THE RELATIVIZED LASCAR GROUPS, TYPE-AMALGAMATIONS, AND ALGEBRAICITY

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Abstract. In this paper we study the relativized Lascar Galois group of a strong type. The group is a quasi-compact connected topological group, and if in addition the underlying theory T is G-compact, then the group is compact. We apply compact group theory to obtain model theoretic results in this note.

For example, we use the divisibility of the Lascar group of a strong type to show that, in a simple theory, such types have a certain model theoretic property that we call divisible amalgamation.

The main result of this paper is that if c is a finite tuple algebraic over a tuple a, the Lascar group of $\mathrm{stp}(ac)$ is abelian, and the underlying theory is G-compact, then the Lascar groups of $\mathrm{stp}(ac)$ and of $\mathrm{stp}(a)$ are isomorphic. To show this, we prove a purely compact group-theoretic result that any compact connected abelian group is isomorphic to its quotient by every finite subgroup.

Several (counter)examples arising in connection with the theoretical development of this note are presented as well. For example, we show that, in the main result above, neither the assumption that the Lascar group of $\mathrm{stp}(ac)$ is abelian, nor the assumption of G-compactness can be removed.

Given a complete theory T, the notion of the Lascar (Galois) group $\mathrm{Gal_L}(T)$ was introduced by D. Lascar (see [1]). The Lascar group only depends on the theory and it is a quasi-compact topological group with respect to a quotient topology of a certain Stone type space over a model ([1] or [9]). More recently, the notions of the *relativized* Lascar groups were introduced in [2] (and studied also in [8] in the context of topological dynamics). Namely, given a type-definable set X in a large saturated model of the theory T, we consider the group of automorphisms restricted to the set X quotiented by the group of restricted automorphisms fixing the Lascar types of the sequences from X of

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length λ. The relativized Lascar groups also only depend on T, on the type that defines X, and on the cardinal λ. These groups are endowed with the quasi-compact quotient topologies induced by the canonical surjective maps from $\mathrm{Gal_L}(T)$ to the relativized Lascar groups.

This paper continues the study started in [2], where a connection between the relativized Lascar groups of a strong type and the first homology group of the strong type was established. If T is G-compact (for example when T is simple) then the relativized Lascar group of a strong type is compact and connected, and we use compact group theory to obtain results presented here.

A common theme for the results in this paper is the connection between group theoretic properties of the Lascar group of a strong type and the model theoretic properties of the type. In Section 1, we use the fact that any compact connected group is divisible to show that any strong type p in a simple theory has a property we call divisible amalgamation. The property would follow from the Independence theorem if p was a Lascar type, but here p is only assumed to be a strong type. Moreover, we give amalgamation criteria for when the Lascar group of the strong type of a model is abelian, and for when the strong type of a model is a Lascar type.

In Section 2, we study morphisms between the relativized Lascar groups. The main goal is to understand the connection between the Lascar group of $\mathrm{stp}(a)$, for some tuple a, and the Lascar group of $\mathrm{stp}(\mathrm{acl}(a))$ or $\mathrm{stp}(ac)$, for a finite tuple $c \in \mathrm{acl}(a)$. We prove that if T is G-compact, then for a finite tuple c algebraic over a, the restriction map from a Lascar group of $\mathrm{stp}(ac)$ to that of $\mathrm{stp}(a)$ is a covering map. In addition, if the Lascar group of $\mathrm{stp}(ac)$ is abelian then this group is isomorphic to that of $\mathrm{stp}(a)$ as topological groups. In order to achieve this, we separately prove a purely compact group theoretical result that any compact connected abelian group is isomorphic to its quotient by a finite subgroup. We also give an example showing that the abelianness of the relativized Lascar group is essential in the isomorphism result.

In Section 3, we mainly present 3 counterexamples: a non G-compact theory where $\mathrm{stp}(a)$ is a Lascar type but $\mathrm{stp}(\mathrm{acl}(a))$ is not a Lascar type; an example showing that in the mentioned isomorphism result in Section 2, the tuple c being finite is essential; and an example answering a question raised in [7], namely a Lie group structure example where an RN-pattern minimal 2-chain is not equivalent to a Lascar pattern 2-chain having the same boundary.

In the remaining part of this section we recall the definitions and terminology of basic notions, which, unless said otherwise, we use throughout this note. We work in a large saturated model $\mathcal{M}(=\mathcal{M}^{\mathrm{eq}})$ of a complete theory T, and we use the standard notation. So $A, B, \ldots$ and $M, N, \ldots$ are small subsets and elementary submodels of $\mathcal{M}$, respectively. Lower-case letters $a, b, \ldots$ will denote tuples of elements from $\mathcal{M}$, possibly infinite. We will explicitly specify when a tuple is assumed to be finite.

To simplify the notation, we state the results for types over the empty set (rather than over $\operatorname{acl}(\emptyset)$). This does not reduce the generality because, after naming a parameter set, we may assume that $\operatorname{dcl}(\emptyset) = \operatorname{acl}(\emptyset)$. We fix a complete *strong* type $p(x) \in S(\emptyset)$ with possibly infinite arity of x. For tuples a, b, we write $a \equiv_A b$ ($a \equiv^s_A b$, resp.) to mean that they have the same (strong, resp.) type over A. Note that $a \equiv^s_A b$ if and only if $a \equiv_{\operatorname{acl}(A)} b$.

Let us recall the definitions of the Lascar groups and types. These are well-known notions in model theory. In particular, the Lascar group depends only on T and is a *quasi-compact* group under the topology introduced in [1] or [9]. Moreover, due to our assumption $\operatorname{dcl}(\emptyset) = \operatorname{acl}(\emptyset)$, the group is *connected* as well. Recall that a topological space is *compact* if it is quasi-compact and Hausdorff.

Definition 0.1.

- $\operatorname{Autf}(\mathcal{M})$ is the normal subgroup of $\operatorname{Aut}(\mathcal{M})$ generated by
$$\{f \in \operatorname{Aut}(\mathcal{M}) \mid f \text{ pointwise fixes some model } M \prec \mathcal{M}\}.$$
- For tuples $a, b \in \mathcal{M}$, we say they have *the same Lascar type*, written $a \equiv^L b$ or $\operatorname{Ltp}(a) = \operatorname{Ltp}(b)$, if there is $f \in \operatorname{Autf}(\mathcal{M})$ such that $f(a) = b$. We say a, b have the same KP(Kim–Pillay)-type (write $a \equiv^{KP} b$) if they are in the same class of any bounded $\emptyset$-type-definable equivalence relation.
- The group $\operatorname{Gal_L}(T) := \operatorname{Aut}(\mathcal{M})/\operatorname{Autf}(\mathcal{M})$ is called the *Lascar (Galois) group* of T.
- We say T is *G-compact* if $\operatorname{Gal_L}(T)$ is compact, equivalently $\{\mathrm{id}\}$ is closed in $\operatorname{Gal_L}(T)$.

Note that the above is the definition of "G-compactness over $\emptyset$", but for convenience throughout this paper we omit "over $\emptyset$."

Fact 0.2. *For tuples a, b, we have $a \equiv^L b$ if and only if the Lascar distance between a, b is finite, i.e., there are finitely many indiscernible sequences $I_1, \ldots, I_n$, and tuples $a = a_0, a_1, \ldots, a_n = b$ such that each of $a_{i-1}I_i$ and a_iI_i is an indiscernible sequence for $i = 1, \ldots, n$.*

Now let us recall, mainly from [2], the definitions of various relativized Lascar groups of p and related facts.

Definition 0.3.
- $\mathrm{Aut}(p) := \{f \upharpoonright p(\mathcal{M}) : \ f \in \mathrm{Aut}(\mathcal{M})\}$;
- for a cardinal $\lambda > 0$, $\mathrm{Autf}^{\lambda}(p) = \mathrm{Autf}^{\lambda}_{\mathrm{fix}}(p) :=$

  $\{\sigma \in \mathrm{Aut}(p) \mid$ for any $\bar{a} = (a_i)_{i<\lambda}$ with $a_i \models p, \ \bar{a} \equiv^L \sigma(\bar{a})\}$;
- $\mathrm{Autf}_{\mathrm{fix}}(p) :=$

  $\{\sigma \in \mathrm{Aut}(p) \mid \bar{a} \equiv^L \sigma(\bar{a})$ where $\bar{a}$ is some enumeration of $p(\mathcal{M})\}$;

  and $\mathrm{Autf}_{\mathrm{res}}(p) := \{f \upharpoonright p(\mathcal{M}) : \ f \in \mathrm{Autf}(\mathcal{M})\}$.

