On the Zeros of Degree One L -functions from the Extended Selberg Class

Haseo Ki and Yoonbok Lee

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Abstract

We investigate the zeros of degree one L -functions from the extended Selberg class off the critical line.

1 Introduction

In [14], Selberg introduced the class $\mathcal{S}$ consists of the functions $F(s)$ satisfying the following conditions.

1. (Dirichlet series) For $\sigma > 1$, $F(s)$ is an absolutely convergent Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \quad (s = \sigma + it).$$

2. (Analytic continuation) For some integer $m \geq 0$, $(s-1)^m F(s)$ is an entire function of finite order.
3. (Functional equation) $F(s)$ satisfies a functional equation of the form

$$\Phi(s) = \omega \bar{\Phi}(1-s),$$

where

$$\Phi(s) = Q^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j) F(s)$$

with $\bar{\Phi}(s) = \overline{\Phi(\bar{s})}$, $Q > 0$, $\lambda_j > 0$, $\operatorname{Re} \mu_j \geq 0$ and $|\omega| = 1$.

4. (Ramanujan hypothesis) For every $\epsilon > 0$, $a(n) \ll n^\epsilon$.

5. (Euler product) For σ sufficiently large,

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s} \quad (s = \sigma + it),$$

where $b(n) = 0$ unless n is a positive power of a prime, and $b(n) \ll n^\theta$ for some $\theta < \frac{1}{2}$.

For a function $F(s)$ in the Selberg class $\mathcal{S}$, we define $d = 2 \sum_j \lambda_j$ to be the degree of the function. We denote by $\mathcal{S}_d$ the subclass of functions with degree d in the Selberg class $\mathcal{S}$. We note that the structure of $\mathcal{S}_d$ has been completely determined for $0 \leq d \leq 1$. From the work of Conrey and Ghosh [4], we have $\mathcal{S}_0 = \{1\}$ and $\mathcal{S}_d = \emptyset$ for $0 < d < 1$. For $d = 1$, the functions $F \in \mathcal{S}_1$ are of forms $F(s) = \zeta(s)$ or $F(s) = L(s + i\theta, \chi)$ with a primitive Dirichlet character χ together with $\theta \in \mathbb{R}$. On the other hand, we denote by $\mathcal{S}^\#$ the extended Selberg class of functions satisfying the conditions 1-3, and we similarly define $\mathcal{S}_d^\#$ as $\mathcal{S}_d$. Theorems 1 and 2 in [10] describe the structure of $\mathcal{S}_d^\#$ for $0 \leq d \leq 1$.

If $d = 0$, the functional equation is $Q^s F(s) = \omega Q^{1-s} \bar{F}(1-s)$. The proof of Theorem 3.1 [4] shows that the Dirichlet series $F(s) = \sum_n \frac{a(n)}{n^s} \in \mathcal{S}_0^\#$ is absolutely convergent in the whole complex plane. Thus, we have

$$\sum_{n=1}^{\infty} a(n) \left(\frac{Q^2}{n} \right)^s = \omega Q \sum_{n=1}^{\infty} \frac{\overline{a(n)}}{n} n^s.$$

We let $q = Q^2$, then $a(n) = 0$ for $n \nmid q$. For $n|q$, we have

$$a(n) = \frac{\omega n}{\sqrt{q}} \overline{a\left(\frac{q}{n}\right)}. \quad (1.1)$$

Theorem A (Theorem 3.1 of [4]). *1. If $0 < d < 1$, then $\mathcal{S}_d^\# = \emptyset$. If $F \in \mathcal{S}_0^\#$, then $q \in \mathbb{N}$, the pair (q, ω) is an invariant of $F(s)$ and $\mathcal{S}_0^\#$ is the disjoint union of the subclasses $\mathcal{S}_0^\#(q, \omega)$, with $q \in \mathbb{N}$ and $|\omega| = 1$.*

2. Every $F \in \mathcal{S}_0^\#(q, \omega)$, with q and ω as above, is a Dirichlet polynomial of the form

$$F(s) = \sum_{n|q} \frac{a(n)}{n^s}.$$

For $d = 1$, we use the notation

$$\begin{aligned} \beta &= \prod_{j=1}^r \lambda_j^{-2\lambda_j}, & \xi &= 2 \sum_{j=1}^r \left(\mu_j - \frac{1}{2} \right) = \eta + i\theta, & q &= \frac{2\pi Q^2}{\beta}, \\ \omega^* &= \omega e^{-i\pi(\eta+1)/2} \left(\frac{Q^2}{\beta} \right)^{i\theta} \prod_{j=1}^r \lambda_j^{-2i\text{Im} \mu_j}. \end{aligned}$$

If χ is a Dirichlet character mod q , we denote by f_χ its conductor, and by χ^* the primitive character inducing χ . We denote by ω_{χ^*} and Q_{χ^*} the ω -factor and the Q -factor in the standard functional equation for $L(s, \chi^*)$, i.e., $\omega_{\chi^*} = \tau(\chi^*)/i^a \sqrt{f_\chi}$, where $\tau(\chi^*)$ is the Gauss sum, $a = 0$ if $\chi(-1) = 1$ and $a = 1$ if $\chi(-1) = -1$, and $Q_{\chi^*} = \sqrt{f_\chi/\pi}$. Moreover, we write

$$\mathfrak{X}(q, \xi) = \left\{ \begin{array}{ll} \{\chi \pmod{q} | \chi(-1) = 1\} & \text{if } \eta = -1 \\ \{\chi \pmod{q} | \chi(-1) = -1\} & \text{if } \eta = 0 \end{array} \right\}.$$

χ_0 denotes the principal character mod q .

