Dispersion of suspended particles in turbulent flow

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The upward dispersion of heavy particles in suspension in turbulent flow was studied using a numerical model. The interaction between the turbulence and the particle diffusion leads to the formation of a horizontal front (or a “lutocline”), across which the diffusion of particles and the propagation of turbulent energy are inhibited. However, as the settling velocity of the particles becomes larger, or as the particle concentration becomes smaller, the interaction weakens, thus suppressing the front formation. One-dimensional model equations for the problem are solved numerically to calculate the evolution of the particle concentration. A criterion for the formation of the front is proposed and the steady depth of the suspension layer is determined.

I. INTRODUCTION

Whenever particles suspended in a fluid are diffused by turbulence, they always affect the turbulence structure itself, because of the generation of the buoyancy flux, the formation of stratification, the interfacial friction between particles and fluid, and other factors. In particular, when the directions of the buoyancy flux and particle diffusion are opposite to each other, the turbulence is suppressed, and the eddy diffusivity is reduced. The local reduction of diffusion is capable of inducing a strong concentration gradient at a certain depth, where the turbulence is suppressed even further, thus leading to the formation of a horizontal front (or a “lutocline,” in oceanographic terminology), across which the diffusion of particles and the propagation of turbulent energy are inhibited.

Figure 1 (a) illustrates a simple example in which a front is formed because of the interaction between turbulence and particle diffusion. Initially, heavy particles are at the bottom of a water-filled tank and turbulence is induced by oscillating a horizontal grid. This causes the particles to be diffused upward where they form a suspension layer of depth $D$, with a well-defined front separating turbulent and nonturbulent layers, as observed in the experiments of E and Hopfinger. Lutoclines of this nature are often observed in very turbid waters, such as shallow estuarine or coastal waters, with unconsolidated mud at the bottom.

As the density or concentration of the particles decreases, however, the effect of diffusion on the turbulence weakens. Also, as the settling velocity of the particles $w_s$ becomes larger, it is more difficult to generate a sufficient upward diffusion of particles to affect the turbulence. In this case, the formation of a front is impossible and the result shows a smooth upward decrease of particle concentration and turbulent energy, as illustrated in Fig. 1(b). This type of concentration distribution has been observed in many sediment-dispersion studies, which regard the eddy diffusivity as an external parameter, independent of particle diffusion; see, for example, Batchelor, Hunt, Smith and McLean, and Dyer.

Although the effects of sediment suspensions on eddy diffusivity have been taken into consideration recently, the pertinent physical mechanisms and the condition under which a given flow leads to the formation of a front [Fig. 1(a)] or to a smoothly varying concentration distribution [Fig. 1(b)] are not yet understood. Furthermore, in the former case, it is important to predict the depth of the lutocline (or the suspension layer). The purpose of the present study is to address these aspects by using a one-dimensional math-
II. FORMULATION OF THE MATHEMATICAL MODEL

The characteristic volume concentration of suspended particles in the ocean is typically of the order of $10^{-3}$ (see, for example, Lumley1). This value is the same for the case of lutocline formation both in the laboratory experiment and in the ocean. On this basis, it is assumed in the following analysis that the concentration is low, therefore the interaction between the particles is neglected and the fluid phase is regarded as incompressible. If it is also assumed that the suspension is made of spheres of uniform size $d$ and density $\rho_p$, the conservation equations of mass and momentum, for both fluid and suspended particles, are given by9-13

$$
\frac{\partial}{\partial t} (1-c) + \frac{\partial}{\partial x_i} \left[ u_i (1-c) \right] = 0,
$$

(1)

$$
\frac{\partial c}{\partial t} + \frac{\partial}{\partial x_i} (u_i c) = 0,
$$

(2)

$$
\frac{\partial}{\partial t} \left[ \rho_f (1-c) u_i \right] + \frac{\partial}{\partial x_j} \left[ \rho_f (1-c) u_i u_j \right]
= -\nabla p + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \gamma c (\bar{u}_{pi} - u_i) - \rho_g^* c \delta_{ij},
$$

(3)

$$
\frac{\partial}{\partial t} \left( \rho_p^* u_i \right) + \frac{\partial}{\partial x_j} \left( \rho_p^* u_i u_j \right) = \gamma c (\bar{u}_{pi} - u_i) - \rho_g^* c \delta_{ij}.
$$

(4)

In Eqs. (1)-(4), $c$ is the volume concentration of the solid particles, $u_i$ is the velocity component of the fluid phase, $u_{pi}$ is the velocity component of the solid-particle phase, $\rho_f$ is the fluid density, $p$ is the pressure, $\mu$ is the molecular viscosity of fluid, $g^*$ is the buoyancy difference between the particle and fluid $[g^* = g(\rho_p - \rho_f)/\rho_f]$, and $\gamma$ is the particle drag coefficient given by Stokes' formula $[\gamma = 6 \pi \mu d / (4 \pi d^3/3)]$. The particles settling velocity $w$, can be obtained from (4) as

$$
w = \rho_g^* / \gamma
$$

by applying $d(c u_{pi})/dt = 0$ and $u_i = 0$. This means that the particles settle with an average velocity $w$ relative to the surrounding fluid, if the interactions between particles are neglected. It is therefore possible to decompose the velocity of the particle into two components, namely, the settling velocity $w$ and the neutral drifting velocity $\bar{u}_{pl}$ following the fluid velocity field, i.e.,

$$
u_{pi} = \bar{u}_{pl} - w_i \delta_{ij}.
$$

(6)

