The zeros of the derivative of the Riemann zeta function near the critical line

Haseo Ki*

Department of Mathematics, Yonsei University, Seoul 120–749, Korea

haseo@yonsei.ac.kr

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Abstract

We study the horizontal distribution of zeros of $\zeta'(s)$ which are denoted as $\rho' = \beta' + i\gamma'$. We assume the Riemann hypothesis which implies $\beta' \geqslant 1/2$ for any non-real zero ρ', equality being possible only at a multiple zero of $\zeta(s)$. In this paper we prove that $\liminf\, (\beta' - 1/2)\log\gamma' \neq 0$ if and only if for any $c > 0$ and $s = \sigma + it$ with $|\sigma - 1/2| < c/\log t$ $(t \geqslant 10)$

$$\frac{\zeta'}{\zeta}(s) = \frac{1}{s-\rho} + O(\log t),$$

where $\rho = 1/2 + i\gamma$ is the closest zero of $\zeta(s)$ to s and the origin. We also show that if $\liminf\, (\beta' - 1/2)\log\gamma' \neq 0$, then for any $c > 0$ and $s = \sigma + it$ $(t \geqslant 10)$, we have

$$\log\zeta(s) = O\left(\frac{(\log t)^{2-2\sigma}}{\log\log t}\right)$$

uniformly for $1/2 + c/\log t \leqslant \sigma \leqslant \sigma_1 < 1$.

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1 Introduction

The Riemann hypothesis (RH) states that the real part of any nonreal zero of the Riemann zeta function $\zeta(s)$ is $1/2$. A. Speiser [14] has a theorem which says that RH is equivalent to the nonexistence of nonreal zeros of $\zeta'(s)$ in Re $(s) < 1/2$.

We let $\rho' = \beta' + i\gamma'$ denote a zero of $\zeta'(s)$ where a sum over ρ' is repeated according to multiplicity. N. Levinson and H. L. Montgomery [9] proved that for nonreal zeros of $\zeta'(s)$ the average value of β' with $0 < \gamma' \leqslant T$ is $1/2 + \log\log T/\log T$. But it is likely that $\beta' - 1/2$ is usually of the order $1/\log T$ rather than $\log\log T/\log T$.

By a result of B. C. Berndt [1], the number of zeros with ordinates less than T is

$$\sum_{0<\gamma'\leqslant T} 1 = \frac{T}{2\pi}\log\frac{T}{4\pi e} + O(\log T).$$

Under RH, K. Soundararajan [13] demonstrated the presence of a positive proportion of zeros of $\zeta'(s)$ in the region $\sigma < 1/2 + \nu/\log T$ for all $\nu \geqslant 2.6$. 'A positive proportion of zeros' means

$$\liminf_{T\to\infty} \frac{1}{\frac{T}{2\pi}\log T} \#\{\rho' : \beta' \leqslant 1/2 + \nu/\log T, 0 < \gamma' \leqslant T\} > 0.$$

We introduce further results related to the theorem of K. Soundararajan. We arrange the zeros of the Riemann zeta function $\zeta(s)$ in the upper half-plane as $\rho_1, \rho_2, \ldots$ with $\rho_n = \beta_n + i\gamma_n$ and

$$0 < \gamma_1 \leqslant \gamma_2 \leqslant \ldots,$$

where a zero of multiplicity m appears precisely m times in this sequence. RH is that $\beta_n = 1/2$ for any $n = 1, 2, 3, \ldots$. If two zeros $\rho_{n_1} \neq \rho_{n_2}$ have the same imaginary part, then $n_1 < n_2$ implies $\beta_{n_1} < \beta_{n_2}$. For $a > 0$, define

$$D^-(a) = \liminf_{T\to\infty} \frac{1}{\frac{T}{2\pi}\log T} \#\{n : \gamma_n \leqslant T,\, \gamma_{n+1} - \gamma_n < \frac{a}{\log T}\}$$

In his remarkable work, Y. Zhang [16] showed that not only unconditionally there exists a $\nu > 0$ such that a positive proportion of zeros of $\zeta'(s)$ are in the region $|\sigma - 1/2| < \nu/\log T$, but also assuming RH and that $D^-(a)$ for any $a > 0$, ν can be arbitrary small. Recently, Feng [5] proved the Zhang's conditional result assuming only the same hypothesis for $D^-(a)$. Thus it is very probable that $\liminf\,(\beta' - 1/2)\log\gamma' = 0$.

Assuming the truth of RH, K. Soundararajan [13] conjectured that the following two statements are equivalent:

(i) $\liminf(\beta' - 1/2)\log\gamma' = 0$;

(ii) $\liminf(\gamma^+ - \gamma)\log\gamma = 0$ where γ^+ is the least ordinate of a zero of $\zeta(s)$ with $\gamma^+ > \gamma$.

Y. Zhang [16] has shown that (ii) implies (i) as follows.

Theorem A. *Assume RH. Let α_1 and α_2 be positive constants satisfying $\alpha_1 < 2\pi$ and*

$$\alpha_2 > \alpha_1 \left(1 - \sqrt{\frac{\alpha_1}{2\pi}}\right)^{-1}.$$

If $\rho = 1/2 + i\gamma$ is a zero of $\zeta(s)$ such that γ is sufficiently large and $\gamma^+ - \gamma < \alpha_1(\log\gamma)^{-1}$, then there exists a zero ρ' of $\zeta'(s)$ such that

$$|\rho' - \rho| < \alpha_2(\log\gamma)^{-1}.$$

In this paper, we consider the converse of Theorem A. Namely is it true that (i) implies (ii)? M. Z. Garaev and C. Y. Yildirim [7] proved a weaker form of the converse; if we assume RH and $\liminf(\beta' - 1/2)\log\gamma'(\log\log\gamma')^2 = 0$, then we have $\liminf_{n\to\infty}(\gamma_{n+1} - \gamma_n)\log\gamma_n = 0$. In this article, we have the following.

