A NOTE ON DIFFERENTIAL OPERATORS OF INFINITE ORDER

YOUNGJOON CHA, HASEO KI AND YOUNG–ONE KIM

Abstract. A sufficient condition for entire functions $f$ and $g$ to be such that the series
$$\sum_{m=0}^{\infty} \frac{f^{(m)}(0)g^{(m)}}{m!}$$
represents an entire function is established; and in that case, the
growth of the resulting function is described.

1. Introduction

This paper is concerned with the growth of entire functions. Let $0 < \rho < \infty$ and
$0 \leq \tau \leq \infty$. An entire function is said to be of growth $(\rho, \tau)$ if it is of order less than $\rho$
or is of order $\rho$ with type not exceeding $\tau$. (For the definitions of order and type, see [1]
or [5].) Therefore an entire function $f$ is of growth $(\rho, \tau)$ with $\tau < \infty$ if and only if for
each $\epsilon > 0$ there is a constant $C > 0$ such that $|f(z)| \leq C \exp((\tau + \epsilon)|z|^\rho)$ for all $z$; and
it is of growth $(\rho, \infty)$ if and only if for each $\epsilon > 0$ there is a constant $C > 0$ such that
$|f(z)| \leq C \exp(|z|^\rho + \epsilon)$ for all $z$. We remark that this notation is due to R. P. Boas [1, p.
8].

When $g$ is an entire function and $m$ a non-negative integer, we denote the $m$-th derivative
of $g$ by $D^m g$, that is, $D^m g = g^{(m)}$. It is well known that if $g$ is of growth $(\rho, \tau)$, then
the same holds for $D^m g$ for all $m$ [5, Chapter 1]. More generally, suppose that $f$ and $g$
are entire functions and one of them is a polynomial. Also suppose that $g$ is of growth $(\rho, \tau)$.
Then it is easy to see that the entire function
$$\sum_{m=0}^{\infty} \frac{f^{(m)}(0)g^{(m)}}{m!} D^m g$$
is also of growth $(\rho, \tau)$. Note that this series is actually a finite sum. We denote this
new entire function by $f(D)g$. For each non-negative integer $n$, let $M_n$ denote the monic
monomial of degree $n$, that is, $M_n(z) = z^n$. Then we have $g = \sum_{n=0}^{\infty} g^{(n)}(0)M_n/n!$.
Moreover, it is easy to see that
$$f(D)g = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} f(D)M_n,$$

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and the series converges absolutely and uniformly on compact sets in the complex plane. Note that the right hand side is an infinite series if and only if $g$ is transcendental and $f$ does not vanish identically.

Now suppose that $f$ and $g$ are entire functions, and that both

$$\sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} D^m g \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} f(D) M_n$$

close to an entire function absolutely and uniformly on compact sets in the complex plane. In this case also, we denote the entire function by $f(D)g$, and say that $f(D)g$ is well defined. There are many results on the operators of the form $f(D)$. See, for example, [3] and the references cited there. See [4] also.

The purpose of this paper is to establish the following theorem.

**Theorem 1.1.** Let $f$ be an entire function of growth $(p, \alpha)$ and $g$ an entire function of growth $(q, \beta)$. Suppose that $p^{-1} + q^{-1} > 1$, or $p^{-1} + q^{-1} = 1$ and $(p\alpha)^{1/p}(q\beta)^{1/q} < 1$. Then $f(D)g$ is well defined. Moreover, we have the following: If $p^{-1} + q^{-1} > 1$, the entire function $f(D)g$ is of growth $(q, \beta)$; and if $p^{-1} + q^{-1} = 1$ and $(p\alpha)^{1/p}(q\beta)^{1/q} < 1$, it is of growth $(q, b\beta)$, where

$$b = \left[1 - \left((p\alpha)^{1/p}(q\beta)^{1/q}\right)^{1-q}\right].$$

It should be remarked that an equivalent version of this theorem is stated and proved in Sikkema's book [6, pp. 89–106], but our statement and proof are much easier to apply and simpler than those of Sikkema.

We will prove Theorem 1.1 in Section 2. In fact, our proof gives a slightly more general result. See Propositions 2.1 and 2.2 below. This paper concludes with an example which shows that the assumptions in the theorem are necessary and the results are best possible with regard to the growth scale we are using (Section 3).

