MULTIFREQUENCY TRANS-ADMITTANCE SCANNER: 
MATHEMATICAL FRAMEWORK AND FEASIBILITY

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Abstract. A trans-admittance scanner (TAS) is a device for breast cancer diagnosis based on 
numerous experimental findings that complex conductivities of breast tumors significantly differ 
from those of surrounding normal tissues. In TAS, we apply a sinusoidal voltage between a hand-
held electrode and a scanning probe placed on the breast skin to make current travel through the 
breast. The scanning probe has an array of electrodes at zero voltage. We measure exit currents 
(Neumann data) through the electrodes that provide a map of trans-admittance data over the breast 
surface. The inverse problem of TAS is to detect a suspicious abnormality underneath the breast skin 
from the measured Neumann data. Previous anomaly detection methods used the difference between 
the measured Neumann data and a reference Neumann data obtained beforehand in the absence of 
abnormality. However, in practice, the reference data is not available and its computation is not possible 
since the inhomogeneous complex conductivity of the normal breast is unknown. To deal with this 
problem, we propose a frequency-difference TAS (fdTAS), in which a weighted frequency difference 
of the trans-admittance data measured at a certain moment is used for anomaly detection. This 
paper provides a mathematical framework and the feasibility of fdTAS by showing the relationship 
between the anomaly information and the weighted frequency difference of the Neumann data.

Key words. breast cancer detection, electrical conductivity, T-Scan, anomaly estimation algo-

AMS subject classifications. 35R30, 34A45, 65N21, 78A30, 78A70

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1. Introduction. A trans-admittance scanner (TAS) is a device for breast cancer 
diagnosis that is based on the consensus that complex conductivity values of breast 
tumors significantly differ from those of surrounding normal tissues [4, 18, 22, 35, 37]. 
For example, T-Scan is a commercially available TAS system that has been suggested 
for adjunctive clinical uses with X-ray mammography to decrease equivocal findings 
and thereby reduce unnecessary biopsies [4]. In TAS, a patient holds a reference elec-
trode with one hand through which a sinusoidal voltage \( V_0 \sin \omega t \) is applied, while a 
scanning probe at the ground potential is placed on the surface of the breast. The 
voltage difference \( V_0 \sin \omega t \) produces electric current flowing through the breast re-

\[ u(x) \] is the complex potential, and \( \sigma \) and \( \epsilon \) denote the conductivity and permittivity, respectively. The scanning probe is 
equipped with a planar array of electrodes, and we measure exit currents (Neumann

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data) \( g = -(\sigma + i\omega\epsilon) \frac{\partial u}{\partial n} \) which reflect electrical properties of tissues under the scan probe. Here, \( \frac{\partial u}{\partial n} \) is the normal derivative of \( u \).

The inverse problem of TAS is to detect a suspicious abnormality in a breast region underneath the probe from measured Neumann data \( g \). All previous anomaly detection methods utilize a difference \( g - g^* \), where \( g^* \) is a reference Neumann data measured beforehand without any anomaly inside the breast region [1, 34]. This difference \( g - g^* \) can be viewed as a kind of background subtraction so that it makes the anomaly apparently visible. However, in practice, it is not available in most cases, and calculating \( g^* \) is not possible since the inhomogeneous complex conductivity of a specific normal breast is unknown. In order for TAS to be more practical, we should avoid using this background difference data \( g - g^* \). Therefore, we propose a frequency difference TAS method which uses a frequency difference of trans-admittance data measured at a certain moment.

To be precise, let the human body occupy a three-dimensional domain \( \Omega \) with a smooth boundary \( \partial \Omega \). Let \( \Gamma \) and \( \gamma \) be portions of \( \partial \Omega \), denoting the probe plane placed on the breast and the surface of the metallic reference electrode, respectively. Through \( \gamma \), we apply a sinusoidal voltage of \( V_0 \sin \omega t \) with its frequency \( f = \omega / 2\pi \) in a range of 50 Hz to 500 kHz. Then the corresponding complex potential \( u_\omega \) at \( \omega \) satisfies the following mixed boundary value problem:

\[
\begin{aligned}
\nabla \cdot ((\sigma + i\omega\epsilon)\nabla u_\omega(x)) &= 0 \quad \text{in } \Omega, \\
u_\omega(x) &= 0, \quad x \in \Gamma, \\
u_\omega(x) &= V_0, \quad x \in \gamma, \\
(\sigma + i\omega\epsilon)\nabla u_\omega(x) \cdot n(x) &= 0, \quad x \in \partial \Omega \setminus (\Gamma \cup \gamma),
\end{aligned}
\]

(1)

where \( n \) is the unit outward normal vector to the boundary \( \partial \Omega \). Note that both \( \sigma = \sigma(x, \omega) \) and \( \epsilon = \epsilon(x, \omega) \) depend on \( \omega \). The scan probe \( \Gamma \) consists of a planar array of electrodes \( E_1, \ldots, E_m \), and we measure exit current \( g_\omega(j) \) through each electrode \( E_j \):

\[
g_\omega(j) := -\int_{E_j} (\sigma + i\omega\epsilon) \nabla u_\omega \cdot n \, ds \quad (j = 1, \ldots, m),
\]

where \( ds \) is the area element.

In the frequency-difference TAS (fdTAS), we apply voltage with two different frequencies \( f_1 = \omega_1 / 2\pi \) and \( f_2 = \omega_2 / 2\pi \) with \( 50 \, \text{Hz} \leq f_1 < f_2 \leq 500 \, \text{kHz} \) and measure two sets of corresponding Neumann data \( g_{\omega_1} \) and \( g_{\omega_2} \) through \( \Gamma \) at the same time. We assume that there exists a region of breast tumor \( D \) beneath the probe \( \Gamma \) so that...
that $\sigma + i\omega\epsilon$ changes abruptly across $\partial D$. (See Remark 2.1.) The inverse problem of fdTAS is to detect the anomaly $D$ beneath $\Gamma$ from a difference between $g_{\omega_1}$ and $g_{\omega_2}$.

In order for any detection algorithm to be practical, we must take into account the following limitations:

(a) Since $\Omega$ differs for each subject, the algorithm should be robust against any change in the geometry of $\Omega$ and also any change in the complex conductivity distribution outside the breast region.

(b) The Neumann data $g_{\omega}$ is available only on a small surface $\Gamma$ instead of the whole surface $\partial \Omega$.

(c) Since the inhomogeneous complex conductivity of the normal breast without $D$ is unknown, it is difficult to obtain the reference Neumann data $g_{\omega}^*$ in the absence of $D$.

These limitations are indispensable to a TAS model in practical situations, and these are the reasons why we try to improve the previous techniques [1, 2, 3, 6, 7, 8, 9, 10, 11, 13, 17, 21, 24, 25, 26, 27, 28, 31, 34] by using frequency difference.