Theorem 1. *Assume RH. Then the following are equivalent:*

(1) $\liminf(\beta' - 1/2)\log\gamma' \neq 0$*;*

(2) *For any $c > 0$ and $s = \sigma + it$ with $|\sigma - 1/2| < c/\log t$ $(t \geqslant 10)$*

$$\frac{\zeta'}{\zeta}(s) = \frac{1}{s - \rho} + O(\log t),$$

where $\rho = 1/2 + i\gamma$ is the closest zero of $\zeta(s)$ to s and the origin;

Corollary 1. *Assume RH and* $\liminf(\beta' - 1/2)\log\gamma' \neq 0$*. Then, for any $c > 0$ and $s = \sigma + it$ $(t \geqslant 10)$, we have*

$$\frac{\zeta'}{\zeta}(s) = O\left((\log t)^{2-2\sigma}\right)$$

uniformly for $1/2 + c/\log t \leqslant \sigma \leqslant \sigma_1 < 1$.

Based on our theorems and Soundararajan's conjecture, we speculate as follows.

Conjecture. *Assume RH. Then the following are equivalent:*

(i)$'$ *For any $c > 0$ and $s = \sigma + it$ $(t \geqslant 10)$, we have*

$$\frac{\zeta'}{\zeta}(s) = O\left((\log t)^{2-2\sigma}\right)$$

uniformly for $1/2 + c/\log t \leqslant \sigma \leqslant \sigma_1 < 1$*;*

(ii)$'$ *The negation of* (ii)*, i.e.,* $\liminf(\gamma^+ - \gamma)\log\gamma \neq 0$.

We briefly introduce why (2) in Theorem 1 doesn't seem possible. We let $s = \sigma + it$ for real numbers, σ, t. It is known in [3, p. 99] and [15, Theorem 9.6(A)] that for $t \geqslant 10$ and $-1 \leqslant \sigma \leqslant 2$ we have

$$\frac{\zeta'}{\zeta}(s) = \sum_{|\gamma - t| \leqslant 1} \frac{1}{s-\rho} + O(\log t), \tag{1.1}$$

where the sum is over the ordinates of the complex zeros $\rho = \beta + i\gamma$ of $\zeta(s)$. However, assuming RH, we know the finer result

$$\frac{\zeta'}{\zeta}(s) = \sum_{|\gamma - t| \leqslant 1/\log\log t} \frac{1}{s-\rho} + O(\log t) \tag{1.2}$$

for $t \geqslant 10$. For this we refer to [15, p. 357 (14.15.2)]. According to the formula (1.2) and its proof, we can see that the formula (2) in Theorem 1 is very unrealistic.

Concerning Corollary 1, it is worth noting that under RH

$$\frac{\zeta'}{\zeta}(s) = O\left((\log t)^{2-2\sigma}\right) \tag{1.3}$$

holds uniformly for $1/2 + c/\log\log t \leqslant \sigma \leqslant \sigma_1 < 1$, where $t \geqslant 10$, $c > 0$ and 'O' depends upon c and σ_1. For the proof of it, we apply the fact [15, (14.14.5)]

$$\log \zeta(z) = O\left(\frac{\log t}{\log\log t}\right) \tag{1.4}$$

on $|s - z| = c/(2\log\log t)$ to the formula

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2\pi i} \int_{|s-z| = \frac{c}{2\log\log t}} \frac{\log \zeta(z)}{(s-z)^2} dz.$$

Then we immediately get (1.3). We note that the same method does not work when $1/2 + c/\log t \leqslant \sigma \leqslant \sigma_1 < 1$, because on $|s - z| = c/(2\log t)$ we do not have (1.4). With this information, it is very likely that on Re $(s) = 1/2 + c/\log t$ $(t \geqslant 10)$,

$$\frac{\zeta'}{\zeta}(s) \neq O(\log t).$$

On the other hand, Corollary 1 follows from a Phragmén-Lindelöf argument, provided that we have

$$\frac{\zeta'}{\zeta}(s) = O(\log t)$$

on Re $(s) = 1/2 + c/\log t$ $(t \geqslant 10)$. We find the behavior of the logarithmic derivative of the Riemann zeta function near Re $(s) = 1/2$ subtle and so we need a deep observation about the Riemann zeta function near the critical line to establish $\liminf(\beta' - 1/2)\log\gamma' = 0$. In fact we will see from Theorem 4 in Section 2 that the behavior of $\zeta'(s)/\zeta(s)$ on Re $(s) = 1/2 + c/\log t$ is very much related to

$$\sum_{0<|\gamma-\tilde{\gamma}|<1} \frac{1}{\gamma - \tilde{\gamma}},$$

where $1/2 + i\gamma, 1/2 + i\tilde{\gamma}$ are complex zeros of $\zeta(s)$ and the sum is over $\tilde{\gamma}$.

From Corollary 1, we can demonstrate the following.

Corollary 2. *Assume RH and* $\liminf(\beta' - 1/2)\log\gamma' \neq 0$. *Then, for any* $c > 0$ *and* $s = \sigma + it$ $(t \geqslant 10)$, *we have*

$$\log\zeta(s) = O\left(\frac{(\log t)^{2-2\sigma}}{\log\log t}\right)$$

uniformly for $1/2 + c/\log t \leqslant \sigma \leqslant \sigma_1 < 1$.

Assuming RH, it is known in [15, p. 355 (14.14.5)] and [15, Theorem 14.14(B)] that for any $c > 0$ and $s = \sigma + it$ $(t \geqslant 10)$, we have

$$\log\zeta(s) = O\left(\frac{(\log t)^{2-2\sigma}}{\log\log t}\right) \tag{1.5}$$

holds uniformly for $1/2 + c/\log\log t \leqslant \sigma \leqslant \sigma_1 < 1$, and there exists an absolute constant $c^* > 0$ depending on c such that

$$-c^* \frac{\log t}{\log\log t} \log\left(\frac{2}{\left(\sigma - \frac{1}{2}\right)\log\log t}\right) < \log|\zeta(s)| < c^* \frac{\log t}{\log\log t} \tag{1.6}$$

holds for $1/2 < \sigma \leqslant 1/2 + c/\log\log t$;

$$\arg\zeta(s) = O\left(\frac{\log t}{\log\log t}\right) \tag{1.7}$$

holds uniformly for $1/2 \leqslant \sigma \leqslant 1/2 + c/\log\log t$. However, as in the proof of (1.3), one cannot relax the condition '$1/2 + c/\log\log t$' of (1.5) as $1/2 + c/\log t$ $(t \to \infty)$. On the other hand, we have Ω-theorems related to $\log\zeta(s)$ near the critical line. For these, we refer to Montgomery's results [11] and [15, p. 209]. In particular, Montgomery showed that assuming RH, for $1/2 \leqslant \sigma < 1$ and any real θ, there is a t with $T^{1/6} \leqslant t \leqslant T$ such that

$$\text{Re}\,\left(e^{-i\theta}\log\zeta(s)\right) \geqslant \frac{1}{20}(\log T)^{1-\sigma}(\log\log T)^{-\sigma}.$$