2. Proof of Theorem 1.1

When $f$ is an entire function, we define $\tilde{f}$ by $\tilde{f} = \sum_{n=0}^{\infty} |f^{(n)}(0)| M_n/n!$. It is well known that the entire functions $f$ and $\tilde{f}$ have the same order and type [5, Chapter I, Theorem 2]. Moreover, it is clear that if $f$ and $g$ are entire functions and

$$\sum_{m=0}^{\infty} \frac{|f^{(m)}(0)|}{m!} D^m \tilde{g}(x) < \infty$$

for all $x \geq 0$, then both $\tilde{f}(D)\tilde{g}$ and $f(D)g$ are well defined, and

$$|f(D)g(z)| \leq \tilde{f}(D)\tilde{g}(|z|) \quad (z \in \mathbb{C}).$$

Therefore, in order to prove Theorem 1.1, we need only to consider the functions $f$ and $g$ such that $f^{(m)}(0), g^{(m)}(0) \geq 0$ for all $m$. In the remainder of this section, we assume that $f$ and $g$ are such entire functions, so that $f = \tilde{f}$ and $g = \tilde{g}$. Then Theorem 1.1 is an immediate consequence of the following three propositions.
Proposition 2.1. Let $\alpha$ be a positive real number, and suppose that $f(x) \leq \exp(\alpha x)$ for all $x \geq 0$. Then for each $\epsilon > 0$ there is a constant $C > 0$ such that

$$\sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} D^m g(x) \leq C g(x + \alpha + \epsilon) \quad (x \geq 0).$$

Proposition 2.2. Let $0 < q \leq 1$, $0 < \beta < \infty$, and suppose that $g(x) \leq \exp(\beta x^q)$ for all $x \geq 0$. Then for each $\epsilon > 0$ there is a constant $C > 0$ such that

$$\sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} D^m g(x) \leq C f(b + \epsilon) \exp(\beta x^q) \quad (x \geq 0),$$

where $b = (e^{1-q} \beta q)^{1/q}$.

Proposition 2.3. Let $p, q, \alpha$ and $\beta$ be positive real numbers such that $p^{-1} + q^{-1} = 1$ and $(p \alpha)^{1/p} (q \beta)^{1/q} < 1$. Suppose that $f(x) \leq \exp(\alpha x^p)$ and $g(x) \leq \exp(\beta x^q)$ for all $x \geq 0$. Then there is a constant $C > 0$ such that

$$\sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} D^m g(x) \leq C x^{q/2} \exp(b \beta x^q) \quad (x \geq 0),$$

where

$$b = \left[1 - (p \alpha)^{1/p} (q \beta)^{1/q}\right]^{1-q}.$$

In our proof of these propositions, we will use the following lemma.

Lemma 2.4. Let $\rho$ and $\tau$ be positive real numbers, and $h$ an entire function such that $|h(z)| \leq \exp(\tau |z|^\rho)$ for all $z \in \mathbb{C}$. Then

$$|h^{(m)}(z)| \leq m! \left(\frac{m}{\tau \rho}\right)^{m/\rho} \exp \left[\tau \left(|z| + \left(\frac{m}{\tau \rho}\right)^{1/\rho}\right)^\rho\right] \quad (m = 1, 2, \ldots, z \in \mathbb{C}).$$

Proof. Let $m$ and $z$ be arbitrary. From Cauchy’s integral formula, we have

$$h^{(m)}(z) = \frac{m!}{2\pi i} \int_{|\zeta-z|=R} \frac{h(\zeta)}{(\zeta-z)^{m+1}} d\zeta \quad (R > 0),$$

and hence

$$|h^{(m)}(z)| \leq m! R^{-m} \exp(\tau(|z| + R)^\rho) \quad (R > 0).$$

Now, substitution of $(m/\tau \rho)^{1/\rho}$ in $R$ yields the desired inequality. \( \square \)

