On the zeros of degree one $L$-functions from the extended Selberg class

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1. Introduction. In [13], Selberg introduced the class $S$ consisting of the functions $F(s)$ satisfying the following conditions.

(1) (Dirichlet series) For $\sigma > 1$, $F(s)$ is an absolutely convergent Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \quad (s = \sigma + it).$$

(2) (Analytic continuation) For some integer $m \geq 0$, $(s - 1)^m F(s)$ is an entire function of finite order.

(3) (Functional equation) $F(s)$ satisfies a functional equation of the form

$$\Phi(s) = \omega \overline{\Phi}(1 - s),$$

where

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s)$$

with $\overline{\Phi}(s) = \overline{\Phi(s)}$, $Q > 0$, $\lambda_j > 0$, $\text{Re} \mu_j \geq 0$ and $|\omega| = 1$.

(4) (Ramanujan hypothesis) For every $\epsilon > 0$, $a(n) \ll n^\epsilon$.

(5) (Euler product) For $\sigma$ sufficiently large,

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s} \quad (s = \sigma + it),$$

where $b(n) = 0$ unless $n$ is a positive power of a prime, and $b(n) \ll n^\theta$ for some $\theta < 1/2$. 

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For a function $F(s)$ in the Selberg class $S$, we define $d = 2 \sum_j \lambda_j$ to be the degree of $F$. We denote by $S_d$ the subclass of functions of degree $d$ in $S$. We note that the structure of $S_d$ has been completely determined for $0 \leq d \leq 1$. From the work of Conrey and Ghosh [4], we have $S_0 = \{1\}$ and $S_d = \emptyset$ for $0 < d < 1$. For $d = 1$, by Kaczorowski and Perelli [9], the functions $F \in S_1$ are of the forms $F(s) = \zeta(s)$ or $F(s) = L(s + i\theta, \chi)$ with a primitive Dirichlet character $\chi$ and $\theta \in \mathbb{R}$. On the other hand, we denote by $S^\#_d$ the subclass of functions of degree $d$ in $S$. We note that the structure of $S^\#_d$ has been completely determined for $0 \leq d \leq 1$. From the work of Conrey and Ghosh [4], we have $S^\#_0 = \{1\}$ and $S^\#_d = \emptyset$ for $0 < d < 1$. For $d = 1$, by Kaczorowski and Perelli [9], the functions $F \in S^\#_1$ are of the forms $F(s) = \zeta(s) = L(s + i\theta, \chi)$ with a primitive Dirichlet character $\chi$ and $\theta \in \mathbb{R}$. On the other hand, we denote by $S^\#_d$ the extended Selberg class of functions satisfying conditions (1)–(3), and we define $S^\#_d$ similarly to $S_d$. Theorems 1 and 2 in [9] describe the structure of $S^\#_d$ for $0 \leq d \leq 1$. If $d = 0$, the functional equation is $Q^*F(s) = \omega Q^{1-s}F(1-s)$. The proof of [9, Theorem 1] shows that the Dirichlet series $F(s) = \sum_{n=1}^{\infty} a(n)/n^s \in S^\#_0$ is absolutely convergent in the whole complex plane. Thus, we have

$$\sum_{n=1}^{\infty} a(n) \left( \frac{Q^2}{n} \right)^s = \omega Q \sum_{n=1}^{\infty} \overline{a(n)} n^{-n^s}. $$

We let $q = Q^2$; then $a(n) = 0$ for $n \nmid q$. For $n \mid q$, we have

$$a(n) = \frac{\omega n}{\sqrt{q}} a \left( \frac{q}{n} \right).$$

**Theorem A** (Theorem 1 of [9]).

1. If $0 < d < 1$, then $S^\#_d = \emptyset$. If $F \in S^\#_0$, then $q \in \mathbb{N}$, the pair $(q, \omega)$ is an invariant of $F(s)$ and $S^\#_0$ is the disjoint union of the subclasses $S^\#_0(q, \omega)$ with $q \in \mathbb{N}$ and $|\omega| = 1$.

2. Every $F \in S^\#_0(q, \omega)$ with $q$ and $\omega$ as above is a Dirichlet polynomial of the form

$$F(s) = \sum_{n|q} \frac{a(n)}{n^s}. $$

For $d = 1$, we use the notation

$$\beta = \prod_{j=1}^{r} \lambda_j^{-2\lambda_j}, \quad \xi = 2 \sum_{j=1}^{r} (\mu_j - 1/2) = \eta + i\theta, \quad q = \frac{2\pi Q^2}{\beta}, $$

$$\omega^* = \omega e^{-i\pi(\eta + 1)/2} \left( \frac{Q^2}{\beta} \right)^{i\theta} \prod_{j=1}^{r} \lambda_j^{-2i \text{Im} \mu_j}. $$

If $\chi$ is a Dirichlet character modulo $q$, we denote by $f_\chi$ its conductor, and by $\chi^*$ the primitive character inducing $\chi$. We denote by $\omega_{\chi^*}$ and $Q_{\chi^*}$ the $\omega$-factor and the $Q$-factor in the standard functional equation for $L(s, \chi^*)$, i.e., $\omega_{\chi^*} = \tau(\chi^*)/a \sqrt{f_\chi}$, where $\tau(\chi^*)$ is the Gauss sum, $a = 0$ if $\chi(-1) = 1$.
and \(a = 1\) if \(\chi(-1) = -1\), and \(Q_{\chi^*} = \sqrt{f_{\chi}/\pi}\). Moreover, we write
\[
\mathcal{X}(q, \xi) = \left\{ \chi \mod q \mid \chi(-1) = 1 \right\} \quad \text{if } \eta = -1,
\]
\[
\left\{ \chi \mod q \mid \chi(-1) = -1 \right\} \quad \text{if } \eta = 0.
\]
\(\chi_0\) denotes the principal character modulo \(q\).