Notice that $\mathrm{Autf}^{\lambda}(p)$, $\mathrm{Autf}_{\mathrm{fix}}(p)$, and $\mathrm{Autf}_{\mathrm{res}}(p)$ are normal subgroups of $\mathrm{Aut}(p)$.
- $\mathrm{Gal}^{\lambda}_{\mathrm{L}}(p) = \mathrm{Gal}^{\mathrm{fix},\lambda}_{\mathrm{L}}(p) := \mathrm{Aut}(p)/\mathrm{Autf}^{\lambda}(p)$; [1]
- $\mathrm{Gal}^{\mathrm{fix}}_{\mathrm{L}}(p) := \mathrm{Aut}(p)/\mathrm{Autf}_{\mathrm{fix}}(p)$, and $\mathrm{Gal}^{\mathrm{res}}_{\mathrm{L}}(p) := \mathrm{Aut}(p)/\mathrm{Autf}_{\mathrm{res}}(p)$.

We will give an example (in Example 2.2) where $\mathrm{Gal}^{1}_{\mathrm{L}}(p)$ and $\mathrm{Gal}^{2}_{\mathrm{L}}(p)$ are distinct. In [2, Remark 3.4], a canonical topology on each of the above groups was defined. With these topologies, they become quotients of the topological group $\mathrm{Gal}_{\mathrm{L}}(T)$.

Fact 0.4. [2] $\mathrm{Autf}^{\omega}(p) = \mathrm{Autf}_{\mathrm{fix}}(p)$*, and, for each* $\lambda(\leq \omega)$, $\mathrm{Gal}^{\lambda}_{\mathrm{L}}(p)$ *does not depend on the choice of a monster model, and is a quasi-compact connected topological group. Hence, if* T *is* G*-compact,* $\mathrm{Gal}^{\lambda}_{\mathrm{L}}(p)$ *is a compact connected group.*

If $p(x)$ *is a type of a model, then* $\mathrm{Autf}^{1}(p) = \mathrm{Autf}_{\mathrm{fix}}(p) = \mathrm{Autf}_{\mathrm{res}}(p)$, $\mathrm{Gal}^{1}_{\mathrm{L}}(p) = \mathrm{Gal}^{\mathrm{fix}}_{\mathrm{L}}(p) = \mathrm{Gal}^{\mathrm{res}}_{\mathrm{L}}(p) \cong \mathrm{Gal}_{\mathrm{L}}(T)$*. The abelianization of* $\mathrm{Gal}^{1}_{\mathrm{L}}(p)$ *(i.e., the group* $\mathrm{Gal}^{1}_{\mathrm{L}}(p)/(\mathrm{Gal}^{1}_{\mathrm{L}}(p))'$*) is isomorphic to the first homology group* $H_1(p)$*.*

The last statement in the above fact explains the connection between the relativized Lascar group and the model theoretic homology group H_1 of the type. Let us recall now the key definitions in the homology theory in model theory. We fix a ternary automorphism-invariant relation $\mathop{\smile\!\!\!\!\!\!|}^*$ between small sets of $\mathcal{M}$ satisfying
- finite character: for any sets A, B, C, we have $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ iff $a \mathop{\smile\!\!\!\!\!\!|}^*_C b$ for any finite tuples $a \in A$ and $b \in B$;
- normality: for any sets A, B and C, if $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$, then $A \mathop{\smile\!\!\!\!\!\!|}^*_C \mathrm{acl}(BC)$;
- symmetry: for any sets A, B, C, we have $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ iff $B \mathop{\smile\!\!\!\!\!\!|}^*_C A$;

[1]Similarly, $\mathrm{Autf}^{\lambda}_{\mathrm{KP}}(p)$ is defined as the group of automorphisms in $\mathrm{Aut}(p)$ fixing the KP-type of any λ-many realizations of p, and $\mathrm{Gal}^{\lambda}_{\mathrm{KP}}(p) := \mathrm{Aut}(p)/\mathrm{Autf}^{\lambda}_{\mathrm{KP}}(p)$. Then in this paper, one may work with $\mathrm{Gal}^{\lambda}_{\mathrm{KP}}(p)$ instead of $\mathrm{Gal}^{\lambda}_{\mathrm{L}}(p)$, and remove the assumption of T being G-compact where that is assumed.

- transitivity: $A \mathop{\smile\!\!\!\!\!\!\mid}^*_B D$ iff $A \mathop{\smile\!\!\!\!\!\!\mid}^*_B C$ and $A \mathop{\smile\!\!\!\!\!\!\mid}^*_C D$, for any sets A and $B \subseteq C \subseteq D$;
- extension: for any sets A and $B \subseteq C$, there is $A' \equiv_B A$ such that $A' \mathop{\smile\!\!\!\!\!\!\mid}^*_B C$.

Throughout this paper we call the above axioms **the basic 5 axioms**. We say that A is $*$-independent from B over C if $A \mathop{\smile\!\!\!\!\!\!\mid}^*_C B$. Notice that there is at least one such relation for any theory, namely, the *trivial independence relation* given by: For any sets A, B, C, put $A \mathop{\smile\!\!\!\!\!\!\mid}^*_B C$. Of course there is a non-trivial such relation when T is simple or rosy, given by forking or thorn-forking, respectively.

Notation 0.5. Let s be an arbitrary finite set of natural numbers. Given any subset $X \subseteq \mathcal{P}(s)$, we may view X as a category where for any $u, v \in X$, $\mathrm{Mor}(u, v)$ consists of a single morphism $\iota_{u,v}$ if $u \subseteq v$, and $\mathrm{Mor}(u, v) = \emptyset$ otherwise. If $f \colon X \to \mathcal{C}_0$ is any functor into some category $\mathcal{C}_0$, then for any $u, v \in X$ with $u \subseteq v$, we let f^u_v denote the morphism $f(\iota_{u,v}) \in \mathrm{Mor}_{\mathcal{C}_0}(f(u), f(v))$. We shall call $X \subseteq \mathcal{P}(s)$ a *primitive category* if X is non-empty and *downward closed*; i.e., for any $u, v \in \mathcal{P}(s)$, if $u \subseteq v$ and $v \in X$ then $u \in X$. (Note that all primitive categories have the empty set $\emptyset \subseteq \omega$ as an object.)

We use now $\mathcal{C}$ to denote the category whose objects are the small subsets of $\mathcal{M}$, and whose morphisms are elementary maps. For a functor $f : X \to \mathcal{C}$ and objects $u \subseteq v$ of X, $f^u_v(u)$ denotes the set $f^u_v(f(u))(\subseteq f(v))$.

Definition 0.6. By a *$*$-independent functor in p*, we mean a functor f from some primitive category X into $\mathcal{C}$ satisfying the following:

(1) If $\{i\} \subseteq \omega$ is an object in X, then $f(\{i\})$ is of the form $\mathrm{acl}(Cb)$ where $b \models p$, $C = \mathrm{acl}(C) = f^{\emptyset}_{\{i\}}(\emptyset)$, and $b \mathop{\smile\!\!\!\!\!\!\mid}^* C$.

(2) Whenever $u (\neq \emptyset) \subseteq \omega$ is an object in X, we have

$$f(u) = \mathrm{acl}\left(\bigcup_{i \in u} f^{\{i\}}_u(\{i\})\right)$$

and $\{f^{\{i\}}_u(\{i\}) |\ i \in u\}$ is $*$-independent over $f^{\emptyset}_u(\emptyset)$.

We let $\mathcal{A}^*_p$ denote the family of all $*$-independent functors in p.

A $*$-independent functor f is called a *$*$-independent n-simplex* in p if $f(\emptyset) = \emptyset$, our named algebraically closed set, and $\mathrm{dom}(f) = \mathcal{P}(s)$ with $s \subseteq \omega$ and $|s| = n + 1$. We call s the *support* of f and denote it by $\mathrm{supp}(f)$.

In the rest we may call a $*$-independent n-simplex in p just an n-*simplex* of p, as far as no confusion arises.