See [11, p. 512]. Recently, from random matrix theory, in the case that $\theta = 0$ in the above Ω-result, it is conjectured in [4] that we have the following:

$$\max_{t\in[0,T]}\left|\zeta\left(\frac{1}{2} + it\right)\right| = \exp\left((1 + o(1))\sqrt{\frac{1}{2}\log T\log\log T}\right).$$

Concerning negative values of $\log|\zeta(s)|$, we observe that by (1.6) and (1.7), it is possible that we have

$$\left|\log\zeta\left(\frac{1}{2}+\frac{1}{\log t}+it\right)\right| > \psi(t)\frac{\log t}{\log\log t}$$

for some arbitrarily large values of t, where $\psi(t)\to\infty$ as $t\to\infty$. A sharp Ω-result like this implies $\liminf(\beta'-1/2)\log\gamma'=0$. We note that it will follow if we show that (1.5) does not hold uniformly for $1/2+1/\log t\leqslant\sigma\leqslant 1/2+1/\log\log t$.

We apply our theorem to mean values of the logarithmic derivative of the Riemann zeta function:

$$\int_0^T\left|\frac{\zeta'}{\zeta}(\sigma+it)\right|^2 dt$$

for $\sigma=1/2+a/\log T$.

We may get the following from a result of A. Selberg [12, equation (1.2)].

Theorem B. *Assume RH. Then*

$$\int_0^T\left|\frac{\zeta'}{\zeta}(\sigma+it)\right|^2 dt\sim\frac{1}{4a^2}T\log^2 T$$

holds where $\sigma=1/2+a/\log T$ *and* $a\to\infty$, $a=o(\log T)$.

We introduce more studies on mean values of the logarithmic derivative of $\zeta(s)$. Following Montgomery [10], let

$$F(\alpha,T)=\frac{1}{\frac{T}{2\pi}\log T}\sum_{0<\gamma,\tilde{\gamma}\leqslant T}T^{i\alpha(\gamma-\tilde{\gamma})}w(\gamma-\tilde{\gamma}),$$

where $\beta+i\gamma$ and $\tilde{\beta}+i\tilde{\gamma}$ are zeros of $\zeta(s)$ and $w(u)=\frac{4}{4+u^2}$. Montgomery conjectured that for any fixed $A>1$,

(MH) $\qquad F(\alpha,T)\sim 1 \qquad$ uniformly for $1\leqslant\alpha\leqslant A$.

Under RH, D. A. Goldston, S. M. Gonek and H. L. Montgomery [8] considered

$$\int_0^T\left|\frac{\zeta'}{\zeta}(\sigma+it)\right|^2 dt$$

for $\sigma=1/2+a/\log T$ as $a\to 0$ and proved the following, provided that (MH) is valid.

Theorem C. *Assume RH and (MH). Then*

$$\int_0^T\left|\frac{\zeta'}{\zeta}(\sigma+it)\right|^2 dt\sim\frac{1}{2a}T\log^2 T$$

holds where $\sigma=1/2+a/\log T$ *and* $a=a(T)\to 0$ *(sufficiently slowly) as* $T\to\infty$.

We have the following as in Theorem 1.

Theorem 2. *Assume RH and* $\liminf(\beta' - 1/2)\log\gamma' \neq 0$. *Then*

$$\int_0^T \left|\frac{\zeta'}{\zeta}(\sigma+it)\right|^2 dt = \frac{1}{2a} T\log^2 T\left(1 - \frac{\log(2\pi e)}{\log T} + O\left(a\log\frac{1}{a}\right) + O\left(\frac{1}{T}\right)\right)$$

holds where $\sigma = 1/2 + a/\log T$ *and* $a \to 0$. *Here 'O' doesn't depend upon* a *and* T.

(MH) implies Montgomery's pair correlation conjecture. That is,

$$\sum_{\substack{0<\gamma,\tilde{\gamma}\leqslant T \\ 0<\gamma-\tilde{\gamma}\leqslant 2\pi\beta/\log T}} 1 \sim \frac{T}{2\pi}\log T \int_0^\beta 1 - \left(\frac{\sin \pi u}{\pi u}\right)^2 du.$$

Clearly (MH) implies

$$\liminf(\gamma^+ - \gamma)\log\gamma = 0.$$

Assuming RH, Theorem A says that

$$\liminf(\beta' - 1/2)\log\gamma' \neq 0 \textbf{ implies } \liminf(\gamma^+ - \gamma)\log\gamma \neq 0.$$

Thus the assumptions of Theorem C and Theorem 2 are contradictory to each other. However, we have the similar conclusion in Theorem C and Theorem 2. Further, a theorem of D. A. Goldston, S. M. Gonek and H. L. Montgomery [8, Theorem 3] says the following.

Theorem D. *Assume RH. Then, for* $\sigma = 1/2 + a/\log T$ *and any fixed* $a > 0$,

$$\int_0^T \left|\frac{\zeta'}{\zeta}(\sigma+it)\right|^2 dt \sim \frac{1-e^{-2a}}{4a^2} T\log^2 T$$

holds as $T \to \infty$ *if and only if the pair correlation conjecture is true.*

Combining Theorem B and Theorem 2, we immediately have the following.

Theorem 3. *Assume RH and* $\liminf(\beta' - 1/2)\log\gamma' \neq 0$. *Then, for* $\sigma = 1/2 + a/\log T$,

$$\int_0^T \left|\frac{\zeta'}{\zeta}(\sigma+it)\right|^2 dt \sim \frac{1-e^{-2a}}{4a^2} T\log^2 T$$

holds where $a \to 0$ *or* $a \to \infty$ *and* $a = o(\log T)$.