**Theorem B** (Theorem 2 of [9]).

1. *If \(F \in S_1^\#\), then \(q \in \mathbb{N}\) and \(\eta \in \{-1, 0\}\). The triple \((q, \xi, \omega^*)\) is an invariant of \(F(s)\), and \(S_1^\#\) is the disjoint union of the subclasses \(S_1^\#(q, \xi, \omega^*)\) with \(q \in \mathbb{N}\), \(\eta \in \{-1, 0\}\), \(\theta \in \mathbb{R}\) and \(|\omega^*| = 1\). Moreover, \(a(n) n^\theta\) is periodic with period \(q\).

2. *Every \(F \in S_1^\#(q, \xi, \omega^*)\) with \(q, \xi\) and \(\omega^*\) as above can be uniquely written as*
\[
F(s) = \sum_{\chi \in \mathcal{X}(q, \xi)} P_\chi(s + i\theta)L(s + i\theta, \chi^*),
\]

*where \(P_\chi \in S_0^\#(q/\chi, \omega^* \bar{\omega}_{\chi^*})\). Moreover, \(P_{\chi_0}(1) = 0\) if \(\theta \neq 0\).*

Bombieri and Hejhal [2] studied the distribution of zeros of the linear combinations \(F(s) = \sum_{j=1}^J b_j e^{i\alpha_j} L_j(s)\) of various \(L\)-functions with the same gamma factor. Assuming an orthonormality condition on \(a_j(p)\) (where \(a_j(n)\) are the coefficients of \(L_j(s)\)), the generalized Riemann hypothesis for \(L_j(s)\) and a weak condition on the spacing of zeros of \(L_j(s)\), they proved that almost all zeros of \(F(s)\) are simple and on the critical line \(\Re s = 1/2\). Hejhal [6] studied the behavior of zeros of \(F(s)\) near the critical line and announced that the true order of the number of zeros of \(F(s)\) in \(\Re s \geq \sigma\), \(T \leq \Im s \leq T + H\) is
\[
H \frac{1}{(\sigma - 1/2)\sqrt{\log \log T}}
\]
for \(1/2 + (\log \log T)\kappa/\log T \leq \sigma \leq 1/2 + (\log T)^{-\delta}\), \(c_1 T^w \leq H \leq c_2 T\), \(\kappa > 2\) with possibly few exceptional \(\{b_j\}_{j=1}^J\). Note that this result for the special case \(J = 2\) was also justified by the same author in [5].

Recently, the second author [11] investigated the off-line zeros of the Epstein zeta function \(E(s, Q)\) associated to the quadratic form \(Q(x, y) = ax^2 + bxy + cy^2\), \(a > 0\), \(b^2 - 4ac < 0\), \(a, b, c \in \mathbb{Z}\). It is a classical example that belongs to the class \(S_2^\#\). We find the number of zeros \(N_E(\sigma_1, \sigma_2; 0, T)\) in the rectangular region \(\sigma_1 < \Re s < \sigma_2\), \(0 < \Im s < T\) to be \(c(\sigma_1, \sigma_2)T + o(T)\) for \(1/2 < \sigma_1 < \sigma_2\), which improves Voronin’s result \(N_E(\sigma_1, \sigma_2; 0, T) \gg T\) for \(1/2 < \sigma_1 < \sigma_2 < 1\) (see [14] or Chapter 7 of [10]) based on the joint distribution for Hecke \(L\)-functions. We observe that one can apply our method to degree one objects.
For $F \in S^\#$, Kaczorowski and Kulas [8] defined the density property to be $N_F(\sigma, T) = o(T)$ for every fixed $1/2 < \sigma < 1$. This property classifies the elements in $S^\#_1$. If $F \in S^\#_1$ has the density property, then $F(s + i\theta) = P(s)L(s, \chi)$ for certain real $\theta$, a Dirichlet polynomial $P \in S^\#_0$ and a primitive Dirichlet character $\chi$. Otherwise, $F(s + i\theta) = \sum_{j \leq J} P_j(s)L(s, \chi_j)$ for $J \geq 2$, $\theta \in \mathbb{R}$, Dirichlet polynomials $P_j \in S^\#_0$ and primitive inequivalent Dirichlet characters $\chi_j$. For $F \in S^\#_1$ violating the density property, they obtain $N_F(\sigma_1, \sigma_2; 0, T) \gg T$ for $1/2 < \sigma_1 < \sigma_2 < 1$. Saias and Weingartner [12] extend their method to the strip $1 < \Re s < 1 + \eta$ for some small $\eta > 0$ and achieve $N_F(\sigma_1, \sigma_2; 0, T) \gg T$ for $1/2 < \sigma_1 < \sigma_2 < 1 + \eta$. Our main purpose is to improve these results by obtaining an asymptotic formula for $N_F(\sigma_1, \sigma_2; 0, T)$.

By Theorems A and B, we can write the function $E(s + i\theta) \in S^\#_1$ as

$$E(s) = \sum_{j=1}^{J} h_j(p_1^{-s}, \ldots, p_k^{-s}) \prod_{p > p_k} \left(1 - \frac{\chi_j(p)}{p^s}\right)^{-1}$$

for some integer $k > 0$, where

$$h_j(x_1, \ldots, x_k) = \tilde{h}_j(x_1, \ldots, x_k) \prod_{l \leq k} (1 - \chi_j(p_l)x_l)^{-1}$$

and $\tilde{h}_j$ is a polynomial of $k$ variables. Let

$$E_n(s) = \sum_{j=1}^{J} h_j(p_1^{-s}, \ldots, p_k^{-s}) \prod_{p_k < p \leq p_n} \left(1 - \frac{\chi_j(p)}{p^s}\right)^{-1}$$

for $n > k$. Then, $E_n(s)$ converges in the mean with index 2 towards $E(s)$ in $[1/2, \infty]$ by Parseval’s identity for almost periodic functions, i.e.,

$$\limsup_{T \to \infty} \frac{1}{T} \int_{1}^{T} \left|E(\sigma + it) - E_n(\sigma + it)^2 d\sigma dt \to 0$$

as $n \to \infty$ for any $1/2 < \alpha < \beta$ (for the method of proof, see Proposition 2.3 of [11]). Applying Lemma 2.3 to $E_n(s)$, we get an asymptotic formula for $N_{E_n}(\sigma_1, \sigma_2; 0, T)$. The theory of mean motions partially preserves this property through the convergence in the mean with index $p > 0$ via Lemma 2.4.