In the fdTAS model, we use a weighted frequency difference of Neumann data $g_{\omega_2} - \alpha g_{\omega_1}$ instead of $g_{\omega_2} - g_{\omega_1}$. The weight constant $\alpha$ is approximately

$$\alpha \approx \frac{\int_{\Gamma} g_{\omega_2} ds}{\int_{\Gamma} g_{\omega_1} ds},$$

and the weight is a crucial factor in the anomaly detection. We should note that the simple difference $g_{\omega_2} - g_{\omega_1}$ may fail to extract the anomaly due to the complicated structure of the solution of the complex conductivity equation. See Remark 3.3. In Theorem 3.2, we explain how $g_{\omega_2} - \alpha g_{\omega_1}$ reflects a contrast in complex conductivity values between the anomaly $D$ and surrounding normal tissues. The approximate representation formula is given in Remark 3.4.

Recently, we published a preliminary experimental validation study of fdTAS in [32]. However, this previous work lacks a mathematical analysis of the method. We therefore describe a rigorous mathematical framework of the fdTAS method in this paper.

2. Mathematical model and the feasibility of fdTAS. We assume that $\sigma$ and $\epsilon$ are isotropic, positive, and piecewise smooth functions in $\Omega$. Let $u_\omega$ be the $H^1(\Omega)$-solution of (1). Denoting the real and imaginary parts of $u_\omega$ by $v_\omega = \Re u_\omega$ and $h_\omega = \Im u_\omega$, the mixed boundary value problem (1) can be expressed as the following coupled system:

$$\begin{align*}
\nabla \cdot (\sigma \nabla v_\omega) - \nabla \cdot (\omega \epsilon \nabla h_\omega) &= 0 \quad \text{in } \Omega, \\
\nabla \cdot (\omega \epsilon \nabla v_\omega) + \nabla \cdot (\sigma \nabla h_\omega) &= 0 \quad \text{in } \Omega, \\
v_\omega &= 0 \quad \text{and} \quad h_\omega = 0 \quad \text{on } \Gamma, \\
v_\omega &= V_0 \quad \text{and} \quad h_\omega = 0 \quad \text{on } \gamma, \\
\mathbf{n} \cdot \nabla v_\omega &= 0 \quad \text{and} \quad \mathbf{n} \cdot \nabla h_\omega = 0 \quad \text{on } \partial \Omega \setminus (\Gamma \cup \gamma).
\end{align*}$$

The measured Neumann data $g_{\omega}$ can be decomposed into

$$g_{\omega}(x) := \mathbf{n} \cdot (-\sigma \nabla v_\omega(x) + \omega \epsilon \nabla h_\omega(x)) + i \mathbf{n} \cdot (-\sigma \nabla h_\omega(x) - \omega \epsilon \nabla v_\omega(x)), \quad x \in \Gamma.$$  

The solution of the coupled system (2) is a kind of saddle point [5, 12], and we have the following relations:

$$V_0 \int_{\Gamma} \Re(g_{\omega}) ds = \min_{v \in \mathcal{H}_{re}} \max_{h \in \mathcal{H}_{im}} \int_{\Omega} \left[ \sigma |\nabla v|^2 - 2\omega \epsilon \nabla v \cdot \nabla h - \sigma |\nabla h|^2 \right] dx.$$
and

\[ V_0 \int_{\Gamma} \Im(g_{\omega}) ds = \min_{v \in H^1_{\text{re}}} \max_{h \in H^1_{\text{im}}} \int_{\Omega} [\omega \epsilon |\nabla v|^2 + 2\sigma \nabla v \cdot \nabla h - \omega \epsilon |\nabla h|^2] \, dx, \]

where \( H_{\text{re}} := \{ v \in H^1(\Omega) : v|_\Gamma = 0, \nabla v|_{\partial\Omega \setminus \partial\gamma} = 0 \} \) and \( H_{\text{im}} := \{ h \in H^1(\Omega) : h|_{\partial\gamma} = 0, \frac{\partial h}{\partial n}|_{\partial\Omega \setminus \partial\gamma} = 0 \} \).

In order to detect a lesion \( D \) underneath the scan probe \( \Gamma \), we define a local region of interest under the probe plane \( \Gamma \) as shown in Figure 2. For simplicity, we let \( x_3 \) be the axis normal to \( \Gamma \) and let the center of \( \Gamma \) be the origin. Hence, the probe region \( \Gamma \) can be approximated as a two-dimensional region \( \Gamma = \{(x_1, x_2, 0) : \sqrt{x_1^2 + x_2^2} < L \} \), where \( L \) is the radius of the scan probe. We set the region of interest inside the breast as a half ball \( \Omega_L = \Omega \cap B_L \) shown in Figure 2, where \( B_L \) is a ball with a radius \( L \) and its center at the origin.

Remark 2.1. We summarize conductivity and permittivity values of normal and tumor tissues in the breast. Both \( \sigma \) and \( \omega \) have a unit of S/m and \( \sigma + i\omega \epsilon = \sigma + i2\pi f \epsilon_0 \epsilon_r \), where \( \epsilon_0 \approx 8.854 \times 10^{-12} \, [\text{F/m}] \) is the permittivity of the free space and \( \epsilon_r \) is a relative permittivity. Note that \( \frac{\omega \epsilon_0}{\sigma} \leq \frac{1}{\pi} \) for a frequency \( f = \omega/2\pi \geq 50 \, \text{kHz} \) [37].

<table>
<thead>
<tr>
<th>( f = \omega/2\pi, [\text{Hz}] )</th>
<th>( \sigma_n, [\text{S/m}] )</th>
<th>( \sigma_c, [\text{S/m}] )</th>
<th>( \omega \epsilon_n, [\text{S/m}] )</th>
<th>( \omega \epsilon_c, [\text{S/m}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \leq 500 )</td>
<td>0.03</td>
<td>0.2</td>
<td>( \ll \sigma_n )</td>
<td>( \ll \sigma_c )</td>
</tr>
<tr>
<td>50 \times 10^3</td>
<td>0.03</td>
<td>0.2</td>
<td>5.6 \times 10^{-4}</td>
<td>1.7 \times 10^{-2}</td>
</tr>
<tr>
<td>100 \times 10^3</td>
<td>0.03</td>
<td>0.2</td>
<td>2.8 \times 10^{-4}</td>
<td>2.2 \times 10^{-2}</td>
</tr>
<tr>
<td>500 \times 10^3</td>
<td>0.03</td>
<td>0.2</td>
<td>1.1 \times 10^{-3}</td>
<td>5.6 \times 10^{-2}</td>
</tr>
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</table>

For a successful anomaly detection, we should carefully choose two frequencies \( \omega_1 \) and \( \omega_2 \). In our TAS system, we choose \( f_1 = \omega_1/2\pi \) and \( f_2 = \omega_2/2\pi \) such that

\[ 50 \, \text{Hz} \leq f_1 \leq 500 \, \text{Hz} \quad \text{and} \quad 50 \, \text{kHz} \leq f_2 \leq 500 \, \text{kHz}. \]

We denote by \( u_1 = v_1 + ih_1 \) and \( u_2 = v_2 + ih_2 \) the complex potentials satisfying (2) at \( \omega_1 \) and \( \omega_2 \), respectively, and let \( g_1 = g_{\omega_1} \) and \( g_2 = g_{\omega_2} \). The fdTAS aims to detect \( D \) from a weighted difference between \( g_1 \) and \( g_2 \).