Apparently, Theorem D and Theorem 3 are similar. However the conclusion of Theorem D says a much stronger statement than that of Theorem 3. We notice that the pair correlation conjecture implies $\liminf(\beta' - 1/2)\log\gamma' = 0$.

2 Proofs of Theorem 1 and Corollaries 1, 2

First, we state a basic fact.

Lemma 2.1. *Let* $T > 0$*. Then, we have:*

(1) *The number of zeros of* $\zeta(s)$ *in* $0 < \mathrm{Im}\,(s) \leqslant T$ *is*

$$\sum_{1<\gamma_n\leqslant T} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T);$$

(2) *The number of zeros of* $\zeta(s)$ *in* $T \leqslant \mathrm{Im}\,(s) \leqslant T+1$ *is* $O(\log T)$*.*

For Lemma 2.1(1), see [3, p. 98] and [15, Theorem 9.4]. Lemma 2.1(2) immediately follows from (1).

We start with the following theorem.

Theorem 4. *Assume RH and* $\liminf(\gamma_{n+1}-\gamma_n)\log\gamma_n > 0$*. Then the following three statements are equivalent:*

(A) $\liminf(\beta'-1/2)\log\gamma' > 0$*;*

(B) *Let* $c > 0$ *and* $s = \sigma + it$*. For sufficiently large* n*, we have*

$$\frac{\zeta'}{\zeta}(s) = \frac{1}{s-\rho_n} + O(\log t),$$

where $\frac{\gamma_{n-1}+\gamma_n}{2} < t \leqslant \frac{\gamma_{n+1}+\gamma_n}{2}$ *and* $|\sigma - 1/2| < c/\log\gamma_n$*;*

(C) $\limsup|M_n|/\log\gamma_n < \infty$*, where*

$$M_n = \sum_{0<|\gamma_m-\gamma_n|\leqslant 1}\frac{1}{\gamma_n-\gamma_m}.$$

Proof of Theorem 4. We may assume that all but finitely many nontrivial zeros of $\zeta(s)$ are simple and on $\mathrm{Re}\,(s) = 1/2$, because we assume RH and $\liminf(\gamma_{n+1}-\gamma_n)\log\gamma_n > 0$.

(C) $\Rightarrow$ (B). We recall that

$$\frac{\zeta'}{\zeta}(s) = O(\log t) + \sum_{|\gamma-t|\leqslant 1}\frac{1}{s-\rho}$$

for $-1 \leqslant \mathrm{Re}\,(s) \leqslant 2$ and $t \geqslant 2$.

Lemma 2.2. *Let $\delta > 0$. Suppose that $\liminf(\gamma_{n+1} - \gamma_n)\log\gamma_n > \delta$. Then we have*

$$\sum_{m\neq n} \frac{1}{(\gamma_n - \gamma_m)^2} = O\left(\frac{\log^2\gamma_n}{\delta^2}\right)$$

for sufficiently large n.

Proof of Lemma 2.2. We write

$$\begin{aligned}\sum_{m\neq n} \frac{1}{(\gamma_m - \gamma_n)^2} =& \sum_{|\gamma_m-\gamma_n|>1} \frac{1}{(\gamma_m - \gamma_n)^2} + \sum_{0<|\gamma_m-\gamma_n|\leqslant 1} \frac{1}{(\gamma_m - \gamma_n)^2} \\ =& I + II.\end{aligned} \tag{2.1}$$

By Lemma 2.1(2), the number of the γ_n between t and $t+1$ is $O(\log t)$ for $t > 1$. Then, for some $C > 0$, we get

$$\begin{aligned} I =& \sum_{k=1}^{\infty} \sum_{\gamma_n+k<\gamma_m\leqslant\gamma_n+k+1} \frac{1}{(\gamma_m - \gamma_n)^2} + \sum_{k=1}^{\infty} \sum_{\gamma_n-k-1\leqslant\gamma_m<\gamma_n-k} \frac{1}{(\gamma_m - \gamma_n)^2} \\ \leqslant& \sum_{k=1}^{\infty} \frac{C\log(\gamma_n+k)}{k^2} + \sum_{\gamma_n-k>1} \frac{C\log(\gamma_n-k)}{k^2} \\ \leqslant& C\sum_{k=1}^{\infty} \frac{\log\gamma_n + \log k}{k^2} + C\sum_{k=1}^{\infty} \frac{\log\gamma_n}{k^2} = O(\log\gamma_n) \end{aligned} \tag{2.2}$$

By the assumption of Lemma 2.2, there exists a positive integer n_1 such that

$$\gamma_{m+1} - \gamma_m \geqslant \frac{\delta}{\log\gamma_m} \qquad \text{for all } m \geqslant n_1.$$

Using this, for sufficiently large n, we have

$$|\gamma_{n+k} - \gamma_n| \geqslant \frac{|k|\delta}{\log(\gamma_n+1)} \geqslant \frac{|k|\delta}{2\log\gamma_n}$$

for $|\gamma_{n+k} - \gamma_n| \leqslant 1$ and $|k| \geqslant 1$. By Lemma 2.1(2) there exists $a > 0$ such that $1 \leqslant |k| \leqslant a\log\gamma_n$. Thus we get

$$II \leqslant \sum_{1\leqslant|k|\leqslant a\log\gamma_n} \frac{1}{\left(\frac{k\delta}{2\log\gamma_n}\right)^2} < \sum_{k=1}^{\infty} \frac{2}{\left(\frac{k\delta}{2\log\gamma_n}\right)^2} = O\left(\frac{\log^2\gamma_n}{\delta^2}\right).$$

We apply this inequality and (2.2) to (2.1) and obtain

$$\sum_{m\neq n} \frac{1}{(\gamma_m - \gamma_n)^2} = O\left(\frac{\log^2\gamma_n}{\delta^2}\right)$$

for sufficiently large n. This proves Lemma 2.2. □

Lemma 2.3. *Choose* $c_1 > 0$ *such that* $\liminf(\gamma_{n+1} - \gamma_n)\log\gamma_n > 2c_1$. *Define* $M_n(t)$ *by*

$$M_n(t) = \sum_{0<|\gamma_m-\gamma_n|\leqslant 1} \frac{1}{t-\gamma_m}$$

for $\gamma_{n-1} + c_1/\log\gamma_n \leqslant t \leqslant \gamma_{n+1} - c_1/\log\gamma_n$. *Then we have*

$$M_n(t) = O(\log t),$$

where the implied constant is absolute.