Proof of Proposition 2.1. From Lemma 2.4, we have

$$f^{(m)}(0) \leq m! \left(\frac{e\alpha}{m}\right)^m \quad (m = 1, 2, \ldots).$$
Then there is a constant $C_1 > 0$ such that
\[ f^{(m)}(0) \leq C_1 \sqrt{m} \alpha^m \quad (m = 1, 2, \ldots), \]
because $m! \sim \sqrt{2\pi m^{m+\frac{1}{2}} e^{-m}}$ for $m \to \infty$. Let $\epsilon > 0$ be arbitrary. Then the last inequality implies that we have
\[ f^{(m)}(0) \leq C(\alpha + \epsilon)^m \quad (m = 0, 1, 2, \ldots) \]
for some constant $C > 0$. Hence
\[ \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} D^m g(x) \leq C \sum_{m=0}^{\infty} \frac{(\alpha + \epsilon)^m}{m!} D^m g(x) \leq C g(x + \alpha + \epsilon) \quad (x \geq 0). \]

**Proof of Proposition 2.2.** From Lemma 2.4, it follows that
\[ g^{(m)}(x) \leq m! \left( \frac{m}{\beta q} \right)^{-m/q} \exp \left[ \beta \left( x + \left( \frac{m}{\beta q} \right)^{1/q} \right)^{q} \right] \quad (m = 1, 2, \ldots, x \geq 0). \]
Since $0 < q \leq 1$, we have $m^{-m/q} \leq m^{-m}$ for $m = 1, 2, \ldots$, and $(x + y)^q \leq x^q + y^q$ for $x, y \geq 0$. Moreover, $m! \sim \sqrt{2\pi m^{m+\frac{1}{2}} e^{-m}}$ for $m \to \infty$. Hence the above inequality implies that there is a constant $C_1 > 0$ such that
\[ g^{(m)}(x) \leq C_1 \sqrt{m} e^{-m} (\beta q)^{m/q} \exp \left( \beta x^q + \frac{m}{q} \right) \quad (m = 1, 2, \ldots, x \geq 0). \]
Since $b = (e^{1-q} \beta q)^{1/q}$, the right hand side of this inequality is equal to $C_1 \sqrt{m} b^m \exp (\beta x^q)$. Thus
\[ g^{(m)}(x) \leq C_1 \sqrt{m} b^m \exp (\beta x^q) \quad (m = 1, 2, \ldots, x \geq 0). \]
Let $\epsilon > 0$ be arbitrary. Then the last inequality implies that there is a constant $C_2 > 0$ such that
\[ g^{(m)}(x) \leq C_2 (b + \epsilon)^m \exp (\beta x^q) \quad (m = 0, 1, 2, \ldots, x \geq 0), \]
and hence
\[ \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} D^m g(x) \leq C_2 \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} (b + \epsilon)^m \exp (\beta x^q) \]
\[ \leq C_2 f(b + \epsilon) \exp (\beta x^q) \quad (x \geq 0). \]

**Proof of Proposition 2.3.** First of all, Lemma 2.4 implies that
\[ f^{(m)}(0) \leq m! \left( \frac{e\alpha p}{m} \right)^{m/p} \quad (m = 1, 2, \ldots). \]
Since \( p^{-1} + q^{-1} = 1 \) and \( \Gamma(s+1) \sim \sqrt{2\pi} s^{s+\frac{1}{2}} e^{-s} \) for \( s \to \infty \), it follows that
\[
q^{-1/2} m! \left( \frac{e}{m} \right)^{m/p} \sim \Gamma \left( \frac{m}{q} + 1 \right) q^{m/q} \quad (m \to \infty).
\]
Hence there is a constant \( C_1 > 0 \) such that
\[
f^{(m)}(0) \leq C_1 \Gamma \left( \frac{m}{q} + 1 \right) \left( (\alpha p)^{1/p} q^{1/q} \right)^m \quad (m = 0, 1, 2, \ldots).
\]
For convenience, we set \( B = (\alpha p)^{1/p} (q\beta)^{1/q} \). Then the last inequality becomes
\[
f^{(m)}(0) \leq C_1 \Gamma \left( \frac{m}{q} + 1 \right) \left( B\beta^{-1/q} \right)^m \quad (m = 0, 1, 2, \ldots),
\]
and hence we have
\[
\sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} g^{(m)}(x) \leq C_1 \sum_{m=0}^{\infty} \frac{\Gamma \left( \frac{m}{q} + 1 \right) (B\beta^{-1/q})^m}{m!} g^{(m)}(x)
\]
\[
= C_1 \sum_{m=0}^{\infty} \frac{g^{(m)}(x)}{m!} (B\beta^{-1/q})^m \int_0^{\infty} e^{-t} t^m dt
\]
\[
= C_1 \int_0^{\infty} e^{-t} \sum_{m=0}^{\infty} \frac{g^{(m)}(x)}{m!} (B(t/\beta)^{1/q})^m dt
\]
\[
= C_1 \int_0^{\infty} e^{-t} g \left( x + B(t/\beta)^{1/q} \right) dt \quad (x \geq 0).
\]
From this, we obtain
\[
(2.1) \quad \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} g^{(m)}(x) \leq C_1 \int_0^{\infty} \exp \left( -t + \beta \left( x + B(t/\beta)^{1/q} \right)^q \right) dt \quad (x \geq 0),
\]
because \( g(x) \leq \exp(\beta x^q) \) for all \( x \geq 0 \).