Definition 0.7. Let $n \geq 0$. We define:

$$S_n(\mathcal{A}_p^*) := \{ f \in \mathcal{A}_p^* \mid f \text{ is an } n\text{-simplex of } p \}$$
$$C_n(\mathcal{A}_p^*) := \text{the free abelian group generated by } S_n(\mathcal{A}_p^*).$$

An element of $C_n(\mathcal{A}_p^*)$ is called an *n-chain* of p. The support of a chain c, denoted by $\mathrm{supp}(c)$, is the union of the supports of all the simplices that appear in c with a non-zero coefficient. Now for $n \geq 1$ and each $i = 0, \ldots, n$, we define a group homomorphism

$$\partial_n^i \colon C_n(\mathcal{A}_p^*) \to C_{n-1}(\mathcal{A}_p^*)$$

by putting, for any n-simplex $f \colon \mathcal{P}(s) \to \mathcal{C}$ in $S_n(\mathcal{A}_p^*)$ where $s = \{s_0 < \cdots < s_n\} \subseteq \omega$,

$$\partial_n^i(f) := f \restriction \mathcal{P}(s \setminus \{s_i\})$$

and then extending linearly to all n-chains in $C_n(\mathcal{A}_p^*)$. Then we define the *boundary map*

$$\partial_n \colon C_n(\mathcal{A}_p^*) \to C_{n-1}(\mathcal{A}_p^*)$$

by

$$\partial_n(c) := \sum_{0 \leq i \leq n} (-1)^i \partial_n^i(c).$$

We shall often refer to $\partial_n(c)$ as the *boundary of c*. Next, we define:

$$Z_n(\mathcal{A}_p^*) := \mathrm{Ker}\ \partial_n$$
$$B_n(\mathcal{A}_p^*) := \mathrm{Im}\ \partial_{n+1}.$$

The elements of $Z_n(\mathcal{A}_p^*)$ and $B_n(\mathcal{A}_p^*)$ are called *n-cycles* and *n-boundaries* in p, respectively. It is straightforward to check that $\partial_n \circ \partial_{n+1} = 0$. Hence we can now define the group

$$H_n^*(p) := Z_n(\mathcal{A}_p^*)/B_n(\mathcal{A}_p^*)$$

called the nth *$*$-homology group* of p.

Notation 0.8.
(1) For $c \in Z_n(\mathcal{A}_p^*)$, $[c]$ denotes the homology class of c in $H_n^*(p)$.
(2) When n is clear from the context, we shall often omit it in ∂_n^i and in ∂_n, writing simply ∂^i and ∂.

Definition 0.9. A 1-chain $c \in C_1(\mathcal{A}_p^*)$ is called a *1-$*$-shell* (or just a 1-shell) in p if it is of the form

$$c = f_0 - f_1 + f_2$$

where f_i's are 1-simplices of p satisfying

$$\partial^i f_j = \partial^{j-1} f_i \quad \text{whenever } 0 \leq i < j \leq 2.$$

Hence, for $\mathrm{supp}(c) = \{n_0 < n_1 < n_2\}$ and $k \in \{0, 1, 2\}$, it follows that

$$\mathrm{supp}(f_k) = \mathrm{supp}(c) \smallsetminus \{n_k\}.$$

Notice that the boundary of any 2-simplex is a 1-shell. Recall that a notion of an *amenable* collection of functors into a category is introduced in [4]. Due to the 5 axioms of $\downarrow^*$, it easily follows that $\mathcal{A}_p^*$ forms such a collection of functors into $\mathcal{C}$. Hence the following corresponding fact holds.

Fact 0.10. *([4] or [2])*

$$H_1^*(p) = \{[c] \mid c \textit{ is a } 1\text{-}*\textit{-shell with } \mathrm{supp}(c) = \{0, 1, 2\}\}.$$

We now recall basic notions and results appeared in [2]. **For the rest of this section we assume that p is the strong type of an algebraically closed set.**

Definition 0.11. (1) Let $f : \mathcal{P}(s) \to \mathcal{C}$ be an n-simplex of p. For $u \subseteq s$ with $u = \{i_0 < \ldots < i_k\}$, we shall write $f(u) = [a_0 \ldots a_k]_u$, where each $a_j \models p$ is an algebraically closed tuple as assumed above, if $f(u) = \mathrm{acl}(a_0 \ldots a_k)$, and $\mathrm{acl}(a_j) = f_u^{\{i_j\}}(\{i_j\})$. So, $\{a_0, \ldots, a_k\}$ is $*$-independent. Of course, if we write $f(u) \equiv [b_0 \ldots b_k]_u$, then it means that there is an automorphism sending $a_0 \ldots a_k$ to $b_0 \ldots b_k$.

(2) Let $s = f_{12} - f_{02} + f_{01}$ be a 1-$*$-shell in p such that $\mathrm{supp}(f_{ij}) = \{n_i, n_j\}$ with $n_i < n_j$ for $0 \le i < j \le 2$. Clearly there is a quadruple (a_0, a_1, a_2, a_3) of realizations of p such that $f_{01}(\{n_0, n_1\}) \equiv [a_0 a_1]_{\{n_0,n_1\}}$, $f_{12}(\{n_1, n_2\}) \equiv [a_1 a_2]_{\{n_1,n_2\}}$, and $f_{02}(\{n_0, n_2\}) \equiv [a_3 a_2]_{\{n_0,n_2\}}$. We call this quadruple a *representation of* s. For any such representation of s, call a_0 an *initial point*, a_3 a *terminal point*, and (a_0, a_3) an *endpoint pair* of the representation.

We summarize some properties of endpoint pairs of 1-shells. We define an equivalence relation $\sim^*$ on the set of pairs of realizations p as follows: For $a, a', b, b' \models p$, $(a, b) \sim^* (a', b')$ if two pairs (a, b) and (a', b') are endpoint pairs of 1-shells s and s' respectively such that $[s] = [s'] \in H_1^*(p)$. We write $\mathcal{E}^* = p(\mathcal{M}) \times p(\mathcal{M})/ \sim^*$. We denote the class of $(a, b) \in p(\mathcal{M}) \times p(\mathcal{M})$ by $[a, b]$. Now, define a binary operation $+_{\mathcal{E}^*}$ on $\mathcal{E}^*$ as follows: For $[a, b], [b', c'] \in \mathcal{E}^*$, $[a, b] +_{\mathcal{E}^*} [b', c'] = [a, c]$, where $bc \equiv b'c'$.

Fact 0.12. *The operation $+_{\mathcal{E}^*}$ is well-defined, and the pair $(\mathcal{E}^*, +_{\mathcal{E}^*})$ forms an abelian group which is isomorphic to $H_1^*(p)$. More specifically, for $a, b, c \models p$ and $\sigma \in \mathrm{Aut}(\mathcal{M})$, we have:*

- $[a, b] + [b, c] = [a, c]$;

- *$[a,a]$ is the identity element;*
- *$-[a,b]=[b,a]$;*
- *$\sigma([a,b]):=[\sigma(a),\sigma(b)]=[a,b]$; and*
- *$f:\mathcal{E}^*\to H_1^*(p)$ sending $[a,b]\mapsto[s]$, where (a,b) is an endpoint pair of s, is a group isomorphism.*

We identify $\mathcal{E}^*$ and $H_1^*(p)$.

Fact 0.13. *$H_1^*(p)$ is isomorphic to G/N where $G:=\mathrm{Aut}(p)$ and N is the normal subgroup of G consisting of all automorphisms fixing setwise all orbits of elements of $p(\mathcal{M})$ under the action of G' (so G/N is independent from the choice of a monster model). Hence $H_1^*(p)$ is independent from the choice of $\mathop{\smile\!\!\!\!\!|}^*$ and we write $H_1(p)$ for $H_1^*(p)$.*

Fact 0.14. *Let p be the fixed strong type of an algebraically closed set.*

(1) Let $\mathop{\smile\!\!\!\!\!|}^*$ be an independence relation satisfying the 5 basic axioms. Let $a,b\models p$. Then the following are equivalent.
   - *(a) $[a,b]=0\in H_1(p)$;*
   - *(b) There is a balanced-chain-walk from a to b, i.e., there are some $n\geq 0$ and a finite sequence $(d_i)_{0\leq i\leq 2n+2}$ of realizations of p satisfying the following conditions:*
     - *(i) $d_0=a$, and $d_{2n+2}=b$;*
     - *(ii) $\{d_j,d_{j+1}\}$ is $*$-independent for each $j\leq 2n+1$; and*
     - *(iii) there is a bijection*
     $$\sigma:\{0,1,\dots,n\}\to\{0,1,\dots,n\}$$
     such that $d_{2i}d_{2i+1}\equiv d_{2\sigma(i)+2}d_{2\sigma(i)+1}$ for $i\leq n$;
   - *(c) There are some $n\geq 0$ and finite sequences $(d_i:0\leq i\leq n)$, $(d_i^j:i<n,\ 1\leq j\leq 3)$ of realizations of p such that $d_0=a$, $d_n=b$, and for each $i<n$, $d_id_i^1\equiv d_i^3d_i^2$, $d_i^1d_i^2\equiv d_{i+1}d_i^3$.*

   In particular, if $a\equiv^L b$ then $[a,b]=0\in H_1(p)$.