If $J = 1$, then we encounter the Riemann hypothesis. Our method does not work in this case, since we are using the Euler product $\log \zeta(s) = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}}$ and this cannot give any information about $\zeta(s) = 0$. Concerning this matter, see Borchsenius and Jessen [3]. From now on, we only consider the case $J > 1$. 
Zeros of degree one L-functions

We consider \( S_1^\#(p, \xi, \omega^*) \) for \( p \) prime or 1. By (1.1) and Theorem B, we have

\[
\tilde{h}_j = a_j(1) \text{ or } a_j(1) + \frac{\omega a_j(1)}{p^{s-1/2}},
\]

and as a result \( \tilde{h}_j \neq 0 \) for \( \Re s > 1/2 \). In this case, the method in \([1]\) works, and we have the following theorem.

**Theorem 1.1.** Let \( E(s + i\theta) \in S_1^\#(p, \xi, \omega^*) \) for \( p \) prime or \( p = 1 \), and \( |\omega| = 1 \), and let \( 1/2 < \sigma_1 < \sigma_2 \). Suppose \( J > 1 \) in (1.2). Then

\[
N_E(\sigma_1, \sigma_2; 0, T) = c(\sigma_1, \sigma_2)T + o(T)
\]
as \( T \to \infty \). The constant \( c(\sigma_1, \sigma_2) \) can be represented as an integral \( \int_{\sigma_1}^{\sigma_2} H_\sigma(0) \, d\sigma \) for the density function \( H_\sigma(x) \) of some distribution \( \mu_\sigma \), and \( c(\sigma_1, \sigma_2) > 0 \) if \( 1/2 < \sigma_1 \leq 1 \). In particular, for \( \sigma_0 > 1/2 \), the number of zeros on the line segment \( \Re s = \sigma_0, 0 < \Im s < T \) is \( o(T) \).

When \( q \) is a prime power, the \( \tilde{h}_j \) are polynomials of the same single variable by Theorems A(2) and B(2). If these polynomials have the same factor with \( cT + o(T) \) zeros on the line segment \( \Re s = \sigma_0, 0 < \Im s < T \) for some \( 1/2 < \sigma_0 < 1 \), then we cannot expect the integral form of the constant \( c(\sigma_1, \sigma_2) \) in general. Indeed, we may take \( \tilde{h}_j(p^{-s}) = 1 + 2p^{3/4-s} + p^{1-2s} \) by letting \( \omega = a(1) = 1 \), and \( a(p) = 2p^{-3/4} \). Then the function \( s \mapsto \tilde{h}_j(p^{-s}) \) has \( \frac{\log pT}{2\pi} + O(1) \) zeros on \( \Re s = \log(p^{3/4} + \sqrt{p^{3/2} - p})/\log p, 0 < \Im s < T \). We still have the following.

**Theorem 1.2.** Let \( E(s + i\theta) \in S_1^\#(q, \xi, \omega^*) \) for \( q \) a prime power, and let \( 1/2 < \sigma_1 < \sigma_2 \). Suppose \( J > 1 \) in (1.2). Then

\[
N_E(\sigma_1, \sigma_2; 0, T) = c(\sigma_1, \sigma_2)T + o(T)
\]
as \( T \to \infty \), and \( c(\sigma_1, \sigma_2) > 0 \) if \( 1/2 < \sigma_1 \leq 1 \). Suppose that the closed interval \([\sigma_1, \sigma_2]\) does not contain the real part of exceptional points satisfying \( h_j = 0 \). Then the constant \( c(\sigma_1, \sigma_2) \) can be represented as an integral \( \int_{\sigma_1}^{\sigma_2} H_\sigma(0) \, d\sigma \) for the density function \( H_\sigma(x) \) of some distribution \( \mu_\sigma \). In this case for \( \sigma_0 \in [\sigma_1, \sigma_2] \), the number of zeros on the line segment \( \Re s = \sigma_0, 0 < \Im s < T \) is \( o(T) \).

For general \( q \), we could also prove a similar theorem, although it is not easy to classify the common zeros of \( \tilde{h}_j \) with multiple variables. We will discuss and prove a general theorem in Section 3.

**2. Lemmas.** We begin with the work of Jessen and Tornhave \([7]\) that concerns zeros of a Dirichlet series in the region of its absolute convergence. For the basic theory of almost periodic functions, we refer to \([1]\).
Lemma 2.1 (Theorem 8 of [7]). A function \( f(s) \) almost periodic in \([\alpha, \beta]\) and not identically zero has no zeros in the substrip \((\alpha \leq \alpha_0 < \beta, \beta_0 \leq \beta)\), if and only if its Jensen function

\[
\varphi(\sigma) = \lim_{T_2-T_1 \to \infty} \frac{1}{T_2-T_1} \int_{T_1}^{T_2} \log |f(\sigma + it)| \, dt
\]

is linear in the interval \((\alpha_0, \beta_0)\).