By setting \( t = \beta x^q u \) and \( H(u) = u - (1 + Bu^{1/q})^q \), we have
\[
(2.2) \quad \int_0^{\infty} \exp \left( -t + \beta \left( x + B(t/\beta)^{1/q} \right)^q \right) dt = \beta x^q \int_0^{\infty} \exp (-\beta x^q H(u)) du.
\]
Since \( q > 1 \) and \( B = (\alpha p)^{1/p} (q\beta)^{1/q} < 1 \), the equation \( H'(u) = 0 \) has exactly one positive root \( u_0 \); and we have \( H(u_0) = -(1 - B^p)^{1-q} = -b \), because \( p^{-1} + q^{-1} = 1 \). Moreover, \( H''(u) > 0 \) for all \( u > 0 \), \( H \) is analytic in the interval \( 0 < u < \infty \) and \( \int_0^{\infty} e^{-H(u)} du < \infty \). Hence, by an application of the Laplace method [2, Chapter 4], we obtain
\[
\int_0^{\infty} \exp (-\beta x^q H(u)) du = \sqrt{\frac{2\pi}{\beta H''(u_0)}} x^{-q/2} \exp \left( b \beta x^q \right) (1 + O(x^{-q})) \quad (x \to \infty).
\]
Now, our assertion follows from (2.1) and (2.2). □
3. An Example

For convenience, we denote by \( \mathcal{G} \) the set of ordered pairs \((p, \alpha)\) of real numbers such that \( p > 1 \) and \( \alpha > 0 \). When \((p, \alpha) \in \mathcal{G}\), we define \((p, \alpha)^* \in \mathcal{G}\) by

\[
(p, \alpha)^* = \left( p(p-1)^{-1}, p^{-1}(p-1)(p\alpha)^{-1/(p-1)} \right).
\]

If \((p, \alpha) \in \mathcal{G}\) and \((p, \alpha)^* = (q, \beta)\), then we have

\[
p^{-1} + q^{-1} = 1 \quad \text{and} \quad (p\alpha)^{1/p}(q\beta)^{1/q} = 1,
\]

and hence \((q, \beta)^* = (p, \alpha)\).

When \((p, \alpha) \in \mathcal{G}\) with \((p, \alpha)^* = (q, \beta)\), we define the entire function \(f_{(p, \alpha)}\) by

\[
f_{(p, \alpha)}(z) = \int_0^\infty \exp(-\beta t^q + zt) \, dt \quad (z \in \mathbb{C}).
\]

A direct calculation shows that

\[
(3.1) \quad f_{(p, \alpha)} = \frac{1}{q} \sum_{n=0}^{\infty} \beta^{-n+1} \Gamma \left( \frac{n+1}{q} \right) \frac{M_n}{n!}.
\]

The following lemma implies that \(f_{(p, \alpha)}\) is of order \(p\) and type \(\alpha\).

**Lemma 3.1.** Let \((p, \alpha) \in \mathcal{G}\). Then there is a constant \(C > 0\) such that

\[
f_{(p, \alpha)}(x) = Cx^{\frac{p}{q}-1} \exp(\alpha x^p) \left( 1 + O \left( x^{-p} \right) \right) \quad (x \to \infty).
\]

**Proof.** Let \((p, \alpha)^* = (q, \beta)\), and \(H(u) = \beta u^q - u\). Then we have \(\int_0^\infty e^{-H(u)} \, du < \infty\), \(H\) is analytic in the interval \(0 < u < \infty\), \(H''(u) > 0\) for all \(u > 0\), \(H(p\alpha) = -\alpha\), \(H'(p\alpha) = 0\), and

\[
-\beta t^q + xt = -x^p H \left( \frac{t}{x^{p-1}} \right) \quad (x > 0, \ t > 0).
\]