(2) The following are equivalent.
   - *(a) $\mathrm{Gal}_{\mathrm{L}}^1(p)$ is abelian;*
   - *(b) For all $a,b\models p$, $[a,b]=0$ in $H_1(p)$ iff $a\equiv^L b$.*

(3) p is a Lascar type (i.e. $a\equiv^L b$ for for any $a,b\models p$) iff $H_1(p)$ is trivial and $\mathrm{Gal}_{\mathrm{L}}^1(p)$ is abelian.

Remark 0.15. (1) Assume that $p(x)$ is the type of a small model. Then for any $M,N\models p$, the equivalence classes of (equality of) Lascar types of M and N are interdefinable: Let $M\equiv^L M'$ and $MN\equiv M'N'$. It suffices to show that $N\equiv^L N'$. Now there is $f\in\mathrm{Autf}(\mathcal{M})$ such that $f(M')=M$. Let $N'':=f(N')$ so that $N''\equiv^L N'$ and $MN\equiv MN''$. Hence $N\equiv_M N''$ and $N\equiv^L N''\equiv^L N'$ as wanted. When p is a type of any tuple we will see that the same holds if $\mathrm{Gal}^1_{\mathrm{L}}(p)$ is abelian (Remark 1.10).

(2) Notice that for tuples $a, b \models p$, there is a commutator f in $\mathrm{Aut}(p)$ such that $f(a) = b$ if and only if there are $d^1, d^2, d^3 \models p$ such that $ad^1 \equiv d^3d^2$ and $bd_3 \equiv d_1d_2$ (**). Fact 0.14(1)(c) is an iterative application of this to that $[a, b] = 0 \in H_1(p)$ if and only if $h(a) = b$ for some h in the commutator subgroup of $\mathrm{Aut}(p)$.

Now we recall the following fact of compact group theory by M. Gotô from [5, Theorem 9.2]: Assume $(F, \cdot)$ is a compact connected topological group. Then F', the commutator subgroup of F, is simply the set of commutators in F, i.e.

$$F' = \{f \cdot g \cdot f^{-1} \cdot g^{-1} \mid f, g \in F\}.$$

It follows that F' is a closed subgroup of F, and both F' and F/F' are compact connected groups as well.

Due to the theorem we newly observe here that if T is G-compact then in Fact 0.14(1)(c), we can choose $n = 1$: There are $d^1, d^2, d^3 \models p$ such that above (**) holds.

1. Amalgamation properties of strong types in simple theories

Note that if T is simple then since T is G-compact, each $\mathrm{Gal}^{\lambda}_{\mathrm{L}}(p)$ is a compact (i.e., quasi-compact and Hausdorff) connected group, so it is divisible (see [5, Theorem 9.35]). **In this section we assume T is simple** (except Remark 1.10 and Example 1.11), and the independence is nonforking independence. It is still an open question whether the strong type p is necessarily a Lascar type. If so, then the following theorem follows easily by the 3-amalgamation of Lascar types in simple theories. But regardless of the answer to the question, p has the following amalgamation property. We *do not assume* here that a realization of p is algebraically closed.

Theorem 1.1 (divisible amalgamation)**.** *Let p be a strong type in a simple theory. Let $a, b \models p$ and $a \perp b$. Then for each $n \geq 1$, there are independent $a = a_0, a_1, \ldots, a_n = b$ such that $a_0a_1 \equiv a_ia_{i+1}$ for every $i < n$.*

Proof. Clearly we can assume $n > 1$. Note that there is $f \in \mathrm{Aut}(p)$ such that $b = f(a)$. Then since $G := \mathrm{Gal}^1_{\mathrm{L}}(p)$ is divisible, there is $h \in \mathrm{Aut}(p)$ such that $[h]^n = [f]$ (in G). Put $c = h(a)$. Now there is $c_1 \equiv^L c$ such that $c_1 \perp ab$. Then there is $h' \in \mathrm{Autf}^1(p)$ such that

$h'(c) = c_1$. Let $g = h' \circ h$. Then $[g] = [h]$ so $[g]^n = [f]$ too, and $g(a) = c_1$.

Claim. We can find additional elements $c_2, \ldots, c_n$ such that $\{a = c_0, c_1, \ldots, c_n, b\}$ is independent, and for each $1 \leq m \leq n$, $c_0c_1 \equiv c_{m-1}c_m$ and there is $h_m \in \mathrm{Aut}(p)$ such that $[h_m] = [g]^m$ in G, and $h_m(a) = c_m$: For an induction hypothesis, assume for $1 \leq m < n$ we have found $a = c_0, c_1, \ldots, c_m$ such that $\{c_0, c_1, \ldots, c_m, b\}$ is independent and $c_0c_1 \equiv c_{i-1}c_i$ for all $1 \leq i \leq m$ and there is $h_m \in \mathrm{Aut}(p)$ such that $[h_m] = [g]^m$ in G, and $h_m(c_0) = c_m$.

Notice now that then $c'_m := g^m(a) \equiv^L h_m(a) = c_m$. Put $c'_{m+1} := g(c'_m) = g^{m+1}(a)$. Then there is c_{m+1} such that $c_mc_{m+1} \equiv^L c'_mc'_{m+1}$, and $\{c_0, c_1, \ldots, c_{m+1}, b\}$ is independent. Since $c_{m+1} \equiv^L c'_{m+1}$, there is $h'' \in \mathrm{Autf}^1(p)$, such that $h''(c'_{m+1}) = c_{m+1}$ and so for $h_{m+1} := h'' \circ g^{m+1}$, we have $h_{m+1}(a) = c_{m+1}$ and $[g]^{m+1} = [h_{m+1}]$ in G. Moreover the equality of types $c_mc_{m+1} \equiv c'_mc'_{m+1} \equiv c_0c_1$ is witnessed by g^m. Hence the claim is proved.

Notice that $c_n = h_n(a)$ with $[h_n] = [g]^n = [f]$. Hence $c_n \equiv^L b = f(a)$. Then, by 3-amalgamation, we find $b' \models \mathrm{tp}(b/a) \cup \mathrm{tp}(c_n/c_1 \ldots c_{n-1})$ with $b' \mathop{\smile\!\!\!\!\!|}\, c_0 \ldots c_{n-1}$. Then the automorphic images of $c_1 \ldots c_{n-1}$ (rename them as $a_1 \ldots a_{n-1}$) under a map sending b' to b over a satisfy the conditions of the theorem. □

Now we assume that p is the type of an algebraically closed tuple. For $a, a' \models p$, we have a better description of $[a, a'] = 0 \in H_1(p)$.

Proposition 1.2. *For $a, a' \models p$, the following are equivalent.*

(1) *$[a, a'] = 0 \in H_1(p)$, equivalently there is h in the commutator subgroup of $\mathrm{Aut}(p)$ such that $h(a) = a'$.*
(2) *There are $b, c, d \models p$ such that each of $\{a, b, c, d\}$, $\{a', b, c, d\}$ is independent, and $ab \equiv cd$, $bd \equiv a'c$.*

Proof. (1)⇒(2) Since T is G-compact, by Remark 0.15(2) there are $b', c', d' \models p$ such that $ab' \equiv c'd'$ and $b'd' \equiv a'c'$. Now there is $b \equiv^L b'$ such that $b \mathop{\smile\!\!\!\!\!|}\, aa'b'c'd'$. Hence, by 3-amalgamation, there is $d_0 \equiv^L d'$ such that $d_0 \models \mathrm{tp}(d'/b) \cup \mathrm{tp}(d''/b')$ and $d_0 \mathop{\smile\!\!\!\!\!|}\, bb'$, where $d''b' \equiv d'b$ and $d'' \equiv^L d' \equiv^L d_0$. By extension there is no harm in assuming that $d_0 \mathop{\smile\!\!\!\!\!|}\, bb'aa'c'd'$.

Now since $a'c' \equiv b'd'$, there are $a''c$ such that $b'd'bd_0 \equiv a'c'a''c$. Hence, $c \equiv^L c'$, $a'' \equiv^L a'$, and $c \mathop{\smile\!\!\!\!\!|}\, a'c'$. Again by extension, we can further assume that $c \mathop{\smile\!\!\!\!\!|}\, aa'c'bd_0$. Moreover, since $ab' \equiv c'd'$, there is $d_1 \equiv^L d'$ such that $ab'b \equiv c'd'd_1$. Then since $c \mathop{\smile\!\!\!\!\!|}\, c'$ with $c \equiv^L c$, and

$d_1 \mathop{\smile\!\!\!\!\!\!\;\;|}\, c'$, again by 3-amalgamation we can assume that $c'd_1 \equiv cd_1$ and $d_1 \mathop{\smile\!\!\!\!\!\!\;\;|}\, cc'$.