Using (6), (3) can be modified to

$$
\frac{\partial}{\partial t} \left[ \rho_f (1-c) u_i \right] + \frac{\partial}{\partial x_j} \left[ \rho_f (1-c) u_i u_j \right]
= -\nabla p + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \gamma c (\bar{u}_{pi} - u_i) - \rho_g^* c \delta_{ij},
$$

(7)

where the last term in the rhs of (7) represents the virtual density increase of the fluid due to the suspension of particles. At low concentrations, $1-c \approx 1$ and, hence, (7) becomes

$$
\frac{\partial}{\partial t} \left( \rho_f u_i \right) + \frac{\partial}{\partial x_j} (\rho_f u_i u_j)
= -\nabla p + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \gamma c (\bar{u}_{pi} - u_i) - \rho_g^* c \delta_{ij},
$$

(8)

If the field variables are decomposed into the usual mean and fluctuating parts associated with turbulent flows, i.e.,

$$
u_i = U_i + \nu_i',
$$

(9)

$$
u_{pi} = U_{pi} + \nu_{pi}',
$$

(10)

$$p = P + p',
$$

(11)

$$c = C + c',
$$

(12)

then the equations for the mean concentration of fluid and suspended particles are

$$
\frac{\partial}{\partial t} \left( 1-C \right) + \frac{\partial}{\partial x_j} \left[ w (1-C) \right] - \frac{\partial}{\partial x_j} \bar{w} c' = 0,
$$

(13)

$$
\frac{\partial}{\partial t} \frac{\partial}{\partial x_j} \left( W_c C \right) - \frac{\partial}{\partial x_j} \bar{w} c' = 0,
$$

(14)

when the flow is horizontally homogeneous without a mean flow. Here, $W, w', W_c, and w_c'$ are the vertical components of the velocity field $U_i, \nu_i', U_{pi}, and \nu_{pi}'$, respectively. Using the fact that

$$W = w - w_c,
$$

(15)

the mean vertical velocity $W$ can be obtained from (13) and (14).37

$$W = \bar{w} C + (\bar{w} c' - \bar{w}_c c').
$$

(16)

Similarly, the equation for the mean turbulent kinetic energy of fluid is

$$\begin{align*}
\frac{\partial}{\partial t} E + W \frac{\partial}{\partial z} \frac{\partial}{\partial z} + \bar{w} \frac{\partial}{\partial z} + \frac{\partial}{\partial z} \left( \frac{E}{\rho_f} \bar{w} + \frac{\bar{w}}{\rho_f} \right) - \epsilon^* - \gamma^* C u_i' (u_i' - u_i') - g^* c' w' - \frac{\partial}{\partial z} (\delta_{ij} \epsilon^*) = 0,
\end{align*}
$$

(17)

where

$$E = \frac{u_i'^2/2}{2}$$

is the averaged turbulent kinetic energy of the fluid, $E' = u_i' u_j'/2$ is the fluctuating kinetic energy of the fluid, $\epsilon^* = \nu (\partial u_i' / \partial x_j)^2$ is the dissipation rate of turbulent kinetic energy, and $\gamma^* = \gamma / \rho_f$. Equation (17) is produced by multiplying (8) by $u_i'$ and then averaging it, and neglecting $\nu \frac{\partial^2 E}{\partial z^2} (\frac{\partial}{\partial z} - \partial E/w' / \partial z)$ and $\bar{c} u_i' (u_i' - u_i')$ [ $\frac{\partial}{\partial z} (\delta_{ij} \epsilon^*)$]. The triple correlation term $\bar{c}' u_i' (u_i' - u_i')$ is expected to be negligible in comparison with $C u_i' (u_i' - u_i')$, because $u_i' (u_i' - u_i')$ is negative most of the time, independent of the concentration gradient, while $\bar{c}' = 0$.

The term representing the work by the interfacial friction in (17) can be calculated for the cases $\gamma^* \sim 1 < \tau$ and $\gamma^* \sim 1 > \tau$, where $\tau$ is the time scale of the turbulent eddies. When $\gamma^* \sim 1 < \tau$, it can be shown that11,15

$$\gamma^* C u_i' (u_i' - u_i') \approx -CE / \tau,
$$

(18)
which is independent of $\gamma^*$. On the other hand, when $\gamma^* \rightarrow \infty$ (see Genchev and Karpuzov\textsuperscript{12})
\[
\gamma^* C \frac{u_i'(u_{ji} - u_{ij})}{u_j} \approx -\gamma^* C E,
\]
because
\[
\frac{u_i'u_j}{u_j'} \ll \frac{u_i}{u_j'}.
\]
The particle size $d$ is typically of the order of $10^{-3}$ to $10^{-2}$ cm when a lutocline is observed,\textsuperscript{12} which gives $\gamma^* \sim 10^{-3}$ to $10^{-2}$ sec$^{-1}$. On the other hand, $\tau$ can be estimated as $\tau \sim 1$ to 10 sec for the oscillating-grid generated turbulence, if $\tau$ is calculated from $t/u$, where $l$ and $u$ are the length and velocity scale of the turbulence. The time scale $\tau$ is expected to be much larger in the ocean.