Theorem B (Theorem 2 of [10]). *1. If $F \in \mathcal{S}_1^\#$, then $q \in \mathbb{N}$ and $\eta \in \{-1, 0\}$. The triple (q, ξ, ω^*) is an invariant of $F(s)$, and $\mathcal{S}_1^\#$ is the disjoint union of the subclasses $\mathcal{S}_1^\#(q, \xi, \omega^*)$ with $q \in \mathbb{N}$, $\eta \in \{-1, 0\}$, $\theta \in \mathbb{R}$ and $|\omega^*| = 1$. Moreover, $a(n)n^{i\theta}$ is periodic with period q .*

2. Every $F \in \mathcal{S}_1^\#(q, \xi, \omega^)$, with q, ξ and ω^* as above, can be uniquely written as*

$$F(s) = \sum_{\chi \in \mathfrak{X}(q, \xi)} P_\chi(s + i\theta) L(s + i\theta, \chi^*),$$

where $P_\chi \in \mathcal{S}_0^\#(q/f_\chi, \omega^* \bar{\omega}_{\chi^*})$. Moreover, $P_{\chi_0}(1) = 0$ if $\theta \neq 0$.

Bombieri and Hejhal [2] studied the distribution of zeros of the linear combination $F(s) = \sum_{j=1}^J b_j e^{i\alpha_j} L_j(s)$ of various L -functions with a same gamma factor. Assuming the orthonormality condition of $a_j(p)$, the generalized Riemann hypothesis for $L_j(s)$ and a weak condition of the spacing of zeros of $L_j(s)$, they proved that almost all zeros of $F(s)$ are simple and on the critical line $\text{Re}(s) = \frac{1}{2}$. Hejhal [7] studied the behavior of zeros of $F(s)$ near the critical line and announced that the true order of the number of zeros of $F(s)$ in $\text{Re } s \geq \sigma$ and $T \leq \text{Im } s \leq T + H$ is

$$\frac{H}{(\sigma - \frac{1}{2}) \sqrt{\log \log T}}$$

for $\frac{1}{2} + \frac{(\log \log T)^\kappa}{\log T} \leq \sigma \leq \frac{1}{2} + (\log T)^{-\delta}$, $c_1 T^w \leq H \leq c_2 T$, $\kappa > 2$ with small possible exceptions of $\{b_j\}_{j=1}^J$. Note that this result for the special case $J = 2$ was also justified by the same author [6].

Recently, the second author [12] investigated the off-line zeros of Epstein zeta function $E(s, Q)$ associated to quadratic form $Q(x, y) = ax^2 + bxy + cy^2$, $a > 0$, $b^2 - 4ac < 0$, $a, b, c \in \mathbb{Z}$. It is a classical example that belongs to the class $\mathcal{S}_2^\#$. We get the number of zeros $N_E(\sigma_1, \sigma_2; 0, T)$ on the rectangular region $\sigma_1 < \text{Re } s < \sigma_2$, $0 < \text{Im } s < T$ to be $c(\sigma_1, \sigma_2)T + o(T)$ for $\frac{1}{2} < \sigma_1 < \sigma_2$ which improves Voronin's result [15] (or Chapter 7 of [11]) based on the joint universality theorem for Hecke L -functions. For $\mathcal{S}_1^\#$, Kaczorowski and Kulas [9] and Saias and Weingartner [13] proved the true order of the number of off-line zeros inside and outside the critical strip, respectively. [9] is based on the joint universality theorem for Dirichlet L -functions inside the critical strip and [13] is based on the idea of

Davenport and Heilbronn [5]. The main purpose of this paper is improving their results by obtaining an asymptotic formula of $N_F(\sigma_1, \sigma_2; 0, T)$ for $F \in \mathcal{S}_1^\#$ by means of the classical theory of mean motions.

By Theorems A and B, we can let the function $E(s + i\theta) \in \mathcal{S}_1^\#$ as

$$E(s) = \sum_{j=1}^J h_j(p_1^{-s}, \dots, p_k^{-s}) \prod_{p > p_k} \left(1 - \frac{\chi_j(p)}{p^s}\right)^{-1} \quad (1.2)$$

for some integer $k > 0$, where

$$h_j(x_1, \dots, x_k) = \tilde{h}_j(x_1, \dots, x_k) \prod_{l \leq k} (1 - \chi_j(p_l)x_l)^{-1}$$

and $\tilde{h}_j$ is a polynomial of k variables. Let

$$E_n(s) = \sum_{j=1}^J h_j(p_1^{-s}, \dots, p_k^{-s}) \prod_{p_k < p \leq p_n} \left(1 - \frac{\chi_j(p)}{p^s}\right)^{-1}$$

for $n > k$. Then, $E_n(s)$ converges in the mean with index 2 towards $E(s)$ in $[\frac{1}{2}, \infty]$ by Parseval's identity for almost periodic functions, i.e.,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_1^T \int_\alpha^\beta |E(\sigma + it) - E_n(\sigma + it)|^2 d\sigma dt \rightarrow 0$$

as $n \rightarrow \infty$ for any reduced strip $\frac{1}{2} < \alpha < \beta$. We may find the way of a proof in Proposition 2.3 of [12]. Applying Lemma 2.3 to $E_n(s)$, we get an asymptotic formula for $N_{E_n}(\sigma_1, \sigma_2; 0, T)$. The theory of mean motions partially preserves this property through the convergence in the mean with index $p > 0$.

If $J = 1$, then we encounter with the well-known conjecture. In this paper, we only consider the case $J > 1$.

We consider $\mathcal{S}_1^\#(p, \xi, \omega^*)$ for a prime p . By (1.1) and Theorem B, we have

$$\tilde{h}_j = a_j(1) \quad \text{or} \quad a_j(1) + \frac{\overline{\omega a_j(1)}}{p^{s-\frac{1}{2}}},$$

and as a result $\tilde{h}_j \neq 0$ for $\text{Re } s > \frac{1}{2}$. In this case, the method in [12] works, and we have the following theorem.