Now the situation is that $aa'b \mathop{\smile\!\!\!\!\!\!\;\;|}\, c$, $d_1 \mathop{\smile\!\!\!\!\!\!\;\;|}\, c$, and $d_0 \mathop{\smile\!\!\!\!\!\!\;\;|}\, aa'b$. Moreover, $d_0 \equiv^L d' \equiv^L d_1$. Hence, by 3-amalgamation, we have $d \models \mathrm{tp}(d_0/aa'b) \cup \mathrm{tp}(d_1/c)$ and $d \mathop{\smile\!\!\!\!\!\!\;\;|}\, aa'bc$. Therefore, each of $\{a, b, c, d\}$, $\{a', b, c, d\}$ is independent. Moreover, due to above combinations

$$ab \equiv c'd_1 \equiv cd_1 \equiv cd,$$

and

$$bd \equiv bd_0 \equiv bd' \equiv b'd'' \equiv b'd_0 \equiv a'c$$

as wanted.

(2)⇒(1) Clear by Remark 0.15(2). □

From now until Theorem 1.8, we assume that $p(x)$ is the (strong) type of a small model, and all tuples $a, b, c, \ldots$ realize p, so all are universes of models. Hence, by Fact 0.4, $\mathrm{Autf}^1(p) = \mathrm{Autf}_{\mathrm{fix}}(p)$ and $(\mathrm{Gal_L}(T) \cong)\, \mathrm{Gal}^{\mathrm{fix}}_{\mathrm{L}}(p) = \mathrm{Gal}^1_{\mathrm{L}}(p)$, which we simply write $\mathrm{Autf}(p)$ and $\mathrm{Gal_L}(p)$, respectively. Moreover, $H_1(p) \cong \mathrm{Gal_L}(p)/(\mathrm{Gal_L}(p))'$ and hence it is a compact connected (so divisible) abelian group.

Definition 1.3. Let $r(x, y), s(x, y)$ be types completing $p(x) \wedge p(y) \wedge x \mathop{\smile\!\!\!\!\!\!\;\;|}\, y$.

(1) We say p has *abelian* (or *commutative*) *amalgamation of* r *and* s, if there are independent $a, b, c, d \models p$ such that $ab, cd \models r$ and $ac, bd \models s$.

We say p has *abelian amalgamation* if so has it for any such completions r and s.

(2) We say p has *reversible amalgamation of* r *and* s if there are independent $a, b, c, d \models p$ such that $ab, dc \models r$ and $ac, db \models s$.

We say p has *reversible amalgamation* if so has it for any such completions r and s.

Lemma 1.4. *The following are equivalent.*

(1) The type p has abelian amalgamation.

(2) Let $r(x, y), s(x, y)$ be any types completing $p(x) \wedge p(y) \wedge x \mathop{\smile\!\!\!\!\!\!\;\;|}\, y$, and let $a, b, c \models p$ be independent such that $ac \models s$ and $ab \models r$. Then there is $d \models p$ independent from abc such that $cd \models r$ and $bd \models s$.

Proof. (1)⇒(2) By (1), there are $b_0, d_0 \models p$ such that $\{a, c, b_0, d_0\}$ is independent, $ab_0, cd_0 \models r$ and $ac, b_0d_0 \models s$. Hence, there is d_1 such that $abd_1 \equiv ab_0d_0$. Now by 3-amalgamation over the model a, there is

$$d \models \mathrm{tp}(d_0/a; c) \cup \mathrm{tp}(d_1/a; b),$$

such that $\{a,b,c,d\}$ is independent. Moreover, $cd \equiv cd_0 \models r$ and $bd \equiv bd_1 \equiv b_0d_0 \models s$, as desired.

(2)⇒(1) Clear. □

By the same proof we obtain the following too.

Lemma 1.5. *The following are equivalent.*

1. *The type p has reversible amalgamation.*
2. *Let $r(x,y), s(x,y)$ be any types completing $p(x) \wedge p(y) \wedge x \mathop{\smile\!\!\!\!\!\mid}\, y$, and let $a,b,c \models p$ be independent such that $ac \models s$ and $ab \models r$. Then there is $d \models p$ independent from abc such that $dc \models r$ and $db \models s$.*

Theorem 1.6. *The following are equivalent.*

1. $\mathrm{Gal_L}(p)(\cong \mathrm{Gal_L}(T))$ *is abelian.*
2. *p has abelian amalgamation.*

Proof. (1)⇒(2) Assume (1). Thus for $G := \mathrm{Aut}(p)$, we have $G' \leq \mathrm{Autf}(p) \quad (*)$. Now let $r(x,y)$ and $s(x,y)$ be some complete types containing $p(x) \wedge p(y) \wedge x \mathop{\smile\!\!\!\!\!\mid}\, y$. There are independent $a, a', b, d, c \models p$ such that $ab, cd \models r$ and $bd, a'c \models s$. Hence there is a commutator f in G' such that $f(a) = a'$. Hence by (*), $a \equiv^L a'$. Then by 3-amalgamation of $\mathrm{Lstp}(a)$, there is

$$a_0 \models \mathrm{tp}(a/bd) \cup \mathrm{tp}(a'/c)$$

such that $\{a_0,b,c,d\}$ independent; $a_0b \equiv ab \models r$; and $a_0c \equiv a'c \models s$. Therefore p has commutative amalgamation.

(2)⇒(1) To show (1), due to Fact 0.14(2) it is enough to prove that if $a, a' \models p$ and $[a,a'] = 0 \in H_1(p) \quad (\dagger)$, then $a \equiv^L a'$. Now assume (2) and (†). Thus, by Proposition 1.2, there are $b,c,d \models p$ such that each of the sets $\{a,b,c,d\}$ and $\{a',b,c,d\}$ is independent, $ab \equiv cd$ and $bd \equiv a'c$. Now by Lemma 1.4 and extension there is $c' \models p$ such that $c' \mathop{\smile\!\!\!\!\!\mid}\, aa'cd$ and $ac' \equiv bd \equiv a'c$ and $c'd \equiv ab \equiv cd$. Then, since d is a model, we have $c \equiv^L c'$, and there is a'' such that $a'c \equiv_d a''c'$, so $a \equiv^L a''$ and $a'' \equiv_{c'} a$. Since c' is a model as well, we conclude that $a' \equiv^L a'' \equiv^L a$. □

A proof similar to that of the above Theorem 1.6 (2)⇒(1) applied to Fact 0.14(1)(b) gives the following as well.

Proposition 1.7. *If p has reversible amalgamation, then* $\mathrm{Gal_L}(p)$ *is abelian (equivalently, p has abelian amalgamation).*

Proof. Assume that p has reversible amalgamation, and $[a,b] = 0 \in H_1(p)$ for $a,b \models p$ (†). As before it is enough to prove that $a \equiv^L b$.

It follows from the extension axiom for Lascar types together with Fact 0.14(1)(b), there are $b' \equiv^L b$ and a finite *independent* sequence $(d_i)_{0\leq i\leq 2n+2}$ of realizations of p satisfying the following conditions:

(i) $d_0 = a$, $d_{2n+2} = b'$; and
(ii) there is a bijection

$$\sigma : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$$

such that $d_{2i}d_{2i+1} \equiv d_{2\sigma(i)+2}d_{2\sigma(i)+1}$ for $i \leq n$.

In other words, there is an independent balanced chain-walk from a to b', and it suffices to prove $a \equiv^L b'$. Notice that reversible amalgamation implies Lemma 1.5(2) which exactly means that in above chain-walk two adjacent edges can be swapped with sign reversed (i.e., for say d_jd_{j+1} and $d_{j+1}d_{j+2}$ there is $d'_{j+1}(\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.3ex\hbox{$\mid$}}\, d_jd_{j+2})$ such that $d_jd'_{j+1} \equiv d_{j+2}d_{j+1}$ and $d'_{j+1}d_{j+2} \equiv d_{j+1}d_j$). By iterating this process one can transfer the chain-walk to another balanced chain-walk $(d'_i)_{0\leq i\leq 2n+2}$ from a to b' such that the walk has a Lascar pattern, i.e., the bijection σ for the new walk is the identity map. Hence it follows $d'_{2i} \equiv_{d'_{2i+1}} d'_{2i+2}$ for $i \leq n$, and since each d'_{2i+1} is a model, we have $a = d'_0 \equiv^L b'$, as wanted. □

A stronger consequence is obtained.

Theorem 1.8. *The strong type p is a Lascar type iff p has reversible amalgamation.*

Proof. (⇒) It follows from 3-amalgamation of Lstp.