Proof of Lemma 2.3. Since

$$M_n'(t) = \sum_{0<|\gamma_m-\gamma_n|<1} \frac{-1}{(t-\gamma_m)^2} < 0,$$

$M_n(t)$ is decreasing. Thus it suffices to consider the endpoints for the proof. By Lemma 2.2, our assumption (C) and the fact [15, Theorem 9.11] that the gaps between ordinates of successive zeros tend to 0, we have

$$\begin{aligned} M_n\left(\gamma_{n+1} - \frac{c_1}{\log\gamma_n}\right) &= \sum_{0<|\gamma_m-\gamma_n|\leqslant 1} \frac{1}{\gamma_{n+1}-\gamma_m-\frac{c_1}{\log\gamma_n}} \\ &= \sum_{0<|\gamma_m-\gamma_{n+1}|\leqslant 1} \frac{1}{\gamma_{n+1}-\gamma_m-\frac{c_1}{\log\gamma_n}} + O(\log\gamma_n) \\ &= \sum_{0<|\gamma_m-\gamma_{n+1}|\leqslant 1} \frac{1}{\gamma_{n+1}-\gamma_m-\frac{c_1}{\log\gamma_n}} - \frac{1}{\gamma_{n+1}-\gamma_m} + \\ &\quad M_{n+1} + O(\log\gamma_n) \\ &= O\left(\frac{1}{\log\gamma_n} \sum_{0<|\gamma_m-\gamma_{n+1}|<1} \frac{1}{(\gamma_{n+1}-\gamma_m)^2}\right) + O(\log\gamma_{n+1}) \\ &= O(\log\gamma_n). \end{aligned}$$

Similarly, we have

$$M_n\left(\gamma_{n-1} + \frac{c_1}{\log\gamma_n}\right) = O(\log\gamma_n).$$

Lemma 2.3 follows. □

Using (1.1), Lemma 2.1(2), Lemma 2.3 and (C), for $\rho_n = 1/2 + i\gamma_n$ and $s = \sigma + it$

$(t \geqslant 10)$, we get

$$\begin{aligned}
\frac{\zeta'}{\zeta}(s) =& \frac{1}{s-\rho_n} + \sum_{\substack{\rho\neq\rho_n \\ 0<|\gamma-t|\leqslant 1}} \frac{1}{s-\rho} + O(\log t) \\
=& \frac{1}{s-\rho_n} + \sum_{0<|\gamma_m - t|\leqslant 1} \frac{1}{s-\rho_m} + \sum_{\substack{|\gamma_m-\gamma_n|>1 \\ 0<|\gamma_m-t|\leqslant 1}} \frac{1}{s-\rho_m} + O(\log t) \\
=& \frac{1}{s-\rho_n} + \sum_{0<|\gamma_m-\gamma_n|\leqslant 1} \frac{1}{s-\rho_m} + O\left(\sum_{0<|\gamma-t|\leqslant 1} 1\right) + O(\log t) \\
=& \frac{1}{s-\rho_n} + \sum_{0<|\gamma_m-\gamma_n|\leqslant 1} \frac{1}{s-\rho_m} - \frac{1}{i(t-\gamma_m)} + \frac{M_n(t)}{i} + O(\log t) \\
=& \frac{1}{s-\rho_n} + O\left(\sum_{m\neq n} \frac{\sigma - \frac{1}{2}}{(\gamma_m-\gamma_n)^2}\right) + O(\log t)
\end{aligned}$$

in $|\sigma - 1/2| < c/\log\gamma_n$. By this and Lemma 2.2, (B) follows.

(B) $\Rightarrow$ (A). We need the following lemma for this.

Lemma 2.4. *Assume RH and* $\liminf(\gamma_{n+1}-\gamma_n)\log\gamma_n > 0$. *Let* $\tilde{\delta}$ *be such that* $0 < \tilde{\delta} < \delta$ *where* δ *is as in Lemma 2.2. Suppose* $\zeta'(\beta' + i\gamma') = 0$ *for sufficiently large* γ'. *If* $|\gamma' - \gamma_n| \geqslant \tilde{\delta}/(2\log\gamma')$ *for all* n, *then we have*

$$\frac{1}{2}\log\gamma' + O(1) = O\left(\left(\beta' - \frac{1}{2}\right)\frac{\log^2\gamma'}{\tilde{\delta}^2}\right).$$

Proof of Lemma 2.4. Using the formula (8) in [3, p. 80], we have

$$\mathrm{Re}\,\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2}\log\pi - \mathrm{Re}\,\frac{1}{s-1} - \frac{1}{2}\mathrm{Re}\,\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)} + \sum_{\rho}\mathrm{Re}\,\frac{1}{s-\rho}, \tag{2.3}$$

where ρ runs through all complex zeros of $\zeta(s)$. We note that any complex zero ρ of $\zeta(s)$ is either $1/2 + i\gamma_n$ or $1/2 - i\gamma_n$ for some n, provided that RH is true. We assumed that for any n,

$$|\gamma_n - \gamma'| > \frac{\tilde{\delta}}{2\log\gamma'}. \tag{2.4}$$

By (2.3) we obtain that for $s = \beta' + i\gamma'$, we get

$$\sum_{\rho}\mathrm{Re}\,\frac{1}{s-\rho} = -\frac{1}{2}\log\pi + \mathrm{Re}\,\frac{1}{s-1} + \frac{1}{2}\mathrm{Re}\,\frac{\Gamma'\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+1\right)}. \tag{2.5}$$

Applying the standard fact [3, p. 73]

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O(\frac{1}{|s|}), \qquad (|\arg(s) - \pi| > \theta > 0)$$

to (2.5), we obtain

$$\sum_\rho \operatorname{Re} \frac{1}{s-\rho} = \frac{1}{2}\log t + O(1). \tag{2.6}$$

