ON THE NUMBER OF COUNTABLE MODELS OF A COUNTABLE NSOP_1 THEORY WITHOUT WEIGHT ω

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Abstract. In this paper, we prove that if a countable non-$\aleph_0$-categorical NSOP_1 theory with nonforking existence has finitely many countable models then there is a finite tuple whose own preweight is ω. This result is an extension of author's theorem on any supersimple theory.

1. Introduction

In this paper, T always is a complete theory in a **countable** language $\mathcal{L}$, and recall that $I(\omega, T)$ denotes the number of non-isomorphic countable models of T. We extend the following theorem of the author for supersimple theories to the context of NSOP_1 theories.

Fact 1.1. [5] *If T is supersimple then $I(\omega, T)$ is either 1 or infinite.*

As well-known Fact 1.1 is an extension of Laclan's result in [7] for superstable theories. Later, Pillay improved Lachlan's result as follows, which is described in [3].

Fact 1.2. *Assume T is stable and $1 < I(\omega, T) < \omega$. Then there is a finite tuple whose own preweight is ω.*

The author indeed proved the same Fact 1.2 for simple theories, which directly implies Fact 1.1 since a supersimple theory can not have a type of finite tuple whose weight is ω.

Our main theorem in this note is the extension of Fact 1.2 for NSOP_1 theories.

Theorem 1.3. *Assume T is $NSOP_1$ holding nonforking existence. If $1 < I(\omega, T) < \omega$, then there is a finite tuple whose own preweight is ω.*

Now we recall basic facts and terminology for this note. As usual we work in a large saturated model. Unless said otherwise, $a, b, c, \dots$ are **finite** tuples, $A, B, C, \dots$ are small sets, and $M, N, \dots$ are elementary submodels from the saturated model. That $a \equiv_A b$ means a, b have the same type over A; and for a sequence a_i $(i < \kappa)$, $a_{<j}$ denotes

$\{a_i \mid i < j\}$. The following (until Fact 1.6) can be found in the literature on simple theories, for example in [6].

Definition 1.4. (1) A formula $\varphi(x, a_0)$ *divides* over A if there is an A-indiscernible sequence $\langle a_i \mid i < \omega\rangle$ such that $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent. A formula *forks* over A if the formula implies some finite disjunction of formulas, each of which divides over A. A type divides/forks over A if the type implies a formula which divides/forks over A. We write $A \mathop{\smile\!\!\!\!\!\!\;|}_B C$ if for any finite $a \in A$, $\mathrm{tp}(a/BC)$ does not fork over B.

(2) We say T is *stable* if nonforking holds uniqueness over models: For any $M \subseteq A$ and $p(x) \in S(M)$, there is the unique extension $q(x) \in S(A)$ of p which does not fork over M.

(3) We say T is *simple* if nonforking satisfies local character: For any a and A, there is $A_0 \subseteq A$ with $|A| \leq |T|$ such that $a \mathop{\smile\!\!\!\!\!\!\;|}_{A_0} A$. Any stable theory is simple. We say T is *supersimple* if for any a and A, there is finite $A_0 \subseteq A$ such that $a \mathop{\smile\!\!\!\!\!\!\;|}_{A_0} A$; and T is *superstable* if T is stable and supersimple.

(4) An A-indiscernible sequence $\langle a_i \mid i < \omega\rangle$ is said to be *Morley* over A (or A-*Morley*) if $a_i \mathop{\smile\!\!\!\!\!\!\;|}_A a_{<i}$ for each $i < \omega$.

Fact 1.5. *(1) (Extension) If $a \mathop{\smile\!\!\!\!\!\!\;|}_A B$ then for any C there is $a' \equiv_{AB} a$ such that $a' \mathop{\smile\!\!\!\!\!\!\;|}_A BC$.*

(2) (Base monotonicity) If $A \mathop{\smile\!\!\!\!\!\!\;|}_B CD$ then $A \mathop{\smile\!\!\!\!\!\!\;|}_{BC} D$.

(3) (Left transitivity lifting) If $B \mathop{\smile\!\!\!\!\!\!\;|}_C D$ and $A \mathop{\smile\!\!\!\!\!\!\;|}_{BC} D$, then $AB \mathop{\smile\!\!\!\!\!\!\;|}_C D$. Hence for a sequence $\langle c_i \mid i < \kappa\rangle$, if $c_i \mathop{\smile\!\!\!\!\!\!\;|}_A c_{<i}$ holds for each $i < \kappa$, then $c_{\geq i} \mathop{\smile\!\!\!\!\!\!\;|}_A c_{<i}$ for all $i < \kappa$.