Now, let us investigate the connection between \( u_1 \) and \( u_2 \) and whether the frequency-difference Neumann data \( g_2 - \alpha g_1 \) has any information of \( D \). Since both \( \sigma \) and \( \epsilon \) depend on \( \omega \) and \( x \), \( \sigma(x, \omega_1) \neq \sigma(x, \omega_2) \) and \( \epsilon(x, \omega_1) \neq \epsilon(x, \omega_2) \). For simplicity, we denote

\[ \sigma_j(x) = \sigma(x, \omega_j) \quad \text{and} \quad \epsilon_j(x) = \epsilon(x, \omega_j), \quad j = 1, 2. \]

There is a cancerous lesion \( D \) inside \( \Omega_L \), and the complex conductivity \( \sigma_j + i\omega_j \epsilon_j \) changes abruptly across \( \partial D \) as in the table in Remark 2.1. To distinguish them,
we denote
\[ \sigma_j = \begin{cases} \sigma_{j,n} & \text{in } \Omega_L \setminus D, \\ \sigma_{j,c} & \text{in } D, \end{cases} \quad \text{and} \quad \epsilon_j = \begin{cases} \epsilon_{j,n} & \text{in } \Omega_L \setminus D, \\ \epsilon_{j,c} & \text{in } D. \end{cases} \]

With the use of this notation, \( u_1 \) and \( u_2 \) satisfy
\[ \begin{cases} \nabla \cdot ((\sigma_1 + i\omega \epsilon_1) \nabla u_1) = 0 & \text{in } \Omega, \\ u_1|_\Gamma = 0, \quad u_1|_\gamma = V_0, \quad \sigma_1 + i\omega \epsilon_1 \frac{\partial u_1}{\partial n}|_{\partial\Omega \setminus (\Gamma \cup \gamma)} = 0, \end{cases} \]
\[ \begin{cases} \nabla \cdot ((\sigma_2 + i\omega \epsilon_2) \nabla u_2) = 0 & \text{in } \Omega, \\ u_2|_\Gamma = 0, \quad u_2|_\gamma = V_0, \quad (\sigma_2 + i\omega \epsilon_2) \frac{\partial u_2}{\partial n}|_{\partial\Omega \setminus (\Gamma \cup \gamma)} = 0. \end{cases} \]

Remark 2.2. Due to the complicated structure of (3) and (4) for the solution \( u_\omega \), it is quite difficult to analyze the interrelation between the complex conductivity contrast \( \nabla(\sigma + i\omega \epsilon) \) and the Neumann data \( g_\omega \). In [1], the authors briefly mentioned that the multifrequency TAS method can be regarded as a straightforward extension of their single-frequency TAS algorithm (Remark 2.3 in [1]). However, the simple frequency-difference data \( g_2 - g_1 \) on \( \Gamma \) may fail to extract the anomaly for more general cases of complex conductivity distributions in \( \Omega \) due to the complicated structure of the solution of (2). To be precise, the use of the weighted difference is essential when the background comprises biological materials with nonnegligible frequency-dependent complex conductivity values. To explain it clearly, consider a homogeneous complex conductivity distribution in \( \Omega \) where \( \sigma(x, \omega) + i\omega \epsilon(x, \omega) \) depends only on \( \omega \). Due to the frequency dependency, the simple difference \( g_2 - g_1 \) is not zero, while \( g_2 - \alpha g_1 \) is zero. Hence, any reconstruction method using \( g_2 - g_1 \) always produces artifacts because \( g_2 - g_1 \) does not eliminate modeling errors. See (26) for an approximation of \( g_2 - g_1 \) in the presence of an anomaly \( D \).

Remark 2.3. In this work, we do not consider effects of contact impedances along electrode-skin interfaces. For details about the contact impedance, please see [20, 36] and other publications cited therein. In TAS, we may adopt a skin preparation procedure and electrode gels to reduce contact impedances. Since we cannot expect complete removal of contact impedances, however, we need to investigate how exit currents are affected by contact impedances of a planar array of electrodes that are kept at the grounded potential. The contact impedance of each electrode leads to a voltage drop across it, and therefore the voltage underneath the electrode-skin interface layer would be slightly different than zero. In other words, when contact impedances are not negligible, the surface area in contact with \( \Gamma \) cannot be regarded as an equipotential surface anymore, and this will result in some changes in exit currents. Future studies are needed to estimate how the contact impedance affects the weighted difference of the Neumann data. We should also investigate experimental techniques, including choice of frequencies, to minimize their effects.

The next observation explains why we should use a weighted difference \( g_2 - \alpha g_1 \) instead of \( g_2 - g_1 \).

Observation 2.4. Denoting \( \eta := \frac{g_2 - i\omega \epsilon_2}{\sigma_1 + i\omega \epsilon_1} \), it follows from a direct computation that \( u_2 - u_1 \) satisfies
\[ \begin{cases} \nabla \cdot ((\sigma_1 + i\omega \epsilon_1) \nabla (u_2 - u_1)) = -((\sigma_1 + i\omega \epsilon_1) \nabla \log \eta) \cdot \nabla u_2 & \text{in } \Omega, \\ (u_2 - u_1)|_{\Gamma \cup \gamma} = 0, \\ (\sigma_1 + i\omega \epsilon_1) \frac{\partial (u_2 - u_1)}{\partial n}|_{\partial\Omega \setminus (\Gamma \cup \gamma)} = 0. \end{cases} \]

For the detection of \( D \), we use the following weighted difference:
\[ g_2 - \alpha g_1 = \eta \left( \sigma_1 + i\omega \epsilon_1 \right) \bm{n} \cdot \nabla (u_2 - u_1) \quad \text{on } \Gamma, \]
where \( \alpha = \eta|_\Gamma \). If \( \nabla \log \eta = 0 \) in (8), \( u_1 = u_2 \) in \( \Omega \) and \( g_2 - \alpha g_1 = 0 \) on \( \Gamma \). In other words, if \( \nabla \log \eta = 0 \) in \( \Omega_L \), it is impossible to detect \( D \) from \( g_2 - \alpha g_1 = 0 \) regardless of contrasts in \( \sigma \) and \( \epsilon \) across \( \partial D \). Any useful information on \( D \) could be found from nonzero \( g_2 - \alpha g_1 \) on \( \Gamma \) when \( |\nabla \log \eta| \) is large along \( \partial D \).