Theorem 1. *Let $E(s + i\theta) \in \mathcal{S}_1^\#(p, \xi, \omega^*)$ for p prime and $|\omega| = 1$, and let $\frac{1}{2} < \sigma_1 < \sigma_2$. Suppose $J > 1$ in (1.2). Then, we have*

$$N_E(\sigma_1, \sigma_2; 0, T) = c(\sigma_1, \sigma_2)T + o(T)$$

as $T \rightarrow \infty$. The constant $c(\sigma_1, \sigma_2)$ can be represented as an integral $\int_{\sigma_1}^{\sigma_2} H_\sigma(0) d\sigma$ for the density function $H_\sigma(x)$ of some distribution μ_σ , and $c(\sigma_1, \sigma_2) > 0$ if $\frac{1}{2} < \sigma_1 \leq 1$. In particular, for $\sigma_0 > \frac{1}{2}$, the number of zeros on the line segment $\text{Re } s = \sigma_0$, $0 < \text{Im } s < T$ is $o(T)$.

When q is a prime power, then $\tilde{h}_j$ are polynomials of the same single variable by Theorems A(2) and B(2). If these polynomials have the same factor with $cT + o(T)$ zeros on the line segment $\operatorname{Re} s = \sigma_0$, $0 < \operatorname{Im} s < T$ for some $\frac{1}{2} < \sigma_0 < 1$, then we cannot expect the integral form of the constant $c(\sigma_1, \sigma_2)$ in general. Indeed, we may take $\tilde{h}_j(p^{-s}) = 1 + 2p^{\frac{3}{4}-s} + p^{1-2s}$ by letting $\omega = a(1) = 1$, and $a(p) = 2p^{-\frac{3}{4}}$. Then, it has $\frac{\log p}{2\pi}T + O(1)$ zeros on $\operatorname{Re} s = \frac{\log\left(p^{\frac{3}{4}} - \sqrt{p^{\frac{3}{2}} - p}\right)}{\log p}$, $0 < \operatorname{Im} s < T$. We still have the following.

Theorem 2. *Let $E(s + i\theta) \in \mathcal{S}_1^\#(q, \xi, \omega^*)$ for q prime power, and let $\frac{1}{2} < \sigma_1 < \sigma_2$. Suppose $J > 1$ in (1.2). Then, we have*

$$N_E(\sigma_1, \sigma_2; 0, T) = c(\sigma_1, \sigma_2)T + o(T)$$

as $T \rightarrow \infty$, and $c(\sigma_1, \sigma_2) > 0$ if $\frac{1}{2} < \sigma_1 \leq 1$. Suppose that the closed interval $[\sigma_1, \sigma_2]$ does not contain the real part of exceptional points satisfying $h_j = 0$. Then, the constant $c(\sigma_1, \sigma_2)$ can be represented as an integral $\int_{\sigma_1}^{\sigma_2} H_\sigma(0) d\sigma$ for the density function $H_\sigma(x)$ of some distribution μ_σ . In this case for $\sigma_0 \in [\sigma_1, \sigma_2]$, the number of zeros on the line segment $\operatorname{Re} s = \sigma_0$, $0 < \operatorname{Im} s < T$ is $o(T)$.

For general q , we could also prove the similar theorem, although it is not easy to classify the common zeros of $\tilde{h}_j$ with multiple variables. We will discuss and prove a general theorem in the section 3.

2 Lemmas

We begin with the work of Jessen and Tornehave [8] that explains zeros of Dirichlet series in the region of its absolute convergence. For the basic theory of almost periodic functions, we refer to [1].

Lemma 2.1 (Theorem 8 of [8]). *A function $f(s)$ almost periodic in $[\alpha, \beta]$ and not identically zero has no zeros in the sub-strip $(\alpha \leq) \alpha_0 < \sigma, \beta_0 (\leq \beta)$, if and only if its Jensen function $\varphi(\sigma)$ is linear in the interval (α_0, β_0) .*

Lemma 2.2 (Theorem 31 of [8]). *For an ordinary Dirichlet series*

$$f(s) = \sum_{n=n_0}^{\infty} \frac{a_n}{n^s}, \quad a_{n_0} \neq 0$$

with the uniform convergence abscissa α , the Jensen function $\varphi(\sigma)$ possesses on every half-line $\sigma > \alpha_1 (> \alpha)$ only a finite number of linearity intervals and a finite number of points of non-differentiability. The values of $\varphi'(\sigma)$ in the linearity intervals belong to the set of numbers $-\log n$, $n \geq n_0$. For $\sigma > (\text{some})\sigma_0$, we have

$$\varphi(\sigma) = -(\log n_0)\sigma + \log |a_{n_0}|.$$

For an arbitrary strip (σ_1, σ_2) , where $\alpha < \sigma_1 < \sigma_2 < \infty$, the relative frequency $H(\sigma_1, \sigma_2)$ of zeros exists and is determined by

$$H(\sigma_1, \sigma_2) = \frac{1}{2\pi}(\varphi'(\sigma_2-) - \varphi'(\sigma_1+)).$$

The following lemma guarantees the existence of second derivative of Jensen functions for almost periodic functions and gives another representation by a certain distribution. The proof can be found in §9 of [3].

Lemma 2.3 (Proposition 2.1 of [12]). *Let $f(s)$ be almost periodic in the strip $[\alpha, \beta]$ and not identically zero. Let ν_σ be the asymptotic distribution function of $f(\sigma + it)$ with respect to $|f'(\sigma + it)|^2$. Suppose ν_σ is absolutely continuous for every σ and its density $G_\sigma(x)$ is a continuous function of x and σ . Then the Jensen function $\varphi_{f-x}(\sigma)$ is twice differentiable with the second derivative $\varphi''_{f-x}(\sigma) = 2\pi G_\sigma(x)$.*

The next lemma is the extension of Lemma 2.3 which is applicable inside the critical strip and which plays the main role of this method.