(⇐) Assume p has reversible amalgamation. Due to Fact 0.14(3) and Proposition 1.7, it suffices to show $H_1(p)$ is trivial. Let an arbitrary $[s] \in H_1(p)$ be given, and let $[a, b'] = [s]$ for $a, b' \models p$. Then, for any $b \equiv^L b'$ with $b \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.3ex\hbox{$\mid$}} ab'$, we have $[a, b] = [a, b'] + [b', b] = [s] + 0 = [s]$. Similarly, for $c \equiv^L b$ with $c \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.3ex\hbox{$\mid$}} ba$, we have $[b, c] = 0$ in $H_1(p)$. Now reversible amalgamation (or Proposition 1.5) says that $[s] = [a, b] + [b, c] = [c, b] + [b, a] = -[s]$. Hence $[s] + [s] = 0$. Since $H_1(p)$ is compact and connected so divisible, any element in $H_1(p)$ is divisible by 2. Therefore, $H_1(p) = 0$ by what we have just proved. □

Question 1.9. Can the same results hold if p is the type of an algebraically closed tuple (not necessarily a model) in a simple theory? The answer to this question is yes if any two Lascar equivalence classes in p are interdefinable, since essentially this property (Remark 0.15(1)) implied the results in this section when p is the type of a model. At least we can show the following remark.

Remark 1.10. Let T be any theory, and let a realization of p be any tuple. If $\mathrm{Gal}^1_{\mathrm{L}}(p)$ is abelian then any two Lascar equivalence classes in

p are interdefinable: Let $a, b \models p$, and $f \in \mathrm{Aut}(p)$. Assume $f(a) \equiv^L a$. We want to show the same holds for b. Now there is $g \in \mathrm{Aut}(p)$ such that $g(a) = b$. Since $\mathrm{Gal}^1_{\mathrm{L}}(p)$ is abelian, we have $f(b) = f(g(a)) \equiv^L g(f(a)) \equiv^L g(a) = b$.

Notice that $\mathrm{Gal}^1_{\mathrm{L}}(p)$ is the group of automorphic permutations of the Lascar classes in p. Hence, if $\mathrm{Gal}^1_{\mathrm{L}}(p)$ is abelian, then $f/\mathrm{Autf}^1(p) \in \mathrm{Gal}^1_{\mathrm{L}}(p)$ is determined by the pair of Lascar classes of c and $f(c)$ for some (any) $c \models p$.

Example 1.11. Let $(\mathcal{M}, <)$ be a monster model of $\mathrm{Th}(\mathbb{Q}, <)$. Then thorn-independence $\mathop{\smile\!\!\!\!\!\!\,\mid}^{*}$ in $\mathcal{M}$ coincides with acl-independence. We can also consider a notion of reversible amalgamation using thorn independence instead of nonforking independence. Note that the Lascar group of $\mathrm{Th}(\mathcal{M})$ is trivial because of o-minimality, but the reversible amalgamation for $\mathop{\smile\!\!\!\!\!\!\,\mid}^{*}$ fails.

Let p be the unique 1-type over $\emptyset(= \mathrm{acl}(\emptyset))$. Consider two types $r(x, y) = \{x < y\}$ and $s(x, y) = \{y < x\}$ completing $p(x) \wedge p(y) \wedge x \mathop{\smile\!\!\!\!\!\!\,\mid}^{*} y$. Then p has no reversible amalgamation of r and s. Suppose $a, b, c, d \models p$ such that $ab, dc \models r$ and $ac, db \models s$. Then $a < b < d < c < a$, and there are no such elements. Thus, a type of a model does not have reversible amalgamation either.

2. Relativized Lascar Galois groups and algebraicity

Recall that throughout we assume that $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$. In Remark 1.8 from [2] it was noticed that if $\mathrm{tp}(a)$ is a Lascar type, and a finite tuple c is algebraic over a then $\mathrm{tp}(ac)$ is also a Lascar type, and hence if additionally T is G-compact then $\mathrm{tp}(\mathrm{acl}(a))$ is also a Lascar type. In other words, if $\mathrm{Gal}^1_{\mathrm{L}}(\mathrm{tp}(a))$ is trivial, then $\mathrm{Gal}^1_{\mathrm{L}}(\mathrm{tp}(\mathrm{acl}(a))$ is trivial when T is G-compact. It seems natural to ask whether, more generally, $\mathrm{Gal}^1_{\mathrm{L}}(\mathrm{tp}(a))$ and $\mathrm{Gal}^1_{\mathrm{L}}(\mathrm{tp}(\mathrm{acl}(a))$ must be always isomorphic in a G-compact T. In general, the answer turns out to be negative (see Section 3), but we obtain some positive results if, instead of looking at $\mathrm{acl}(a)$, we add only a finite part of it to a.

For the whole Section 2, we fix the following notation.

Notation 2.1. Assume that $p = \mathrm{tp}(a)$ and $\bar{p} = \mathrm{tp}(ac)$, where $c \subseteq \mathrm{acl}(a)$ is finite. Consider the natural projection $\pi_\lambda : \mathrm{Gal}^\lambda_{\mathrm{L}}(\bar{p}) \to \mathrm{Gal}^\lambda_{\mathrm{L}}(p)$, and put $\pi := \pi_1$. We denote the kernel of π by K. Denote by E the relation of being Lascar-equivalent (formally, E depends on the length of tuples on which we consider it). Let $ac_1, \dots, ac_n$ be a tuple of representatives of all E-classes in $\bar{p}$ in which the first coordinate is equal to

a (so $n \leq$ the number of realizations of $\mathrm{tp}(c/a)$), and for any $a' \models p$, let $a'c_1^{a'}, \ldots, a'c_n^{a'}$ be its conjugate by an automorphism sending a to a'.

Example 2.2. We give an example of a structure and a type p such that $\mathrm{Gal}_{\mathrm{L}}^1(p)$ and $\mathrm{Gal}_{\mathrm{L}}^2(p)$ are distinct. Let P be a Euclidean plane, say $\mathbb{R}\times\mathbb{R}$. For $n > 0$, define $R_n(xy, zw)$ on P such that $R_n(ab, cd)$ iff $a \neq b$, $c \neq d$, and either lines $L(ab)$ containing a, b and $L(cd)$ are parallel, or the smaller angle between $L(ab)$ and $L(cd)$ is $\leq \frac{\pi}{2n}$. Consider a model $M = (P; R_n(xy, zw))_{0<n}$. Then $F(xy, zw) := \bigwedge_{0<n} R_n(xy, zw)$ is an $\emptyset$-type-definable bounded equivalence relation in $T = \mathrm{Th}(M)$, and each class corresponds to a class of lines whose slopes are infinitesimally close. Let p be the unique complete 1-type over $\emptyset$. Indeed p is a Lascar type, so $\mathrm{Gal}_{\mathrm{L}}^1(p)$ is trivial. On the other hand, a rotation of (a saturated) P around a point in P belongs to $\mathrm{Autf}^1(p)$ but does not belong to $\mathrm{Autf}^2(p)$. One can show that $\mathrm{Gal}_{\mathrm{L}}^2(p)$ is a circle group.

Let us recall basic definitions and facts on covering maps between topological groups (see for example [5]).

Remark 2.3.

- Let X, Y be topological spaces, and let $f : X \to Y$ be a continuous surjective map. We call f a *(k-)covering map* if for each $y \in Y$ there is an open set V containing y such that $f^{-1}(V)$ is a union of (k-many) disjoint open sets in X, and f induces a homeomorphism between each such open set and V. We call f a *local homeomorphism* if for any $x \in X$, there is an open set U containing x such that $f \restriction U : U \to f(U)$ is a homeomorphism. Obviously a covering map is a local homeomorphism. Conversely, if both X, Y are compact, then a local homeomorphism is a covering map. We call X a *covering space* of Y if there is a covering map from X to Y.
- Let G be topological group, and let H be a normal subgroup of G such that G/H with its quotient topology is also a topological group, and the projection map $\mathrm{pr} : G \to G/H$ is open continuous homomorphism. Recall that pr is a covering map iff H is a discrete subgroup. If G is compact, then H is discrete iff H is finite, and hence iff pr is a covering map.

  Assume F is another topological group and $f : G \to F$ is a continuous surjective homomorphism. If G is compact and F is Hausdorff, then f induces an isomorphism between compact topological groups $G/\mathrm{Ker}(f)$ and F.
- Assume T is G-compact. By above, π_λ is a covering homomorphism iff the kernel of π_λ is finite. In particular, we shall show that $\pi = \pi_1$ is a covering homomorphism (Corollary 2.8).

If T is G-compact then the Lascar equivalence E is $\emptyset$-type-definable, and we can assume any formula φ in E is symmetric (i.e., $\varphi(\bar{z}, \bar{w}) \models \varphi(\bar{w}, \bar{z})$).

