A CLASSIFICATION OF 2-CHAINS HAVING 1-SHELL BOUNDARIES IN ROSY THEORIES

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Abstract. We classify, in a non-trivial amenable collection of functors, all 2-chains up to the relation of having the same 1-shell boundary. In particular, we prove that in a rosy theory, every 1-shell of a Lascar strong type is the boundary of some 2-chain, hence making the 1st homology group trivial.

We also show that, unlike in simple theories, in rosy theories there is no upper bound on the minimal lengths of 2-chains whose boundary is a 1-shell.

1. Introduction

In [5],[6], J. Goodrick, A. Kolesnikov and the first author developed a homology theory for any amenable collection of functors in a very general context. But the most interesting examples appear in model theory. Namely, given any strong type $p \in S(A)$ in a rosy theory $T$, we may assign a non-trivial amenable collection of functors preserving thorn-independence and compute the corresponding homology groups. By the general theory, if $T$ has $n$-complete amalgamation ($n \geq 2$) over $A = acl(A)$ then the $(n - 1)$-th homology group of $p \in S(A)$ consists of $(n - 1)$-shells with the support $n + 1 = \{0, \ldots, n\}$. Hence, in any simple $T$ (where, due to 3-amalgamation, every 1-shell is the boundary of some 2-simplex), the 1st homology group is trivial. But the question remained whether the same would hold in rosy theories. In this paper, we show that the answer is yes (as long as $p$ is a Lascar type). A crucial ingredient in our proof is the fact that $a$ and $b$ realize the same Lascar type if and only if their Lascar distance is finite, i.e., $d_A(a, b) < \omega$. In the proof, the number of 2-simplices involved in a 2-chain having the 1-shell boundary is proportional to $d_A(a, b)$. Therefore one may guess that, there does not exist a uniform bound for the minimal lengths of 2-chains with 1-shell boundaries for various Lascar types in rosy theories.

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contrary to the case of simple theories where the bound is 1, due to 3-amalgamation. A series of rosy examples in [2] where the Lascar distances increase are candidates. However in order to confirm that in each example that a candidate 2-chain has the expected minimal length, we need to rule out all other possibilities. For this goal we start to classify all the 2-chains having the same 1-shell boundary in a very general amenable context. The classification also has its own research interests. We obtain some interesting and important results in regard to the classification.

There are basically two operations on the class of 2-chains preserving the length and boundary of a chain. The first one is called the crossing (CR-)operation and the second one is called the renaming-of-support (RS-)operation. Two 2-chains are said to be equivalent if one is obtained from the other by finitely many applications of the two operations.

In the remainder of this section, we recall the definitions of an amenable family of functors and the corresponding homology groups introduced in [5],[6]. We thank Hyeung-Joon Kim and John Goodrick for their valuable suggestions and comments.

**Notation.** Throughout the paper, $s$ denotes an arbitrary finite set of natural numbers. Given any subset $X \subseteq \mathcal{P}(s)$, we may view $X$ as a category where for any $u, v \in X$, $\text{Mor}(u, v)$ consists of a single morphism $\iota_{u,v}$ if $u \subseteq v$, and $\text{Mor}(u, v) = \emptyset$ otherwise. If $f : X \to \mathcal{C}$ is any functor into some category $\mathcal{C}$ then for any $u, v \in X$ with $u \subseteq v$, we let $f^u$ denote the morphism $f(\iota_{u,v}) \in \text{Mor}_\mathcal{C}(f(u), f(v))$. We shall call $X \subseteq \mathcal{P}(s)$ a primitive category if $X$ is non-empty and downward closed, i.e., for any $u, v \in \mathcal{P}(s)$, if $u \subseteq v$ and $v \in X$ then $u \in X$. (Note that all primitive categories have the empty set $\emptyset \subset \omega$ as an object.)

**Remark/Definition 1.1.** Given any primitive categories $X$ and $Y$, define

$$X + Y := \{ t \cup k \mid t \in X, k \in Y \}$$

which is clearly a primitive category itself containing $X$ and $Y$ as subcategories. And, for any $t \in X$, define

$$X_t := \{ k \in X \mid t \cap k = \emptyset \} \quad \text{and} \quad X|_t := \{ k \in X_t \mid t \cup k \in X \}$$

both of which are clearly primitive subcategories of $X$. Observe:

1. $X|_t \subseteq X_t \subseteq X$
2. $X \subseteq X_t + \mathcal{P}(t)$
3. $X|_t = \bigcup \{ \mathcal{P}(u \setminus t) \mid t \subseteq u \in X \}$. 
Moreover, it is easy to check that the following are equivalent: 
\[ X = X_t + \mathcal{P}(t) \iff X_t = X |_t \iff X = \bigcup \{ \mathcal{P}(u) \mid t \subseteq u \subseteq X \}. \]
If one of these equivalent conditions holds, we shall say that \( X \) splits at \( t \).

For any functor \( f : X \to C \) to some category \( C \) and for any \( t \in X \), the localization of \( f \) at \( t \) is the functor \( f |_t : X |_t \to C \) defined as follows: for any \( u \subseteq v \in X |_t \), \( (f |_t)^u_v = f |_{t \cap v}^{u \cap t} \) and \( f |_t(v) = f(t \cup v) \).

**Definition 1.2.** Let \( X \subseteq \mathcal{P}(s) \) and \( Y \subseteq \mathcal{P}(t) \) be any primitive categories (where \( s, t \) are some finite sets of natural numbers). And let \( f : X \to C \) and \( g : Y \to C \) be any functors to some category \( C \).

1. We say that \( f \) and \( g \) are isomorphic if there is an order-preserving bijection \( \tau : s \to t \) such that \( Y = \{ \tau(u) \mid u \in X \} \) and there is a family of isomorphisms \( \{ h_u : f(u) \to g(\tau(u)) \mid u \in X \} \subseteq \text{Mor}(C) \) such that, for any \( u \subseteq v \in X \),
   \[ h_v \circ f^u_v = g^{\tau(u)}_v \circ h_u. \]

2. We say that \( f \) and \( g \) are permutations of each other if there is a bijection \( \sigma : s \to t \) (not necessarily order-preserving) such that \( Y = \{ \sigma(u) \mid u \in X \} \) and, for any \( u \subseteq v \in Y \), \( g(v) = f(\sigma^{-1}(v)) \) and \( (g |_t)^u_v = f^{\sigma^{-1}(u)}_{\sigma^{-1}(v)} \). In this case, we write \( g = f \circ \sigma^{-1} \).

Note that, if \( f \) and \( g \) are permutations of each other via an order-preserving map \( \sigma : s \to t \), then \( f \) and \( g \) are isomorphic.

**Definition 1.3.** Let \( \mathcal{A} \) be a non-empty family of functors from various primitive categories into some fixed category \( C \). We say that \( \mathcal{A} \) is amenable if it satisfies the following properties:

1. (Closed under isomorphism and permutation) If \( f \in \mathcal{A} \) then any functor \( g \) which is isomorphic to \( f \) or is a permutation of \( f \) also belongs to \( \mathcal{A} \).

2. (Closed under restriction and union) For any functor \( f : X \to C \) from some primitive category \( X \) into \( C \),
   \[ f \in \mathcal{A} \iff \text{for every } t \in X, f | \mathcal{P}(t) \in \mathcal{A}. \]

3. (Closed under localization) If \( f : X \to C \) is any functor in \( \mathcal{A} \) then for every \( t \in X \), \( f |_t : X |_t \to C \) is also in \( \mathcal{A} \).

4. (Extensions of localizations are localizations of extensions) Let \( f : X \to C \) be any functor in \( \mathcal{A} \) which splits at some \( t \in X \). Then whenever \( f |_t \) can be extended to some functor \( g : Z \to C \) in \( \mathcal{A} \) where \( t \cap \bigcup Z = \emptyset \), \( f \) can be extended to some functor \( h : \mathcal{P}(t) + Z \to C \) in \( \mathcal{A} \) such that \( h |_t = g \).
Definition 1.4. By a \((regular)\ n\text{-simplex}\) in a category \(C\), we mean a functor \(f: \mathcal{P}(s) \to C\) where \(s \subseteq \omega\) has the size \(n + 1\). We call \(s\) the \(support\) of \(f\) and denote it by \(\text{supp}(f)\).