When $\gamma^* \ll \tau$, (14), (16), and (17) can be rewritten
\[
\frac{\partial C}{\partial t} + \frac{\partial}{\partial x} (WC) - w_i \frac{\partial C}{\partial x} = \frac{\partial}{\partial z} \left( K^* \frac{\partial C}{\partial z} \right),
\]
\[
\frac{\partial E}{\partial t} + W \frac{\partial E}{\partial z} + \frac{2E}{3} \frac{\partial W}{\partial z} = \frac{\partial}{\partial z} \left( K^* \frac{\partial E}{\partial z} \right) - \frac{CE}{\tau} + R K^* \frac{\partial C}{\partial z},
\]
where the eddy diffusivity for the parametrization of turbulent diffusion has been introduced, and the turbulence has been assumed to be isotropic (i.e., $w_i^2 = 2E/3$). Here, $K^*$, $K^*_s$, and $K^*_f$ represent the eddy diffusivities of the fluid, the solid particles, and the turbulent kinetic energy, respectively.

The eddy diffusivity for turbulent kinetic energy $K^*_E$ can be related to $K^*$ by\textsuperscript{16,17}
\[
K^*_E = K^*/\sigma_E,
\]
where $\sigma_E$ is an empirical constant of the order of unity.

On the other hand, for the case of turbulent diffusion of solid particles, the cross-trajectory effects of solid particles have been found to reduce the eddy diffusivity of the solid particles in comparison with that of the fluid, whereas the effect of particle inertia may not be important.\textsuperscript{18-23} Csanady\textsuperscript{19} proposed that $K^*_s$ is related to $K^*$ by
\[
K^*_s/K^* = [1 + (\beta w_i)^2/E]^{-1/2},
\]
where $\beta$ is an empirical constant representing the ratio of Eulerian and Lagrangian time scales. The evaluations of $\beta$ from the experimental data, however, show rather large variations such as 1–10.\textsuperscript{19,20,23}

The eddy diffusivity and the dissipation rate of turbulence can be modeled as
\[
K^* = c_p E^{1/2} l,\]
\[
e^* = c_D E^{3/2} l^{-1},
\]
where $c_p$ and $c_D$ are constants.\textsuperscript{16,17}

If the parameters are nondimensionalized using the characteristic velocity $U$, length scale $L$ of the turbulence, and the characteristic concentration $M$ as
\[
\tilde{E} = E/U, \quad \tilde{t} = t/L, \quad \tilde{z} = z/L, \quad \tilde{t} = t/(L/U),
\]
\[
\tilde{\tau} = \tau/(L/U), \quad \tilde{K}^* = K^*/(UL), \quad \tilde{K}^*_s = K^*_s/(UL), \quad \tilde{e}^* = e^*/(U^3/L), \quad \tilde{C} = C/M, \quad \tilde{W} = W/(w_i M),
\]
then
\[
(21) \quad (22) \quad \text{and} \quad (23) \quad \text{give (in the following equations, the carets on the nondimensional variables are dropped)}
\]
\[
\frac{\partial C}{\partial t} + w_i M \frac{\partial}{\partial z} (WC) - w_i \frac{\partial C}{\partial z} = \frac{\partial}{\partial z} \left( K^* \frac{\partial C}{\partial z} \right),
\]
\[
\frac{\partial E}{\partial t} + w_i M \frac{\partial E}{\partial z} + 2E \frac{2W}{U} \frac{\partial W}{\partial z} = \frac{\partial}{\partial z} \left( K^* \frac{\partial E}{\partial z} \right) - e^* + \frac{g^* M L}{U^2} K^* \frac{\partial C}{\partial z}.
\]

When the low-concentration case (i.e., $M \ll 1$) is considered, the advection term and the interfacial friction term in (29) and (30) can be neglected. On the other hand, the buoyancy flux term, whose magnitude is characterized by $g^* LM / U^2$, can have a significant value when $U$ becomes small as the distance from the turbulence source increases and $g^* \gg 1$. For the case of $\gamma^* \rightarrow \infty$, the interfacial friction term given by (19) is again negligible when $M \ll 1$, because
\[
\gamma^* C E \sim \frac{U^3}{L} \frac{\gamma^*}{U/L} M \ll \frac{U^3}{L}.
\]
Therefore, in the case $M \ll 1$, (29) and (30) can be rewritten as
\[
\frac{\partial C}{\partial t} - R^* \frac{\partial C}{\partial z} = \frac{\partial}{\partial z} \left( K^* \frac{\partial C}{\partial z} \right),
\]
\[
\frac{\partial E}{\partial t} = \frac{\partial}{\partial z} \left( K^* \frac{\partial E}{\partial z} \right) - e^* + G^* K^* \frac{\partial C}{\partial z},
\]
where
\[
R^* = \frac{w_i}{U},
\]
\[
G^* = g^* LM / U^2,
\]
and
\[
K^*_s/K^* = [1 + (\beta R^*)^2/E]^{-1/2}.
\]