Fact 1.6. *Assume T is simple then the following holds.*

(1) (Existence) For any a and A, we have that $a \mathop{\smile\!\!\!\!\!\!\;|}_A A$. Hence for any a_0 and A, there is an A-Morley sequence $\langle a_i \mid i < \omega\rangle$.

(2) A formula divides over a set iff the formula forks over the set.

(3) $\varphi(x, a_0)$ divides over A iff for some/any Morley $\langle a_i \mid i < \omega\rangle$ over A, $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent.

(4) (Symmetry) For any A, B, C we have $A \mathop{\smile\!\!\!\!\!\!\;|}_B C$ iff $C \mathop{\smile\!\!\!\!\!\!\;|}_B A$.

(5) (Transitivity) For any $B \subseteq C \subseteq D$, we have $A \mathop{\smile\!\!\!\!\!\!\;|}_B D$ iff $A \mathop{\smile\!\!\!\!\!\!\;|}_B C$ and $A \mathop{\smile\!\!\!\!\!\!\;|}_C D$.

(6) (Type-amalgamation over a model) Assume $A_0 \mathop{\smile\!\!\!\!\!\!\;|}_M A_1$, $c_0 \equiv_M c_1$, and $c_i \mathop{\smile\!\!\!\!\!\!\;|}_M A_i$ for $i = 0, 1$. Then there is $c \equiv_{MA_i} c_i$ such that $c \mathop{\smile\!\!\!\!\!\!\;|}_M A_1A_2$.

Recently, Kaplan and Ramsey found in [4] that all in Fact 1.6 (but transitivity, particularly base monotonicity) still hold over models in

NSOP_1 theories with the so-called *Kim-independence*. The 1-strong order property (SOP_1) is introduced by Shelah in [9], and a nice criterion for SOP_1 is given in [1].

Definition 1.7. [9]

(1) We say T has *SOP_1* if there are a formula $\varphi(x,y)$ and tuples c_α ($\alpha \in 2^{<\omega}$) such that, for each $\beta \in 2^\omega$, $\{\varphi(x, c_{\beta\lceil m}) \mid m \in \omega\}$ is consistent; and $\{\varphi(x, c_{\alpha^\frown\langle 1\rangle}), \varphi(x, c_\gamma)\}$ is inconsistent whenever $\alpha^\frown\langle 0\rangle$ is an initial segment of $\gamma \in 2^{<\omega}$.
(2) We say T is *$NSOP_1$* if T does not have SOP_1. Any smple theory is NSOP_1.

Fact 1.8. [1] *T has SOP_1 iff there are a sequence $\langle a_i c_i \mid i < \omega\rangle$ and a formula $\varphi(x,y)$ such that*

(1) $a_i \equiv_{(ac)_{<i}} c_i$ for each $i < \omega$,
(2) $\{\varphi(x, a_i) \mid i < \omega\}$ is consistent, while
(3) $\{\varphi(x, c_i) \mid i < \omega\}$ is k-inconsistent for some $k \geq 2$.

Definition 1.9. (Assume T satisfies nonforking existence over A, i.e. for any c, $c \mathop{\smile\!\!\!\!\!\!\mid}_A A$.) A formula $\varphi(x, a_0)$ *Kim-divides* over A if there is an A-Morley sequence $\langle a_i \mid i < \omega\rangle$ such that $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent. A formula *Kim-forks* over A if the formula implies some finite disjunction of formulas, each of which Kim-divides over A. A type Kim-divides/forks over A if the type implies a formula which Kim-divides/forks over A. We write $A \mathop{\smile\!\!\!\!\!\!\mid}^K_B C$ if for any finite $a \in A$, $\mathrm{tp}(a/BC)$ does not Kim-fork over B. Obviously $A \mathop{\smile\!\!\!\!\!\!\mid}_B C$ implies $A \mathop{\smile\!\!\!\!\!\!\mid}^K_B C$. Due to Fact 1.6(3), T is simple then $\mathop{\smile\!\!\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\!\!\mid}^K$. Indeed the converse holds as well.