Lemma 2.4. *Assume T is G-compact. There is a formula $\alpha(xy, x'y') \in E(xy, x'y')$ such that if $\bar{p}(xy) \wedge \bar{p}(x'y') \wedge \alpha(xy, x'y') \wedge E(x, x')$ holds then $\models E(xy, x'y')$.*

Proof. Notice that the type

$$E(xy, x'y') \wedge \exists zwz'w' \equiv xyx'y'(\bigvee_{1 \leq i \neq j \leq n} (E(zw, ac_i) \wedge E(z'w', ac_j)))$$

is inconsistent, and choose $\alpha(xy, x'y') \in E(xy, x'y')$ so that the type

$$\alpha(xy, x'y') \wedge \exists zwz'w' \equiv xyx'y'(\bigvee_{1 \leq i \neq j \leq n} (E(zw, ac_i) \wedge E(z'w', ac_j)))$$

is inconsistent. Now, if $\bar{p}(xy) \wedge \bar{p}(x'y') \wedge \alpha(xy, x'y') \wedge E(x, x')$ holds, then we can find $c'a'c''$ such that $ac'a'c'' \equiv xyx'y'$, and then $a' \equiv^L a$ so $a'c'' \equiv^L ac_i$ for some i, and also $ac' \equiv^L ac_j$ for some j, but, by the choice of α, $i = j$ so $ac' \equiv^L a'c''$, so $xy \equiv^L x'y'$. ☐

Theorem 2.5. *If T is G-compact, then K is finite.*

Proof. Assume T is G-compact. Since E is transitive, there is $\phi(xy, x'y') \in E(xy, x'y')$ such that

$$\phi(xy, zw) \wedge \phi(x'y', zw) \vdash \alpha(xy, x'y') \quad (*),$$

where $\alpha \in E$ is the formula given by Lemma 2.4.

Let $(a_\ell)_{\ell \in I}$ be a small set of representatives of E-classes of realizations of p. For any $a'c' \models \bar{p}$, there are $\ell \in I$ and $1 \leq j \leq n$ such that $\models E(a'c', a_\ell c_j^{a_\ell})$, so $\models \phi(a'c', a_\ell c_j^{a_\ell})$. Hence, by compactness, there are $a_0, \ldots, a_k \models p$ such that

$$\bar{p}(xy) \vdash \bigvee\{\phi(xy, a_i c_j^{a_i}) \mid i \leq k;\ 1 \leq j \leq n\} \quad (**).$$

Claim 2.6. *Let $f/\operatorname{Autf}^1(\bar{p}) \in K$. For each $a_i c_j^{a_i}$ chosen above, there is a unique $a_i c_{j'}^{a_i}$ with $1 \leq j' \leq n$ such that $\phi(f(a_i c_j^{a_i}), a_i c_{j'}^{a_i})$ holds. Such a j' does not depend on the choice of a representative of $f/\operatorname{Autf}^1(\bar{p})$.*

Proof. Notice that $f/\operatorname{Autf}^1(\bar{p}) \in K$ implies $E(f(a_i), a_i)$, so $E(f(a_i c_j^{a_i}), a_i c_{j'}^{a_i})$ holds for some j'. Now if $\phi(f(a_i c_j^{a_i}), a_i c_{j''}^{a_i})$ holds as well, then due to $(*)$ (with the symmetry of the formulas) and Lemma 2.4, we must have $j' = j''$. The second statement of the claim follows similarly. ☐

Claim 2.7. *Let $f/\operatorname{Autf}^1(\bar{p})$, $g/\operatorname{Autf}^1(\bar{p}) \in K$. Assume that the permutations of tuples $a_i c_j^{a_i}$ by f and g described in above claim are the same. Then $f/\operatorname{Autf}^1(\bar{p}) = g/\operatorname{Autf}^1(\bar{p})$:*

Proof. Let $a'c' \models \bar{p}$. By $(**)$, there is some $a_i c_j^{a_i}$ such that $\phi(a'c', a_i c_j^{a_i})$ holds. Hence, $\phi(f(a'c'), f(a_i c_j^{a_i}))$ and $\phi(g(a'c'), g(a_i c_j^{a_i}))$ hold. Moreover, by the previous claim with our assumption, there is $j' \leq n$ such that $\phi(f(a_i c_j^{a_i}), a_i c_{j'}^{a_i})$ and $\phi(g(a_i c_j^{a_i}), a_i c_{j'}^{a_i})$ hold.

Then, again due to $(*)$ and Lemma 2.4, $f(a_i c_j^{a_i}) \equiv^L g(a_i c_j^{a_i})$ since $f(a_i) \equiv^L g(a_i) \equiv^L a_i$. Hence, there is $h \in \operatorname{Autf}^1(\bar{p})$ such that $f(a_i c_j^{a_i}) = hg(a_i c_j^{a_i})$. Now by $(*)$ again, $\alpha(f(a'c'), hg(a'c'))$ holds, and, since $f(a') \equiv^L hg(a') \equiv^L a'$, we have $f(a'c') \equiv^L hg(a'c') \equiv^L g(a'c')$. We conclude that $f/\operatorname{Autf}^1(\bar{p}) = g/\operatorname{Autf}^1(\bar{p})$. □

As there are only finitely many permutations of $a_i c_j^{a_i}$ ($i \leq k$, $1 \leq j \leq n$), K is finite. □

By Remark 2.3, we have the following.

Corollary 2.8. *If T is G-compact then $\pi : \operatorname{Gal}^1_{\mathrm{L}}(\bar{p}) \to \operatorname{Gal}^1_{\mathrm{L}}(p)$ is a covering homomorphism.*

Example 2.9. Consider the following model

$$M = ((M_1, S_1, \{g_n^1 \mid 0 < n\}), (M_2, S_2, \{g_n^2 \mid 0 < n\}), \delta),$$

a 2-sorted structure. Here M_1, M_2 are disjoint unit circles. For $i = 1, 2$, S_i is a ternary relation on M_i such that $S_i(a, b, c)$ holds iff a, b, c are distinct and b comes before c going clockwise around M_i from a; and g_n^i is the clockwise rotation of M_i by $\frac{2\pi}{n}$-radians. The map $\delta : M_1 \to M_2$ is a double covering, i.e. if we identify each M_i as the unit circle in xy-plane centered at 0, then δ is given by $(\cos t, \sin t) \mapsto (\cos 2t, \sin 2t)$. By arguments similar to those described in [2] for M_i, it follows that $T = \operatorname{Th}(M)$ is G-compact, and, in T, $\emptyset = \operatorname{acl}^{\mathrm{eq}}(\emptyset)$. Let $\mathcal{M}_1, \mathcal{M}_2$ be saturated models of M_1, M_2 respectively. For any $a_i, a_i' \in \mathcal{M}_i$, we have $a_i \equiv a_i'$; and $a_i \equiv^L a_i'$ iff they are infinitesimally close. Now, given $a \in M_2$, there are two antipodal $c_1, c_2 \in M_1$ such that $\delta(c_i) = a$. Then $c_i \in \operatorname{acl}(a)$ and $ac_1 \equiv ac_2$, but $ac_1 \not\equiv^L ac_2$. For $p = \operatorname{tp}(a)$ and $\bar{p} = \operatorname{tp}(ac_1)$, $\pi : \operatorname{Gal}^1_{\mathrm{L}}(\bar{p}) \to \operatorname{Gal}^1_{\mathrm{L}}(p)$ is a 2-covering homomorphism of circle groups. Notice that for any $n > 2$,

$$\delta(y) = x \wedge \delta(y') = x' \wedge (S_1(y, y', g_n^1(y)) \vee S_1(y', y, g_n^1(y')))$$

serves as the formula $\alpha(xy, x'y')$ in Lemma 2.4.

Question 2.10. If T is G-compact, then is the kernel of the projection $\pi_\lambda : \mathrm{Gal}^\lambda_{\mathrm{L}}(\bar{p}) \to \mathrm{Gal}^\lambda_{\mathrm{L}}(p)$ finite as well for $\lambda > 1$? Is T being G-compact essential in Theorem 2.5?

We have a partial answer of the second question above. Namely, we also get that K is finite when we replace the assumption of G-compactness by abelianness of $\mathrm{Gal}^1_{\mathrm{L}}(\bar{p})$.

Proposition 2.11. *If* $\mathrm{Gal}^1_{\mathrm{L}}(\bar{p})$ *is abelian, then* $|K| = n$.

Proof. If $f/\operatorname{Autf}^1(\bar{p}) \in K$, then $f(a) \equiv^L a$, so there is another representative f' fixing a. Now, since $\mathrm{Gal}^1_{\mathrm{L}}(\bar{p})$ is abelian, by Remark 1.10, the class of an automorphism f' fixing a depends only on the number of Lascar types of $f'(ac) = af'(c)$ for which, due to our choice, there are exactly n possibilities. □

We investigate the epimorphism π further when there are nicer conditions with G-compact T, and we will obtain a better version of Theorem 2.5. Recall some known facts first.