In this paper, the case of the low-concentration situation described by (32) and (33) will be considered as the simplest model to explain the formation of the lutocline mentioned in the Introduction. To solve the equations, however, the information on the velocity and length scale of turbulence is required. Shear-free turbulence generated by oscillating grids, on which the current model is based, has been extensively studied in the laboratory\textsuperscript{24-25} and its numerical simulation has been studied recently by Noh and Fernando.\textsuperscript{26} The experimental data show that the integral length scale and the rms velocity of turbulence in a homogeneous fluid are given by
\[
l_i = a_i (z + z_0),
\]
\[
u = a_s S (z + z_0)^{-1},
\]
where $f$ and $S$ are the frequency and the stroke of the grid oscillations, respectively, $z$ is the distance from the midplane of the grid oscillations, and $a_i$, $a_s$, and $z_0$ are constants that depend on the grid geometry.

Experimental and theoretical studies\textsuperscript{27-29} have shown that, in stably stratified fluids, the growth of the vertical
length scale of turbulence $l_\nu$ is limited by the buoyancy length scale $l_b$ ($=u/N$, where $N$ is the Brunt–Väisälä frequency). On the basis of these results, it is assumed that $l_b$ has the form

$$l_b = l_n/\left[1 + (l_n/c_i l_b)^2\right]^{1/2},$$

where $c_i$ is a constant of order one, and $l_b$ can be calculated by

$$l_b = -(2E/3)^{1/2}/[G* l_b (\partial C/\partial z)].$$

If the length scale is rescaled by $l_i = l_b/a_i$, (26) and (27) can be rewritten as

$$K* = c_{i*} E^{1/2} l_i,$$

$$\epsilon = c_{\mu*} E^{1/2} l_i^{1-1},$$

where $c_{i*} = a_i c_{\mu}$ and $c_{\mu*} = a_i c_{\mu}$. A similar rescaling of time by $t_i = c_{\mu*}/\sigma_{E}$ changes (32) and (33) to

$$\frac{1}{\sigma_{E}} \frac{\partial C}{\partial t_i} - R \frac{\partial C}{\partial z} = \frac{\partial}{\partial z} \left(K_S \frac{\partial C}{\partial z}\right),$$

$$\frac{\partial E}{\partial t_i} = \frac{\partial}{\partial z} \left(K \frac{\partial E}{\partial z}\right) - \left(c_{\mu} \sigma_{E}/c_{\mu*}\right) \epsilon + GK \frac{\partial C}{\partial z},$$

where

$$K = E^{1/2} l_i,$$

$$\epsilon = E^{1/2} l_i^{1-1},$$

$$R = c_{\mu*} R*,$$

$$G = \sigma_{E} R*,$$

$$K_S = E^{1/2} l_i/\left[1 + (\beta R)^2/E \right]^{1/2}.$$  

The empirical constant $(c_{\mu} \sigma_{E})/c_{\mu*}$ appearing in (44) can be obtained by substituting the experimental results of (37) and (38) into the steady-state form of (44) with $G = 0$, which gives $(c_{\mu} \sigma_{E})/c_{\mu*} = 6$. In this way, the empirical constants $c_{\mu}$, $c_{\mu*}$, and $a_i$ are removed from (43) and (44) and they are incorporated to redefine the parameters $R*$ and $G*$ as in (47) and (48) and the time scale $t_i$.

Furthermore, (39) can be also rewritten as

$$l_i = l_n/\left[1 + c_{R} (l_n/l_b)^2\right]^{1/2},$$

where $l_{n1} = l_n/a_i$ and $c_{R} = (a_i/c_i)^2$.

### III. NUMERICAL METHOD

A one-dimensional implicit finite difference method was employed to solve (43) and (44), together with (45)–(49) for different values of $R$ and $G$.

For the calculation, the characteristic velocity and length scales are defined by $U = E^{1/2}$ at $z = 0$ and $L = l_n$ at $z = 0$. The characteristic concentration is defined by

$$M = \int_0^\infty C \, dz,$$

in the steady state.

Because $K$ varies with space and time, the form

$$\frac{\partial}{\partial z} \left(K \frac{\partial}{\partial z} f_i \right) = \frac{1}{2(\Delta z)^2} \left[ (K_{i+1} + K_i) (f_{i+1} - f_i) \right]$$

$$- (K_i + K_{i-1}) (f_i - f_{i-1})$$

was used for $f = E$ and $C$. Using (52), (43) can be expressed as

$$\frac{1}{\sigma_{E}} \frac{\partial C}{\partial t_i} \bigg|_{i} = - \frac{\partial J}{\partial z}$$

$$= - (1/\Delta z) (J_{i+1/2} - J_{i-1/2}),$$

where

$$J_{i+1/2} = (1/2 \Delta z) \left[ (K_i + K_{i+1}) (C_{i+1} - C_i) \right]$$

$$- R \Delta z (C_{i+1} - C_i)$$

is the concentration flux at the level $i + 1/2$.