An A-indiscernible sequence $\langle b_i \mid i < \omega\rangle$ is called $\mathop{\smile\!\!\!\!\!\!\mid}^K$-*Morley* over A (in $p(x)$) if $b_i \mathop{\smile\!\!\!\!\!\!\mid}^K_A b_{<i}$ holds for each $i < \omega$ (and $p(x) = \mathrm{tp}(a_i/A)$).

Note that nonforking existence holds over any model since any type over a model is finitely satisfiable over the model.

Fact 1.10. [4] *Let T be $NSOP_1$.*

(1) (Kim's lemma for $\mathop{\smile\!\!\!\!\!\!\mid}^K$ over a model) $\varphi(x, a_0)$ Kim-divides over M iff for any Morley sequence $\langle a_i \mid i < \omega\rangle$ over M, $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent.
(2) A formula Kim-divides over a model iff the formula Kim-forks over the model.
(3) (Extension for $\mathop{\smile\!\!\!\!\!\!\mid}^K$ over a model) If $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M B$ then for any C there is $a' \equiv_{MB} a$ such that $a' \mathop{\smile\!\!\!\!\!\!\mid}^K_M BC$.

(4) *(Symmetry for* $\mathop{\smile\!\!\!\!\!\!|}^{K}$ *over a model) For any* A, C *we have* $A \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} C$ *iff* $C \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} A$.

(5) *(Type-amalgamation for* $\mathop{\smile\!\!\!\!\!\!|}^{K}$ *over a model) Assume* $A_0 \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} A_1$, $c_0 \equiv_M c_1$, *and* $c_i \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} A_i$ *for* $i = 0, 1$. *Then there is* $c \equiv_{MA_i} c_i$ *such that* $c \mathop{\smile\!\!\!\!\!\!|}^{K}_{M} A_1 A_2$.

In a joint work [2], it is now proved that Fact 1.10 still holds over any set as far as nonforking existence holds. Due to Fact 1.6(1), the class of NSOP_1 theories with nonforking existence fully contains that of simple theories. Moreover all the typical nonsimple NSOP_1 examples described in [4] (namely, the random parameterized equivalence relations, ω-free PAC fields, and an infinite dimensional vector space over an algebraically closed field equipped with a symmetric alternating bilinear form) have nonforking existence. Even we conjecture that any NSOP_1 T has existence.

Fact 1.11. [2] *Assume* T *is* $NSOP_1$ *with nonforking existence(Fact 1.6(1)).*

(1) *(Kim's lemma for* $\mathop{\smile\!\!\!\!\!\!|}^{K}$*)* $\varphi(x, a_0)$ *Kim-divides over* A *iff for any Morley sequence* $\langle a_i \mid i < \omega \rangle$ *over* A, $\{\varphi(x, a_i) \mid i < \omega\}$ *is inconsistent.*

(2) *A formula Kim-divides over some set iff the formula Kim-forks over the set.*

(3) *(Extension for* $\mathop{\smile\!\!\!\!\!\!|}^{K}$*) If* $p(x)$ *is a type over* B *which does not Kim-forks over* A, *then there is a completion* $q(x) \in S(AB)$ *which does not Kim-fork over* A. *In particular if* $a \mathop{\smile\!\!\!\!\!\!|}^{K}_{A} B$ *then for any* C *there is* $a' \equiv_{AB} a$ *such that* $a' \mathop{\smile\!\!\!\!\!\!|}^{K}_{A} BC$.

(4) *(Symmetry for* $\mathop{\smile\!\!\!\!\!\!|}^{K}$*) For any* A, B, C *we have* $A \mathop{\smile\!\!\!\!\!\!|}^{K}_{B} C$ *iff* $C \mathop{\smile\!\!\!\!\!\!|}^{K}_{B} A$.

(5) *(Chain condition for* $\mathop{\smile\!\!\!\!\!\!|}^{K}$*) Let* $a \mathop{\smile\!\!\!\!\!\!|}^{K}_{A} b_0$, *and let* $I = \langle b_i \mid i < \omega \rangle$ *be* $\mathop{\smile\!\!\!\!\!\!|}^{K}$*-Morley over* A. *Then there is* $a' \equiv_{Ab_0} a$ *such that* $a' \mathop{\smile\!\!\!\!\!\!|}^{K}_{A} I$ *and* I *is* $a'A$*-indiscernible.*

From now on for simplicity, **we assume that any NSOP_1 theory in this note has nonforking existence.**

In addition to Fact 1.11, type-amalgamation over sets for Lascar types are proved for any NSOP_1 theory, but we omit to state it as we will not use the property. Instead we will use Fact 1.11(5).