For chosen frequencies \( \omega_1 \) and \( \omega_2 \), we can assume that \( \sigma \) and \( \epsilon \) are approximately constant in the normal breast region \( \Omega_L \backslash \bar{D} \) and also in the cancerous region \( D \). Hence, if \( \eta \) changes abruptly across \( \partial D \), we roughly have

\[
\nabla \log \eta \approx 0 \quad \text{in} \quad \Omega_L \backslash \bar{D} \quad \text{and} \quad |\nabla \log \eta| = \infty \quad \text{on} \quad \partial D,
\]

and therefore the term \((\sigma_1 + i\omega_1\epsilon_1)\nabla \log \eta \cdot \nabla u_2 \) in (8) is supported on \( \partial D \) in the breast region \( \Omega_L \). This explains that the difference \( g_2 - \alpha g_1 \) on \( \Gamma \) can provide the information of \( \partial D \). Take note that the inner product \( \nabla \log \eta \cdot \nabla u_2 \) is to be interpreted in a suitable distributional sense if the coefficients jump at \( \partial D \).

3. Mathematical analysis for fdTAS.

3.1. Representation formula. Observation 2.4 in the previous section roughly explains how \( D \) is related to \( g_2 - \alpha g_1 \). In this section, the observation will be justified rigorously in a simplified model. We assume that \( \sigma_{j,n}, \sigma_{j,c}, \epsilon_{j,n}, \) and \( \epsilon_{j,c} \) are constants. According to the table in Remark 2.1, the change of the conductivity due to the change of frequency is small, so we assume that

\[
\sigma_{1,n} = \sigma_{2,n} := \sigma_n \quad \text{and} \quad \sigma_{1,c} = \sigma_{2,c} := \sigma_c.
\]

Since the breast region of interest is relatively small compared with the entire body \( \Omega \), we may assume that \( \Omega \) is the lower half space \( \Omega = \mathbb{R}^3_- := \{x = (x_1, x_2, x_3) \mid x_3 < 0\} \) and \( \gamma = \infty \).

Suppose that \( v_j \) and \( h_j \) are \( H^1 \)-solutions of the following coupled system for \( j = 1 \) and 2:

\[
\begin{align*}
\nabla \cdot (\sigma \nabla v_j) - \nabla \cdot (\omega_j \epsilon_j \nabla h_j) &= 0 \quad \text{in} \quad \Omega = \mathbb{R}^3_-,
\n\nabla \cdot (\omega_j \epsilon_j \nabla v_j) + \nabla \cdot (\sigma \nabla h_j) &= 0 \quad \text{in} \quad \Omega = \mathbb{R}^3_-,
\nv_j &= 1 \quad \text{and} \quad h_j = 0 \quad \text{on} \quad \Gamma,
\mathbf{n} \cdot \nabla v_j &= 0 \quad \text{and} \quad \mathbf{n} \cdot \nabla h_j &= 0 \quad \text{on} \partial \Omega \backslash \Gamma.
\end{align*}
\]

Let \( u_j = v_j + ih_j \). Then \( V_0(1 - u_j) \) can be viewed as a solution of (7) with \( \Omega = \mathbb{R}^3_- \) and \( \gamma = \infty \).

Let us introduce a key representation formula explaining the relationship between \( D \) and the weighted difference \( g_2 - \alpha g_1 \). For each \( x \in \mathbb{R}^3 \backslash \Gamma \), we define

\[
\Psi(x, y) = \Phi(x, y) + \Phi(x, y^+) + \varphi(x, y),
\]

where \( y^+ = (y_1, y_2, -y_3) \) is the reflection point of \( y \) with respect to the plane \( \{y_3 = 0\} \) and \( \varphi(x, \cdot) \) is the \( H^1(\mathbb{R}^3 \backslash \Gamma) \)-solution of the following PDE:

\[
\begin{align*}
\Delta_y \varphi(x, y) &= 0, \quad y \in \mathbb{R}^3 \backslash \Gamma,
\varphi(x, y) &= \frac{1}{2\pi|x-y|}, \quad y \in \Gamma,
\varphi(x, y) &= 0 \quad \text{as} \quad |y| \to \infty.
\end{align*}
\]

The following theorem explains an explicit relation between \( D \) and \( \Im(g_2 - \alpha g_1) \).
where

\[ x \in \Omega \]

Putting \((13)\) into \((12)\) and then applying the following formula:

\[ \begin{align*}
1 - \Re(g_2 - \alpha g_1)(x) &= \int_D \nabla_y \frac{\partial \Phi(x, y)}{\partial x_3} \cdot \Theta(y) dy \\
&\quad + \frac{\partial}{\partial x_3} \int_{\partial \Omega \setminus \Gamma} \frac{\partial \Phi(x, y)}{\partial y_3} \left[ \int_D \nabla_z \Psi(y, z) \cdot \Theta(z) dz \right] ds, \ x \in \Gamma,
\end{align*} \]

where

\[ \Theta(y) = \frac{\sigma_n - \sigma_c}{\sigma_n} \nabla(h_2 - h_1)(y) + \frac{\omega_2(\epsilon_{2,n} - \epsilon_{2,c})}{\sigma_n} \nabla(v_2 - v_1)(y) - \Re(\beta \nabla u_1(y)) \]

and

\[ \beta = \frac{i}{1 + i \frac{\omega_1 \epsilon_{1,n}}{\sigma_n}} \left( \left[ \frac{\omega_2 \epsilon_{2,n}}{\sigma_n} \left( \frac{\epsilon_{2,c}}{\sigma_c} - \frac{\epsilon_{2,n}}{\sigma_n} \right) - \frac{\omega_1 \epsilon_{2,n}}{\sigma_n} \left( \frac{\epsilon_{1,c}}{\sigma_c} - \frac{\epsilon_{1,n}}{\sigma_n} \right) \right] - i \frac{\omega_1 \omega_2 \epsilon_{1,n} \epsilon_{2,n}}{\sigma_n^2} \left( \frac{\epsilon_{1,c}}{\sigma_c} - \frac{\epsilon_{2,c}}{\epsilon_{2,n}} \right) \right). \]