We have

$$\begin{aligned}\sum_\rho \operatorname{Re} \frac{1}{s-\rho} =& \sum_{n=1}^\infty \frac{\beta' - \frac{1}{2}}{\left(\beta' - \frac{1}{2}\right)^2 + (\gamma' - \gamma_n)^2} + \frac{\beta' - \frac{1}{2}}{\left(\beta' - \frac{1}{2}\right)^2 + (\gamma' + \gamma_n)^2} \\ =& \sum_{n=1}^\infty \frac{\beta' - \frac{1}{2}}{\left(\beta' - \frac{1}{2}\right)^2 + (\gamma' - \gamma_n)^2} + O\left(\beta' - \frac{1}{2}\right).\end{aligned}$$

Thus, by this and (2.6), we obtain

$$\frac{1}{2}\log\gamma' + O(1) = O\left(\sum_{n=1}^\infty \frac{\beta' - \frac{1}{2}}{(\gamma' - \gamma_n)^2}\right). \tag{2.7}$$

Using (2.4) and Lemma 2.2, we have

$$\sum_{n=1}^\infty \frac{\beta' - \frac{1}{2}}{(\gamma' - \gamma_n)^2} = O\left(\left(\beta' - \frac{1}{2}\right)\frac{(\log\gamma')^2}{\tilde{\delta}^2}\right).$$

We insert this to (2.7) and then we get

$$\frac{1}{2}\log\gamma' + O(1) = O\left(\left(\beta' - \frac{1}{2}\right)\frac{(\log\gamma')^2}{\tilde{\delta}^2}\right).$$

This proves Lemma 2.4. □

Suppose

$$\liminf(\beta' - 1/2)\log\gamma' = 0.$$

Then this and Lemma 2.4 implies that we have sequences $\langle\epsilon_k\rangle$, $\langle\rho_{n_k}\rangle$ and $\langle\rho'_k\rangle$ such that

$$\rho_{n_k} = 1/2 + i\gamma_{n_k},\ \zeta'(\rho'_k) = 0,\ |\rho'_k - \rho_{n_k}| < \frac{\epsilon_k}{\log\gamma_{n_k}} \text{ and } \epsilon_k \to 0\ \ (\epsilon_k > 0). \tag{2.8}$$

Using (B), we get

$$\frac{1}{\rho'_k - \rho_{n_k}} + O(\log\gamma_{n_k}) = 0.$$

Thus we obtain that for some $c_1 > 0$,

$$|\rho'_k - \rho_{n_k}| > \frac{c_1}{\log\gamma_{n_k}}.$$

Note that c_1 doesn't depend on ϵ_k's. By this and (2.8), we obtain

$$\frac{\epsilon_k}{\log \gamma_{n_k}} > \frac{c_1}{\log \gamma_{n_k}}.$$

But this is a contradiction, since $\epsilon_k \to 0$ and $c_1 > 0$ is a fixed real number. Thus (A) follows.

(A) $\Rightarrow$ (C). Suppose

$$\limsup \frac{|M_n|}{\log \gamma_n} = \infty.$$

We re-write (1.1) as

$$\frac{\zeta'}{\zeta}(s) = \sum_{0<|\gamma_m-\gamma_n|\leqslant 1} \frac{1}{s-\rho_m} + \frac{1}{s-\rho_n} + O(\log \gamma_n)$$

where we will take $|s-\rho_n| \leqslant \epsilon/\log \gamma_n$ with ϵ an arbitrarily small positive real number. For such s, it follows that

$$(s-\rho_n)\frac{\zeta'}{\zeta}(s) - (s-\rho_n)\sum_{0<|\gamma_m-\gamma_n|\leqslant 1} \frac{1}{s-\rho_m} = 1 + O\left(|s-\rho_n|\log \gamma_n\right) = 1 + O(\epsilon).$$

Since we are assuming $\limsup \frac{|M_n|}{\log \gamma_n} = \infty$, we have

$$1 + O(\epsilon) < \frac{\epsilon}{\log \gamma_n}(|M_n| + u\log \gamma_n),$$

with any real constant u, for infinitely many n. Hence for such n and for $|s-\rho_n| \leqslant \epsilon/\log \gamma_n$, we obtain

$$\left|(s-\rho_n)\frac{\zeta'}{\zeta}(s) - (s-\rho_n)\sum_{0<|\gamma_m-\gamma_n|\leqslant 1} \frac{1}{s-\rho_m}\right| < \frac{\epsilon}{\log \gamma_n}(|M_n| + u\log \gamma_n).$$

As in the proof of (C) $\Rightarrow$ (B), using Lemma 2.2 when $|s-\rho_n| \leqslant \epsilon/\log \gamma_n$, we have

$$\left|\sum_{0<|\gamma_m-\gamma_n|\leqslant 1} \frac{1}{s-\rho_m} - \frac{M_n}{i}\right| \ll \frac{\epsilon}{\delta^2}\log \gamma_n.$$

So, with an appropriate constant u (the above estimate gives $u \ll \frac{\epsilon}{\delta^2}$), we obtain

$$\left|(s-\rho_n)\frac{\zeta'}{\zeta}(s) - (s-\rho_n)\sum_{0<|\gamma_m-\gamma_n|\leqslant 1} \frac{1}{s-\rho_m}\right| < \left|(s-\rho_n)\sum_{0<|\gamma_m-\gamma_n|\leqslant 1} \frac{1}{s-\rho_m}\right|$$

on $|s-\rho_n| = \epsilon/\log \gamma_n$ for infinitely many n. We note that

$$(s-\rho_n)\sum_{0<|\gamma_m-\gamma_n|\leqslant 1} \frac{1}{(s-\rho_m)}$$

has a zero at $s = \rho_n$. Thus Rouché's theorem implies that for some s' in $|s - \rho_n| < \epsilon/\log\gamma_n$,

$$(s' - \rho_n)\frac{\zeta'}{\zeta}(s') = 0,$$

i.e. $\zeta'(s') = 0$. Therefore $\liminf(\beta' - 1/2)\log\gamma' = 0$. Hence (A) implies (C). We have completed the proof of Theorem 4. □

Now we prove Theorem 1 and Corollaries 1, 2.