Definition 1.5. Let \(\mathcal{A}\) be an amenable family of functors into some category \(C\). Let \(B \in \text{Ob}(C)\). If \(f\) is a functor in \(\mathcal{A}\) such that \(f(\emptyset) = B\), we shall say that \(f\) is \(over\) \(B\). And we define:

\[
S_n(\mathcal{A}; B) := \{ f \in \mathcal{A} \mid f \text{ is a regular } n\text{-simplex over } B \}
\]

\[
C_n(\mathcal{A}; B) := \text{the free abelian group generated by } S_n(\mathcal{A}; B).
\]

The elements of \(C_n(\mathcal{A}; B)\) are called the \(n\text{-chains over } B\) in \(\mathcal{A}\). For each \(i = 0, \ldots, n\), we define a group homomorphism

\[
\partial_i^n: C_n(\mathcal{A}; B) \to C_{n-1}(\mathcal{A}; B)
\]

by letting, for any \(n\text{-simplex } f: \mathcal{P}(s) \to C\) in \(S_n(\mathcal{A}; B)\) where \(s = \{s_0 < \cdots < s_n\}\),

\[
\partial_i^n(f) := f \upharpoonright \mathcal{P}(s \setminus \{s_i\})
\]

and then extending linearly to all \(n\text{-chains in } C_n(\mathcal{A}; B)\). Then we define the \(boundary map\)

\[
\partial_n: C_n(\mathcal{A}; B) \to C_{n-1}(\mathcal{A}; B)
\]

by

\[
\partial_n(c) := \sum_{0 \leq i \leq n} (-1)^i \partial_i^n(c).
\]

We shall often refer to \(\partial_n(c)\) as the \(boundary\) of \(c\). Next, we define:

\[
Z_n(\mathcal{A}; B) := \text{Ker } \partial_n
\]

\[
B_n(\mathcal{A}; B) := \text{Im } \partial_{n+1}.
\]

The elements of \(Z_n(\mathcal{A}; B)\) and \(B_n(\mathcal{A}; B)\) are called \(n\text{-cycles}\) and \(n\text{-boundaries}\), respectively. It is straightforward to check

\[
\partial_{n-1} \circ \partial_n = 0.
\]

Hence we may define

\[
H_n(\mathcal{A}; B) := Z_n(\mathcal{A}; B)/B_n(\mathcal{A}; B)
\]

called the \(n\text{-th } (simplicial)\ homology group of } \mathcal{A} \text{ over } B\).

Notation 1.6. \(\begin{enumerate}
\item For \(c \in Z_n(\mathcal{A}; B)\), \([c]\) denotes the coset of \(B_n(\mathcal{A}; B)\) containing \(c\).
\item When \(n\) is clear from context, we shall often omit \(n\) from \(\partial_i^n\) and \(\partial_n\), writing simply as \(\partial^i\) and \(\partial\).
\end{enumerate}\)
(3) When we write an \( n \)-chain \( c \in C_n(\mathcal{A}; B) \) as 

\[
c = \sum_{i=1}^{k} n_i f_i
\]

we shall assume, unless stated otherwise, that \( n_i \)'s are nonzero integers and \( f_i \)'s are distinct \( n \)-simplices. (This form is called the standard form of a chain.) For such an \( n \)-chain \( c \), we define the length of \( c \) and the support of \( c \) as \(|c| \coloneqq \sum_{i=1}^{k} |n_i| \) and \( \text{supp}(c) \coloneqq \bigcup_{i=1}^{k} \{\text{supp}(f_i)\} \), respectively.

(4) For \( c, d \in C_n(\mathcal{A}; B) \), we say that \( d \) is a subchain (or subsummand) of \( c \) if they are in the standard forms 

\[
c = \sum_{i=1}^{k} n_i f_i \quad \text{and} \quad d = \sum_{i \in J} m_i f_i,
\]

where \( J \subseteq \{1, \ldots, k\} \) and, for each \( i \in J \), \( n_i \cdot m_i > 0 \) and \(|m_i| \leq |n_i|\).

\textbf{Remark/Definition 1.7.} Let \( c \) be any \( n \)-chain and let \( d \) be a subsummand of \( c \). For any \( n \)-chain \( d' \), we shall say that the \( n \)-chain 

\[
c' \coloneqq c - d + d'
\]

is obtained by replacing the subsummand \( d \) in \( c \) by \( d' \). Note that, if \(|d'| \leq |d|\) then \(|c'| \leq |c|\).

\textbf{Remark/Definition 1.8.} Given any bijection \( \sigma : \omega \to \omega \) (not necessarily order-preserving), we may induce an automorphism \( \sigma^*: C_n(\mathcal{A}; B) \to C_n(\mathcal{A}; B) \) for each \( n \) as follows: for any \( n \)-chain \( c = \sum n_i f_i \in C_n(\mathcal{A}, B) \), where each \( f_i \) is an \( n \)-simplex with \( s_i := \text{supp}(f_i) = \{s_{i,0} < \cdots < s_{i,n}\} \), we let \( \sigma_i := \sigma \upharpoonright s_i \) and \( t_i := \sigma_i(s_i) = \{t_{i,0} < \cdots < t_{i,n}\} \). We define 

\[
\sigma^*(c) \coloneqq \sum n_i |\sigma_i| f_i \circ \sigma_i^{-1}
\]

(see Definition 1.2(2)) with \(|\sigma_i| := \text{sign}(\sigma'_i) \coloneqq (\pm 1)\) where \( \sigma'_i \in \text{Sym}(n+1) \) such that for \( j \leq n \), \( \sigma_i(s_{i,j}) = t_{i,\sigma'(j)} \). For example 

\[
\sigma^*(f_i) = |\sigma_i| f_i \circ \sigma_i^{-1}.
\]

Moreover, \( \sigma^* \) commutes with the boundary map, i.e., \( \partial \circ \sigma^* = \sigma^* \circ \partial \). This can be verified inductively by first checking the case where \( \sigma \) is a transposition.

Next we define the amalgamation properties. For \( n = \{0, \ldots, n-1\} \), we let \( \mathcal{P}^-(n) \coloneqq \mathcal{P}(n) \setminus \{n\} \). i.e., \( \mathcal{P}^-(n) \) is the set of all the proper subsets of \( n \).
Definition 1.9. Let $\mathcal{A}$ be an amenable family of functors into a category $\mathcal{C}$.

1. $\mathcal{A}$ has $n$-amalgamation $(n \geq 1)$ if every functor $f: \mathcal{P}^-(n) \to \mathcal{C}$ in $\mathcal{A}$ can be extended to some functor $g: \mathcal{P}(n) \to \mathcal{C}$ in $\mathcal{A}$.
2. $\mathcal{A}$ has $n$-complete amalgamation (written $n$-CA) if it has $k$-amalgamation for every $1 \leq k \leq n$.
3. $\mathcal{A}$ has strong 2-amalgamation if, whenever $f: \mathcal{P}(s) \to \mathcal{C}$ and $g: \mathcal{P}(t) \to \mathcal{C}$ are simplices in $\mathcal{A}$ which agree on $\mathcal{P}(s \cap t)$, then there exists some simplex $h: \mathcal{P}(s \cup t) \to \mathcal{C}$ in $\mathcal{A}$ extending both $f$ and $g$.

Remark 1.10. It is easy to verify that, for any amenable family $\mathcal{A}$:

1. strong 2-amalgamation $\Rightarrow$ 2-amalgamation.
2. (1-amalgamation + strong 2-amalgamation) $\Rightarrow$ $\mathcal{A}$ has $n$-simplices for every $n \geq 0$.

Definition 1.11. An amenable family of functors is called non-trivial if it has 1-amalgamation and strong 2-amalgamation (in particular, it has 2-CA).

Definition 1.12. An $n$-chain $c \in C_n(\mathcal{A}; B)$ is called an $n$-shell if it is in the form

$$c = \pm \sum_{0 \leq i \leq n+1} (-1)^i f_i$$

where $f_i$’s are $n$-simplices satisfying

$$\partial^i f_j = \partial^{i-1} f_i \quad \text{whenever } 0 \leq i < j \leq n+1.$$  

We define $E_n(\mathcal{A}; B) := \{ c \in C_n(\mathcal{A}; B) \mid c \text{ is an } n\text{-shell} \}$.

It is straightforward to verify the following proposition.

Proposition 1.13.  

1. $E_n(\mathcal{A}; B) \subset Z_n(\mathcal{A}; B)$.
2. For every $f \in S_n(\mathcal{A}; B)$, $\partial_n(f) \in E_{n-1}(\mathcal{A}; B)$.
3. If $c = \pm \sum_{0 \leq i \leq n+1} (-1)^i f_i$ is any $n$-shell, then $|\text{supp}(c)| = n + 2$.
   Moreover, there exists a unique functor $g: \mathcal{P}^- (\text{supp}(c)) \to \mathcal{C}$ in $\mathcal{A}$ extending all the $f_i$’s. More precisely, if we let $\text{supp}(c) = \{ s_0 < \cdots < s_{n+1} \}$, then $g \mid \mathcal{P}(\text{supp}(c) \setminus \{ s_i \}) = f_i$ for each $i$.
4. $\mathcal{A}$ has $(n+2)$-amalgamation if and only if for any $n$-shell $c$, there exists some $(n+1)$-simplex $d$ such $c = \pm \partial(d)$.

Definition 1.14. An amenable family of functors has weak 3-amalgamation if each 1-shell is the boundary of some 2-chain $c$ with $|c| \leq 3$. 

The following result due to [5], [6] illustrates the importance of the notion of shell.

**Fact 1.15.** [5][6] Let $\mathcal{A}$ be any non-trivial amenable family of functors. If $\mathcal{A}$ has $(n+1)$-CA for some $n \geq 1$, then

$$H_n(\mathcal{A}; B) = \{ [c] \mid c \in E_n(\mathcal{A}; B), \supp(c) = \{0, \ldots, n+1\}\}.$$ 

In particular,

1. $H_1(\mathcal{A}; B) = 0 \iff E_1(\mathcal{A}; B) \subset B_1(\mathcal{A}; B)$
2. If $\mathcal{A}$ has weak 3-amalgamation then $H_1(\mathcal{A}; B) = 0$.

In the remainder of the paper, $\mathcal{A}$ shall denote a non-trivial amenable family of functors into a category $\mathcal{C}$.