Lemma 2.4 (Theorem 1 of [3]). *Let $-\infty \leq \alpha < \alpha_0 < \beta_0 < \beta \leq \infty$ and let $f_1(s), f_2(s), \dots$ be a sequence of functions almost periodic in $[\alpha, \beta]$ converging uniformly in $[\alpha_0, \beta_0]$ towards a function $f(s)$. Suppose that none of the functions is identically zero and $f(s)$ may be continued as a regular function in the half strip $\alpha < \sigma < \beta$, $t > \gamma_0$, and that $f_n(s)$ converges in mean with an index $p > 0$ towards $f(s)$ in $[\alpha, \beta]$. Then the Jensen function*

$$\varphi_f(\sigma) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{\gamma}^T \log |f(\sigma + it)| dt$$

exists uniformly in $[\alpha, \beta]$ for some $\gamma > \gamma_0$, and $\varphi_{f_n}(\sigma)$ converges uniformly in $[\alpha, \beta]$ towards $\varphi_f(\sigma)$ as $n \rightarrow \infty$. The function $\varphi_f(\sigma)$ is convex in (α, β) , and for every strip (σ_1, σ_2) where $\alpha < \sigma_1 < \sigma_2 < \beta$, the two relative frequencies of zeros defined by

$$\underline{H}_f(\sigma_1, \sigma_2) = \liminf_{T \rightarrow \infty} \frac{1}{T} N_f(\sigma_1, \sigma_2; \gamma, T),$$

$$\overline{H}_f(\sigma_1, \sigma_2) = \limsup_{T \rightarrow \infty} \frac{1}{T} N_f(\sigma_1, \sigma_2; \gamma, T),$$

satisfy the inequalities

$$\frac{1}{2\pi}(\varphi'_f(\sigma_2-) - \varphi'_f(\sigma_1+)) \leq \underline{H}_f(\sigma_1, \sigma_2) \leq \overline{H}_f(\sigma_1, \sigma_2) \leq \frac{1}{2\pi}(\varphi'_f(\sigma_2+) - \varphi'_f(\sigma_1-))$$

Suppose further that $\varphi_f(\sigma)$ is twice differentiable. Then, we have

$$N_f(\sigma_1, \sigma_2; 0, T) = \frac{T}{2\pi} \int_{\sigma_1}^{\sigma_2} \varphi''_f(\sigma) d\sigma + o(T)$$

for $\alpha < \sigma_1 < \sigma_2 < \beta$ as $T \rightarrow \infty$.

Together with the above lemmas, we investigate the Fourier transforms of certain distributions. For our purpose, we need the two more lemmas.

We use the following notations:

$$\begin{aligned}\mathfrak{L}_n(\sigma, \Theta; \chi_j) &= L_k(\sigma, \theta; \chi_j) L_{k,n}(\sigma, \vartheta; \chi_j), \\ L_k(\sigma, \theta; \chi_j) &= h_j(p_1^{-\sigma} e^{2\pi i \theta_1}, \dots, p_k^{-\sigma} e^{2\pi i \theta_k}), \\ L_{k,n}(\sigma, \vartheta; \chi_j) &= \prod_{k < l \leq n} \left(1 - \frac{\chi_j(p_l) e^{2\pi i \vartheta_l}}{p_l^\sigma}\right)^{-1}, \\ M_{n,\sigma}(\vartheta) &= (\log L_{k,n}(\sigma, \vartheta; \chi_1), \dots, \log L_{k,n}(\sigma, \vartheta; \chi_J)), \\ E_{n,\sigma}(\Theta) &= \sum_{j=1}^J \mathfrak{L}_n(\sigma, \Theta; \chi_j)\end{aligned}$$

for $n > k$, $\Theta = (\theta, \vartheta) \in [0, 1]^n$, $\theta = (\theta_1, \dots, \theta_k) \in [0, 1]^k$ and $\vartheta = (\vartheta_{k+1}, \dots, \vartheta_n) \in [0, 1]^{n-k}$. Let $\mu_{n,\sigma}$ be the distribution function of $E_{n,\sigma}$ with respect to $\left|\frac{\partial}{\partial \sigma} E_{n,\sigma}\right|^2$. Then, its Fourier transform is

$$\hat{\mu}_{n,\sigma}(y) = \int_{[0,1]^n} e^{i \sum_j \mathfrak{L}_n(\sigma, \Theta; \chi_j) \cdot y} \left| \sum_j \mathfrak{L}'_n(\sigma, \Theta; \chi_j) \right|^2 d\Theta.$$

Lemma 2.5. For $\sigma > \frac{1}{2}$, $\delta > 0$ and $j \leq J$, we define

$$A_{j,\sigma}(\delta) = \{\theta \in [0, 1]^k : |\tilde{h}_j(p_1^{-\sigma} e^{2\pi i \theta_1}, \dots, p_k^{-\sigma} e^{2\pi i \theta_k})| < \delta\}.$$

Then, for any integer $K \leq J$ we have

$$\hat{\mu}_{n,\sigma}(y) \ll \left| \bigcap_{r_1 < \dots < r_K \leq J} (A_{r_1,\sigma}(\delta) \cup \dots \cup A_{r_K,\sigma}(\delta)) \right| + |\delta y|^{-K}$$

as $|y| \rightarrow \infty$, where the corresponding constant does not depend on n .