Proof. Due to Green’s identity, \((\sigma_n + i \omega_2 \epsilon_{2,n})(u_2 - u_1)\) has integral representation:

for each \(x \in \Omega\),

\[ (\sigma_n + i \omega_2 \epsilon_{2,n})(u_2 - u_1)(x) \]

\[ = (\sigma_n + i \omega_2 \epsilon_{2,n}) \int_{\partial \Omega \setminus \Gamma} \frac{\partial \Phi(x, y)}{\partial n} (u_2 - u_1)(y) ds + \int_{\Gamma} \Phi(x, y)(g_2 - \alpha g_1)(y) ds \]

\[ + \int_{\partial D} \Phi(x, y)(\sigma_n + i \omega_2 \epsilon_{2,n}) \left( \frac{\partial(u_2 - u_1)}{\partial n} \right)_+ - \frac{\partial(u_2 - u_1)}{\partial n} \right)_-(y) ds. \]

Here, we denote \(\frac{\partial u_j}{\partial n} \right)_+ = n \cdot \nabla u_j^+ |_{\partial D}\), where \(u_j^+ = u_j|_{\Omega \setminus \Gamma}\) and \(u_j^- = u_j|_{\Gamma}\). From the transmission condition,

\[ (\sigma_n + i \omega_2 \epsilon_{j,n}) \frac{\partial u_j}{\partial n} \right)_+ = (\sigma_c + i \omega_2 \epsilon_{j,c}) \frac{\partial u_j}{\partial n} \right)_-, \ j = 1, 2 \quad \text{on} \ \partial D. \]

It follows that

\[ \begin{align*}
\frac{\sigma_n + i \omega_2 \epsilon_{2,n}}{\sigma_n} \left( \frac{\partial(u_2 - u_1)}{\partial n} \right)_+ &= \frac{\sigma_c + i \omega_2 \epsilon_{2,c}}{\sigma_c} \frac{\partial(u_2 - u_1)}{\partial n} \right)_- + \beta \frac{\partial u_1}{\partial n} \right-_\quad \text{on} \ \partial D.
\end{align*} \]

Putting \((13)\) into \((12)\) and then applying \(-\frac{\partial}{\partial x_3}\) to both sides of \((12)\) yield for each \(x \in \Gamma\)

\[ \begin{align*}
\frac{1}{2\sigma_n} \Im(g_2 - \alpha g_1)(x) &= \int_D \nabla_y \frac{\partial \Phi(x, y)}{\partial x_3} \cdot \left[ \frac{\sigma_n - \sigma_c}{\sigma_n} \nabla(h_2 - h_1) \\
&\quad + \frac{\omega_2(\epsilon_{2,n} - \epsilon_{2,c})}{\sigma_n} \nabla(v_2 - v_1) - \Im(\beta \nabla u_1) \right] dy + \Xi(x),
\end{align*} \]

where

\[ \Xi(x) = -\frac{\partial}{\partial x_3} \int_{\partial \Omega \setminus \Gamma} \frac{\partial \Phi(x, y)}{\partial y_3} \left( (h_2 - h_1)(y) + \frac{\omega_2 \epsilon_{2,n}}{\sigma_n} (v_2 - v_1)(y) \right) ds. \]
From the definition of \( \Psi(x, y) \), it is easy to see that \( \Psi(x, y) \) satisfies

\[
\begin{align*}
\Delta_x \Psi(x, y) &= \delta(x - y), \quad x, y \in \mathbb{R}^3, \\
\Psi(x, y) &= 0, \quad x \in \Gamma, \; y \in \mathbb{R}^3, \\
\frac{\partial \Psi(x, y)}{\partial x_3} &= 0, \quad x \in \partial \Omega \setminus \Gamma, \; y \in \mathbb{R}^3, \\
\Psi(x, y) &= 0 \; \text{as} \; |x - y| \to \infty.
\end{align*}
\]

In order to relate \( \Xi(x) \) with \( D \), we repeat the argument in (12) with \( \Phi \) replaced by \( \Psi \):

\[
(h_2 - h_1)(y) + \frac{\omega_2 \epsilon_{2,n}}{\sigma_n} (v_2 - v_1)(y) = \frac{1}{\sigma_n} \Im \{(\sigma_n + i \omega_2 \epsilon_{2,n}) (u_2 - u_1)(y)\}
\]

\[
= 3 \int_{\partial D} \Psi(y, z) \frac{\sigma_n}{\sigma_n} \left( \frac{\partial (u_2 - u_1)}{\partial n} \bigg|_+ - \frac{\partial (u_2 - u_1)}{\partial n} \bigg|_- \right) (z) \, dz, \quad y \in \partial \Omega \setminus \Gamma.
\]

The above identity and the jump condition (13) lead to

\[
(h_2 - h_1)(y) + \frac{\omega_2 \epsilon_{2,n}}{\sigma_n} (v_2 - v_1)(y) = -\int_D \nabla_2 \Psi(y, z) \cdot \Theta(z) \, dz, \quad y \in \partial \Omega \setminus \Gamma.
\]

This completes the proof. \( \Box \)

Now, let us derive a constructive formula extracting \( D \) from the representation formula (11) under some reasonable assumptions. We assume that

\[
\hat{D} \subset \Omega_{L/2}, \quad D = B_\delta(\xi), \quad \text{and} \quad \delta \leq \text{dist}(D, \Gamma) \leq C_1 \delta,
\]

where \( C_1 \) is a positive constant, \( B_\delta \) is a ball with the radius \( \delta \) and the center \( \xi \), and \( \frac{\delta}{L} \leq \frac{1}{10} \). Suppose we choose \( \frac{\omega_1}{\pi} \approx 50 \text{ Hz} \) and \( \frac{\omega_2}{\pi} \approx 100 \text{ kHz} \). Then the experimental data in Remark 2.1 shows \( \frac{\omega_1 \epsilon_{1,n}}{\sigma_n} \approx \frac{1}{100} \) and \( \frac{\omega_2 \epsilon_{2,n}}{\sigma_n} \leq \frac{1}{10000} \). Hence, in practice, we can view

\[
\frac{\omega_1 \epsilon_{1,n}}{\sigma_n} \approx 0, \quad (\delta/L)^3 \approx 0, \quad \left( \frac{\omega_2 \epsilon_{2,n}}{\sigma_n} \right)^2 \approx 0.
\]

Based on the experimental data in Remark 2.1, we assume that

\[
\max \left\{ \frac{\epsilon_{j,n}}{\epsilon_{j,c}}, \frac{\sigma_n}{\sigma_c} \right\} \leq \kappa_1, \quad \frac{\omega_2 \epsilon_{2,n}}{\sigma_n} \leq \frac{\sigma_n}{\sigma_c}, \quad \frac{\sigma_c}{\sigma_n} \leq \kappa_3,
\]

where \( \kappa_1 \) and \( \kappa_2 \) are positive constants less than \( \frac{1}{2} \) and \( \kappa_3 \) is a positive constant less than 10. Taking advantage of these, we can simplify the representation formula (11).