Now we begin to talk about the prototypical examples of an amenable family of functors: complete types in rosy theories. In the sequel we work with a large saturated model $\mathcal{M} = \mathcal{M}^{eq}$ and its theory $T$ which is rosy. Recall that a theory is called rosy if there is a ternary independence relation $\downarrow$ on the small sets of its model, satisfying the basic independence properties. (See [1], [4] for the precise definition.) We take $\downarrow$ here to be thorn-independence. Any simple or $o$-minimal theory is known to be rosy. Moreover, if a simple theory $T$ has elimination of hyperimaginaries then non-forking independence is equal to thorn-independence. **So we assume that any simple $T$ in this paper has elimination of hyperimaginaries.** (Of course this is just for convenience as we can work in $\mathcal{M}^{heq}$ without the assumption.) In particular, we assume that 3-amalgamation holds over any algebraically closed set in simple $T$.

We fix any algebraically closed small subset $B \subseteq \mathcal{M}$ and consider the category $\mathcal{C}_B$ whose objects are all the small subsets of $\mathcal{M}$ containing $B$, and whose morphisms are elementary maps over $B$ (i.e., fixing $B$ pointwise). We also fix any $p(x) \in S(B)$ (where $x$ could be an infinite tuple). When $f$ is any functor from a primitive category $X$ into $\mathcal{C}_B$ and $u \subseteq v \in X$, we shall abbreviate $f^u_v(f(u))$ as $f^u_v(u)$.

**Definition 1.16.** By a closed independent functor in $p(x)$, we mean a functor $f$ from some primitive category $X$ into $\mathcal{C}_B$ satisfying the following:

1. Whenever $\{i\} \subset \omega$ is an object in $X$, we can choose a realization $b \models p(x)$ such that, if we let $C := f^\emptyset_{\{i\}}(\emptyset)$ then $f(\{i\}) = acl(Cb)$ and $b \downarrow_B C$. 


(2) Whenever \( u(\neq \emptyset) \subset \omega \) is an object in \( X \), we have
\[
f(u) = \acl \left( \bigcup_{i \in u} f^{(i)}_{u}(\{i\}) \right)
\]
and \( \{f^{(i)}_{u}(\{i\})\} \) \( i \in u \) is independent over \( f^{0}_{u}(\emptyset) \).

We let \( \mathcal{A}(p) \) be the family of all closed independent functors in \( p \).

**Fact 1.17.** [6] \( \mathcal{A}(p) \) is a non-trivial amenable family of functors.

**Notation 1.18.** We shall abbreviate \( S_{n}(\mathcal{A}(p); B), C_{n}(\mathcal{A}(p); B), \ldots \) and \( S_{n} \mathcal{A}(p), C_{n} \mathcal{A}(p); \ldots \) We shall also abbreviate \( H_{n}(\mathcal{A}(p); B) \) simply as \( H_{n}(p) \). Other than this, we use standard notation. For example \( a \equiv_{A} b \) denotes \( tp(a/A) = tp(b/A) \); and \( a \equiv^{L}_{A} b \) denotes \( Ltp(a/A) = Ltp(b/A) \), i.e., the Lascar (strong) types of \( a, b \) over \( A \) are the same.

2. \( H_{1}(p) \) in Rosy Theories

If a theory \( T \) is simple then due to 3-amalgamation and Fact 1.15, we know \( H_{1}(p) = 0 \). In this section we show the same holds for any rosy \( T \) as long as \( p \) is a Lascar type.

Let \( f : X \to C_{B} \) be any functor in \( \mathcal{A}(p) \) with \( f(\emptyset) = B \). If \( u \in X \) with \( u = \{i_{0} < \cdots < i_{k}\} \), we shall write \( f(u) = [a_{0}, \ldots, a_{k}] \), where \( a_{j} \models p \), \( f(u) = acl(B, a_{0} \cdots a_{k}) \), and acl\((a_{j}B) = f^{(i)}_{u}(\{i_{j}\}) \). Thus \( \{a_{0}, \ldots, a_{k}\} \) is independent over \( B \).

**Theorem 2.1.** If \( B \) is a model, then \( \mathcal{A}(p) \) has weak 3-amalgamation over \( B \) (so \( H_{1}(p) = 0 \)).

**Proof.** Let \( f = a_{12} - a_{02} + a_{01} \) be any 1-shell in \( E_{1} \mathcal{A}(p) \) where each \( a_{ij} : P(\{i, j\}) \to C_{B} \) is a 1-simplex. We want to find a 2-chain \( g \) with length 3 such that \( \partial g = f \). For this goal there is no harm in assuming that \( a_{01}(\{1\}) = [a] = a_{12}(\{1\}) \) and \( a_{12}(\{2\}) = [b] = a_{02}(\{2\}) \). Let \( a_{01}(\emptyset) := [c] \) and \( a_{02}(\emptyset) := [c'] \), and let \( q \) be a coher of \( p \) over \( Babccc' \). Choose any \( c'' \models q \). Then \( c'' \models Babccc' \) (see [4]) and \( cc'' \equiv_{B} c'c'' \). Now let \( g := a_{123} - a_{023} + a_{013} \) where \( a_{ij3} \) are 2-simplices having support \( \{i, j, 3\} \) extending \( a_{ij} \) such that \( a_{123}(\{1, 2, 3\}) = [a, b, c''] \), \( a_{023}(\{0, 2, 3\}) = [c', b, c''] \), \( a_{013}(\{0, 1, 3\}) = [c, a, c''] \). Hence we may assume \( \partial^{3}(a_{023}) = \partial^{3}(a_{123}) \) and \( \partial^{3}(a_{013}) = \partial^{3}(a_{123}) \). But \( cc'' \equiv_{B} c'c'' \) implies that we may further assume \( \partial^{3}(a_{013}) = \partial^{3}(a_{023}) \). Therefore \( \partial g = f \) as desired. \( \square \)

**Remark 2.2.** Of course the same proof shows that weak 3-amalgamation (over a model) holds not only in \( \mathcal{A}(p) \) but more generally inside \( \mathcal{M} \) (with arbitrary vertices).
Lemma 2.3. Let $b \equiv_B a$ and $d \equiv_B b$ (called the Lascar distance between $a$ and $b$). Recall the fact that $a \equiv_B b$ if $d_B(a, b) < \omega$ in any rosy theory.

**Proof.** Extend $I$ to $I'$ indiscernible over $B$ having a sufficiently large length. Then by the extension axiom there is $c' \equiv_B c_0$ such that $c' \subseteq I'$. Moreover, by the pigeonhole principle, there are $a_i, a_j \in I'$ ($i < j$) such that $c'a_i \equiv_B c' a_j$. Now, by $B$-indiscernibility, there is $c$ such that $c a_0 a_1 \equiv_B c' a_i a_j$. Then $c$ is the desired tuple. \qed

**Theorem 2.4.** Suppose that $p$ is a Lascar strong type. Then $H_1(p) = 0$.

**Proof.** For notational simplicity we let $B = \emptyset$. As in the proof of Theorem 2.1, given any 1-shell $f = a_{12} - a_{02} + a_{01}$ in $E_1 A(p)$ where each $a_{ij} : \mathcal{P}(\{i, j\}) \to C_B$ is a 1-simplex, we want to find a 2-chain $g$ such that $\partial g = f$. Again there is no harm in assuming that $a_{01}(\{1\}) = [a] = a_{12}(\{1\})$ and $a_{12}(\{2\}) = [b] = a_{02}(\{2\})$. Let $a_{01}(\emptyset) := [c]$ and $a_{02}(\emptyset) := [c']$. By extension we can further assume $a, b, c, c'$ is independent. Now, $c, c' \models p$ and let $d(c, c') = n$. So there are $c = c_0, \ldots, c_n = c'$ such that $c c_{i+1}$ begins an indiscernible sequence, for $i < n$. We can further assume that $a b \subseteq c c_{n-1}$; so $ab \subseteq c_0 \cdots c_n$. Then by Lemma 2.3, there are $e_i \models p$ ($i < n$) such that $c c_{i+1} \subseteq e_i$ and $e_i c_i \equiv e_i c_{i+1}$ (*). Again by extension we suppose $a b \subseteq c c_{i+1} e_i$, so that each of the $\{a, c_i, e_i\}, \{a, c_{i+1}, e_i\}$ is independent. Moreover each $a, e_{n-1}, b$, $\{e_{n-1}, c_i, b\}$ is independent as well (**).

Now there is $g_0 := g_0^+ - g_0^-$ where $g_0^+, g_0^-$ are 2-simplices with support $\{0, 1, 3\}$ such that $g_0^+ = [c_0, a, e_0]$ and $g_0^- = [c_1, a, e_0]$; $\partial_0 g_0^+ = \partial_0 g_0^-$; $\partial_1 g_0^+ = \partial_1 g_0^-$ (this is possible by (**)); and $g_0^+$ extends $a_{01}$ (i.e., $\partial_2^2 g_0^+ = a_{01}$). Hence $\partial g_0 = a_{01} - \partial_2^2 g_0^-$. 

By iteration we can find $g_i := g_i^+ - g_i^-$ \ ((0 < i < n - 1) where $g_i^+, g_i^-$ are 2-simplices with support $\{0, 1, 3\}$ such that $g_i^+ = [c_{i+1}, a, e_i]$ and $g_i^- = [c_i, a, e_i]$; $\partial_0^i g_i^+ = \partial_0^i g_i^-$; $\partial_1^i g_i^+ = \partial_1^i g_i^-$ (this again is possible by (**)); and $\partial_2^i g_i^+ = \partial_2^i g_i^-$. Therefore we have $\partial (g_0 + \cdots + g_{n-2}) = a_{01} - \partial_2^2 g_{n-2}$.