Proof. We write

$$\hat{\mu}_{n,\sigma}(y) = \sum_{l_1, l_2} \int_{[0,1]^n} e^{i \sum_j \mathfrak{L}_n(\sigma, \Theta; \chi_j) \cdot y} \mathfrak{L}'_n(\sigma, \Theta; \chi_{l_1}) \overline{\mathfrak{L}'_n(\sigma, \Theta; \chi_{l_2})} d\Theta.$$

Define set functions

$$\begin{aligned}\lambda_{n,\sigma;l_1,l_2}(B) &= \int_{M_{n,\sigma}^{-1}(B)} \frac{L'_{k,n}}{L_{k,n}}(\sigma, \vartheta; \chi_{l_1}) \overline{\frac{L'_{k,n}}{L_{k,n}}(\sigma, \vartheta; \chi_{l_2})} d\vartheta \\ \lambda_{n,\sigma;l}(B) &= \int_{M_{n,\sigma}^{-1}(B)} \frac{L'_{k,n}}{L_{k,n}}(\sigma, \vartheta; \chi_l) d\vartheta \\ \lambda_{n,\sigma}(B) &= |M_{n,\sigma}^{-1}(B)|\end{aligned}$$

for any Borel sets $B \subset \mathbb{C}^J$. By the identity

$$a\bar{b} = \frac{1}{4} \sum_{m=1}^4 i^m |a + i^m b|^2 \quad a, b \in \mathbb{C},$$

one can prove that $\hat{\mu}_{n,\sigma}(y)$ is a linear combination of at most four absolutely continuous distribution functions. (See [12] for details.) Thus, we denote corresponding densities $G_{n,\sigma;l_1,l_2}(x)$, $G_{n,\sigma;l}(x)$, $G_{n,\sigma}(x)$ of $\lambda_{n,\sigma;l_1,l_2}$, $\lambda_{n,\sigma;l}$, $\lambda_{n,\sigma}$, respectively. By Theorem 6 of [3], the densities $G_{n,\sigma;l_1,l_2}(x)$, $G_{n,\sigma;l}(x)$, $G_{n,\sigma}(x)$ have a majorant of the form $K e^{-\lambda|x|^2}$, and their partial derivatives of order $\leq q$ have a majorant of the form $K_q e^{-\lambda|x|^2}$ for $n \geq n_q$. Thus, we have

$$\hat{\mu}_{n,\sigma}(y) = \sum_{l_1,l_2} \int_{[0,1]^k} \int_{\mathbb{C}^J} e^{i \sum_j (L_k(\sigma,\theta;\chi_j) e^{x_j}) \cdot y + x_{l_1} + \bar{x}_{l_2}} \mathfrak{G}_{n,\sigma;l_1,l_2}(x, \theta) dx d\theta,$$

where

$$\begin{aligned} \mathfrak{G}_{n,\sigma;l_1,l_2}(x, \theta) = & L'_k(\sigma, \theta; \chi_{l_1}) \overline{L'_k(\sigma, \theta; \chi_{l_2})} G_{n,\sigma}(x) + L'_k(\sigma, \theta; \chi_{l_1}) \overline{L_k(\sigma, \theta; \chi_{l_2})} G_{n,\sigma;l_2}(x) \\ & + L_k(\sigma, \theta; \chi_{l_1}) \overline{L'_k(\sigma, \theta; \chi_{l_2})} G_{n,\sigma;l_1}(x) + L_k(\sigma, \theta; \chi_{l_1}) \overline{L_k(\sigma, \theta; \chi_{l_2})} G_{n,\sigma;l_1,l_2}(x). \end{aligned}$$

We only consider the first term $L'_k(\sigma, \theta; \chi_{l_1}) \overline{L'_k(\sigma, \theta; \chi_{l_2})} G_{n,\sigma}(x)$, since the others can be treated similarly. If $\theta \notin A_{j,\sigma}(\delta)$ for K -many j , integration by parts with respect to such x_j gives

$$\int_{\mathbb{C}^J} e^{i \sum_j (L_k(\sigma,\theta;\chi_j) e^{x_j}) \cdot y + x_{l_1} + \bar{x}_{l_2}} G_{n,\sigma}(x) dx = O(|\delta y|^{-K}). \quad (2.1)$$

For the other θ , we give a trivial upperbound by its measure, and we have

$$\hat{\mu}_{n,\sigma}(y) \ll \left| \bigcap_{r_1 < \dots < r_K \leq J} (A_{r_1,\sigma}(\delta) \cup \dots \cup A_{r_K,\sigma}(\delta)) \right| + |\delta y|^{-K},$$

where the corresponding constant does not depend on n as $y \rightarrow \infty$.

So, it is enough to show the equation (2.1). We decompose

$$\begin{aligned} & \int_{\mathbb{C}^J} e^{i \sum_j (L_k(\sigma,\theta;\chi_j) e^{x_j}) \cdot y + x_{l_1} + \bar{x}_{l_2}} G_{n,\sigma}(x) dx \\ &= \sum_{m \in \mathbb{Z}^J} \int_{(\mathbb{R} \times [0, 2\pi])^J} e^{i \sum_j e^{x_j} \cdot (\overline{L_k(\sigma,\theta;\chi_j)} y) + x_{l_1} + \bar{x}_{l_2}} G_{n,\sigma}(x + 2\pi m i) dx. \end{aligned}$$