**Theorem 3.2.** Under the assumptions (15) and (17), the imaginary part of the weighted frequency difference \( g_2 - \alpha g_1 \) can be expressed as

\[
\frac{1}{2\sigma_n} \Im (g_2 - \alpha g_1)(x) = \int_D \frac{\partial}{\partial x_3} \frac{(x - y) \cdot \hat{\Theta}(y)}{4\pi|x - y|^3} \, dy + \text{Error}(x), \quad x \in \Gamma_{L/2},
\]

where

\[
\hat{\Theta} = \frac{\sigma_n - \sigma_c}{\sigma_n} \nabla h_2 - \frac{\omega_2 \epsilon_{2,n}}{\sigma_n} \left( \epsilon_{2,n} - \frac{\sigma_n}{\sigma_c} \right) \nabla v_1
\]
and the error term \( \text{Error}(x) \) is estimated by

\[
|\text{Error}(x)| \leq \left[ \frac{\omega^2 \epsilon_n^2}{\sigma_n} P_1 \left( \left( \frac{\epsilon_{2,c}}{\epsilon_{2,n}} - \frac{\sigma_c}{\sigma_n} \right) \right) \right] \frac{\delta^3}{L^3} + \left( \frac{\omega^1 \epsilon_{1,n}}{\sigma_n} P_1 \left( \left( \frac{\epsilon_{1,c}}{\epsilon_{1,n}} - \frac{\sigma_c}{\sigma_n} \right) \right) \right) \frac{\delta^3}{L^3}.
\]

Here, \( P_n(\lambda) \) is a polynomial function of order \( n \) such that \( P_n(0) = 0 \) and its coefficients depend only on \( \kappa_j, j = 1, 2, 3 \).

**Proof.** From the transmission conditions of \( u_\omega \) across \( \partial D \), we have

\[
\frac{\partial h_\omega}{\partial n} + \frac{\sigma_c}{\sigma_n} \frac{\partial h_\omega}{\partial n} = \frac{\omega \epsilon_n}{\sigma_n} \frac{\sigma_c}{\sigma_n} \frac{\partial v_{\omega}}{\partial n} + \frac{\omega \epsilon_n}{\sigma_n} \frac{\sigma_c}{\sigma_n} \frac{\partial h_\omega}{\partial n}.
\]

Since \( h_\omega \) satisfies the mixed boundary condition with \( h_\omega|_{\Gamma} = 0 \) and \( \frac{\partial h_\omega}{\partial n}|_{\partial \Omega|_{\Gamma}} = 0 \), we have the following estimate:

\[
\int_\Omega \left( \chi_{\Omega|D} + \frac{\sigma_c}{\sigma_n} \chi_D \right) |\nabla h_\omega|^2 \leq \int_\Omega \left( \frac{\omega \epsilon_n}{\sigma_n} \frac{\sigma_c}{\sigma_n} \frac{\partial v_{\omega}}{\partial n} \right) \left( \frac{\epsilon_{2,c}}{\epsilon_{2,n}} - \frac{\sigma_c}{\sigma_n} \right) ||\nabla v_{\omega}||_{L^2(D)} ||\nabla h_\omega||_{L^2(D)} + \int_\Omega \left( \frac{\omega \epsilon_n}{\sigma_n} \frac{\sigma_c}{\sigma_n} \frac{\partial h_\omega}{\partial n} \right) \left( \frac{\epsilon_{2,c}}{\epsilon_{2,n}} - \frac{\sigma_c}{\sigma_n} \right) ||\nabla h_\omega||_{L^2(D)}.
\]

This gives

\[
||\nabla h_\omega||_{L^2(D)} \leq \left( \frac{\epsilon_{2,c}}{\epsilon_{2,n}} - \frac{\sigma_c}{\sigma_n} \right) \frac{1}{||\nabla v_{\omega}||_{L^2(D)}} \left( \frac{\omega \epsilon_n}{\sigma_n} \frac{\sigma_c}{\sigma_n} \right) ||\nabla v_{\omega}||_{L^2(D)}.
\]

Since \( ||\nabla v_{\omega}||_{L^2(D)} \leq C \sqrt{|D|} \), where \( C \) depends only on \( \kappa_3 \), we obtain

\[
||\nabla h_\omega||_{L^2(D)} \leq \left( \frac{\omega \epsilon_n}{\sigma_n} \frac{\sigma_c}{\sigma_n} \right) \sqrt{|D|}.
\]

We also use the jump condition for \( v_\omega \):

\[
\frac{\partial v_{\omega}}{\partial n} + \frac{\sigma_c}{\sigma_n} \frac{\partial v_{\omega}}{\partial n} = \frac{\omega \epsilon_n}{\sigma_n} \frac{\sigma_c}{\sigma_n} \frac{\partial v_{\omega}}{\partial n} + \frac{\omega \epsilon_n}{\sigma_n} \frac{\sigma_c}{\sigma_n} \frac{\partial h_\omega}{\partial n}.
\]

Applying the same process as in (19), we obtain

\[
||\nabla v_{\omega} - \nabla u_0||_{L^2(D)} \leq \left( \frac{\omega \epsilon_n}{\sigma_n} \frac{\sigma_c}{\sigma_n} \right) \sqrt{|D|},
\]

where \( u_0 = u_\omega \) with \( \omega = 0 \). From (22), we get

\[
||\nabla v_2 - \nabla v_1||_{L^2(D)} \leq ||\nabla v_2 - \nabla u_0||_{L^2(D)} + ||\nabla v_1 - \nabla u_0||_{L^2(D)} \leq \left( \frac{\omega \epsilon_n^2}{\sigma_n} \frac{\sigma_c}{\sigma_n} \right) \sqrt{|D|}.
\]

Hence, it follows from (20) and (23) that

\[
||\Theta - \hat{\Theta}||_{L^2(D)} \leq \left( \frac{\omega \epsilon_n^2}{\sigma_n} \frac{\sigma_c}{\sigma_n} \right) \frac{1}{||\nabla v_{\omega}||_{L^2(D)}} \left( \frac{\epsilon_{1,c}}{\epsilon_{1,n}} - \frac{\sigma_c}{\sigma_n} \right) \left( \frac{\epsilon_{1,c}}{\epsilon_{1,n}} - \frac{\sigma_c}{\sigma_n} \right) \sqrt{|D|}.
\]
From the Schwarz inequality,
\[
\left| \int_D \frac{\partial}{\partial x} \frac{(x - y)}{4\pi|x - y|^2} \cdot \left( \Theta(y) - \tilde{\Theta}(y) \right) dy \right| \leq \frac{C\sqrt{|D|}}{|x - \xi|^3} \| \Theta - \tilde{\Theta} \|_{L^2(D)}
\]
\[
\leq \left( \frac{\omega_1 \epsilon_{1,n} p_1}{\sigma_n} \left( \left| \frac{\epsilon_{1,c}}{\epsilon_{1,n}} \right| - \frac{\sigma_c}{\sigma_n} \right) + \left( \frac{\omega_2 \epsilon_{2,n}}{\sigma_n} \right)^2 \left( \frac{\epsilon_{2,c}}{\epsilon_{2,n}} - \frac{\sigma_c}{\sigma_n} \right) \right) \frac{\delta^3}{|x - \xi|^3}.
\]