Remark 1.12. (1) Assume T is NSOP_1 and let $p(x) \in S(A)$. Then that '$\langle x_i \mid i < \omega \rangle$ is a sequence of realizations of p such that $x_i \mathop{\smile\!\!\!\!\!\!|}^{K}_{A} x_{<i}$ for each $i < \omega$' is A-type-definable by $\bigwedge_{i<\omega} p(x_i) \cup \Gamma(x_0, x_1, \dots)$ where

$$\Gamma(x_0, x_1, \dots) := \{\neg\varphi(x_0, \dots, x_n, x_{n+1}) \in \mathcal{L}(A) \mid \varphi(x_0, \dots, x_n, a) \text{ Kim-divides over } A \text{ for some/any } a \models p\}.$$

Hence clearly that $\langle x_i \mid i < \omega \rangle$ is a $\mathop{\smile\kern-0.6em\mid}^K$-Morley sequence over A in p is A-type-definable as well.

(2) Notice that contrary to simple theory context, that $\langle c_i \mid i < \omega \rangle$ is $\mathop{\smile\kern-0.6em\mid}^K$-Morley over A in NSOP$_1$ T *need not* imply

$$c_i \mathop{\smile\kern-0.6em\mid}^K_A \{c_j \mid j \neq i\}$$

for all $i \in \omega$, since transitivity for $\mathop{\smile\kern-0.6em\mid}^K$ does not hold.

Now we are ready to talk about the notion of weight.

Definition 1.13. Assume T is NSOP$_1$. We say a finite tuple c (or its type) has *own preweight* ω if there are $b_i \equiv c$ $(i < \omega)$ such that $c \not\mathop{\smile\kern-0.6em\mid}^K b_i$, and $b_i \mathop{\smile\kern-0.6em\mid}^K b_{<i}$ for all $i < \omega$.

For more development of the weight notion in simple theories, see [6]. As pointed out in Remark 1.12(2), in Definition 1.13, $\{b_i\}_i$ need not be fully $\mathop{\smile\kern-0.6em\mid}^K$-independent.

Recall that T is supersimple iff there do not exist c and sets A_i $(i < \omega)$ such that $A_i \subseteq A_{i+1}$ and $c \not\mathop{\smile\kern-0.6em\mid}_{A_i} A_{i+1}$ for any i. But if T is NSOP$_1$, only one direction is clear: If there do not exist c and sets A_i $(i < \omega)$ such that $A_i \subseteq A_{i+1}$ and $c \not\mathop{\smile\kern-0.6em\mid}^K_{A_i} A_{i+1}$ for any i (*), then for any c and A there is $A_0 \subseteq A$ with $|A_0| \leq |T|$ such that $c \not\mathop{\smile\kern-0.6em\mid}^K_{A_0} A$.

In NSOP$_1$ T, if transitivity lifting (i.e. $a \mathop{\smile\kern-0.6em\mid}^K_A B$ and $a \mathop{\smile\kern-0.6em\mid}^K_{AB} C$ implies $a \mathop{\smile\kern-0.6em\mid}^K_A BC$) holds then (*) implies that there is no $p(x) \in S(\emptyset)$ whose own preweight is ω.

Example 1.14. (1) Consider the typical stable but nonsuperstable theory. Namely T is the theory in $\mathcal{L} = \{E_i(x,y) \mid i < \omega\}$ saying that each binary E_i is an equivalence relation only having infinitely many infinite classes such that for each $j > i$, E_j is finer than E_i and each E_i-class contains infinitely many E_j-classes. Notice that T is a small (i.e. $S(\emptyset)$ is countable) non-$\aleph_0$-categorical theory. But there is no finite tuple whose own preweight is ω.

(2) Due to our Theorem 1.3, a necessary condition for an NSOP$_1$ theory to have $1 < I(\omega, T) < \omega$ is that T should be small and having a finite tuple with own preweight ω. Herwigh constructed such an example of a stable theory [3].