Lemma 2.2 (Theorem 31 of [7]). For an ordinary Dirichlet series

\[
f(s) = \sum_{n=n_0}^{\infty} \frac{a_n}{n^s}, \quad a_{n_0} \neq 0,
\]

with the uniform convergence abscissa \(\alpha\), the Jensen function \(\varphi(\sigma)\) has on every half-line \(\sigma > \alpha_1 \) \((> \alpha)\) only a finite number of linearity intervals and a finite number of points of non-differentiability. The values of \(\varphi'(\sigma)\) in the linearity intervals belong to the set of numbers \(-\log n, n \geq n_0\). For \(\sigma > \) (some) \(\sigma_0\), we have

\[
\varphi(\sigma) = -(\log n_0)\sigma + \log |a_{n_0}|.
\]

For an arbitrary strip \((\sigma_1, \sigma_2)\), where \(\alpha < \sigma_1 < \sigma_2 < \infty\), the relative frequency \(H(\sigma_1, \sigma_2)\) of zeros exists and is determined by

\[
H(\sigma_1, \sigma_2) = \frac{1}{2\pi} (\varphi'(\sigma_2-) - \varphi'(\sigma_1+)).
\]

The following lemma guarantees the existence of the second derivative of Jensen functions for almost periodic functions and gives another representation by a certain distribution. The proof can be found in §9 of [3].

Lemma 2.3 (Proposition 2.1 of [11]). Let \(f(s)\) be almost periodic in the strip \([\alpha, \beta]\) and not identically zero. Let \(\nu_\sigma\) be the asymptotic distribution function of \(f(\sigma + it)\) with respect to \(|f'(\sigma + it)|^2\). Suppose \(\nu_\sigma\) is absolutely continuous for every \(\sigma\) and its density \(G_\sigma(x)\) is a continuous function of \(x\) and \(\sigma\). Then the Jensen function \(\varphi_{f-x}(\sigma)\) is twice differentiable with

\[
\varphi''_{f-x}(\sigma) = 2\pi G_\sigma(x).
\]

The next lemma is an extension of Lemma 2.3 which is applicable inside the critical strip and which plays the main role in this method.

Lemma 2.4 (Theorem 1 of [3]). Let \(-\infty \leq \alpha < \alpha_0 < \beta_0 < \beta \leq \infty\) and let \(f_1(s), f_2(s), \ldots\) be a sequence of functions almost periodic in \([\alpha, \beta]\) converging uniformly in \([\alpha_0, \beta_0]\) towards a function \(f(s)\). Suppose that none of the functions is identically zero and \(f(s)\) may be continued as a regular function in the half-strip \(\alpha < \sigma < \beta, t > \gamma_0\), and that \(f_n(s)\) converges in
mean with an index \( p > 0 \) towards \( f(s) \) in \( [\alpha, \beta] \). Then the Jensen function
\[
\varphi_f(\sigma) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \log |f(\sigma + it)| \, dt
\]
eexists uniformly in \( [\alpha, \beta] \) for some \( \gamma > \gamma_0 \), and \( \varphi_f(n) \) converges uniformly in \( [\alpha, \beta] \) towards \( \varphi_f(\sigma) \) as \( n \to \infty \). The function \( \varphi_f(\sigma) \) is convex in \( (\alpha, \beta) \), and for every strip \( (\sigma_1, \sigma_2) \) where \( \alpha < \sigma_1 < \sigma_2 < \beta \), the two relative frequencies of zeros defined by
\[
\mathcal{H}_f(\sigma_1, \sigma_2) = \lim_{T \to \infty} \frac{1}{T} \sum_{n=1}^T \varphi'_f(\sigma) \, d\sigma + o(T)
\]
satisfy the inequalities
\[
\frac{1}{2\pi} (\varphi'_f(\sigma_2) - \varphi'_f(\sigma_1)) \leq \mathcal{H}_f(\sigma_1, \sigma_2) \leq \mathcal{H}_f(\sigma_1, \sigma_2)
\]
\[
\leq \frac{1}{2\pi} (\varphi'_f(\sigma_2) + \varphi'_f(\sigma_1)).
\]
Suppose further that \( \varphi_f(\sigma) \) is twice differentiable. Then
\[
N_f(\sigma_1, \sigma_2; 0, T) = T \int_{\sigma_1}^{\sigma_2} \varphi''_f(\sigma) \, d\sigma + o(T)
\]
for \( \alpha < \sigma_1 < \sigma_2 < \beta \) as \( T \to \infty \).

Together with the above lemmas, we investigate the Fourier transforms of certain distributions. We need two more lemmas, in which we use the following notation:
\[
\mathcal{L}_n(\sigma, \Theta; \chi_j) = L_k(\sigma, \vartheta; \chi_j) L_{k,n}(\sigma, \vartheta; \chi_j),
\]
\[
L_k(\sigma, \vartheta; \chi_j) = h_j(p_1^{-\sigma} e^{2\pi i \vartheta_1}, \ldots, p_k^{-\sigma} e^{2\pi i \vartheta_k}),
\]
\[
L_{k,n}(\sigma, \vartheta; \chi_j) = \prod_{k < l \leq n} \left( 1 - \frac{\chi_j(p_1 e^{2\pi i \vartheta_1})}{p_l^{\sigma}} \right)^{-1},
\]
\[
M_{n,\sigma}(\vartheta) = (\log L_k, n(\sigma, \vartheta; \chi_1), \ldots, \log L_{k,n}(\sigma, \vartheta; \chi_j)),
\]
\[
E_{n,\sigma}(\Theta) = \sum_{j=1}^J \mathcal{L}_n(\sigma, \Theta; \chi_j)
\]
for \( n > k \), \( \Theta = (\vartheta_1, \vartheta) \in [0, 1]^n \), \( \vartheta = (\vartheta_1, \ldots, \vartheta_k) \in [0, 1]^k \) and \( \vartheta = (\vartheta_{k+1}, \ldots, \vartheta_n) \in [0, 1]^{n-k} \). Let \( \mu_{n,\sigma} \) be the distribution function of \( E_{n,\sigma} \) with respect to \( |\vartheta| \, E_{n,\sigma}|^2 \). Its Fourier transform is
\[
\hat{\mu}_{n,\sigma}(y) = \int_{[0,1]^n} e^{i \sum_j \mathcal{L}_n(\sigma, \Theta; \chi_j) \cdot y} \left| \sum_{j=1}^J \mathcal{L}_n'(\sigma, \Theta; \chi_j) \right|^2 d\Theta.
\]
LEMMA 2.5. For σ > 1/2, δ > 0 and j ≤ J, define

\[ A_{j,\sigma}(\delta) = \{ \theta \in [0, 1]^k : |\hat{\nu}_j(p_1^{-\sigma} e^{2\pi i \theta_1}, \ldots, p_k^{-\sigma} e^{2\pi i \theta_k})| < \delta \} \]