The rest of the proof is similar to that of Theorem 2.1. We put $g_{n-1} := g_{n-1}^+ - a_{023} + a_{123}$ where $a_{j23}$ is a 2-simplex with support $\{j, 2, 3\}$ extending $a_{j2}$ such that $a_{023}(\{0, 2, 3\}) = [c_n, b, e_{n-1}], a_{123}(\{1, 2, 3\}) = [a, b, e_{n-1}]$ (see (**)). Also $g_{n-1}^+$ is a 2-simplex with $g_{n-1}^+ = [c_n, b, e_{n-1}]$. \qed
Moreover again by (*), we have \( \partial^2g_{n-1} = \partial^2a_{023} \). Thus it follows
\[
\partial g_{n-1} = \partial^2g_{n-1} - a_{02} + a_{12} = \partial^2g_{n-2} - a_{02} + a_{12}.
\]

Therefore for \( g := g_0 + \cdots + g_{n-1} \), we have \( \partial g = f \) as desired. \( \square \)

3. A classification of 2-chains with a 1-shell boundary

In this section, we bring our attention back to a non-trivial amenable family of functors \( \mathcal{A} \) and classify 2-chains of \( \mathcal{A} \) having 1-shell boundaries. Basically we show that any 2-chain having a 1-shell boundary is equivalent to one of two types of 2-chains, called the \( NR \)-type and the \( RN \)-type.

We start by introducing two operations on 2-chains called the crossing operation and the renaming-of-support operation, respectively. For any distinct real numbers \( x \) and \( y \), we shall abbreviate the open interval \( (\min\{x, y\}, \max\{x, y\}) \) as \( [x, y] = [y, x] \).

**Definition 3.1.** Let \( v \in C_2(\mathcal{A}; B) \) be a 2-chain and let \( w := \epsilon_1 \alpha_1 + \epsilon_2 \alpha_2 \) be a subsummand of \( v \), where \( \alpha_i \)'s are 2-simplices with for \( i = 1, 2 \), \( \epsilon_i = \pm 1 \), \( \text{supp}(\alpha_i) = \{\ell_1, \ell_2, k_i\} \) (\( k_i, \ell_i \) being all distinct numbers) such that \( \alpha_1 \) and \( \alpha_2 \) agree on the intersection of their domains, namely \( \mathcal{P}(\{\ell_1, \ell_2\}) \). Further assume that, if we let \( \gamma := \alpha_i \mid \mathcal{P}(\{\ell_1, \ell_2\}) \), then \( \gamma \) does not appear in \( \partial(w) \), i.e., the two \( \gamma \) terms in \( \partial(w) \) have opposite signs and cancel each other.

Now by strong 2-amalgamation, there exists some 3-simplex \( \mu \) extending both \( \alpha_i \). For \( i = 1, 2 \), let \( \beta_i := \mu \mid \mathcal{P}(\{k_2, \ell_1\}) \) and
\[
w' := \begin{cases} 
\epsilon_2 \beta_1 + \epsilon_1 \beta_2 & \text{if } \epsilon_1 \epsilon_2 = -1, \text{and exactly one of } k_2, \ell_1 \text{ belongs to } [(k_1, \ell_2)] \\
\epsilon_1 \beta_1 + \epsilon_2 \beta_2 & \text{otherwise.}
\end{cases}
\]

Then the operation of replacing the subsummand \( w \) in \( v \) by \( w' \) is called the crossing operation (or simply CR-operation).

**Example 3.2.** Let \( f_0, f_1, f_2, f_3 \) be 2-simplices with \( \text{supp}(f_i) = \{0, 1, 2, 3\} \setminus \{i\} \). Assume that \( f_i \) and \( f_j \) agree on their intersection, for every pair \( i, j \). Consider the 2-chain \( c = f_0 - f_1 + f_2 \). Then we can apply the CR-operation to the subsummand \( f_0 - f_1 \) to obtain a new 2-chain
\[
c' = (-f_2 + f_3) + f_2 \text{ or simply } f_3.
\]

This example illustrates in particular that a CR-operation may not be reversible. i.e., once we apply a CR-operation to a 2-chain, we may not be able to recover the original 2-chain by applying more CR-operations (unless we allow 2-chains to be written redundantly as \( f_3 - f_2 + f_2 \)).
Next, we define an operation on \( n \)-chains called the renaming-of-support operation.

**Definition 3.3.** Let \( c \) be an \( n \)-chain in \( C_n(\mathcal{A}; B) \) and let \( d \) be a subsummand of \( c \). Let \( j \in \text{supp}(d) \) such that \( j \notin \text{supp}(\partial_n(d)) \). (In this situation, we say that \( d \) has a vanishing support, namely \( j \), in its boundary.) Choose any \( k \notin \text{supp}(c) \) and any bijection \( \sigma : \omega \to \omega \) which sends \( j \mapsto k \) but which fixes the rest of the elements in \( \text{supp}(c) \). Then the operation of replacing the subsummand \( d \) in \( c \) by \( \sigma^*_n(d) \) is called the renaming-of-support operation (or simply RS-operation). (See Remark/Definition 1.8 to recall the definition of \( \sigma^*_n \).)

**Remark 3.4.** When we apply the CR- and RS-operation to some subsummand of an \( n \)-chain \( c \), the resulting \( n \)-chain has the same boundary as \( c \) (guaranteed by the fact that \( \sigma^*_n \) commutes with the boundary map \( \partial \)) and has a shorter or equal length as \( c \) (by Remark/Definition 1.7 and the clear fact that \( \sigma^*_n \) preserves the lengths of \( n \)-chains).

**Remark/Definition 3.5.** A 2-chain \( c \) is called proper if its length \( |c| \) does not change after any finitely many applications of CR/RS-operations to its subsummands. It is clear that any 2-chain may be reduced to a proper 2-chain after finitely many applications of the two operations. Any CR-operation (also RS-operation) applied to any proper 2-chain is in fact reversible. This allows us to define an equivalence relation \( \sim \) among proper 2-chains by: \( c \sim c' \iff c \) can be obtained from \( c' \) by finitely many applications of the CR/RS-operations to its subsummands. Note that \( c \sim c' \) implies \( \partial(c) = \partial(c') \) and \( |c| = |c'| \).

We are now ready to introduce the notions of renameable type and non-renameable type for 2-chains having 1-shell boundaries.

**Definition 3.6.** Let \( \alpha \) be a 2-chain having a 1-shell boundary.

1. We say \( \alpha \) is of renameable type (or simply RN-type) if some subsummand of \( \alpha \) has a vanishing support. Otherwise, \( \alpha \) is said to be of non-renameable type (or simply NR-type).
2. \( \alpha \) is called minimal if it is proper, and for any proper \( \alpha' \) equivalent to \( \alpha \), there does not exist any subsummand \( \beta \) of \( \alpha' \) such that \( \partial(\beta) = 0 \).

**Remark 3.7.** Suppose that \( \alpha \) is a 2-chain having a 1-shell boundary.

1. Note that \( \alpha \) is of NR-type iff none of the CR or RS-operation is applicable to \( \alpha \), i.e. nothing else is equivalent to \( \alpha \) except \( \alpha \) itself. So an NR-type chain is minimal.
As was the case in Example 3.2, an RN-type $\alpha$ can sometimes be transformed to an NR-type by CR-operations. But if $\alpha$ is proper then its RN/NR-type is preserved under equivalence.

(2) We can always find some minimal 2-chain $\alpha'$ such that $\partial(\alpha) = \partial(\alpha')$. Such an $\alpha'$ can be obtained from $\alpha$ by finitely many applications of CR/RS-operations and deleting subsummands having trivial boundary.

There is a 2-chain $\beta$ with $|\beta| = 5$ having a 1-shell boundary such that any subsummand of $\beta$ does not have the trivial boundary but $\beta'$ with $|\beta'| = 5$ obtained from $\beta$ by the CR-operation has a subsummand with the boundary 0.

(3) If $\alpha$ is minimal then any $\alpha'$ equivalent to $\alpha$ is minimal as well (of course $|\alpha| = |\alpha'|$ and $\partial(\alpha) = \partial(\alpha')$ too).

**Notation.** Let $f$ be any simplex. For any subset $\{j_0, \ldots, j_k\} \subseteq \supp(f)$, we shall abbreviate $f \upharpoonright P(\{j_0, \ldots, j_k\})$ as $f_{j_0 \cdots j_k}$. Also, given a chain $c = \sum_{i \in I} n_i f_i$ (in its standard form), and any subset $\{j_0, \ldots, j_k\} \subseteq \supp(c)$, we shall write $c_{j_0 \cdots j_k}$ to denote the subchain $\sum_{i \in j} n_i f_i$, where $J := \{i \in I \mid \supp(f_i) = \{j_0, \ldots, j_k\}\}$.

**Example 3.8.** Of course any 2-simplex is of NR-type. The following is an NR-type 2-chain with length 5: Let $\alpha = a_1 + a_2 + a_3 - a_4 - a_5$ be a 2-chain with 2-simplices $a_i$ having $\supp(a_i) = \{0, 1, 2\}$ such that:

- $a_{1,2}^1, a_{2,2}^1 = a_{1,2}^2, a_{3,2}^1 = a_{1,2}^2$ are distinct;
- $a_{2,2}^2, a_1^3 = a_{2,2}^2, a_3^2 = a_{2,2}^3$ are distinct;
- and so are $a_3^{0,1}, a_1^{0,1} = a_4^{0,1}, a_2^{0,1} = a_5^{0,1}$.