2. Kim-forking and isolation

In order to prove Theorem 1.3, we will take the similar pattern of the proof for Fact 1.1 in [5].

We first recall Pillay's notion of semi-isolation ([3],[8]), and figure our its relationship with Kim-forking in NSOP$_1$ theories. We say $\mathrm{tp}(b/a)$ is *semi-isolated* if there is a formula $\varphi(x,a)$ in $\mathrm{tp}(b/a)$ such that $\models \varphi(x,a) \rightarrow \mathrm{tp}(b)$.

Fact 2.1. *(1) If $\mathrm{tp}(b/a)$ is isolated, then $\mathrm{tp}(b/a)$ is semi-isolated.*
(2) If $\mathrm{tp}(c/b)$ and $\mathrm{tp}(b/a)$ are semi-isolated, then $\mathrm{tp}(c/a)$ is semi-isolated.

We cite a proof of the the following folklore for self-containedness.

Fact 2.2. *Suppose that $\mathrm{tp}(b/a)$ is isolated, whereas $\mathrm{tp}(a/b)$ is nonisolated. Then $\mathrm{tp}(a/b)$ is nonsemi-isolated.*

Proof. Let $\mathrm{tp}(b/a)$ be isolated by $\varphi(x,a)$ (*). To lead a contradiction assume that $\psi(b,y)$ semi-isolates $\mathrm{tp}(a/b)$. Now since $\mathrm{tp}(a/b)$ is nonisolated, there is an $\mathcal{L}$-formula $\phi(x,y)$ such that $\varphi(b,y)\wedge\psi(b,y)\wedge\phi(b,y)$ and $\varphi(b,y)\wedge\psi(b,y)\wedge\neg\phi(b,y)$ are both consistent, while both imply $\mathrm{tp}(a)$. Hence $\varphi(x,a)\wedge\phi(x,a)$ and $\varphi(x,a)\wedge\neg\phi(x,a)$ are both consistent, contradicting (*). $\square$

The following is the key proposition describing a relationship between isolation and Kim-dividing in NSOP$_1$ theories.

Proposition 2.3. *Assume that T is NSOP$_1$. Let $a \equiv b$. Assume $\mathrm{tp}(b/a)$ is semi-isolated, but $\mathrm{tp}(a/b)$ is nonsemi-isolated. Then $a \not\mathop{\smile\!\!\!\!\!\!\;\mid}^K b$.*

Proof. Suppose not, so that $a \mathop{\smile\!\!\!\!\!\!\;\mid}^K b$.

Claim 1. There is $c \models q = \mathrm{tp}(a)$ such that $b \mathop{\smile\!\!\!\!\!\!\;\mid}^K ac$ and $ba \equiv cb$: Choose $c_0 \models q$ such that $ba \equiv c_0 b$. Hence $a \mathop{\smile\!\!\!\!\!\!\;\mid}^K b$ and $b \mathop{\smile\!\!\!\!\!\!\;\mid}^K c_0$. Thus for some Morley sequence $I = \langle b_i \mid i < \omega\rangle$ with $b = b_0$, there is c_0' such that $c_0 b \equiv c_0' b$ and I is a-indiscernible and c_0'-indiscernible as well. Now by compactness we can assume the length of I is arbitrary large and then by the pigeonhole principle and Ramsey's Theorem, we can assume there is an infinite subsequence I' of I such that I' is ac_0'-indiscernible. Then by an a-automorphims sending I' to I, we can clearly find a desired c.

Now put $c_0 b_0 a_0 = cba$. By extension for $\mathop{\smile\!\!\!\!\!\!\;\mid}^K$, it easily follows that there are $c_i b_i a_i \equiv cba$ $(i < \omega)$ such that $a_{i+1} c_i \equiv ba$ (*), and moreover $(cba)_i \mathop{\smile\!\!\!\!\!\!\;\mid}^K (cba)_{<i}$.

Claim 2. Let $\varphi(x,a)$ semi-isolate $\text{tp}(b/a)$. Then $\{\varphi(c_i,x)\wedge\varphi(x,a_i) \mid i<\omega\}$ is 2-inconsistent: If it were not 2-inconsistent, then there is d such that $\varphi(d,a_j)$ and $\varphi(c_i,d)$ for some $j>i$. Therefore clearly $\text{tp}(d/a_j)$ and $\text{tp}(c_i/d)$ are both semi-isolated, and hence again by Fact 2.1(2), so does $\text{tp}(c_i/a_j)$. Now since $\text{tp}(a_j/a_{i+1})$ is semi-isolated by (*), once more Fact 2.1(2) implies $\text{tp}(c_i/a_{i+1})$ is semi-isolated. But since $\text{tp}(c_ia_{i+1})=\text{tp}(ab)$, it leads a contradiction. Hence the claim is proved.