Proof of Theorem 1. Assume that

$$\liminf(\beta' - 1/2)\log\gamma' \neq 0.$$

Then RH implies that $\liminf(\beta' - 1/2)\log\gamma'$ is positive. Then, by Theorem A, we obtain $\liminf(\gamma_{n+1} - \gamma_n)\log\gamma_n > 0$. Thus, by Theorem 4, we have (1) ⇒ (2).

Assume (2) is true. If there exist multiple zeros for $\zeta(s)$, we immediately get a contradiction from (2). Thus all zeros of $\zeta(s)$ are simple. Suppose that

$$\liminf(\gamma_{n+1} - \gamma_n)\log\gamma_n = 0.$$

Then there exists a sequence of natural numbers $\langle n_k\rangle$ with $n_k < n_{k+1}$ such that

$$(\gamma_{n_k+1} - \gamma_{n_k})\log\gamma_{n_k} \to 0$$

as $k \to \infty$. Using (2), we have

$$\begin{aligned}\frac{\zeta'}{\zeta}(s) =& \frac{1}{s - \rho_{n_k+1}} + O(\log t)\\ =& \frac{1}{s - \rho_{n_k}} + O(\log t)\end{aligned}$$

at $s = 1/2 + i(\gamma_{n_k+1} + \gamma_{n_k})/2$. By this, we obtain

$$\frac{1}{s - \rho_{n_k+1}} - \frac{1}{s - \rho_{n_k}} = O(\log\gamma_{n_k})$$

or

$$\frac{1}{\gamma_{n_k+1} - \gamma_{n_k}} = O(\log\gamma_{n_k}).$$

Namely, we have

$$\frac{1}{(\gamma_{n_k+1} - \gamma_{n_k})\log\gamma_{n_k}} = O(1).$$

This is a contradiction, for $\lim_{k\to\infty}(\gamma_{n_k+1} - \gamma_{n_k})\log\gamma_{n_k} = 0$. Hence, by Theorem 4, we have (2) ⇒ (1).

Thus Theorem 1 follows. □

Proof of Corollary 1. Assume (1). We fix $c > 0$. Then, by Theorem 4 (B), we have

$$\frac{\zeta'}{\zeta}(s) = O\left(\log(|t|+3)\right) \tag{2.9}$$

on $\sigma = 1/2 + c/\log(|t|+3)$. Using this, it is not hard to see that Corollary 1 follows from a Phragmén-Lindelöf argument. We give the detailed proof for convenience. For the following version of the Phragmén-Lindelöf Theorem, we refer to [2, p. 138].

Phragmén-Lindelöf Theorem. *Let G be a simply connected region and let f be an analytic function on G. Suppose there is an analytic function $\varphi : G \to \mathbb{C}$ which never vanishes and is bounded on G. If M is a constant and $\partial_\infty G = A \cup B$ such that*

(a) *for every a in A, $\limsup_{s\to a} |f(s)| \leqslant M$;*

(b) *for every b in B, and $\eta > 0$, $\limsup_{s\to b} |f(s)||\varphi(s)|^\eta \leqslant M$;*

then $|f(s)| \leqslant M$ for all s in G.

Here $\partial_\infty G = \partial G =$ the boundary of G if G is bounded, $\partial_\infty G = \partial G \cup \{\infty\}$ if G is unbounded and the limit superior of $f(s)$ as $s \to a$, is defined by

$$\limsup_{s\to a} |f(s)| = \lim_{r\to 0^+} \sup\{|f(s)| : s \in G \cap B(a,r)\},$$

where $B(a,r) = \{s \in \mathbb{C} : |s-a| < r\}$. If $a = \infty$, $B(a,r)$ is the ball in the metric of $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

We define sets G_1 and G_2 by

$$G_1 = \{s \in \mathbb{C} : \sigma \geqslant \frac{1}{2} + \frac{c}{\log(|t|+3)}\} \text{ and } G_2 = \{s \in \mathbb{C} : 0 \leqslant \sigma \leqslant 1+c,\ |t| \leqslant 1\}.$$

We define G by

$$G = G_1 \cap (\mathbb{C} - G_2).$$

We note that $1 \notin G$. We define $f(s)$ by

$$f(s) = \frac{\zeta'(s)}{\zeta(s)\log s}$$

for $s \in G$. Then, $f(s)$ is analytic on the region G. We choose $\varphi(s) = \exp(-\sqrt{s})$. Clearly, the function $\varphi : G \to \mathbb{C}$ is an analytic function which never vanishes and is bounded on G. By (2.9), there exists $M > 0$ such that we have

$$|f(s)| \leqslant M \tag{2.10}$$

on the boundary ∂G.

Claim. *Assume RH. We have*

$$\frac{\zeta'}{\zeta}(s) = O\left(\log^2(|t|+3)\right) \qquad (s=\sigma+it\in G).$$

Proof of Claim. We may suppose $\sigma < 2$ and $t \geqslant 10$. Using Lemma 2.1(2), (1.1) and the fact that $|\sigma - 1/2| \geqslant c/\log(|t|+3)$ for $s \in G$, we get

$$\frac{\zeta'}{\zeta}(s) = \sum_{|t-\gamma|<1} \frac{1}{s-\rho} + O(\log t) = O\left(\log t \sum_{|t-\gamma|<1} 1\right) + O(\log t) = O\left(\log^2 t\right).$$

Thus, Claim follows. □

Let $\eta > 0$. Then, by Claim, we conclude that in G, we have

$$\limsup_{s\to\infty} |f(s)||\varphi(s)|^\eta = O\left(\limsup_{r\to\infty} \log r \exp\left(-\eta\sqrt{r}\cos\frac{\pi}{4}\right)\right) = 0.$$