Then $\alpha$ is of NR-type with a 1-shell boundary $a_{1,2}^1 - a_{2,2}^2 + a_3^{0,1}$.

Before stating our first main theorem of the classification, we introduce a notion called chain-walk which will be used in our proof.

**Remark 3.9.** Recall that if $\alpha$ is a 2-chain with a 1-shell boundary, then its length is always an odd positive number.

For the rest of this section, we fix a 1-shell boundary $f_{12} - f_{02} + f_{01}$ with $\supp(f_{jk}) = \{j < k\}$.

**Definition 3.10.** Let $\alpha$ be a 2-chain having the boundary $f_{12} - f_{02} + f_{01}$. A subchain $\beta = \sum_{i=0}^{m} \epsilon_i b_i$ of $\alpha$ (where $\epsilon_i = \pm 1$ and $b_i$ is a 2-simplex, for each $i$) is called a chain-walk in $\alpha$ from $f_{01}$ to $-f_{02}$ if

1. there are non-zero numbers $k_0, \ldots, k_{m+1}$ (not necessarily distinct) such that $k_0 = 1$, $k_{m+1} = 2$, and for $i \leq m$, $\supp(b_i) = \{k_i, k_{i+1}, 0\}$;
satisfy conditions (1)-(3). By a section \( \beta \) walk representation. For example, a reordering of terms in a representation. Notice that a chain-walk may have more than one.

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(2) \((\partial \epsilon_0 b_0)^{0, 1} = f_{01}, (\partial \epsilon_m b_m)^{0, 2} = -f_{02}; \) and
(3) for \(0 \leq i < m,\\)
\((\partial \epsilon_i b_i)^{0, k_{i+1}} + (\partial \epsilon_{i+1} b_{i+1})^{0, k_{i+1}} = 0.\\)

The sum \(\sum_{i=0}^{m} \epsilon_i b_i\) with its order is called a representation of the chain-walk \(\beta\). Unless said otherwise a chain-walk is written in the form of a representation. Notice that a chain-walk may have more than one representation. For example, a reordering of terms in \(\beta\) above may also satisfy conditions (1)-(3). By a section of the chain-walk \(\beta\), we shall mean a subchain of \(\beta\) in the form

\[\beta' := \sum_{i=j}^{m'} \epsilon_i b_i \quad \text{for some } 0 \leq j < m' \leq m\]

and the sequence \(\langle k_j, k_{j+1}, \ldots, k_{m'}, k_{m'+1} \rangle\) is called the walk sequence of \(\beta'\). A chain-walk \(\beta\) in \(\alpha\) is called maximal (in \(\alpha\)) if it has the maximal possible length. We say \(\alpha\) is centered at 0 if some (hence every) maximal chain-walk in \(\alpha\) from \(f_{01}\) to \(-f_{02}\) is, as a chain, equal to \(\alpha\).

We similarly define such notions as a chain-walk in \(\alpha\) from \(-f_{02}\) to \(f_{12}\), \(\alpha\) is centered at 2, and so on.

**Remark 3.11.** In the definition above, if \(\beta\) is a chain-walk in \(\alpha\) from \(f_{01}\) to \(-f_{02}\), then \(0 \in \text{supp}(b_i)\) for all \(i\), but \(0 \notin \text{supp}(\partial \beta - f_{01} + f_{02})\); and the walk sequence of \(\beta\) is a sequential arrangement of \((\text{supp}(b_i) \setminus \{0\})\)'s without repetition of the overlapped support.

Note now that given any 2-chain \(\alpha\) as in the definition above, since there are only finitely many 2-simplex terms in \(\alpha\), we can always find a chain-walk say, from \(f_{01}\) to \(-f_{02}\): We start with any 2-simplex term \(b\) in \(\alpha\) such that \(\partial^2 b = f_{01}\) and then keep finding a term in \(\alpha\) (with the coefficient) cancelling out adjacent 1-simplex boundaries. This process must stop with a term having \(-f_{02}\) as its boundary.

Even if 0 is in the support of every simplex term of \(\alpha\), it need not be centered at 0: Let \(\alpha = c_0 + c_1 - c_2\) such that \(\partial c_0 = g_{12} - f_{02} + f_{01};\)
\(\partial c_1 = f_{12} - g_{02} + g_{01};\) and \(\partial(-c_2) = -g_{12} + g_{02} - g_{01},\) where \(f_{ij} \neq g_{ij}\). Then \(c_0\) itself is a maximal chain-walk in \(\alpha\) from \(f_{01}\) to \(-f_{02}\). Note that \(\alpha = c_0 + c_1 - c_2\) is not a chain-walk from \(f_{01}\) to \(-f_{02}\), whereas it is a chain-walk from \(f_{12}\) to \(f_{01}\), i.e, \(\alpha\) is centered at 1.

**Lemma 3.12.** Let \(\alpha\) be a 2-chain with the 1-shell boundary \(f_{12} - f_{02} + f_{01}\). Let \(\beta = \sum_{i=0}^{m} \epsilon_i b_i\) be a chain-walk in \(\alpha\), say from \(-f_{02}\) to \(f_{12}\). Assume there is a section \(\beta' = \sum_{i=j}^{m'} \epsilon_i b_i\) of \(\beta\) such that \(\text{supp}(b_i) = \{2, k_i, k_{i+1}\}\),
either \( k_i \neq k_{m' + 1} \) for all \( i = j, \ldots, m' \); or \( k_i \neq k_j \) for all \( i = j + 1, \ldots, m' + 1 \). Then by finitely many applications of CR-operations to \( \beta' \), we obtain a simplex \( c \) with \( \text{supp}(c) = \{2, k_j, k_{m' + 1}\} \) such that, for some \( \epsilon = \pm 1 \), \( \beta' := \sum_{i=0}^{m'-2} \epsilon_ib_i + \epsilon c + \sum_{i>m'} \epsilon_ib_i \) is still a chain-walk from \(-f_{02}\) to \( f_{12}\).

**Proof.** When \( j = m' \), there is nothing to prove. Assume the lemma holds when \( m' - j = n \). Let us show that the lemma holds when \( m' - j = n + 1 \). Assume \( k_i \neq k_{m' + 1} \) for all \( i = j, \ldots, m' \). Then we can apply the CR-operation to \( \epsilon_{m'-1}b_{m'-1} + \epsilon_m'b_{m'} \), and we get \( \epsilon'_{m'-1}b'_{m'-1} + \epsilon'_m'b'_{m'} \) with \( \text{supp}(b'_{m'-1}) = \{2, k_{m'-1}, k_{m' + 1}\} \), having the same boundary. Due to the induction hypothesis applied to \( \sum_{i=j}^{m'-2} \epsilon_ib_i + \epsilon'_{m'-1}b'_{m'-1} \), we are done. When \( k_i \neq k_j \) for all \( i = j + 1, \ldots, m' + 1 \), we apply the CR-operation to \( \epsilon jb_j + \epsilon_j+1b_{j+1} \), and similarly we are done. \( \square \)

**Remark/Definition 3.13.** In Lemma 3.12, we call \( \beta'' \), a reduct of \( \beta \). The walk sequence of \( \beta'' \) is also called a reduct of the walk sequence of \( \beta \). So given a chain-walk its reducts are also chain-walks, which are obtained by the repeated applications of the CR-operation as described in Lemma 3.12.

**Theorem 3.14.** Let \( \alpha \) be a minimal 2-chain with the boundary \( f_{12} - f_{02} + f_{01} \).

1. Assume \( \alpha \) is of NR-type. Then \( |\alpha| = 1 \) or \( |\alpha| \geq 5 \). If \( |\alpha| \geq 5 \) then any chain-walk in \( \alpha \) from \( f_{01} \) to \( -f_{02} \) is of the form \( \sum_{i=0}^{2n}(-1)^ia_i \), which is as a chain equal to \( \alpha \) such that \( f_{12} = a^{12}_{2j} \) for some \( 1 \leq j \leq n - 1 \).

2. \( \alpha \) is of RN-type iff \( \alpha \) is equivalent to a 2-chain \( \alpha' = a_0 + \sum_{i=1}^{2n-1} \epsilon_i a_i + a_{2n} \) (\( n \geq 1 \)) which is a chain-walk from \( f_{01} \) to \( -f_{02} \) such that \( \partial^0a_{2n} = f_{12} \), \( \partial^1(a_{2n}) = -f_{02} \) and \( \text{supp}(a_{2n}) = \{0, 1, 2\} \). (The representation of \( \alpha' \) is called standard.)

**Proof.** (1) As mentioned in Remark 3.11, a chain-walk \( \beta \) in \( \alpha \) from \( f_{01} \) to \( -f_{02} \) exists. Now since \( \alpha \) is of NR-type, \( \text{supp}(\alpha) = \{0, 1, 2\} \). If \( |\beta| < |\alpha| \) then it follows \( \alpha - \beta \) has a vanishing support 0, a contradiction. Hence \( |\alpha| = |\beta| \) and \( \alpha = \beta \). Suppose now that \( |\alpha| = 3 \). So the chain-walk is \( a_0 - a_1 + a_2 = \alpha \) and either \( \partial^0a_0 = f_{12} \) or \( \partial^0a_2 = f_{12} \). Then
either $\partial^0 a_1 = \partial^0 a_2$ or $\partial^0 a_0 = \partial^0 a_1$. In either case, the subchain of $\alpha$ has a vanishing support $1$ or $2$, a contradiction. Hence $|\alpha| = 1$ or $\geq 5$.