Now, it remains to study the last term in (11). Using the Schwarz inequality, it is easy to see that
\[
\left| \frac{\partial}{\partial x} \int_{\partial D \setminus \Gamma} \frac{\partial \Phi(x,y)}{\partial y} \int_D \nabla_x \Psi(y,z) \cdot \Theta(z) dz \right| \leq \frac{\delta^3}{L^2} \| \Theta \|_{L^2(D)}.
\]

This completes the proof. \( \square \)

Remark 3.3. According to Theorem 3.2, (21), and (23),
\[
\frac{1}{2\sigma_n^3} (g_2 - \alpha g_1) = 0 \quad \text{when} \quad \left| \frac{\epsilon_{j,c}}{\epsilon_{j,n}} - \frac{\sigma_c}{\sigma_n} \right| = 0, \quad j = 1, 2.
\]

Hence, even if \( \epsilon_{2,c} \) and \( \epsilon_{1,c} \) are quite different, we cannot extract any information of \( D \) when \( \left| \frac{\epsilon_{i,c}}{\epsilon_{i,n}} - \frac{\sigma_c}{\sigma_n} \right| = 0, \quad j = 1, 2 \). On the other hand, even if \( \epsilon_{2,c} = \epsilon_{1,c} \), we can extract the information of \( D \) whenever \( \left| \frac{\epsilon_{i,c}}{\epsilon_{i,n}} - \frac{\sigma_c}{\sigma_n} \right| \neq 0, \quad j = 1, 2 \).

Remark 3.4. Based on (18), we can derive the following simple approximate formula for the reconstruction of \( D \):
\[
\frac{1}{2\sigma_n^3} (g_2 - \alpha g_1)(x) \approx \frac{\omega_2 \epsilon_{2,n}}{\sigma_n} \left( \frac{3\sigma_n^2}{2\sigma_n + \sigma_c} \right) \left( \frac{\epsilon_{2,c}}{\epsilon_{2,n}} - \frac{\sigma_c}{\sigma_n} \right) \partial_{x_3} U(\xi)
\]
\[
\times |D| \left( \frac{\delta^3}{4\pi|x - \xi|^5} \right) - \frac{(x_1 - \xi_1)^2 - (x_2 - \xi_2)^2}{4\pi|x - \xi|^5}, \quad x \in \Gamma_{L/2},
\]

where \( U \) is the solution of (10) in the absence of anomaly at \( \omega = 0 \). Note that the difference \( g_2 - g_1 \) can be approximated by
\[
\frac{1}{2\sigma_n^3} (g_2 - g_1)(x) \approx \frac{\omega_2 \epsilon_{2,n}}{2\sigma_n^2} g_1(x) + i \frac{\omega_2 \epsilon_{2,n}}{\sigma_n} \left( \frac{3\sigma_n^2}{2\sigma_n + \sigma_c} \right) \left( \frac{\epsilon_{2,c}}{\epsilon_{2,n}} - \frac{\sigma_c}{\sigma_n} \right) \partial_{x_3} U(\xi)
\]
\[
\times |D| \left( \frac{\delta^3}{4\pi|x - \xi|^5} \right) - \frac{(x_1 - \xi_1)^2 - (x_2 - \xi_2)^2}{4\pi|x - \xi|^5}, \quad x \in \Gamma_{L/2},
\]

and therefore any detection algorithm using the above approximation will be disturbed by the term \( \frac{\omega_2 \epsilon_{2,n}}{2\sigma_n^2} g_1 \).

We will prove the approximation (25) roughly. From [1], we have
\[
\left| \nabla U(y) - \partial_{x_3} U(\xi) e_3 \right| \leq C \frac{\delta}{L} \sqrt{|D|}, \quad y \in D,
\]

where \( e_3 = (0, 0, 1) \). Now let \( V \) be the \( H^1 \)-solution of the following PDE:
\[
\begin{cases}
\Delta V = 0 & \text{in } \Omega \setminus \partial D, \\
\frac{\sigma_n}{\sigma_c} \frac{\partial V}{\partial n} |_+ - \frac{\partial V}{\partial n} |_- = -n \cdot e_3 & \text{on } \partial D, \\
V |_{\Gamma} = 0 & \text{and } \frac{\partial V}{\partial n}|_{\partial D \setminus \Gamma} = 0.
\end{cases}
\]
Using the same process as in (19) and (27), we obtain

$$\|\nabla u_0 - \nabla U(y) - \partial_{x_3} U(\xi) (\frac{\sigma_n}{\sigma_c} - 1) \nabla V\|_{L^2(D)} \leq C \frac{\delta}{L} \sqrt{|D|},$$

where $C$ depends only on $\kappa_3$. From (22), (27), and (29), we have

$$\|\nabla \nu_\omega - \partial_{x_3} U(\xi) \left( e_3 + \left( \frac{\sigma_n}{\sigma_c} - 1 \right) \nabla V \right)\|_{L^2(D)}$$

$$\leq \left( \left( \frac{\omega \epsilon_n}{\sigma_n} \right)^2 P_2 \left( \frac{\epsilon_c}{\sigma_n} - \frac{\sigma_c}{\sigma_n} \right) + C \frac{\delta}{L} \right) \sqrt{|D|}.$$

Applying the same process as in (19) and the identities (20) and (29), we obtain

$$\|\nabla h_2 - \frac{\omega \epsilon_n}{\sigma_n} \sigma_n (\frac{\sigma c}{\sigma_n} - \frac{\epsilon_c}{\sigma_n} \partial_{x_3} U(\xi) \left( 1 + \left( \frac{\sigma_n}{\sigma_c} - 1 \right) \partial_{x_3} V(\xi) \right) \nabla V \|_{L^2(D)}$$

$$\leq \left( \frac{\omega \epsilon_n}{\sigma_n} \right)^2 P_3 \left( \frac{\epsilon_c}{\sigma_n} - \frac{\sigma_c}{\sigma_n} \right) \| \nabla v_2 - \partial_{x_3} U(\xi) \left( e_3 + \left( \frac{\sigma_n}{\sigma_c} - 1 \right) \partial_{x_3} V(\xi) e_3 \right) \|_{L^2(D)}.$$

From (30), we get

$$\|\nabla h_2 - \frac{\omega \epsilon_n}{\sigma_n} \sigma_n (\frac{\sigma c}{\sigma_n} - \frac{\epsilon_c}{\sigma_n} \partial_{x_3} U(\xi) \left( 1 + \left( \frac{\sigma_n}{\sigma_c} - 1 \right) \partial_{x_3} V(\xi) \right) \nabla V \|_{L^2(D)}$$

$$\leq \left( \frac{\omega \epsilon_n}{\sigma_n} \right)^3 P_3 \left( \frac{\epsilon_c}{\sigma_n} - \frac{\sigma_c}{\sigma_n} \right) \| \nabla v_2 - \partial_{x_3} U(\xi) \left( e_3 + \left( \frac{\sigma_n}{\sigma_c} - 1 \right) \partial_{x_3} V(\xi) e_3 \right) \|_{L^2(D)}.$$

From [1], $\nabla V$ in $D$ is approximated by

$$\nabla V|_D \approx \frac{\sigma_c}{2\sigma_n + \sigma_c} \left( 1 - \frac{r^3}{16\pi|\xi|^3} \right) e_3 \approx \frac{\sigma_c}{2\sigma_n + \sigma_c} e_3.$$

The approximation (25) follows immediately from Theorem 3.2, (30), (31), and (32).