With this and (2.10), we see that the functions f and φ fulfill the conditions of Phragmén-Lindelöf Theorem. Hence $|f(s)| \leqslant M$ for $s \in G$. Namely, we obtain

$$\frac{\zeta'}{\zeta}(s) = O(\log|s|)$$

for $s \in G$. In particular, we have

$$\frac{\zeta'}{\zeta}(s) = O(\log t)$$

uniformly for $1/2 + c/\log t \leqslant \sigma \leqslant 1/2 + c/\log\log t$ ($t \geqslant 10$). From this, (1.3) and the fact that for $1/2 + c/\log t \leqslant \sigma \leqslant 1/2 + c/\log\log t$,

$$\log t = O\left((\log t)^{2-2\sigma}\right),$$

we complete the proof of Corollary 1. □

Proof of Corollary 2. Using (1.5) and Corollary 1, we have

$$\begin{aligned}
\log\zeta(s) =& \log\zeta(\sigma_1+it) - \int_\sigma^{\sigma_1} \frac{\zeta'(\widetilde{\sigma}+it)}{\zeta(\widetilde{\sigma}+it)} d\widetilde{\sigma} \\
=& O\left(\frac{(\log t)^{2-2\sigma_1}}{\log\log t}\right) + O\left(\int_\sigma^{\sigma_1} (\log t)^{2-2\widetilde{\sigma}} d\widetilde{\sigma}\right) \\
=& O\left(\frac{(\log t)^{2-2\sigma_1}}{\log\log t}\right) + O\left(\frac{(\log t)^{2-2\sigma}}{\log\log t}\right) \\
=& O\left(\frac{(\log t)^{2-2\sigma}}{\log\log t}\right)
\end{aligned}$$

for $1/2 + c/\log t \leqslant \sigma \leqslant \sigma_1$. Thus, we have proved Corollary 2. □

3 Proof of Theorem 2

We let $T > 2$ and $\sigma = 1/2 + a/\log T$ for a small a. We set

$$a_n = \frac{\gamma_n + \gamma_{n-1}}{2}.$$

Theorem 4 implies that under the assumptions for Theorem 2, we obtain that for $s = \sigma + it$ and $n = 2, 3, \ldots$,

$$\frac{\zeta'}{\zeta}(s) = \frac{1}{s - \rho_n} + O(\log t) \tag{3.1}$$

for $0 \leqslant \sigma - 1/2 < 1/\log\gamma_n$ and $a_n < t \leqslant a_{n+1}$. We write

$$\begin{aligned}\int_1^T \left|\frac{\zeta'}{\zeta}(\sigma+it)\right|^2 dt =& \sum_{2\leqslant n\leqslant n_1}\int_{a_n}^{a_{n+1}}\left|\frac{\zeta'}{\zeta}(\sigma+it)\right|^2 dt + \int_{a_{n_1+1}}^T \left|\frac{\zeta'}{\zeta}(\sigma+it)\right|^2 dt + O(1)\\ =& I + II + O(1),\end{aligned}$$

where $a_{n_1+1} < T < a_{n_1+2}$. Using (3.1), it is easy to see that

$$II = \int_{a_{n_1+1}}^T \left|\frac{1}{(\sigma-1/2)+i(t-\gamma_{n_1+1})} + O(\log T)\right|^2 dt = O\left(\frac{\log T}{a}\right) + o(\log^2 T).$$

We insert (3.1) into I and then we get

$$\begin{aligned}I =& \sum_{2\leqslant n\leqslant n_1}\int_{a_n}^{a_{n+1}}\left|\frac{1}{(\sigma-1/2)+i(t-\gamma_n)} + O(\log T)\right|^2 dt\\ =& 2\sum_{1<\gamma_n<T}\frac{1}{\sigma-1/2}\tan^{-1}\frac{\gamma_{n+1}-\gamma_n}{2(\sigma-1/2)} +\\ & O\left(\log T\sum_{1<\gamma_n<T}\log\left(1+\frac{\gamma_{n+1}-\gamma_n}{2(\sigma-1/2)}\right)\right) + O\left(T\log^2 T\right).\end{aligned} \tag{3.2}$$

Since $\liminf(\gamma_{n+1}-\gamma_n)\log\gamma_n > 0$, there is $\alpha > 0$ such that

$$\frac{\gamma_{n+1}-\gamma_n}{2(\sigma-1/2)} \geqslant \frac{\alpha}{2a}$$

is large as $a \to 0$. Using this, Lemma 2.1(1) and the fact that $\tan^{-1} x = \frac{\pi}{2} + O(1/x)$ as $x \to \infty$, we get

$$\sum_{1<\gamma_n<T}\tan^{-1}\frac{\gamma_{n+1}-\gamma_n}{2(\sigma-1/2)} = \frac{T}{4}\log\frac{T}{2\pi e} + O(aT\log T) + O(\log T). \tag{3.3}$$

We recall that there exists $A > 0$ such that

$$\#\{n : 0 < \gamma_n \leqslant T, \gamma_{n+1}-\gamma_n \geqslant \frac{\lambda}{\log T}\} = O\left(T\log T e^{-A\lambda^{\frac{1}{2}}(\log\lambda)^{-\frac{1}{4}}}\right),$$

uniformly for $\lambda \geqslant 2$. For this we refer to [6] and [15, p. 246]. Using this formula and Lemma 2.1(1), we can see that

$$\sum_{1<\gamma_n<T} \log\left(1+\frac{\gamma_{n+1}-\gamma_n}{2(\sigma-1/2)}\right) = O\left(T\log T\sum_{m=2}^{\infty}\log\left(1+\frac{m}{2a}\right)e^{-Am^{\frac{1}{2}}(\log m)^{-\frac{1}{4}}}\right)$$
$$= O\left(T\log T\log\frac{1}{a}\right).$$

We insert this and (3.3) into (3.2) and then we obtain

$$I = \frac{1}{2a}T\log^2 T - \frac{\log(2\pi e)}{2a}T\log T + O\left(\log\frac{1}{a}(T\log^2 T)\right) + O\left(\frac{\log^2 T}{a}\right).$$

Using I and II, we get

$$\int_1^T \left|\frac{\zeta'}{\zeta}(\sigma+it)\right|^2 dt = \frac{1}{2a}T\log^2 T\left(1-\frac{\log(2\pi e)}{\log T} + O\left(a\log\frac{1}{a}\right) + O\left(\frac{1}{T}\right)\right).$$

Hence Theorem 2 follows.

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