When $|\alpha| \geq 5$, all we need to show is that $f_{12} \neq \partial^0 a_0$ and $f_{12} \neq \partial^0 a_{2n}$. If $f_{12} = \partial^0 a_0$ then $\alpha - a_0$ has a vanishing support $1$, a contradiction. Hence $f_{12} \neq \partial^0 a_0$. Similarly, we can show $f_{12} \neq \partial^0 a_{2n}$.

(2) $(\Leftarrow)$ It follows $\text{supp}(\partial(\alpha' - a_{2n})) = \{0, 1\}$, i.e., $\alpha' - a_{2n}$ has a vanishing support, so $\alpha'$ is of RN-type. Since CR/RS-operations preserve the minimality and the chain types, $\alpha$ is also an RN-type.

$(\Rightarrow)$ We prove this in a series of claims. Note that $|\alpha| \geq 3$.

Claim 1. There is a 2-chain $\alpha_1 \sim \alpha$ which is centered at $2$ such that $|\text{supp}(\alpha_1)| > 3$.

Proof of Claim 1. Let $\alpha_2 := \alpha$ if $|\text{supp}(\alpha)| > 3$. Otherwise since $\alpha$ is of RN-type, we can apply RS-operations to obtain some $\alpha_2 \sim \alpha$ with $|\text{supp}(\alpha_2)| > 3$. Now there is $\beta := \sum_{i \in I} \epsilon_i b_i$, a maximal chain-walk in $\alpha_2$ from $-f_{02}$ to $f_{12}$. If $\beta = \alpha_2$ we put $\alpha_1 := \alpha_2$ and we are done. Otherwise let $\gamma := \alpha_2 - \beta$, and then $\gamma$ has a vanishing support $2$ in its boundary. By applying the RS-operation to $\gamma$ we find $\gamma'$ with $2 \notin \text{supp}(\gamma')$ such that $\alpha_2 \sim \alpha_2' := \beta + \gamma'$.

Assume now inductively we can find a desired $\alpha_1 \sim \alpha_2'$ when $|\gamma'| = m$. Let $|\gamma'| = m + 1$. Note that $f_{12} - f_{02} + f_{01} \neq \partial(\beta)$, since otherwise $\partial(\gamma') = 0$ contradicting the minimality of $\alpha_2'$. Hence there is $i_0 \in I$ with $\text{supp}(b_{i_0}) = \{2, n_0, n_1\}$ such that $b_{i_0}^{n_0, n_1} \neq f_{01}$ with a coefficient, stays in $\partial(\beta)$. Therefore there must be a term $\epsilon_{j_0} b_{j_0}$ ($\epsilon_{j_0} \in \{1, -1\}$) in $\gamma'$ such that $(2 \notin \text{supp}(b_{j_0}) = \{n_0, n_1, n_2\}$ and $b_{i_0}^{n_0, n_1} = b_{j_0}^{n_0, n_1}$ is cancelled out in $\partial(\epsilon_{i_0} b_{i_0} + \epsilon_{j_0} b_{j_0})$. Now applying the CR-operation to $\epsilon_{i_0} b_{i_0} + \epsilon_{j_0} b_{j_0}$, we get $\epsilon_{i_0}' b_{i_0}' + \epsilon_{j_0}' b_{j_0}'$ with $\text{supp}(b_{i_0}') = \{2, n_0, n_2\}$, $\text{supp}(b_{j_0}') = \{2, n_1, n_2\}$, preserving the boundary. Then from $\beta$, we obtain $\beta'$ by substituting $\epsilon_{i_0}' b_{i_0}' + \epsilon_{j_0}' b_{j_0}'$ for $\epsilon_{i_0} b_{i_0}$. Notice that $\beta'$ is still a chain-walk from $-f_{02}$ to $f_{12}$ while $\alpha_2' \sim \alpha_2'' := \beta' + (\gamma' - \epsilon_{j_0} b_{j_0})$. Hence by the induction hypothesis there is a desired $\alpha_1 \sim \alpha_2''$. We have proved Claim 1.

Claim 2. There is a 2-chain $\alpha_2 \sim \alpha_1$ that has a 1-simplex term $c$ (with the coefficient $1$) such that $\text{supp}(c) = \{0, 1, 2\}$, and $f_{12} - f_{02} = \partial^0(c) - \partial^1(c)$.

Proof of Claim 2. For notational simplicity, let $\{0, 1, 2, 3\} \subseteq \text{supp}(\alpha_1)$, and write $\alpha_1 = \sum_{i=0}^{2n} \epsilon_i c_i$, a chain-walk from $-f_{02}$ to $f_{12}$. So for some $j_0 \leq 2n$, we have $\epsilon_{j_0} = 1$, $\text{supp}(c_{j_0}) = \{0, 1, 2\}$, and $\partial^2(c_{j_0}) = f_{01}$.

Let $\beta_0 := \sum_{i=0}^{j_0-1} \epsilon_i c_i$, and $\beta_1 := \sum_{i=j_0+1}^{2n} \epsilon_i c_i$. We shall find the desired $c$.
Figure 1. A standard RN-type 2-chain

(by applying the process in Lemma 3.12 and finding reducts of chain-walks, starting from $\alpha_1$. Each time, the reduced chain-walk together with the deleted terms is equivalent to $\alpha_1$.

Case 1) $3 \notin \text{supp}(\beta_1)$: So $3 \in \text{supp}(\beta_0)$. Now let $I_0 := \langle 0, \ldots, 3, \ldots, 0 \rangle$ be the walk sequence of $\beta_0$, and let $I_1 = \langle 1, \ldots, 1 \rangle$ be the walk sequence of $\beta_1$. So $I_0I_1$ is the walk sequence of $\alpha_1$. Now $I_0 = J_0J_1$ such that $J_1$ starts with 3 but all other components $\neq 3$. Then due to Lemma 3.12 (applied to $J_1I_1$), we can find $\gamma_1$, a reduct of $\alpha_1$, whose walk sequence is $J_0\langle 3, 1 \rangle$. Now $J_0$ can be reduced to $\langle 0, 1 \rangle$, and we are done.

If 3 is not in $J_0$ then again by Lemma 3.12, we can further find a reduct of $\gamma_1$ whose walk sequence is $\langle 0, 3, 1 \rangle$, then again further reduce it with the walk sequence $\langle 0, 1 \rangle$, and we are done.

If 3 is in $J_0$ then in general, by finding a sequence of all 3’s in $J_0$ and applying Lemma 3.12, we can reduce $J_0$ to a sequence of the form $J_0’ = \langle 03, k_3, k_23, \ldots ; k_{\ell} \rangle$ where each $k_i \neq 3$. If none of the $k_i$’s is 0 then by applying Lemma 3.12 again to $J_0’\langle 3, 1 \rangle$ we directly reduce it to $\langle 0, 1 \rangle$ and we are done. Otherwise, one of the $k_i$’s is 0, and we can similarly reduce $J_0’$ to a sequence of the form $\langle 03, 03, \ldots ; k_{\ell} \rangle$. Now the reduced walk sequence is $\langle 03, 03, \ldots ; k_{\ell}; 3, 1 \rangle$. If $k_{\ell} \neq 1$ then it can directly be reduced to $\langle 0, 1 \rangle$ and we are done. If $k_{\ell} = 1$ then it can be reduced to $\langle 0, 1 ; 3, 1 \rangle$ and further reduced to $\langle 0, 3, 1 \rangle$ and to $\langle 0, 1 \rangle$, so we are done.

Case 2) $3 \notin \text{supp}(\beta_0)$: Then $3 \in \text{supp}(\beta_1)$ and the proof will be similar to Case 1.

Case 3) $3 \in \text{supp}(\beta_0) \cap \text{supp}(\beta_1)$: By an argument similar to that in Case 1, the walk sequence of $\alpha_1$ can be in general reduced to $I = \langle 03, 03, \ldots ; k, 3 \rangle\langle 0, 1 \rangle\langle 3, k'; 31, \ldots, 31 \rangle$. Now by the argument in the last part of the proof of Case 1, $\langle 03, 03, \ldots ; k, 3 \rangle\langle 0, 1 \rangle$ can be reduced to $\langle 0, 1 \rangle$. Hence $I$ can be reduced to $\langle 0, 1 \rangle\langle 3, k'; 31, \ldots, 31 \rangle$. Then by
the same argument it can finally be reduced to ⟨0, 1⟩, and we have proved Claim 2.

Now lastly we simply take a chain-walk γ from $f_{01}$ to $-f_{02}$ in $\alpha_2$ terminating with $c$ (= the 1-simplex described in Claim 2). Then by an argument similar to that in the proof of Claim 1, we repeatedly apply the CR-operation to $\gamma$ (while keeping $c$ unchanged), and obtain a desired $\alpha' \sim \alpha_2$ centered at 0 forming a chain-walk from $f_{01}$ to $-f_{02}$. Then we take the reverse order of the representation of the chain-walk $\alpha'$.

In an upcoming paper [7], it is shown that for any minimal 2-chain whose boundary is a 1-shell, there is an equivalent 2-chain which has the same boundary with support size three.

### 4. Examples

This section is devoted to exhibiting a certain family of examples of 2-chains of types in rosy theories whose boundaries are 1-shells. The existence of these examples implies that, in rosy theories, there is no uniform bound for the minimal lengths of 2-chains having 1-shell boundaries.