Now by compactness applying to the type-definability described in Remark 1.12(1), there is some $\mathop{\smile\!\!\!\!\!\!|}^K$-Morley sequence $\langle c'_0b'_0a'_0 \mid i<\omega\rangle$ over A such that $c'_0b'_0a'_0=cba$ and $\{\varphi(c'_i,x)\wedge\varphi(x,a'_i) \mid i<\omega\}$ is 2-inconsistent. Note now that $b\models\varphi(c,x)\wedge\varphi(x,a)$. Then due to the chain condition for $\mathop{\smile\!\!\!\!\!\!|}^K$ in Fact 1.11, we must have $b \not\mathop{\smile\!\!\!\!\!\!|}^K ac$, contradicting Claim 1. Therefore we must have $a \not\mathop{\smile\!\!\!\!\!\!|}^K b$. □

Corollary 2.4. *Assume that T is NSOP$_1$, and we let $a\equiv b$. If $\text{tp}(b/a)$ is isolated, and $\text{tp}(a/b)$ is nonisolated, then $a \not\mathop{\smile\!\!\!\!\!\!|}^K b$.*

3. Proof of Theorem 1.3

In this section, T is non-$\aleph_0$-categorical. Again we quote a proof of the following fact for completion.

Fact 3.1. *(Folklore) Suppose that $I(\omega,T)$ is finite. Then there is a tuple a and a prime model M over a such that $p(x):=\text{tp}(a)$ is nonisolated and all the types of finite tuples are realized in M. Moreover there is a tuple b in M such that, $b\equiv a$ and $\text{tp}(a/b)$ is nonisolated.*

Proof. Let $q_0,q_1.q_2,...$ be an enumeration of complete types in $S(\emptyset)$. Suppose that $e_i\models q_i$ and $d_i=e_0e_1\dots e_i$. Now there is a prime model N_i over d_i for each $i<\omega$. Thus for some $j<\omega$, $N_j(=M)$ is isomorphic to N_i for infinitely many $i\geq j$. Therefore the prime model M over $d_j(=a)$ realizes every complete type over $\emptyset$. As M is not prime over $\emptyset$, $\text{tp}(a)$ is not isolated.

Now since $T(a)$ is again non-$\aleph_0$-categorical, for some tuple u, $\text{tp}(u/a)$ is nonisolated. Let $u'b(\in M)$ realize $\text{tp}(ua)$. Then as $\text{tp}(u'/b)$ is nonisolated, M is not prime over b. Since M is prime over a, $\text{tp}(a/b)$ must not be isolated. □

We are ready to prove Theorem 1.3. We keep the notation in Fact 3.1. Assume further that T is NSOP$_1$.

Claim 3.2. *There are two realizations a_1,a_0 of p in M such that $a_1 \mathop{\smile\!\!\!\!\!\!|} a_0$, and both $\text{tp}(a_0/a_1),\text{tp}(a_1/a_0)$ are nonisolated.*

Proof. Due to nonforking existence and extension, there is $c \models p$ such that $c \mathop{\smile\!\!\!\!\!\!\!\;|}\, ab$, and hence $c \mathop{\smile\!\!\!\!\!\!\!\;|}^K ab$. Now, by Fact 2.2, $\mathrm{tp}(a/b)$ is nonsemi-isolated. Hence, by Fact 2.1, either $\mathrm{tp}(a/c)$ or $\mathrm{tp}(c/b)$ must be non-isolated. Since $c \mathop{\smile\!\!\!\!\!\!\!\;|}^K ab$, if $\mathrm{tp}(a/c)$ is nonisolated then so is $\mathrm{tp}(c/a)$, by Corollary 2.4, The same holds when $\mathrm{tp}(c/b)$ is nonisolated. Now choose a_1a_0 in M such that $a_1a_0 \equiv ca$ or cb. ☐

We continue the proof with the selected tuples. Note now that $\mathrm{tp}(a/a_0)$, $\mathrm{tp}(a/a_1)$ are both nonisolated (†), since if say $\mathrm{tp}(a/a_0)$ were isolated, then M is prime over a_0 and so $\mathrm{tp}(a_1/a_0)$ would be isolated, a contradiction. Therefore again by Corollalry 2.4, we have $a \not\mathop{\smile\!\!\!\!\!\!\!\;|}^K a_0$ and $a \not\mathop{\smile\!\!\!\!\!\!\!\;|}^K a_1$. We are ready to claim the following which says that p has its own preweight ω, so finishes our proof of Theorem 1.3.