**Remark 3.5.** Our reconstruction algorithm is based on the approximation formula (25). In practice, we may not have a priori knowledge of the background conductivities. In that case, $\alpha$ is unknown. But $\alpha$ can be evaluated approximately by the ratio of the measured Neumann data as follows:

$$\alpha = \frac{\int_D g_2 \omega_2 \, ds}{\int_D \omega_2 \, ds} + \frac{\int_D \left( (1 - \alpha) \sigma_c + i(\omega_2 \epsilon_2 - \alpha \omega_1 \epsilon_1) \right) \nabla u_2 \cdot \nabla u_1 \, dx}{\int_{\Gamma_1} (\sigma + i\omega_1 \epsilon_1) |\nabla u_1|^2 \, ds}.$$

Hence, we may choose $\alpha \approx \frac{\int_D g_2 \omega_2 \, ds}{\int_D \omega_2 \, ds}.$

We can prove the identity (33) for a bounded domain $\Omega$. Using $u_1|_{\gamma} = u_2|_{\gamma} = V_0$, we have

$$\int_{\Gamma} (g_2 - \alpha g_1) \, ds = - \int_{\gamma} (g_2 - \alpha g_1) \, ds = - \frac{1}{V_0} \int_{\gamma} (g_2 u_1 - \alpha g_1 u_1) \, ds$$

$$= - \frac{1}{V_0} \int_{\partial \Omega} (g_2 - \alpha g_1) u_1 \, ds$$

$$= \frac{1}{V_0} \int_{\Omega} ((\sigma + i\omega_1 \epsilon_2) \nabla u_2 - \alpha (\sigma + i\omega_1 \epsilon_1) \nabla u_1) \cdot \nabla u_1 \, dx$$

$$= \frac{1}{V_0} \int_{\Omega} \left[ \alpha (\sigma + i\omega_1 \epsilon_1) (\nabla u_2 - \nabla u_1) \cdot \nabla u_1 ight. - \left. (1 - \alpha) \sigma + i(\omega_2 \epsilon_2 - \alpha \omega_1 \epsilon_1) \right] \nabla u_2 \cdot \nabla u_1 \, dx$$

$$= \frac{1}{V_0} \int_D ((1 - \alpha) \sigma + i(\omega_2 \epsilon_2 - \alpha \omega_1 \epsilon_1)) \nabla u_2 \cdot \nabla u_1 \, dx.$$
The identity (33) follows from the fact that

$$V_0 \int_\Gamma g_{\omega_i} ds = \int_\Omega (\sigma + i \omega_1 \epsilon_1) |\nabla u_1|^2 dx.$$ 

3.2. Numerical simulations. In fdTAS [32], we use a weighted frequency difference of Neumann data $g_2 - \alpha g_1$ instead of the simple difference $g_2 - g_1$. Theorems 3.1 and 3.2 show that the weight $\alpha$ and the difference $(\frac{\omega_2 \epsilon_2 c}{\epsilon_2 n} - \frac{\omega_1 \epsilon_1 c}{\epsilon_1 n})$ are important factors in detecting anomaly $D$.

In order to test the observations in Theorem 3.2 and Remark 3.3, we consider a cubic model $\Omega := [0, 0.12] \times [0, 0.12] \times [0, 0.12]$ m$^3$ with the probe region $\Gamma := \{(x, y, 0.12) : \sqrt{x^2 + y^2} < 0.03\}$ and the reference electrode $\gamma := \{(x, y, z) \in \Omega : z = 0\}$. We assume that $\Omega \setminus D$ and $D$ are homogeneous with frequency-independent conductivity values $\sigma_n = 0.03$ S/m and $\sigma_c = 0.2$ S/m. For permittivity values, we set $\omega_1 \epsilon_{1,n} = \omega_1 \epsilon_{1,c} = 0$ and $\omega_2 \epsilon_{2,n} = 3 \times 10^{-4}$ S/m. Numerical simulations are performed for a cube-shaped anomaly $D$ centered at $(0.06, 0.06, 0.12 - 0.009)$ in meters with its side length of 0.006 m.

Figure 3 shows the images of $g_2 - \alpha g_1$ with three different values of $\omega_2 \epsilon_2 c$ that are chosen so that the corresponding $\mu = (\frac{\omega_2 \epsilon_2 c}{\epsilon_2 n} - \frac{\omega_1 \epsilon_1 c}{\epsilon_1 n})$ is positive, zero, or negative, respectively. This setup allows us to observe that $g_2 - \alpha g_1$ is influenced by $\mu$ and there is an interesting relation between them. As we discussed in Remark 3.3, $\mu = 0$ implies $g_2 - \alpha g_1$ providing no information on $D$ even if $\epsilon_c$ changes a lot with respect to frequency. On the other hand, even if the permittivities $\epsilon_n$ and $\epsilon_c$ do not change with frequency, we can extract information on the anomaly from $g_2 - \alpha g_1$ as far as $\mu \neq 0$.

Figure 4 shows the vector fields of complex potential $\nabla u_2$ corresponding to three different values of $\omega_2 \epsilon_2 c$ as before. In the plots, solid lines are equipotential lines of $u_2$, and arrows indicate the direction and magnitude of electric field $-\nabla u_2$. Figure 4
Fig. 4. Equipotential lines and electric field streamlines in the slice \{0.06, y, z : 0.04 < y < 0.08, 0.08 < z < 0.12\}: Real and imaginary parts of the complex potential \(u_2\) with three different values of \(\omega_2\), as above. Imaginary part plots are individually scaled as (a) \(10^{-3}\), (b) \(10^{-6}\), (c) \(10^{-9}\) and real part plots are shown by using the same scale.

illustrates that the electric field direction of the imaginary part changes as the sign of \(\mu\) changes. We believe that the nonzero vector field is due to computational errors when \(\mu = 0\).

REFERENCES