We recall the examples described in [2]. For a positive integer $n$, consider a (saturated) structure $M_n = (|M_n|; S, g_n)$, where $|M_n|$ is a circle; $S$ is a ternary relation such that $S(a, b, c)$ holds iff $a, b, c$ are distinct and $b$ comes before $c$ going around the circle clockwise starting at $a$; and $g_n$ is a rotation (clockwise) by $2\pi/n$-radians. When $n$ is obvious from context, $g_n$ is often written as $g$. The following Fact 4.1, 4.2 are from [2].

**Fact 4.1.**

1. $\text{Th}(M_n)$ has the unique 1-complete type $p_n(x)$ over $\emptyset$, which is isolated by the formula $x = x$.

2. $\text{Th}(M_n)$ is $\aleph_0$-categorical and has quantifier-elimination.

3. For any subset $A \subset M_n$, $\text{acl}(A) = \text{dcl}(A) = \bigcup_{0 \leq i < n} g_n^i(A)$ (in the home-sort), where $g_n^i = g_n \circ \cdots \circ g_n$.

4. For each $a \in M_n$ with $n > 1$, and an integer $i$, $S(g^i(a), x, g^{i+1}(a))$ isolates a complete type over $a$.

In what follows, we assume $n > 1$.

**Fact 4.2.**

1. There are $a, b \in M_n$ such that $d(a, b) > n/2$.

2. For any $a, b \in M_n$, the following are equivalent:
   
   (i) $a, b$ begin some $\emptyset$-indiscernible sequence,
A some union of finitely many arcs on

\[ \text{(1)} \]

Theorem 4.3. (1) \( \text{Th}(M_n) \) has weak elimination of imaginaries.
(2) \( \text{Th}(M_n) \) is rosy having thorn \( U \)-rank 1 with a trivial pregeometry.

Proof. (1) We claim that if a set \( D \) in \( (M_n)^k \) is definable over \( A_0 \) and \( A_1 \) respectively where \( A_i = \text{acl}(A_i) = \text{dcl}(A_i) \) (in the home-sort) then it is definable over \( B := A_0 \cap A_1 \): We sketch the proof of the claim by freely using Fact 4.1. Let \( k = 1 \). Due to quantifier elimination, \( D \) is some union of finitely many arcs on \( M_n \). Clearly each end-point of a connected component of \( D \) is in \( \text{dcl}(A_i) \) and so in \( B \) as well. Hence \( D \) is indeed \( B \)-definable. Now for induction, assume the claim holds for \( k - 1 \). We want to show it holds for \( k \). Suppose that \( \varphi_i(x_1, \ldots, x_k, \bar{a}_i) \) defines \( D \) where \( \bar{a}_i \in A_i \). Then, for each element \( b \), the set \( D_b \) defined by \( \varphi_i(x_1, \ldots, x_{k-1}, b, \bar{a}_i) \) is definable over \( Bb \), by the induction hypothesis. But due to \( \aleph_0 \)-categoricity (so there are only finitely many formulas over \( \emptyset \) up to equivalence), it easily follows that for each \( y \), \( \varphi_i(x_1, \ldots, x_{k-1}, y, \bar{a}_i) \) is definable over \( B \), i.e. \( D \) is definable over \( B \) as we wanted.

Now let \( E(\bar{x}, \bar{y}) \) be an \( \emptyset \)-definable equivalence relation on \( (M_n)^k \). For \( \bar{a} \in (M_n)^k \), let \( \bar{a}' \) denote a finite tuple of algebraic closure of \( \bar{a} \) in the home-sort. Let \( \bar{b} \) be the maximal subtuple of \( \bar{a}' \) which is algebraic over \( \bar{a}/E \). Thus there is \( \bar{a}'' \equiv_{\text{acl}(\bar{a}/E)} \bar{a}' \) such that \( \bar{b} = \bar{a}' \cap \bar{a}'' \) as sets. Hence due to the claim, \( \bar{a}/E \in \text{dcl}^n(\bar{b}) \) and \( \bar{b} \in \text{acl}(\bar{a}/E) \). We have proved (1).

(2) Due to (1), \( \text{Th}(M_n) \) is rosy having thorn \( U \)-rank 1 as pointed out in [4]. Notice that \( M_n \) has the same pregeometry as the \( n \)-copies of a half-closed interval, and so \( M_n \) forms a trivial pregeometry with its algebraic closures.

\[ \Box \]

Definition 4.4. Let \( a, b \in M_n \) be any elements with \( \text{acl}(a) \neq \text{acl}(b) \).

(1) We define the \( S \)-distance of \( b \) from \( a \), denoted by \( \text{Sd}(a, b) \) as follows: \( \text{Sd}(a, b) = k \) iff \( M_n \models S(g^k(a), b, g^{k+1}(a)) \). For integers \( k < l \), we write \( k \leq \text{Sd}(a, b) \leq l \) if \( M_n \models S(g^k(a), b, g^{l+1}(a)) \).

(2) We define the \( \hat{S} \)-distance of \( b \) from \( a \), denoted by \( \hat{\text{Sd}}(a, b) \), as similar manner as \( \text{Sd}(a, b) \), using the formula
\[ \hat{S}(x, y, z) \equiv (x \neq z \land S(x, y, z)) \lor (x = z \land x \neq y). \]
Remark 4.5. Let \( x, y, z \in M_n \) have mutually disjoint algebraically closures. Then for any \( k, l, m \in \mathbb{Z} \),

1. \( Sd(y, x) = -Sd(x, y) - 1 \);
2. (a) for \( l - k \not\equiv -1 \pmod{n} \), if \( k \leq Sd(x, y) \leq l - 1 \), and \( Sd(y, z) = m \), then \( m + k \leq Sd(x, z) \leq m + l \);
(b) for \( l - k \equiv -1 \pmod{n} \), if \( k \leq Sd(x, y) \leq l - 1 \), and \( Sd(y, z) = m \), then \( m + k \leq Sd(x, z) \leq m + l \).

(1)’ \( \hat{Sd}(y, x) = -\hat{Sd}(x, y) - 1 \);
(2)’ for \( k \not\equiv l \pmod{n} \), if \( k \leq \hat{Sd}(x, y) \leq l - 1 \), and \( \hat{Sd}(y, z) = m \), then \( m + k \leq \hat{Sd}(x, z) \leq m + l \).

Lemma 4.6. Let \( k \) and \( l_0, \ldots, l_m \) be fixed integers and \( L_i := \sum_{i=0}^{j} l_i \). Let \( a \) and \( d_0, \ldots, d_{m+1} \) be elements in \( M_n \) such that

\begin{equation}
(*)_m \colon \hat{Sd}(a, d_0) = k, \; \hat{Sd}(d_i, d_{i+1}) = l_i, \; 0 \leq i \leq m.
\end{equation}

Then

\[ k + L_m \leq \hat{Sd}(a, d_{m+1}) \leq k + L_m + m + 1. \]

Moreover, by choosing appropriate elements for \( a \) and \( d_0, \ldots, d_{m+1} \), the quantity \( \hat{Sd}(a, d_{m+1}) \) can be made to be any integer in \([k + L_m, k + L_m + m + 1]\) \((**)_m\).

Proof. We show this using induction on \( m \). For \( m = 0 \), by Remark 4.5(2)’, it follows from \((*)_0\) that

\[ k + l_0 \leq \hat{Sd}(a, d_1) \leq k + l_0 + 1. \]

Moreover it is not hard to see \((**)_0\) holds.

Now assume the lemma holds for \( m - 1 \) with \( m + 1 < n \). Let us show the lemma for \( m \). For \( i \leq m + 1 \), \( a, d_i \in M_n \) are given which satisfy \((*)_m\). Firstly, by the induction hypothesis for \( m - 1 \),

\[ k + L_{m-1} \leq \hat{Sd}(a, d_m) \leq k + L_{m-1} + m. \]

Since \( m + 1 < n \),

\[ k + L_{m-1} \leq Sd(a, d_m) \leq k + L_{m-1} + m. \]

Then again by Remark 4.5(2)’,

\[ k + L_m \leq \hat{Sd}(a, d_{m+1}) \leq k + L_m + m + 1. \]

Secondly, we show the moreover part. Fix \( L_m \leq j \leq L_m + m + 1 \) and \( a' \in M_n \). If \( j = L_m \), then \( j - L_m = L_{m-1} \) and due to the induction hypothesis, there are \( d'_0, \ldots, d'_m \) that satisfy \((*)_{m-1}\) and

\[ \hat{Sd}(a', d'_m) = k + j - L_m. \]
So, \( \tilde{Sd} (a', g^{l_m}(d'_m)) = k + j \), and \( M_n \models S (g^{l_m}(d'_m), d'_{m+1}, g^{k+j+1}(a')) \) for some \( d'_{m+1} \in M_n \). Thus

\[
\tilde{Sd} (d'_m, d'_{m+1}) = l_m, \quad \tilde{Sd} (a', d'_{m+1}) = k + j.
\]

So, \( a' \) and \( d'_i \) for \( i \leq m + 1 \) satisfy the required condition. Now for \( j > L_m \), the proof is similar to the case \( j = L_m \) except that we replace \( j - l_m \) by \( j - l_m - 1 \) and take \( d'_{m+1} \) in \( M_n \) such that

\[
M_n \models S(g^{k+j}(a'), d'_{m+1}, g^{l_m+1}(d'_m)).
\]

\[\square\]

Now, let \( \mathcal{A}(p_n) \) be the family of all the closed independent functors in \( p_n \). We follow the notation given at the beginning of Section 2: given a closed independent functor \( f \) over \( \emptyset \) in \( p_n \) with \( u = \{i_0 < \cdots < i_k\} \in \text{dom}(f) \), we write \( f(u) = [a_0, \ldots, a_k] \), where \( a_j \in M_n \), \( f(u) = \text{acl}(a_0, \ldots, a_k) \), and \( \text{acl}(a_j) = f^{(i_j)}_a(\{i_j\}) \). When we write \( f(u) \equiv [b_0, \ldots, b_k] \), it of course means that \([a_0, \ldots, a_k] \equiv [b_0, \ldots, b_k] \).