The initial conditions for $E$ and $l_i$ were taken to be

$$E(0,z) = (z + 1)^{-?,}$$

$$l_i (0,z) = z + 1,$$

which are the case of a homogeneous fluid [see (37) and (38)]. The concentration $C$ was assumed to be concentrated at the bottom ($z = 0$) initially, viz.,

$$C(0,z) = \begin{cases} 1, & \text{at } z = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The boundary conditions for $E$ were given by

$$E(t,0) = 1,$$

$$\frac{\partial E(t,H)}{\partial z} = 0,$$

where $z = H$ is the maximum height of the calculation.

In this problem, the boundary condition for $C$ at the bottom ($z = 0$) is important mainly to determine the total concentration within the fluid. To avoid complexity, it is assumed that erosion and sedimentation do not exist at the bottom and, therefore, the total concentration remains constant. The incorporation of the effects of erosion and sedimentation can be accomplished without much difficulty, if an empirical formula such as that given by Markofsky et al. or Parker et al. is applied. The analytical treatment of the problem is very complicated, however, because of the dependence on the nature of particles and the boundary. For the case of no sedimentation and erosion at the bottom, the vertical flux of particle concentration vanishes at the bottom where the particle settling is balanced by a diffusional flux away from the bottom. For the calculations, this is given by

$$J_{i+1/2} = 0$$

from (54), when the cell-interface boundary condition is applied. It is found from (55) that the variation of $C$ at $z = 0$ can be written as

$$\int_0^{1/2 \Delta z} \frac{\partial C}{\partial z} \bigg|_{i} = \int_{1/2 \Delta z}^{1/2 \Delta z} \frac{1}{2(\Delta z)^2} (K_2 + K_1) (C_2 - C_1)$$

$$+ \frac{K}{2\Delta z} (C_2 + C_1),$$

and this leads to the conservation of the total concentration with time, i.e.,

$$\sum_{i=1}^{\infty} C_i = \text{const} (= M).$$

Furthermore, it is assumed that $\partial C/\partial z = 0$ at $z = H$. 

Equations (43)–(50) still include three empirical constants, namely, $\beta$, $c_R$, and $\sigma_E$. The calculation was made with $K_s = K$ corresponding to the case $(\beta R^2)/E \ll 1$. The value $c_R = 0.1$ was used, as obtained for the oscillating-grid generated turbulence in stratified fluids. The effects of the empirical constants $\beta$ and $c_R$ will be discussed separately in Sec. IV D.

Finally, for the calculation of (43), $\sigma_E = 1$ has been used following the suggestions by Sonin for the oscillating grid-generated turbulence. It is important, however, to notice that the choice of $\sigma_E$ does not affect the results of the steady-state distribution of $E$ and $C$, because it disappears in the steady-state equations of (43) and (44) [or (64) and (65) in Sec. IV E].

IV. RESULTS

A. Evolution of turbulent kinetic energy and concentration fields

Owing to the turbulence, particles that are initially at the bottom ($z = 0$) are diffused upward. Diffusion and its effects on turbulence appear in various ways, depending on the values of $R$ and $G$.

Figures 2(a)–2(c) show the evolution of concentration ($C$) distribution with time for different values of $R$ ($= 10^{-1}$, $10^{-2}$, $10^{-3}$), when $G = 10^{-3}$. The graphs correspond to times $t_i = n \Delta t_i$ ($n = 1, \ldots, 10, \Delta t_i = 400$). It is clearly shown that a front is formed for $R = 10^{-2}$ and $10^{-3}$ [Figs. 2(a) and 2(b)], but it does not appear for $R = 10^{-1}$ [Fig. 2(c)]. For the former case [Figs. 2(a) and 2(b)], it is also found that the concentration gradient within the suspension layer increases with $R$. Furthermore, the concentration distribution approaches an equilibrium state more quickly for larger $R$.

Figures 3(a)–3(c) show the corresponding evolution of the turbulent kinetic energy ($E$). For the case of $R = 10^{-2}$ or $10^{-3}$, the energy decreases rapidly with $z$ to a minimum at the point corresponding to the depth of the suspension layer $D$ [cf. Figs. 2(a) and 2(b)]. This implies that, at this depth, the energy decay due to buoyancy flux is dominant in the rhs of (44). For $z > D$, the initial turbulence still persists, but
decays slowly because of dissipation with the evolution of the front. Once the front forms, this region cannot receive energy, as the front inhibits exchange of energy across it. The case with $R = 10^{-1}$, where a well-defined front is absent, is shown in Fig. 3(c). In this case, the point of minimum $E$ does not appear.