Claim 3.3. *There is a set $\{a_u \models p \mid u \in X\}$ where*

$$X = \{u \in 2^{<\omega} \mid u = 0^{m+1} = \overbrace{0\ldots0}^{m+1} \text{ or } 0^m1 \text{ for some } m < \omega\}$$

such that, for each $m < \omega$,

(1) $a_1a_0a \equiv a_{0^m1}a_{0^{m+1}}a_{0^m}$
(2) $a_{0^m1} \mathop{\smile\!\!\!\!\!\!\!\;|}\{a_u \mid u \in X$ *having* 0^{m+1} *an initial segment*$\}$
(3) $a_{0^m1} \mathop{\smile\!\!\!\!\!\!\!\;|}^K a_1a_{01}\ldots a_{0^{m-1}1}$, *and*
(4) $a \not\mathop{\smile\!\!\!\!\!\!\!\;|}^K a_u$ *for all* $u \in X$.

We prove the claim using induction. Given $k < \omega$, assume that we have selected $A = \{a_u \models p \mid u \in X \text{ and } |u| \le k+1\}$ satisfying above (1)-(4) for each $m \le k$. Note that a_1a_0a satisfies the initial condition for $k = 0$. We will find appropriate $a_{0^{k+1}1}, a_{0^{k+2}} \models p$ holding (1)-(4) for $k+1$.

First choose $d_1 = a_{0^{k+1}1}$, $d_0 = a_{0^{k+2}} \models p$ such that $d_1d_0a_{0^{k+1}} \equiv a_1a_0a$. Now by (1) for $k+1$, we have $a_{0^k1} \mathop{\smile\!\!\!\!\!\!\!\;|}\, a_{0^{k+1}}$. Hence by extension we can assume that $a_{0^k1} \mathop{\smile\!\!\!\!\!\!\!\;|}\, d_1d_0a_{0^{k+1}}$. Moreover by (2) for $k+1$ and nonforking extension we can further assume (by iteratively moving d_1d_0 while fixing A pointwise) that (2) also holds on $Aa_{0^{k+1}1}a_{0^{k+2}} = Ad_1d_0$ for each $m \le k+1$. Then with this $d_1d_0 = a_{0^{k+1}1}a_{0^{k+2}}$, the rest can be shown.

Namely, by (2), for each $m \le k+1$, we have

$$a_{0^m1} \mathop{\smile\!\!\!\!\!\!\!\;|}\{a_{0^n1} \mid m < n \le k+1\}.$$

Thus by Fact 1.5, we have that for each $n \le k+1$,

$$\{a_{0^m1} \mid m < n\} \mathop{\smile\!\!\!\!\!\!\!\;|}\, a_{0^n1},$$

and hence by symmetry for $\mathop{\smile\!\!\!\!\!\!\;|}^K$, it follows

$$a_{0^n1} \mathop{\smile\!\!\!\!\!\!\;|}^K \{a_{0^m1} \mid m < n\}.$$

We have shown that (3) holds on $Aa_{0^{k+1}1}a_{0^{k+2}}$. Now for a_1, we already know $a \not\mathop{\smile\!\!\!\!\!\!\;|}^K a_1$, For other a_u $(u \in X)$, due to (1) and Fact 2.1, we have that $\mathrm{tp}(a_u/a)$ is semi-isolated. However $\mathrm{tp}(a/a_u)$ is nonsemi-isolated since if it were so then again by Fact 2.1, $\mathrm{tp}(a/a_0)$ is semi-isolated contradicting above (†) and Fact 2.2. Therefore by Fact 2.3, $a \not\mathop{\smile\!\!\!\!\!\!\;|}^K a_u$. We have proved (4) and so complete the proof of Theorem 1.3.

References

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