[
\tau(\chi_j) = \sum_{m=1}^{4} \chi_j(m)e^{\frac{2\pi mi}{5}}, \quad j = 1, 2.
\]

Then we have \( \tau(\chi_1)\tau(\chi_2) = -5 \), the two Dirichlet \( L \)-functions \( L(s, \chi_j) \) \((j = 1, 2)\) are entire and satisfy the functional equations

\[
\left(\frac{\pi}{5}\right)^{-\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s, \chi_j) = \tau(\chi_j) \left(\frac{\pi}{5}\right)^{-\frac{1-s}{2}} \Gamma\left(\frac{2-s}{2}\right) L(1-s, \chi_j), \quad j = 1, 2;
\]

see [1, pp. 69–71]. These functional equations readily imply

\[
\left(\frac{\pi^2}{5^2}\right)^{-\frac{s}{2}} \Gamma^2\left(\frac{s+1}{2}\right) L(s, \chi_1)L(s, \chi_2)
\]

\[
= \left(\frac{\pi^2}{5^2}\right)^{-\frac{1-s}{2}} \Gamma^2\left(\frac{2-s}{2}\right) L(1-s, \chi_1)L(1-s, \chi_2).
\]

Recall from [8, pp. 282–283] that for some \( \theta \in (0, \pi/4) \), we have

\[
\left(\frac{\pi}{5}\right)^{-\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) l_\theta(s) = \left(\frac{\pi}{5}\right)^{-\frac{1-s}{2}} \Gamma\left(\frac{2-s}{2}\right) l_\theta(1-s),
\]

where

\[
l_\theta(s) = \frac{1}{2} \sec \theta \left[ e^{-i\theta} L(s, \chi_1) + e^{i\theta} L(s, \chi_2) \right] = 1 + \frac{\tan \theta}{2^s} - \frac{\tan \theta}{3^s} - \frac{1}{4^s} + \frac{1}{6^s} + \cdots.
\]

Note that \( l_\theta(s) \) is entire. We fix an integer \( m \geq 3 \) and set

\[
L_1(s) = L(s, \chi_1)L(s, \chi_2)l_\theta^m(s) \quad \text{and} \quad L_2(s) = L^2(s, \chi_1)L^2(s, \chi_2)l_\theta^{m-2}(s).
\]

From the functional equations for \( L(s, \chi_1)L(s, \chi_2) \) and \( l_\theta(s) \), we have

\[
\left(\frac{\pi^{m+2}}{5^{m+2}}\right)^{-\frac{s}{2}} \Gamma^{m+2}\left(\frac{s+1}{2}\right) L_j(s) = \left(\frac{\pi^{m+2}}{5^{m+2}}\right)^{-\frac{1-s}{2}} \Gamma^{m+2}\left(\frac{2-s}{2}\right) L_j(1-s),
\]

\[
j = 1, 2.
\]

We see that \( L_1(s) \) and \( L_2(s) \) are in \( \mathcal{S}^\sharp \), also \( L_1^{-1}(0) = L_2^{-1}(0) \), but \( L_1 \neq L_2 \).

**Case (iii) \( c \neq 0 \) and \( d = 0 \).**

We set \( c = 1 \),

\[
L_1(s) = 1 + \frac{\sqrt{6}}{2^s} + \frac{2}{4^s},
\]
\[ L_2(s) = 1 + \frac{3\sqrt{6}}{2^s} + \frac{18\sqrt{6}}{4^s} + \frac{6\sqrt{6}}{8^s} + \frac{4}{16^s}. \]

It is easy to see that \( L_1(s) \) and \( L_2(s) \) are in \( \mathcal{S}^2 \) and we have
\[ L_2(s) - 1 = \frac{4^s}{2} (L_1(s) - 1)^3. \]

We therefore obtain \( L_1^{-1}(1) = L_2^{-1}(1) \), but \( L_1 \not\equiv L_2 \).

This completes the proof of part II of the theorem.

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