Hence an application of the Laplace method [2, Chapter 4] gives

\[
f_{(p, \alpha)}(x) = \sqrt{\frac{2\pi}{H''(p\alpha)}} x^{\frac{p}{q}-1} \exp(\alpha x^p) \left( 1 + O \left( x^{-p} \right) \right) \quad (x \to \infty). \quad \Box
\]

Let \((p, \alpha), (q, \beta) \in \mathcal{G}\). It follows from Stirling’s formula and (3.1) that if \(p^{-1} + q^{-1} < 1\), or \(p^{-1} + q^{-1} = 1\) and \((p\alpha)^{1/p}(q\beta)^{1/q} \geq 1\), then

\[
\sum_{m=0}^{\infty} \frac{D^m f_{(p, \alpha)}(0)}{m!} D^m f_{(q, \beta)}(0) = \infty.
\]
Therefore the assumptions in Theorem 1.1 are necessary.

Next, suppose that \( p^{-1} + q^{-1} > 1 \). Then \( f_{(p,\alpha)}(D)f_{(q,\beta)} \) is well defined, by Theorem 1.1. From (3.1) and Lemma 3.1, there is a constant \( C > 0 \) such that

\[
f_{(p,\alpha)}(D)f_{(q,\beta)}(x) \geq Cx^{\frac{q}{2} - 1}\exp(\beta x^q)
\]

for all sufficiently large \( x > 0 \). From this inequality and Theorem 1.1, it follows that the entire function \( f_{(p,\alpha)}(D)f_{(q,\beta)} \) is of order \( q \) and type \( \beta \).

Finally, suppose that \( p^{-1} + q^{-1} = 1 \) and \( (p\alpha)^{1/p}(q\beta)^{1/q} < 1 \). Then \( (q, \beta)^* = (p, \alpha_1) \) for some \( \alpha_1 \) with \( \alpha < \alpha_1 \). Since \( f_{(p,\alpha)}(D)f_{(q,\beta)} \) is well defined by Theorem 1.1, we have

\[
f_{(p,\alpha)}(D)f_{(q,\beta)}(z) = \int_0^\infty f_{(p,\alpha)}(t)\exp(-\alpha_1 t^p + zt) \, dt \quad (z \in \mathbb{C}).
\]

Let \( \alpha_2 < \alpha \) be arbitrary. Then there is a constant \( C_1 > 0 \) such that

\[
f_{(p,\alpha)}(t) \geq C_1 \exp(\alpha_2 t^p) \quad (t \geq 0),
\]

by Lemma 3.1, and hence we have

\[
(3.2) \quad f_{(p,\alpha)}(D)f_{(q,\beta)}(x) \geq C_1 \int_0^\infty \exp(-\alpha_1 - \alpha_2 t^p + xt) \, dt \quad (x \geq 0).
\]

Again, by Lemma 3.1, there is a constant \( C_2 > 0 \) such that

\[
(3.3) \quad \int_0^\infty \exp(-\alpha_1 - \alpha_2 t^p + xt) \, dt = C_2 x^{\frac{q}{2} - 1}\exp(b_2 x^q) \left(1 + O(x^{-q})\right)
\]

\[
(x \to \infty),
\]

where

\[
b_2 = \left[1 - \left((p\alpha_2)^{1/p}(q\beta)^{1/q}\right)^{p}\right]^{1-q}.
\]

Since we can take \( \alpha_2 \) arbitrarily close to \( \alpha \), (3.2) and (3.3) imply that the entire function \( f_{(p,\alpha)}(D)f_{(q,\beta)} \) is of order \( q \) and type \( b\beta \), where

\[
b = \left[1 - \left((p\alpha)^{1/p}(q\beta)^{1/q}\right)^{p}\right]^{1-q}.
\]

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References


**Department of Mathematics, Sejong University, Seoul 143–747, Korea**  
*E-mail address*: yjcha@sejong.ac.kr

**Department of Mathematics, Yonsei University, Seoul 120–749, Korea**  
*E-mail address*: haseo@yonsei.ac.kr

**Department of Mathematics, Sejong University, Seoul 143–747, Korea**  
*E-mail address*: kimyo@sejong.ac.kr