Then for any integer K ≤ J we have

\[ \hat{\mu}_{n,\sigma}(y) \ll \bigcap_{r_1 < \cdots < r_K \leq J} (A_{r_1,\sigma}(\delta) \cup \cdots \cup A_{r_K,\sigma}(\delta)) + |\delta y|^{-K} \]

as \(|y| \to \infty\), where the corresponding constant does not depend on n.

Proof. We write

\[ \hat{\mu}_{n,\sigma}(y) = \sum_{l_1,l_2[0,1]^n} e^{i \sum_j L_n(\sigma,\theta_j)\cdot y} \frac{L'_{n,\sigma}(\sigma,\theta;\chi_{l_1})}{L_{n,\sigma}(\sigma,\theta;\chi_{l_2})} d\theta. \]

Define set functions

\[ \lambda_{n,\sigma;l_1,l_2}(B) = \left\{ \frac{L'_{k,n}(\sigma,\theta;\chi_{l_1})}{L_{k,n}(\sigma,\theta;\chi_{l_2})} \right\} \bigcap_{M_{n,\sigma}(B)} \]

\[ \lambda_{n,\sigma;l}(B) = \left\{ \frac{L'_{k,n}(\sigma,\theta;\chi_{l_1})}{L_{k,n}(\sigma,\theta;\chi_{l_2})} \right\} \bigcap_{M_{n,\sigma}(B)} \]

\[ \lambda_{n,\sigma}(B) = \big| M_{n,\sigma}^{-1}(B) \big|, \]

for any Borel set \( B \subset \mathbb{C}^J \). Applying the identity

\[ ab = \frac{1}{4} \sum_{m=1}^{4} i^m |a + i^m b|^2, \quad a, b \in \mathbb{C}, \]

one can prove that \( \hat{\mu}_{n,\sigma}(y) \) is a linear combination of at most four absolutely continuous distribution functions. (See [11] for details.) We denote by \( G_{n,\sigma;l_1,l_2}(x) \), \( G_{n,\sigma;l}(x) \), \( G_{n,\sigma}(x) \) the densities of \( \lambda_{n,\sigma;l_1,l_2}, \lambda_{n,\sigma;l}, \lambda_{n,\sigma} \), respectively. By Theorem 6 of [3], all these densities have majorants of the form \( K e^{-\lambda|x|^2} \), and their partial derivatives of order \( \leq q \) have majorants of the form \( K_q e^{-\lambda|x|^2} \) for \( n \geq n_q \). Thus,

\[ \hat{\mu}_{n,\sigma}(y) = \sum_{l_1,l_2 [0,1]^n} e^{i \sum_j (L_k(\sigma,\theta;\chi_j) e^{x_j}) \cdot y + x_{l_1} + x_{l_2} \mathcal{G}_{n,\sigma;l_1,l_2}(x,\theta)} dx d\theta, \]

where

\[ \mathcal{G}_{n,\sigma;l_1,l_2}(x,\theta) \]

\[ = L_k'(\sigma,\theta;\chi_{l_1})L_k'(\sigma,\theta;\chi_{l_2})G_{n,\sigma}(x) + L_k(\sigma,\theta;\chi_{l_1})L_k(\sigma,\theta;\chi_{l_2})G_{n,\sigma;l_1,l_2}(x) \]

\[ + L_k(\sigma,\theta;\chi_{l_1})L_k'(\sigma,\theta;\chi_{l_2})G_{n,\sigma;l}(x) + L_k(\sigma,\theta;\chi_{l_1})L_k'(\sigma,\theta;\chi_{l_2})G_{n,\sigma;l_1,l_2}(x). \]

We only consider the first term \( L_k'(\sigma,\theta;\chi_{l_1})L_k'(\sigma,\theta;\chi_{l_2})G_{n,\sigma}(x) \), since the
others can be treated similarly. If \( \theta \notin A_{j,\sigma}(\delta) \) for \( K \)-many \( j \), we will prove
\[
(2.1) \quad \int_{\mathbb{C}^j} e^{i \sum_j (L_k(\sigma,\theta;\chi_j)e^{x_j})y+x_1+x_2} G_{n,\sigma}(x) \, dx = O(|\delta y|^{-K}).
\]
For the other \( \theta \), we give a trivial upper bound by the measure of the set of those \( \theta \):
\[
\hat{\mu}_{n,\sigma}(y) \ll \left| \bigcap_{r_1<\cdots<r_K\leq j} (A_{r_1,\sigma}(\delta) \cup \cdots \cup A_{r_K,\sigma}(\delta)) \right| + |\delta y|^{-K},
\]
where the corresponding constant does not depend on \( n \) as \( y \to \infty \).