Changing of variables $e^{x_j} = r_j e^{z_j}$ with Jacobian r_j^{-1} gives

$$= \sum_{m \in \mathbb{Z}^J} \int_{[0, 2\pi]^J} \int_{(0, \infty)^J} e^{i \sum_j r_j e^{z_j} \cdot (\overline{L_k(\sigma,\theta;\chi_j)} y) + z_{l_1} - \bar{z}_{l_2}} r_{l_1} r_{l_2} \prod_j r_j^{-1} G_{n,\sigma}(\log r + i(z + 2\pi m)) dr dz,$$

where $r = (r_1, \dots, r_J)$, $z = (z_1, \dots, z_J)$, and $\log r = (\log r_1, \dots, \log r_J)$. Consider the integration

$$\begin{aligned} & \int_0^{2\pi} \int_0^\infty e^{ir_j e^{z_j} \cdot (\overline{L_k(\sigma, \theta; \chi_j)} y) + z_{l_1} - z_{l_2}} r_{l_1} r_{l_2} r_j^{-1} G_{n, \sigma}(\log r + i(z + 2\pi m)) dr_j dz_j \\ &= \int_0^{2\pi} \int_0^\infty e^{ir_j |L_k(\sigma, \theta; \chi_j) y| \cos(z_j - \alpha_j) + z_{l_1} - z_{l_2}} r_{l_1} r_{l_2} r_j^{-1} G_{n, \sigma}(\log r + i(z + 2\pi m)) dr_j dz_j \end{aligned}$$

for some α_j . For $\theta \notin A_{j, \sigma}(\delta)$, we integrate by parts with respect to z_j for $|\cos(z_j - \alpha_j)| < \frac{1}{2}$, and with respect to r_j for $|\cos(z_j - \alpha_j)| > \frac{1}{2}$. With the uniform upperbound $K_q e^{-\lambda|x|^2}$ of partial derivatives of G of order $\leq q$, we have the equation (2.1). □

Lemma 2.6. $\hat{\mu}_{n, \sigma}(y)$ converges uniformly for every disc $|y| \leq a$ and $\frac{1}{2} < \sigma_1 \leq \sigma \leq \sigma_2$

Proof. By definition, we have

$$\hat{\mu}_{n+1, \sigma}(y) = \int_{[0,1]^n} \int_0^1 e^{iE_{n+1, \sigma}(\Theta, u) \cdot y} \left| \frac{\partial}{\partial \sigma} E_{n+1, \sigma}(\Theta, u) \right|^2 du d\Theta.$$

We get

$$\begin{aligned} & \int_0^1 e^{iE_{n+1, \sigma}(\Theta, u) \cdot y} \left| \frac{\partial}{\partial \sigma} E_{n+1, \sigma}(\Theta, u) \right|^2 du = \int_0^1 e^{iE_{n+1, \sigma}(\Theta, u) \cdot y} du \left| \frac{\partial}{\partial \sigma} E_{n, \sigma}(\Theta) \right|^2 \\ &+ \int_0^1 e^{iE_{n+1, \sigma}(\Theta, u) \cdot y} 2\operatorname{Re} \left[\frac{\partial}{\partial \sigma} \overline{E_{n, \sigma}(\Theta)} e^{2\pi i u} \frac{\partial}{\partial \sigma} \sum_{j=1}^J h_j(\dots) \prod_{k < j \leq n} (\dots)^{-1} \frac{\chi_j(p_{n+1})}{p_{n+1}^\sigma} \right] du \\ &+ O\left(\frac{F_n(\sigma, \Theta)^2}{p_{n+1}^{2\sigma}}\right), \end{aligned}$$

where

$$F_n(\sigma, \Theta) = \sum_{j=1}^J \prod_{k < l \leq n} \left| 1 - \frac{\chi_j(p_l)}{p_l^\sigma} e^{2\pi i \vartheta_l} \right|^{-1}.$$

By $e^{ix} = 1 + ix + O(|x|^2)$ ($x \in \mathbb{R}$), we have

$$\begin{aligned} & \int_0^1 e^{iE_{n+1, \sigma}(\Theta, u) \cdot y} du \\ &= \int_0^1 e^{iE_{n, \sigma}(\Theta) \cdot y} (1 + i(E_{n+1, \sigma}(\Theta, u) - E_{n, \sigma}(\Theta)) \cdot y) du + O\left(\frac{F_n(\sigma, \Theta)^2}{p_{n+1}^{2\sigma}}\right) \\ &= e^{iE_{n, \sigma}(\Theta) \cdot y} + O\left(\frac{F_n(\sigma, \Theta)^2}{p_{n+1}^{2\sigma}}\right). \end{aligned}$$

By $e^{ix} = 1 + O(|x|)$ ($x \in \mathbb{R}$), we have

$$\begin{aligned} \int_0^1 e^{iE_{n+1,\sigma}(\Theta,u) \cdot y \pm 2\pi i u} du &= \int_0^1 e^{iE_{n,\sigma}(\Theta) \cdot y \pm 2\pi i u} du + O\left(\frac{F_n(\sigma, \Theta)}{p_{n+1}^\sigma}\right) \\ &= O\left(\frac{F_n(\sigma, \Theta)}{p_{n+1}^\sigma}\right). \end{aligned}$$

Combining the above equations yields

$$\begin{aligned} \int_0^1 e^{iE_{n+1,\sigma}(\Theta,u) \cdot y} \left| \frac{\partial}{\partial \sigma} E_{n+1,\sigma}(\Theta, u) \right|^2 du &= e^{iE_{n,\sigma}(\Theta) \cdot y} \left| \frac{\partial}{\partial \sigma} E_{n,\sigma}(\Theta) \right|^2 \\ &\quad + O\left(\frac{F_n(\sigma, \Theta)^2 + F_n(\sigma, \Theta)^3 + F_n(\sigma, \Theta)^4}{p_{n+1}^{2\sigma}} \log p_{n+1}\right). \end{aligned}$$

Thus, we have

$$\hat{\mu}_{n+1,\sigma}(y) - \hat{\mu}_{n,\sigma}(y) = O(p_{n+1}^{-2\sigma} \log p_{n+1})$$

and

$$\hat{\mu}_{m,\sigma}(y) - \hat{\mu}_{n,\sigma}(y) = O(p_n^{1-2\sigma_1})$$

for $m > n > k$. Thus, Lemma 2.6 follows. $\square$

3 Main results

We divide the cases $J = 2$ and $J \geq 3$. For $J = 2$, our function is the sum of two spoiled Euler products $f_1(s) + f_2(s)$. We then apply the theory of value distribution for $f_1(s)$ and $\frac{f_2}{f_1}(s)$.