By Theorem 4.3, it is equivalent to saying \( a_0 \cdots a_k \equiv b_0 \cdots b_k \).

**Remark 4.7.** Let \( \tau = \sum_{i=0}^{m} \epsilon_i t_i \) (\( t_i \) 2-simplex) be a chain-walk (in \( p_n \)) from \( f_{01} \) to \( -f_{02} \) such that \( D_i = \text{supp}(t_i) = \{0, k_i, k_{i+1}\} \) with \( k_0 = 1, k_{m+1} = 2 \). Then putting together the triangles \( t_0(D_0), \ldots, t_m(D_m) \) side by side centered at 0, we can find elements \( a \) and \( d_0, \ldots, d_{m+1} \) in \( M_n \) such that for \( 0 \leq i \leq m \),

\[
t_i(D_i) \equiv \begin{cases} [a, d_i, d_{i+1}] & \text{if } k_i < k_{i+1} \\ [a, d_{i+1}, d_i] & \text{if } k_i > k_{i+1}. \end{cases}
\]

Combining the classification results in Section 3 and Lemma 4.6, we will show that there does not exist any finite upper bound for the minimal lengths of 2-chains with 1-shell boundaries in the types \( p_n \).

**Theorem 4.8.** Let \( \mathcal{A} \) be a non-trivial amenable collection and let \( s \) be a 1-shell. Define \( B(s) \), and \( B(\mathcal{A}) \) as follows:

1. \( B(s) := \min \{ |\tau| : \tau \text{ is a (minimal) } 2\text{-chain and } \partial(\tau) = s \} \).

   (If \( s \) is not the boundary of any 2-chain, define \( B(s) := -\infty \).)

2. \( B(\mathcal{A}) := \max \{ B(s) : s \text{ is a 1-shell of } \mathcal{A} \} \).

Let \( n > 1 \) and let \( s = s_{12} - s_{02} + s_{01} \) be a 1-shell from \( \mathcal{A}(p_n) \) with \( \text{supp}(s_{i j}) = \{i, j\} \). Then there are \( a, b, c, c' \) in \( M_n \) and some integers \( k_1, k_2, k_3 \) with \( 0 \leq k_i < n \) such that,

- \( \hat{Sd} (a, c) = k_1, \hat{Sd} (a, b) = k_2, \) and \( \hat{Sd} (b, c') = k_3; \)
- \( s_{01}(\{0, 1\}) \equiv [a, c], s_{02}(\{0, 2\}) \equiv [a, b], \) and \( s_{12}(\{1, 2\}) \equiv [c', b]. \)
Let $0 \leq k_4(< n) \equiv k_2 - (k_1 - k_3) \pmod{n}$ and let
$$n_s := \min\{2(n - k_4) - 1, 2k_4 + 1\}.$$ 

Then
$$B(s) = n_s.$$

Moreover, taking $k_1 = 0$, $k_2 = 0$, and $k_3 = \left\lfloor \frac{n}{2} \right\rfloor$, we get $n_s \geq n - 1$ and $B(\mathcal{A}(p_n)) \geq n - 1$. Therefore $\lim_{n \to \infty} B(\mathcal{A}(p_n)) = \infty$.

**Proof.** (1) $B(s) \geq n_s$: By Theorem 2.4 and Corollary 3.14, there is a chain-walk $\tau = \sum_{i=0}^{2m} (-1)^i t_i$ from $s_{01}$ to $-s_{02}$ and $\partial(\tau) = s$. We want to show $|\tau| \geq n_s$. Suppose not, i.e., $|\tau| = 2m + 1 < n - 1$. By Remark 4.7, there are $d_i$'s ($0 \leq i \leq 2m + 1$) in $M_n$ such that $ac \equiv ad_0$, $d_{2m+1} = b$; and

- $\hat{S}(d_0, d_1) = l_0$, $\hat{S}(d_{2m-1}, d_{2m}) = l_{2m}$ for some integers $l_i$;
- $t_0\{0, k_0, k_1\} \equiv [a, d_0, d_1]$, $t_{2j-1}\{0, k_{2j-1}, k_{2j}\} \equiv [a, d_{2j}, d_{2j-1}]$, and $t_{2j}\{0, k_{2j}, k_{2j+1}\} \equiv [a, d_{2j+1}, d_{2j+1}]$ for $1 \leq j \leq m$.

Now $\partial \tau = s$ implies $\partial^0 t_{2j_0} = s_{12}$ for some $0 \leq j_0 \leq 2m$; and for any $0 \leq j_1 \neq j_0 \leq m$ there is $0 \leq j_2 \neq j_0 \leq m$ (indeed a bijection) such that $\partial^0 t_{2j_1} = \partial^0 t_{2j_2 + 1}$. So

- $\hat{S}(d_{2j_0}, d_{2j_0 + 1}) = -k_3 - 1$; and
- $[d_{2j_1}, d_{2j_1 + 1}] \equiv [d_{2j_2 + 2}, d_{2j_2 + 1}]$.

By Remark 4.5(1)', $\hat{S}(d_{2j_1}, d_{2j_1 + 1}) = -\hat{S}(d_{2j_2}, d_{2j_2 + 1}) - 1$. Therefore $l_{2j_0} = -k_3 - 1$ and $l_{2j_2 + 1} = -l_{2j_1} - 1$, so $\sum_{j=0}^{2m} l_j = -k_3 - m - 1$. Hence due to Lemma 4.6 and $2m + 1 < n - 1$, we have $k_1 - k_3 - m - 1 \leq \hat{S}(a, b) \leq k_1 - k_3 + m$. Thus

$$\hat{S}(a, b) = k_2; \text{ and } k_1 - k_3 - m - 1 \leq \hat{S}(a, b) \leq k_1 - k_3 + m.$$ 

We rewrite it as

$$\hat{S}(g^{k_1-k_3}(a), b) = k_2 - (k_1 - k_3); \text{ and } -m - 1 \leq \hat{S}(g^{k_1-k_3}(a), b) \leq m.$$ 

We can replace $k_2 - (k_1 - k_3)$ by $k_4$ and we have $n - (m + 1) < k_4 + 1$ or $m + 1 > k_4$. In either case, we have $m \geq \min\{n - k_4 - 1, k_4\}$. Therefore $2m + 1 \geq 2 \min\{n - k_4 - 1, k_4\} + 1 = \min\{2(n - k_4) - 1, 2k_4 + 1\} = n_s$, a contradiction. We have proved $B(s) \geq n_s$.

(2) $B(s) \leq n_s$: We construct a chain-walk $\gamma = \sum_{i=0}^{n_s - 1} r_i$ with $\text{supp}(\gamma) = \{0, 1, 2\}$ and $\partial \gamma = s$ as follows: Note that since $n_s$ is odd, $m_s := (n_s - 1)/2$ is an integer. Also note that, if we let $N_1 := k_1 - k_3 - m_s - 1$ and $N_2 := k_1 - k_3 + m_s$, then $k_2 \equiv N_i \pmod{n}$ ($i = 1$ or 2). Hence we
have \( \hat{S}_d(a, b) = N_1 \) or \( \hat{S}_d(a, b) = N_2 \). Applying Lemma 4.6 with \( k_1 \) and 
\( l_0, \ldots, l_{2m_s} \) such that 
\( l_{2i+1} = -l_{2i} - 1 \) for \( 0 < i < m_s \) and 
\( l_{2m_s} = -k_3 - 1 \), 
we obtain 
\( \sum_{i=0}^{2m_s} l_i = L_{2m_s} = -m_s - k_3 - 1 \), and 
\( L_{2m_s} + 2m_s + 1 = m_s - k_3 \).
Therefore if \( j \) is chosen to be such that 
\( j = N_1 - k_1 \) or \( j = N_2 - k_1 \), and 
by applying \((**)_2m_s\) in Lemma 4.6, we can find 
\( d'_0, \ldots, d'_{2m_s+1} (= d'_{n_s}) \) such that 
\( \hat{S}_d(a, d'_0) = k_1 \), 
\( \hat{S}_d(d'_i, d'_{i+1}) = l_i \) for \( 0 < i < 2m_s \), and 
\( \hat{S}_d(a, d'_{n_s}) = k_2 \).
Hence clearly we have a desired 2-chain 
\( \gamma = \sum_{i=0}^{n_s-1} r_i \) such that 
\[
 r_i(\{0, 1, 2\}) \equiv \begin{cases} 
 [a, d'_i, d'_{i+1}] & \text{if } i \equiv 0 \pmod{2} \\
 [a, d'_{i+1}, d'_i] & \text{if } i \equiv 1 \pmod{2}.
\end{cases}
\]
\( \square \)

**Corollary 4.9.** For each \( n \geq 5 \), \( A(p_n) \) does not have weak 3-amalgamation.

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