So, it is enough to prove (2.1). We decompose
\[
\int_{\mathbb{C}^j} e^{i \sum_j (L_k(\sigma,\theta;\chi_j)e^{x_j})y+x_1+x_2} G_{n,\sigma}(x) \, dx
\]
\[
= \sum_{m \in \mathbb{Z}^j} \int_{(\mathbb{R} \times [0,2\pi])^j} e^{i \sum_j r_je^{x_j} \cdot (L_k(\sigma,\theta;\chi_j)y)+x_1+x_2} G_{n,\sigma}(x+2\pi mi) \, dx.
\]
Changing variables \( e^{x_j} = r_je^{z_j} \) with Jacobian \( r_j^{-1} \) shows that the above equals
\[
\sum_{m \in \mathbb{Z}^j} \int_{[0,2\pi]^j} \int_{(0,\infty)^j} e^{i \sum_j r_je^{z_j} \cdot (L_k(\sigma,\theta;\chi_j)y)+z_1-z_2r_1r_2} \prod_j r_j^{-1} G_{n,\sigma}(\log r + i(z + 2\pi m)) \, dr \, dz,
\]
where \( r = (r_1,\ldots,r_J) \), \( z = (z_1,\ldots,z_J) \), and \( \log r = (\log r_1,\ldots,\log r_J) \).

Consider the integral
\[
\int_{0}^{2\pi} \int_{0}^{\infty} e^{ir_je^{z_j} \cdot (L_k(\sigma,\theta;\chi_j)y)+z_1-z_2r_1r_2} r_j^{-1} G_{n,\sigma}(\log r + i(z + 2\pi m)) \, dr \, dz
\]
\[
= \int_{0}^{2\pi} \int_{0}^{\infty} e^{ir_je^{z_j} |\cos(z_j-\alpha_j)|+z_1-z_2r_1r_2} r_j^{-1} G_{n,\sigma}(\log r + i(z + 2\pi m)) \, dr \, dz
\]
for some \( \alpha_j \). For \( \theta \notin A_{j,\sigma}(\delta) \), we integrate by parts with respect to \( z_j \) for \( |\cos(z_j-\alpha_j)| < 1/2 \), and with respect to \( r_j \) for \( |\cos(z_j-\alpha_j)| > 1/2 \). With the uniform upper bound \( K_q e^{-\lambda|x|^2} \) of partial derivatives of \( G \) of order \( \leq q \), we obtain (2.1). \( \blacksquare \)

**Lemma 2.6.** \( \hat{\mu}_{n,\sigma}(y) \) converges uniformly for every disc \(|y| \leq a \) and \( 1/2 < \sigma_1 \leq \sigma \leq \sigma_2 \).

**Proof.** By definition, we have
\[
\hat{\mu}_{n+1,\sigma}(y) = \int_{[0,1]^n} \left| \frac{\partial}{\partial \sigma} E_{n+1,\sigma}(\Theta, u) \right|^2 du d\Theta.
\]
We get
\[\int_0^1 e^{iE_{n+1,\sigma}(\Theta, u)} \left| \frac{\partial}{\partial \sigma} E_{n+1,\sigma}(\Theta, u) \right|^2 du \]
\[= \int_0^1 e^{iE_{n+1,\sigma}(\Theta, u)} y \left| \frac{\partial}{\partial \sigma} E_{n,\sigma}(\Theta) \right|^2 du + \int_0^1 e^{iE_{n+1,\sigma}(\Theta, u)} y \times 2 \Re \left[ \frac{\partial}{\partial \sigma} E_{n,\sigma}(\Theta) e^{2\pi i u} \right] \times \left| \frac{\partial}{\partial \sigma} \sum_{j=1}^J h_j(\ldots) \prod_{k<j \leq n} (\ldots)^{-1} \chi_j(p_{n+1}) \right| du \]
\[+ O\left( \frac{F_n(\sigma, \Theta)^2}{p_{n+1}^2} \right),\]
where
\[F_n(\sigma, \Theta) = \sum_{j=1}^J \prod_{k<l \leq n} \left| 1 - \frac{\chi_j(p_l)}{p_l^\sigma} e^{2\pi i \vartheta_l} \right|^{-1}.\]

As \(e^{ix} = 1 + ix + O(|x|^2)\) \((x \in \mathbb{R})\), we have
\[\int_0^1 e^{iE_{n+1,\sigma}(\Theta, u)} y du \]
\[= \int_0^1 e^{iE_{n,\sigma}(\Theta)} y (1 + i(E_{n+1,\sigma}(\Theta, u) - E_{n,\sigma}(\Theta)) \cdot y) du + O\left( \frac{F_n(\sigma, \Theta)^2}{p_{n+1}^2} \right)\]
\[= e^{iE_{n,\sigma}(\Theta)} y + O\left( \frac{F_n(\sigma, \Theta)^2}{p_{n+1}^2} \right).\]

Since \(e^{ix} = 1 + O(|x|)\) \((x \in \mathbb{R})\), we have
\[\int_0^1 e^{iE_{n+1,\sigma}(\Theta, u)} y \pm 2\pi i u du = \int_0^1 e^{iE_{n,\sigma}(\Theta)} y \pm 2\pi i u du + O\left( \frac{F_n(\sigma, \Theta)}{p_{n+1}^\sigma} \right)\]
\[= O\left( \frac{F_n(\sigma, \Theta)}{p_{n+1}^\sigma} \right).\]

Combining the above equalities yields
\[\int_0^1 e^{iE_{n+1,\sigma}(\Theta, u)} y \left| \frac{\partial}{\partial \sigma} E_{n+1,\sigma}(\Theta, u) \right|^2 du = e^{iE_{n,\sigma}(\Theta)} y \left| \frac{\partial}{\partial \sigma} E_{n,\sigma}(\Theta) \right|^2 \]
\[+ O\left( \frac{F_n(\sigma, \Theta)^2 + F_n(\sigma, \Theta)^3 + F_n(\sigma, \Theta)^4}{p_{n+1}^2 \log p_{n+1}} \right).\]

Thus, we have
\[\hat{\mu}_{n+1,\sigma}(y) - \hat{\mu}_{n,\sigma}(y) = O(p_{n+1}^{-2\sigma} \log p_{n+1})\]
and
\[ \mu_{m,\sigma}(y) - \mu_{n,\sigma}(y) = O(p_n^{1-2\sigma}) \]
for \( m > n > k \). Hence, Lemma 2.6 follows. ■

3. Main results. We consider separately the cases \( J = 2 \) and \( J \geq 3 \). For \( J = 2 \), our function is the sum of two spoiled Euler products \( f_1(s) + f_2(s) \). We then apply the theory of value distribution for \( f_1(s) \) and \( f_2(s) \).