Proposition 3.1. *Let $J = 2$ and $\frac{1}{2} < \sigma_1 < \sigma_2$. Suppose that $h_j(p_1^{-\sigma} e^{2\pi i \theta_1}, \dots, p_k^{-\sigma} e^{2\pi i \theta_k}) \neq 0$ for $j = 1, 2$, $\sigma_1 \leq \sigma \leq \sigma_2$, and $\theta \in [0, 1]^k$. Then, we have*

$$N_E(\sigma_1, \sigma_2; 0, T) = T \int_{\sigma_1}^{\sigma_2} H_\sigma(-1) d\sigma + o(T),$$

where $H_\sigma(x)$ is the density for some distribution function μ_σ . Moreover, $H_\sigma(x) > 0$ for $\frac{1}{2} < \sigma \leq 1$.

Proof. By Lemma 2.4, $\varphi_{E_n}(\sigma)$ converges uniformly to $\varphi_E(\sigma)$ on $[\frac{1}{2}, \infty)$. If $\varphi_E(\sigma)$ is twice differentiable, then we have

$$N_E(\sigma_1, \sigma_2; 0, T) = \frac{T}{2\pi} \int_{\sigma_1}^{\sigma_2} \varphi_E''(\sigma) d\sigma + o(T).$$

By direct calculation, we have

$$\varphi_{E_n}(\sigma) = \varphi_{h_2}(\sigma) + \varphi_{\tilde{L}_{n+1}}(\sigma),$$

where

$$\tilde{L}_n(s) = \frac{h_1}{h_2}(p_1^{-s}, \dots, p_k^{-s}) \prod_{p_k < p \leq p_n} \frac{1 - \frac{\chi_2(p)}{p^s}}{1 - \frac{\chi_1(p)}{p^s}}.$$

By Lemma 2.1, we have $\varphi''_{h_2}(\sigma) = 0$ for $\sigma_1 \leq \sigma \leq \sigma_2$. For $\tilde{L}_n$, the method in chapter II in [3] works. Define

$$\tilde{L}_{n,\sigma}(\Theta) = \frac{h_1}{h_2}(p_1^{-\sigma} e^{2\pi i \theta_1}, \dots, p_k^{-\sigma} e^{2\pi i \theta_k}) \prod_{k < l \leq n} \frac{1 - \frac{\chi_2(p_l) e^{2\pi i \vartheta_l}}{p_l^\sigma}}{1 - \frac{\chi_1(p_l) e^{2\pi i \vartheta_l}}{p_l^\sigma}},$$

$$\mu_{n,\sigma}(B) = \int_{\tilde{L}_{n,\sigma}^{-1}(B)} \left| \frac{\partial}{\partial \sigma} \tilde{L}_{n,\sigma}(\Theta) \right|^2 d\Theta$$

for any Borel set $B \subset \mathbb{C}$ and $n > k$, $\Theta = (\theta_1, \dots, \theta_k, \vartheta_{k+1}, \dots, \vartheta_n) \in [0, 1]^n$. Applying Theorems 5–10 in [3] with some modification, we prove that the absolutely continuous distributions $\mu_{n,\sigma}$ converge to a distribution μ_σ with a density $H_\sigma(x)$ and $\varphi''_{\tilde{L}_{n+1}}(\sigma) = 2\pi H_\sigma(-1) > 0$ for $\frac{1}{2} < \sigma \leq 1$. $\square$

For the case $J \geq 3$, we cannot do the same thing as $J = 2$. However, by virtue of the method in [12], we obtain the following.

Proposition 3.2. *Let $J \geq 3$ and $\frac{1}{2} < \sigma_1 < \sigma_2$. Suppose that $h_j(p_1^{-\sigma} e^{2\pi i \theta_1}, \dots, p_k^{-\sigma} e^{2\pi i \theta_k}) \neq 0$ for $j = l_1, l_2, l_3$, $\sigma_1 \leq \sigma \leq \sigma_2$, and $\theta \in [0, 1]^k$. Then, we have*

$$N_E(\sigma_1, \sigma_2; 0, T) = T \int_{\sigma_1}^{\sigma_2} H_\sigma(0) d\sigma + o(T),$$

where $H_\sigma(x)$ is the density for some distribution function μ_σ .

Proof. By Lemma 2.4, $\varphi_{E_n}(\sigma)$ converges uniformly to $\varphi_E(\sigma)$ on $[\frac{1}{2}, \infty)$. If $\varphi_E(\sigma)$ is twice differentiable, then we have

$$N_E(\sigma_1, \sigma_2; 0, T) = \frac{T}{2\pi} \int_{\sigma_1}^{\sigma_2} \varphi''_E(\sigma) d\sigma + o(T).$$

By Lemma 2.5 with

$$\delta = \min \left\{ |\tilde{h}_j(p_1^{-\sigma} e^{2\pi i \theta_1}, \dots, p_k^{-\sigma} e^{2\pi i \theta_k})| \mid j = l_1, l_2, l_3, \sigma_1 \leq \sigma \leq \sigma_2, \theta \in [0, 1]^k \right\} > 0,$$

we have $\hat{\mu}_{n,\sigma}(y) \ll |y|^{-3}$ and this implies that $\mu_{n,\sigma}$ is absolutely continuous and its density $H_{n,\sigma}(x)$ is continuous. Let $\nu_{n,\sigma}$ be the asymptotic distribution of $E_n(\sigma + it)$ with respect to $|E'_n(\sigma + it)|^2$. Since $\hat{\mu}_{n,\sigma}(y) = \hat{\nu}_{n,\sigma}(y)$ by Kronecker's theorem, $\mu_{n,\sigma} = \nu_{n,\sigma}$ and $H_{n,\sigma}$ is their common density. By Lemma 2.3, $\varphi''_{E_n-x}(\sigma) = 2\pi H_{n,\sigma}(x)$. By Lemma 2.6, $H_{n,\sigma}(x)$

converges to $H_\sigma(x)$ which is the density of some distribution $\mu_\sigma = \lim_{n \rightarrow \infty} \mu_{n,\sigma}$. Therefore, we have

$$N_E(\sigma_1, \sigma_2; 0, T) = T \int_{\sigma_1}^{\sigma_2} H_\sigma(0) d\sigma + o(T).$$