It is also observed that $E$ increases with time below the front ($z < D$), thus approaching its initial value when the front is formed [Figs. 3(a) and 3(b)]; otherwise, however, $E$ decreases with time at all depths [Fig. 3(c)]. This can be easily understood by considering the particle distributions for the front-formation and no front-formation cases. In the latter case, $C$ decreases with $z$ exponentially and the energy is extracted from the entire fluid column because of the buoyancy flux. In the former, however, $C$ tends to be uniform (i.e., nonstratified) within the suspension layer since the front plays a role similar to that of a solid boundary.

Figures 4 and 5 show the evolution of $C(z,t)$ and $E(z,t)$ for the cases $G = 10^{-4}$ and $10^{-2}$ when $R = 10^{-2}$; $\Delta t$, used here is the same as that for the previous case. Note that, for
FIG. 5. The evolution of turbulent energy distribution with time at $R = 10^{-2}$, when (a) $G = 10^{-2}$ and (b) $G = 10^{-4}$.

$G = 10^{-4}$, $E$ is little changed from the initial state and no evidence for the front formation exists at large times. For $G = 10^{-2}$ case, however, the formation of a front is evident. 

Careful inspection of Figs. 2 and 4 also reveals interesting features with regard to the concentration gradients, which can be used to delineate the criterion for the formation/nonformation of a front. When there is no front, $-dC/dz$, at large times, decreases to zero monotonically with $z$. On the contrary, when a front is formed, $-dC/dz$, at larger times, first increases with $z$, and then decreases rapidly to zero after a certain depth. The depth of the inflection point at which $d^2C/dz^2 = 0$ (i.e., the depth of the maximum concentration gradient) can be treated as the depth of the suspension layer $D$, when a sharp front is generated. 

B. A criterion for the formation of a front

In Sec. IV A, it has been shown that the front disappears with increasing $K$ or decreasing $G$. In this section, the condition under which the front is formed will be investigated.

The steady-state forms of (43) and (44) are given by

$$K \frac{dC}{dz} + RC = A_1,$$

$$\frac{d}{dz} \left( K \frac{dE}{dz} \right) - \epsilon + GK \frac{dC}{dz} = 0,$$  \hspace{1cm} (64)  \hspace{1cm} (65)

where a constant $A_1$ is found to be zero, because the particle flux disappears at $z = 0$ [see (53) and (61)]. It is also assumed that $K_s = K$, as in the previous section.

If $E$ is replaced by $K^2 I_1^3$, and $I_1$ is given by (57), (65) can be rewritten as

$$\frac{d}{dz} \left( K \frac{dE}{dz} \right) - 6 \frac{K^3}{(z + 1)^3} - GRC = 0.$$  \hspace{1cm} (66)

To derive (66), the effect of stratification on the integral length scale [see (50)] is neglected, because it becomes important only after the formation of a significant front, and the buoyancy flux term in (65) is replaced using (64). Rearranging the first and the second term on the lhs of (66), one obtains the form

$$\frac{d}{dz} \left( \frac{1}{(z + 1)^3} \frac{dK^3}{dz} \right) = \frac{3}{2} GRC(z + 1)^{-3}.$$  \hspace{1cm} (67)

To solve (67), however, the vertical distribution of $C$ must be known. It has been shown in Sec. IV A that $d^2C/dz^2$ is negative up to the front when a front is formed, and it is positive otherwise. This makes it possible to assume that

$$\frac{d^2C}{dz^2} = 0,$$  \hspace{1cm} (68)

or

$$\frac{dC}{dz} = A_2$$  \hspace{1cm} (69)

at the critical condition until $C$ becomes zero, where the value of a constant $A_2$ can be obtained from (64) as $A_2 = -RC_0$ since $K(z = 0) = 1$.

The vertical distribution of the concentration is then assumed to be given by

$$C = \begin{cases} C_0(1 - Rz), & z < R^{-1}, \\ 0, & z > R^{-1}, \end{cases}$$  \hspace{1cm} (70)

where $C_0 = C(z = 0)$. The value of $C_0$ can be obtained from (51) as

$$C_0 = R.$$  \hspace{1cm} (71)

At the critical condition, it is also expected that

$$K(z = R^{-1}) = 0,$$  \hspace{1cm} (72)

due to the otherwise, the particles would be diffused upward. When the front is formed, $K$ becomes zero at a depth smaller than $z = R^{-1}$. On the other hand, $K$ remains positive at all depths, if the front is not formed.

By using (70)–(72), the integration of (67) yields

$$K^3 = 1 + \frac{3}{4}GR^2 \left[ - \frac{1}{4} (1 - R)(z^4 + 4z^3 + 6z^2 + 4z) + \frac{R}{(5/4)} (z^3 + 5z^2 + 10z + 10z^2 + 5z) \right]$$  \hspace{1cm} (73)

when $z < R^{-1}$. The constants of integration are selected to
satisfy $K(z = 0) = 1$ and $dK/dz (G = 0) = 0$. The critical value of $G$, $G_c$, is then found from (72) as

$$G_c = \sqrt{\frac{2}{R}} \left[ 1 + \frac{3R}{2} + O(R^2) \right]^{-1},$$

(74)

where $R \ll 1$.