**Proposition 3.1.** Let \( J = 2 \) and \( 1/2 < \sigma_1 < \sigma_2 \). Suppose that \( h_j(p_1^{-\sigma}e^{2\pi i \theta_1}, \ldots, p_k^{-\sigma}e^{2\pi i \theta_k}) \neq 0 \) for \( j = 1, 2 \), \( \sigma_1 \leq \sigma \leq \sigma_2 \), and \( \theta \in [0, 1]^k \). Then
\[ N_E(\sigma_1, \sigma_2; 0, T) = T \int_{\sigma_1}^{\sigma_2} H_\sigma(-1) d\sigma + o(T), \]
where \( H_\sigma(x) \) is the density of some distribution function \( \mu_\sigma \). Moreover, \( H_\sigma(x) > 0 \) for \( 1/2 < \sigma \leq 1 \).

**Proof.** By Lemma 2.4, \( \varphi_{E_n}(\sigma) \) converges uniformly to \( \varphi_E(\sigma) \) on \([1/2, \infty)\). If \( \varphi_E(\sigma) \) is twice differentiable, then
\[ N_E(\sigma_1, \sigma_2; 0, T) = \frac{T}{2\pi} \int_{\sigma_1}^{\sigma_2} \varphi_E''(\sigma) d\sigma + o(T). \]
By direct calculation,
\[ \varphi_{E_n}(\sigma) = \varphi_{h_2}(\sigma) + \varphi_{\tilde{L}_{n+1}}(\sigma), \]
where
\[ \tilde{L}_{n}(s) = \frac{h_1}{h_2} (p_1^{-s}, \ldots, p_k^{-s}) \prod_{p_k < p \leq p_n} \frac{1 - \chi_2(p)/p^s}{1 - \chi_1(p)/p^s}. \]
By Lemma 2.1, we have \( \varphi_{h_2}''(\sigma) = 0 \) for \( \sigma_1 \leq \sigma \leq \sigma_2 \). For \( \tilde{L}_n \), the method in Chapter II of [3] works. Define
\[ \tilde{L}_{n,\sigma}(\Theta) = \frac{h_1}{h_2} (p_1^{-\sigma}e^{2\pi i \theta_1}, \ldots, p_k^{-\sigma}e^{2\pi i \theta_k}) \prod_{k < l \leq n} \frac{1 - \chi_2(p_l)e^{2\pi i \theta_l}/p_l^\sigma}{1 - \chi_1(p_l)e^{2\pi i \theta_l}/p_l^\sigma}, \]
\[ \mu_{n,\sigma}(B) = \int_{\tilde{L}_{n,\sigma}(\Theta)} \left| \frac{\partial}{\partial \sigma} \tilde{L}_{n,\sigma}(\Theta) \right|^2 d\Theta \]
for any Borel set \( B \subset \mathbb{C} \) and \( n > k \), \( \Theta = (\theta_1, \ldots, \theta_k, \theta_{k+1}, \ldots, \theta_n) \in [0, 1]^n \). Applying Theorems 5–10 in [3] with some modifications, we deduce that the absolutely continuous distributions \( \mu_{n,\sigma} \) converge to a distribution \( \mu_\sigma \) with a density \( H_\sigma(x) \) and \( \varphi_{\tilde{L}_{n+1}}''(\sigma) = 2\pi H_\sigma(-1) > 0 \) for \( 1/2 < \sigma \leq 1 \).

For the case \( J \geq 3 \), we cannot do the same thing as for \( J = 2 \). However, by the method of [11], we obtain the following.
Proposition 3.2. Let \( J \geq 3 \) and \( 1/2 < \sigma_1 < \sigma_2 \). Suppose that 
\[
h_j(p_1^{-\sigma}e^{2\pi i \theta_1}, \ldots, p_k^{-\sigma}e^{2\pi i \theta_k}) \neq 0
\]
for \( j = l_1, l_2, l_3, \sigma_1 \leq \sigma \leq \sigma_2 \), and \( \theta \in [0, 1]^k \). Then
\[
N_E(\sigma_1, \sigma_2; 0, T) = T \int_{\sigma_1}^{\sigma_2} H_\sigma(0) \, d\sigma + o(T),
\]
where \( H_\sigma(x) \) is the density of some distribution function \( \mu_\sigma \).

Proof. By Lemma 2.4, \( \varphi_{E_n}^\sigma(\sigma) \) converges uniformly to \( \varphi_E(\sigma) \) on \([1/2, \infty)\). If \( \varphi_E(\sigma) \) is twice differentiable, then
\[
N_E(\sigma_1, \sigma_2; 0, T) = \frac{T}{2\pi} \int_{\sigma_1}^{\sigma_2} \varphi_E''(\sigma) \, d\sigma + o(T).
\]
By Lemma 2.5 with
\[
\delta = \min\{|h_j(p_1^{-\sigma}e^{2\pi i \theta_1}, \ldots, p_k^{-\sigma}e^{2\pi i \theta_k})| \mid j = l_1, l_2, l_3, \sigma_1 \leq \sigma \leq \sigma_2, \theta \in [0, 1]^k\} > 0,
\]
we have \( \hat{\mu}_{n,\sigma}(y) \ll |y|^{-3} \) and this implies that \( \mu_{n,\sigma} \) is absolutely continuous and its density \( H_{n,\sigma}(x) \) is continuous. Let \( \nu_{n,\sigma} \) be the asymptotic distribution of \( E_n(\sigma + it) \) with respect to \( |E_n(\sigma + it)|^2 \). Since \( \hat{\mu}_{n,\sigma}(y) = \hat{\nu}_{n,\sigma}(y) \) by Kronecker’s theorem, \( \mu_{n,\sigma} = \nu_{n,\sigma} \) and \( H_{n,\sigma} \) is their common density. By Lemma 2.3, \( \varphi_{E_n-x}''(\sigma) = 2\pi H_{n,\sigma}(x) \). By Lemma 2.6, \( H_{n,\sigma}(x) \) converges to \( H_\sigma(x) \) which is the density of some distribution \( \mu_\sigma = \lim_{n \to \infty} \mu_{n,\sigma} \). Therefore,
\[
N_E(\sigma_1, \sigma_2; 0, T) = T \int_{\sigma_1}^{\sigma_2} H_\sigma(0) \, d\sigma + o(T). \quad \blacksquare
\]