□

By Lemma 2.2, each Dirichlet polynomial $h_j(p_1^{-s}, \dots, p_k^{-s})$ has at most finite number of linearity intervals of its Jensen function $\varphi_{h_j}(\sigma)$ in $[\frac{1}{2}, \infty)$ and say $\mathfrak{J}_j$ the union of the intervals. By Lemmas 2.3 and 2.4 and almost periodicity, it means that h_j has no zero in $\mathfrak{J}_j$. We let $\varsigma_j = \inf \mathfrak{J}_j \geq \frac{1}{2}$, and ς_E be the third small ς_j , more precisely, $\varsigma_E = \varsigma_{l_3}$ when $\varsigma_{l_1} \leq \varsigma_{l_2} \leq \varsigma_{l_3} \leq \dots$ is the linear order of $\varsigma_1, \dots, \varsigma_J$. By combining Lemma 2.4 and Proposition 3.2, we have the following theorem.

Theorem 3. *Let $J \geq 3$ and $\varsigma_E < \sigma_1 < \sigma_2$. Suppose that $\sigma_1, \sigma_2 \in \mathfrak{J}_j$ for at least three j . Then, we have*

$$N_E(\sigma_1, \sigma_2; 0, T) = \frac{T}{2\pi} (\varphi'_E(\sigma_2-) - \varphi'_E(\sigma_1+)) + o(T).$$

Suppose further that $[\sigma_1, \sigma_2] \subset \mathfrak{J}_j$ for at least three j . Then, we have

$$N_E(\sigma_1, \sigma_2; 0, T) = T \int_{\sigma_1}^{\sigma_2} H_\sigma(0) d\sigma + o(T),$$

where $H_\sigma(x)$ is the density for some distribution μ_σ . In this case, for $\sigma_1 < \sigma_0 < \sigma_2$, the number of zeros of $E(s)$ on the line segment $\operatorname{Re} s = \sigma_0$ and $0 < \operatorname{Im} s < T$ is $o(T)$.

If each $\tilde{h}_j$ is non-vanishing on $\operatorname{Re} s > \frac{1}{2}$, Theorem 3 holds.

Theorem 4. *Let $J \geq 3$ and $\frac{1}{2} < \sigma_1 < \sigma_2$. Suppose that $\tilde{h}_j \neq 0$ for $\operatorname{Re} s > \frac{1}{2}$. Then, we have*

$$N_E(\sigma_1, \sigma_2; 0, T) = T \int_{\sigma_1}^{\sigma_2} H_\sigma(0) d\sigma + o(T), \quad (3.1)$$

where $H_\sigma(x)$ is the density for a distribution μ_σ . For $\sigma_0 > \frac{1}{2}$, the number of zeros of $E(s)$ on the line segment $\operatorname{Re} s = \sigma_0$ and $0 < \operatorname{Im} s < T$ is $o(T)$.

As a consequence, we prove Theorem 1.

We consider the case $\tilde{h}_j$ is one variable polynomial. In this case, $\tilde{h}_j(p^{-s}) = 0$ if and only if $p^{-s} = \beta_1, \dots, \beta_m$ for some m . Thus, each line segment $\operatorname{Re} s = -\frac{\log |\beta_j|}{\log p}$ and $0 < \operatorname{Im} s < T$ contains $cT + O(1)$ zeros of $\tilde{h}_j(p^{-s})$. So, we may not have the equation (3.1) for $E(s)$. However, if we disregard these exceptional points, we have the following theorem.

Theorem 5. Let $J \geq 3$ and $\frac{1}{2} < \sigma_1 < \sigma_2$. Let $E(s) = \sum_{j \leq J} \tilde{h}_j(p_1^{-s}, \dots, p_k^{-s}) L(s, \chi_j)$, where each $\tilde{h}_j$ is a polynomial of one variable. Then, we have

$$N_E(\sigma_1, \sigma_2; 0, T) = \frac{T}{2\pi} (\varphi'_E(\sigma_2-) - \varphi'_E(\sigma_1+)) + o(T).$$

Let $\mathfrak{J} = \bigcup_{l_1 < l_2 < l_3 \leq J} (I_{l_1} \cap I_{l_2} \cap I_{l_3})$ is $(\frac{1}{2}, \infty)$ except finitely many points. If $[\sigma_1, \sigma_2] \subset \mathfrak{J}$, then we have

$$N_E(\sigma_1, \sigma_2; 0, T) = T \int_{\sigma_1}^{\sigma_2} H_\sigma(0) d\sigma + o(T),$$

where $H_\sigma(x)$ is the density for some distribution μ_σ . For $\sigma_0 \in \mathfrak{J}$, the number of zeros of $E(s)$ on the line segment $\operatorname{Re} s = \sigma_0$ and $0 < \operatorname{Im} s < T$ is $o(T)$.

As a consequence, we prove Theorem 2.

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Department of mathematics, Yonsei University, Seoul, 120-749, Korea
 Korea Institute for Advanced Study, 207-43 Cheongnyangni-dong, Dongdaemun-gu, Seoul 130-722, Korea
E-mail address : Haseo Ki (haseo@yonsei.ac.kr)

Department of mathematics, POSTECH, Pohang, Gyungbuk 790-784, Korea
E-mail address : Yoonbok Lee (leeyb131@gmail.com)