Figure 6 shows how the numerically computed depth of the suspension layer $D$ (defined in Sec. IV A), changes with time for different values of $R$ when $G = 2^{5} \times 10^{-5}$. For the case of front formation ($R < 2^{7} \times 10^{-4}$), $D$ grows monotonically to its limiting value. On the other hand, if the front is not formed ($R > 2^{9} \times 10^{-4}$), $D$ approaches $z = 0$ quickly, and stays there.

Figure 7 represents the dependence of the critical $G$ at which the front disappears with $R$, estimated from the plots such as in Fig. 6. For a given $R$, the maximum/minimum value of $G$ computed without/with a front is indicated. The dotted line represents $G_c$ obtained from (74). Good agreement is found between the numerical and analytical results.

C. The depth of the suspension layer

It is well known that, when a stabilizing buoyancy flux is imposed on the surface of a turbulent fluid, a horizontal front is formed at a certain depth, at which rapid decreases in turbulent kinetic energy and density occur. A typical example of this phenomenon is the formation of the summer thermocline in the ocean. It has been also observed in the laboratory experiments by Kantha and Long,37 Hopfinger and Linden,38 and Noh and Long39 and in the numerical simulation by Noh and Fernando.26

In such cases, the steady depth of the front (or the thickness of the turbulent-mixed layer) is estimated by a local balance between the rate of generation of turbulence and the stabilizing buoyancy flux $Q$, i.e.,

$$u^3/\rho \sim Q.$$

(75)

Based on (37) and (38) for oscillating-grid turbulence, it has been suggested,37-39 by experimental evidence, that the depth of the front should be proportional to the modified Monin–Obukhov length scale $L_Q$, given by

$$L_Q = A_3 (K'_0/Q_0)^{1/4},$$

(76)

where $Q_0 = Q(z = 0)$ is the buoyancy flux at the surface (for the present flow configuration $Q_0$ is imposed at the bottom) and $A_3$ is a constant. The validity of (76) for the case of buoyancy fluxes induced by suspensions has been experimentally demonstrated by E and Hopfinger.1 In what follows, the extension of (76) to the solid-particle diffusion case is investigated.

Equation (64) can be used to find the buoyancy flux ($= g^* \bar{c} \bar{w}$) in the steady state as

$$Q = -G K_s \frac{dC}{dz},$$

(77)

$$= GRC.$$  

(78)

If $Q_0$ given by (78) is inserted into (76), it can be shown that

$$L_Q = A_3 (K'_0/GRC_0)^{1/4}.$$  

(79)

Since the concentration within the suspension layer is rather uniform at small $R$ [see, for example, Fig. 2(a)], $C_0$ can be estimated as

$$C_0 = L_Q^{-1},$$

(80)

and (79) can be rewritten as

$$L_Q = A_4 G^{-1/3} R^{-1/3},$$

(81)

where $A_4 (= A_3^{1/3})$.

In the calculation, the equilibrium depth of the suspension layer $D_Q$ was defined as

$$D_Q = \lim_{t \to \infty} D(t).$$

(82)

Close inspection of Fig. 6 shows that, at large times, the rate of growth of the suspension layer $u_s (= dD/dt)$ can be written as $u_s = u_0 \alpha^{\alpha (t - t_0)}$, where $u_0 = u_s (t = t_0)$ and $\alpha (\epsilon < 1)$ is a constant; hence it is possible to write

$$D_Q = D_0 + u_0 \int_{t_0}^{\infty} \alpha^{\alpha (t - t_0)} dt = D_0 - \frac{u_0}{\ln \alpha},$$

(83)
where $D_0 = D (t = t_0)$. In the calculation of $D_0$, $t_0$ was selected so that $- (u_0 / \ln \alpha D_0 < 0.01$; the typical $t_0$ was about $10^4$, but, for small $G$ and $R$, it was necessary to select $t_0 = 4 \times 10^4$. In order to ensure that the upper-boundary effect was negligible, the value of $H$ was selected so that $D / H < 0.2$.

Figure 8 shows the variation of $D_0$ with $G$ for constant $R$ values, whereas Fig. 9 shows the dependence of $D_0$ with $R$ at constant $G$. The results show good agreement with (81) as $R$ and $G$ become smaller. From the data for the smallest $R$ and $G$ values used in the calculation, it is possible to estimate $A_4 = 3.6$. At larger values of $R$ and $G$, however, $D_0$ decreases faster than expected from (81).

The deviation at larger $R$ is easily understood from the fact that the concentration gradient within $D$ in such cases becomes significant [e.g., Fig. 2(b)], thus invalidating the assumption (80). On the other hand, if $G$ is large, (37) and (38) cannot be used as the proper length and velocity scales to obtain $L_D$, as was done in deriving (76).