By Lemma 2.2, each Dirichlet polynomial \( h_j(p_1^{-s}, \ldots, p_k^{-s}) \) has at most a finite number of linearity intervals of its Jensen function \( \varphi_{h_j}(\sigma) \) in \([1/2, \infty)\). Let \( \mathcal{J}_j \) be the union of those intervals. By Lemmas 2.3 and 2.4 and almost periodicity, \( h_j \) has no zero in \( \mathcal{J}_j \). We let \( \zeta_j = \inf \mathcal{J}_j \geq 1/2 \), and \( \zeta_E \) be the third smallest \( \zeta_j \), more precisely, \( \zeta_E = \zeta_3 \) when \( \zeta_{l_1} \leq \zeta_{l_2} \leq \zeta_{l_3} \leq \cdots \) is the linear order of \( \zeta_1, \ldots, \zeta_J \). By combining Lemma 2.4 and Proposition 3.2, we obtain the following theorem.

Theorem 3.3. Let \( J \geq 3 \) and \( \zeta_E < \sigma_1 < \sigma_2 \). Suppose that \( \sigma_1, \sigma_2 \in \mathcal{J}_j \) for at least three \( j \). Then
\[
N_E(\sigma_1, \sigma_2; 0, T) = \frac{T}{2\pi} (\varphi_E'(\sigma_2) - \varphi_E'(\sigma_1)) + o(T).
\]
Suppose further that \([\sigma_1, \sigma_2] \subset \mathcal{J}_j \) for at least three \( j \). Then
\[
N_E(\sigma_1, \sigma_2; 0, T) = T \int_{\sigma_1}^{\sigma_2} H_\sigma(0) \, d\sigma + o(T),
\]
where $H_\sigma(x)$ is the density of some distribution $\mu_\sigma$. In this case, for $\sigma_1 < \sigma_0 < \sigma_2$, the number of zeros of $E(s)$ on the line segment $\Re s = \sigma_0$, $0 < \Im s < T$ is $\mathcal{O}(T)$.

If each $\tilde{h}_j$ is non-vanishing on $\Re s > 1/2$, the conclusion of Theorem 3.3 holds.

**Theorem 3.4.** Let $J \geq 3$ and $1/2 < \sigma_1 < \sigma_2$. Suppose that $\tilde{h}_j \neq 0$ for $\Re s > 1/2$. Then

\[
N_E(\sigma_1, \sigma_2; 0, T) = T \int_{\sigma_1}^{\sigma_2} H_\sigma(0) d\sigma + \mathcal{O}(T),
\]

where $H_\sigma(x)$ is the density of a distribution $\mu_\sigma$. For $\sigma_0 > 1/2$, the number of zeros of $E(s)$ on the line segment $\Re s = \sigma_0$ and $0 < \Im s < T$ is $\mathcal{O}(T)$.

As a consequence, we obtain Theorem 1.1.

We now consider the case when $\tilde{h}_j$ is a one-variable polynomial. Then it has only finitely many solutions, say $\beta_1, \ldots, \beta_m \in \mathbb{C}$. So $\tilde{h}_j(p^{-s}) = 0$ if and only if $p^{-s} = \beta_i$ for some $i$. Thus, each line segment $\Re s = -\log |\beta_j|/\log p$, $0 < \Im s < T$ contains $cT + O(1)$ zeros of $\tilde{h}_j(p^{-s})$. So we may not have the equation (3.1) for $E(s)$. However, if we disregard these exceptional points, we obtain the following theorem.

**Theorem 3.5.** Let $J \geq 3$ and $1/2 < \sigma_1 < \sigma_2$. Let

\[
E(s) = \sum_{j \leq J} \tilde{h}_j(p_1^{-s}, \ldots, p_k^{-s})L(s, \chi_j),
\]

where each $\tilde{h}_j$ is a polynomial of one variable. Then

\[
N_E(\sigma_1, \sigma_2; 0, T) = \frac{T}{2\pi} (\varphi'_E(\sigma_2) - \varphi'_E(\sigma_1)) + \mathcal{O}(T).
\]

Suppose $\mathcal{I} = \bigcup_{l_1 < l_2 < l_3 \leq J} (I_{l_1} \cap I_{l_2} \cap I_{l_3})$ is $(1/2, \infty)$ minus finitely many points. If $[\sigma_1, \sigma_2] \subset \mathcal{I}$, then

\[
N_E(\sigma_1, \sigma_2; 0, T) = T \int_{\sigma_1}^{\sigma_2} H_\sigma(0) d\sigma + \mathcal{O}(T),
\]

where $H_\sigma(x)$ is the density of some distribution $\mu_\sigma$. For $\sigma_0 \in \mathcal{I}$, the number of zeros of $E(s)$ on the line segment $\Re s = \sigma_0$, $0 < \Im s < T$ is $\mathcal{O}(T)$.

As a consequence, we obtain Theorem 1.2.

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