D. The effects of $\beta$ and $c_R$

The calculations with $\beta = 1$ and 5 show that the critical values of $G$ are not affected by the presence of the cross-trajectory effect when $R$ is small, but the formation of the front becomes more favorable as $R$ becomes larger, as shown in Fig. 10. The depth of suspension layer $D_0$ also decreases more rapidly as $R$ increases, because the effective turbulent diffusivity decreases. The variations of $D_0$ with $R$ are shown in Fig. 11 for the cases of $\beta = 0$, 1, and 5 at $G = 2 \times 10^{-5}$.

Noh and Fernando found that the relation (76) remains valid regardless of the value $c_R$, but the constant $A_4$ tends to decrease with $c_R$ from the calculation of the depth of thermocline. Similarly, it was found that the constant $A_4$ decreases slightly with $c_R$; for example, $A_4 \approx 2.3$ when $c_R = 1.0$. On the other hand, $c_R$ does not affect the formation of the front, as expected from the fact that the effect of stratification on turbulence length scale becomes important only after the significant front is formed.
V. CONCLUDING REMARKS

The aim of this paper was to investigate, using a numerical model, the upward dispersion of heavy solid particles in turbulent fluids when the interaction between turbulence and particle diffusion is taken into consideration. The specific case of oscillating-grid induced shear-free turbulence acting on particles, which is initially at the bottom of a tank, is considered. It is shown that the dispersion process is governed by the nondimensional parameters $R$ (which represents the ratio of the settling velocity and the rms turbulent velocity) and $G$ (which is equivalent to a Richardson number).

Depending on $R$ and $G$, two flow situations were observed. When $G$ is less than a critical value $G_c = 2.2R^2$, no well-defined front is formed, and the particle concentration decreases continuously away from the grid. When $G > G_c$, however, a front appears across which the diffusion of particles and the propagation of turbulent energy is inhibited. In the case of front formation, the magnitude of the concentration gradient $(= -dc/dz)$ is found to increase up to the front and disappears above it, whereas it decreases continuously in the absence of the front.

At small $G$ and $R$, the equilibrium depth of the front $D_Q$ is estimated as $D_Q = A_4 R^{-1/3} R^{-1/3}$ with $A_4 \approx 3.6$. At large $G$ and $R$ values, however, $D_Q$ decreases faster than the prediction of the $1/3$ power law.

The cross-trajectory effect on the turbulent diffusion of solid particles is found to decrease both values of $G_c$ and $D_Q$ at large $R$ values ($R > 0.1$).

The above results need to be expressed in terms of measurable parameters $G^*$ and $R^*$ for the comparison with the experimental results, which differ from $R$ and $G$ by proportional constants [see (47) and (48)]. The main results (74) and (81) is then represented as

$$ G^* = \frac{3}{v} \left[ \frac{1}{2} \left( \frac{a_1}{c_p} \right)^2 \right] R^2. $$

$$ L_Q = A_4 \sigma \epsilon R^{-1/3} \left( \frac{a_1}{c_p} \right)^{1/3} G^* - \frac{1}{2} R^{1/3} - \frac{1}{2}. $$

From the comparison of the data for the heat flux and the change of the vertical temperature distribution of Hopfinger and Linden's experiment, it is possible to obtain $a_1/c_p = 1$ (see, for example, Noh and Fernando16). Furthermore, $\sigma_{G^*} = 1$ can be used, as discussed in Sec. II.

In E and Hopfinger's experiment, the typical values of the parameters are $w_i \sim 0.2 \text{ cm sec}^{-1}$, $H \sim 10 \text{ cm sec}^{-1}$, $L \sim 1 \text{ cm}$, $g^* \sim 10^5 \text{ cm sec}^{-2}$, and $M \sim 10^{-1}$. From these values, it can be found that $R^* \sim 2.0 \times 10^{-2}$ and $G^* \sim 1.0$, and these conditions clearly belong to the case of the front formation.

Finally, it is important to note the limitations of the present results in the context of geophysical and engineering problems.

(a) The results presented here are made for shear-free turbulence generated at a horizontal boundary. Examples of such situations include the turbulence generated by wave breaking at the sea surface and the agitation of two-phase fluid during chemical reactions. It has also been found that, in many instances, oceanic sediment dispersion is dominated by bottom-generated turbulence, which is the present results can be applicable. In the presence of mean shear, however, turbulent energy is produced at all depths and the resulting flow situation will differ significantly from that treated here.

(b) In the present study, the total depth of water is assumed to be infinite. Under some situations, however, the depth of water $H$ must be taken into consideration. For example, Wolanski et al.2 suggested that lutoclines do not appear in estuaries when $D_Q$ is larger than $H$.

(c) When the concentration gradient at the front becomes larger, especially when $R$ is small and $G$ is large, internal wave breaking and entrainment at the front must be taken into consideration (see, for example, Turner16).

(d) The model can also be extended to the case where sedimentation and erosion at the bottom exist by modifying the boundary condition at the bottom. The current results will be still valid, however, if the definition of $M$ is modified to the total volume of suspended particles per unit area in the steady state.

(e) Finally, it needs to be mentioned that the theoretical model is developed for the case of low concentrations, as discussed in Sec. II. As the concentration increases, several other effects such as interfacial friction and the volume fraction of the solid phase should be considered too.

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