Remark 2.12.

(1) Let X be an $\emptyset$-type-definable set, and let F be a bounded $\emptyset$-type-definable equivalence relation on X. Recall the *logic topology* on X/F: A subset of X/F is closed if and only if its pre-image in X is type-definable over some parameters. It follows that X/F is compact with the logic topology.

(2) Assume T is G-compact. By the natural embedding, $G := \mathrm{Gal}^1_{\mathrm{L}}(p)$ can be considered as a subgroup of $\mathrm{Homeo}(p(\mathcal{M})/E)$, where $p(\mathcal{M})/E$ is equipped with the logic topology. Moreover, for $x := a/E$ and $G_x := \{g \in G \mid g.x = x\}$ (the stabilizer subgroup of G at x, we have that G/G_x as a homogeneous coset space with the quotient topology and $p(\mathcal{M})/E$ with the logic topology are homeomorphic. Needless to say, analogous facts hold for $\bar{p}$.

(3) Let X, Y be topological spaces. Recall that if X is path-connected then so is any quotient space of X. Moreover, if X is a connected covering space of Y, then X is path-connected iff so is Y.

(4) Given a covering map $\delta : X \to Y$, the *unique path-lifting property* says that for any $x_0 \in X$, $y_0 \in Y$ with $\delta(x_0) = y_0$, and any path γ in Y starting at y_0 (i.e. $\gamma : [0,1] \to Y$ is continuous and $\gamma(0) = y_0$), there is a unique path γ' starting at x_0 such that $\delta \circ \gamma' = \gamma$.

Proposition 2.13. *Assume T is G-compact and consider the canonical restriction map $\delta : \bar{p}(\mathcal{M})/E \to p(\mathcal{M})/E$ of compact spaces $\bar{p}(\mathcal{M})/E$, $p(\mathcal{M})/E$ equipped with the logic topology. Then δ is an n-covering map.*

Proof. Choose ϕ as in the proof of Theorem 2.5 and fix $a'c' \models \bar{p}$. Define $D := \bar{p}(\mathcal{M})\backslash\{a''c''/E : a''c'' \models \bar{p} \ \wedge \neg\phi(a''c'', a'c')\}$. Then D is an open neighborhood of $a'c'/E$ in $\bar{p}(\mathcal{M})/E$. To show that δ is a covering map, by Remark 2.3, it is enough to see that δ is injective on D. So choose two pairs $a^0c^0, a^1c^1 \models \bar{p}$ such that $a^0c^0/E, a^1c^1/E \in D$ and $\delta(a^0c^0/E) = \delta(a^1c^1/E)$. Then, by the first condition, $\models \phi(a^0c^0, a'c') \wedge \phi(a^1c^1, a'c')$, and, by the second one, $E(a^0, a^1)$. We conclude by the choice of ϕ that $a^0c^0/E = a^1c^1/E$. That δ is n-covering is clear due to our choice of n. □

Observe the following remark on covering maps.

Remark 2.14. Given topological spaces X, Y, suppose that $\delta : X \to Y$ is a k-covering. Put $F := \{f \in \mathrm{Homeo}(X) : \delta \circ f = \delta\}$.

(1) If Y is path-connected, then $|F| \leq k!$ and this bound is optimal.
(2) If X is path-connected (thus so is Y), then $|F| \leq k$.

Proof. We prove (1) first. It is enough to show that if $f(\in F)$ fixes some fiber $\delta^{-1}(y)$ with $y \in Y$ pointwise (*), then f is identity. Fix $x' \in X$ and let $y' = \delta(x')$. Now there is a path α from y to y'. Then, by the unique path-lifting property, there is the unique path β starting at x, say, and ending at x' such that $\delta \circ \beta = \alpha$. Hence, $\delta(x) = y$. Now, since $f \in F$, we have $\delta \circ (f \circ \beta) = \delta \circ \beta = \alpha$ too. Moreover due to (*), $f(x) = x$ and by the uniqueness, we have $f \circ \beta = \beta$, so $f(x') = x'$. Therefore f is the identity map as desired.

To see that the bound is optimal, consider $X = \bigcup_{1 \leq i \leq k}(i-1, i)$ and $Y = (0, 1)$. Define a covering map $\delta : X \to Y, x \mapsto x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer which is less than or equal to x. Then, each permutation on $\{1, \dots, k\}$ induces a homeomorphism of X fixing each fibers of δ and in this case, $|F| = k!$.

For (2), it is enough to show that if $f \in F$ fixes a point x in X, then f is the identity. Suppose $f \in F$ fixes $x \in X$. If X is path-connected, then for each $x' \in X$, there is a unique path β from x to x' due to the unique path-lifting property, so f is the identity by a similar argument as in the proof of (1). □

By Remarks 2.12(2)(3), 2.14 and Proposition 2.13, we immediately get the following:

Corollary 2.15. *Assume T is G-compact. If $\bar{p}(\mathcal{M})/E$ or $p(\mathcal{M})/E$ is path-connected (each of which holds if $\mathrm{Gal}^1_{\mathrm{L}}(p)$ is path-connected, for example a Lie group), then $|K| = n$.*

Now we show a purely compact abelian group theoretical result, which implies Corollary 2.18.

Lemma 2.16. *Let G be an abelian compact connected topological group, and let F be its finite subgroup. Then G and G/F are isomorphic as topological groups.*

Proof. Recall that given a finite group F and a prime number q dividing the order of F, there is a subgroup of F of order q. Hence, we can assume that F has a prime order q (applying the prime order case and quotienting out finitely many times to obtain the conclusion for any finite F).

We can present G as $\varprojlim_I G_i$ with some directed system $(I, \leq)$ and continuous homomorphisms $f_{i,j} : G_i \to G_j$ for $j \leq i \in I$, where each G_i is an abelian connected Lie group, hence a torus. We can assume each $f_{i,j}$ is surjective. Since F has a prime order, it is generated by a single element, say $a = (a_i)_{i \in I}$. Also, replacing I by $I_{i_0} := \{i \in I |\ i_0 \leq i\}$ for an appropriate i_0, we can assume that I has the least element i_0, and that each F_i (i.e., the projection of F onto the i-th coordinate) has order q. We have that

$$G/F = \varprojlim_I H_i,$$

where $H_i = G_i/F_i$, with maps $k_{i,j} : H_i \to H_j$ induced by $f_{i,j}$. Moreover, for each $i \in I$, we let S_i be the unique one-dimensional subtorus of G_i containing a_i.

Claim 2.17. *We can find subtori $T_i < G_i$, $i \in I$, such that:*

(1) G_i is the direct sum of T_i and S_i (in other words, T_i intersects S_i trivially, and $G_i = T_i + S_i$), and

(2) $f_{i,j}[T_i] \subseteq T_j$ for each $j \leq i$.

Proof. For $i = i_0$, we choose any torus T_{i_0} such that $G_{i_0} = T_{i_0} \oplus S_{i_0}$ (we can do this just because S_{i_0} is a subtorus of G_{i_0}).

Now, take any other $i \in I$. Put $V_i = f^{-1}_{i,i_0}[T_{i_0}]$. Since both a_i and a_{i_0} have order q and $f_{i,i_0}(a_i) = a_{i_0}$, the map f_{i,i_0} maps S_i isomorphically onto S_{i_0}. Hence, $S_i \cap V_i = \{e\}$ (if x was a nontrivial element in the intersection, then $f_{i,i_0}(x)$ would be a nontrivial element in $T_{i_0} \cap S_{i_0}$). Also, for any $x \in G_i$, there are $y \in T_{i_0}$ and $z \in S_{i_0}$ such that $f_{i,i_0}(x) = y + z$, so choosing $w \in S_i$ such that $f_{i,i_0}(w) = z$, we get that $f_{i,i_0}(x - w) = y \in T_{i_0}$ and $x \in S_i + V_i$. Hence, G_i is the direct sum of S_i and V_i. Note that V_i is a closed (so compact) subgroup of G_i.

Now let T_i be the identity component of the group V_i. Then T_i is a torus of the same dimension as V_i intersecting S_i trivially, so $\dim(T_i + S_i) = \dim(G_i)$. Hence, by the connectedness of G_i, we get that $G_i = T_i \oplus S_i$.

To see that $f_{i,j}[T_i] \subseteq T_j$ for $j \leq i$, notice first that $f_{i,j}[T_i] \subseteq V_j$. Now, since $f_{i,j}$ is continuous and T_i is connected, $f_{i,j}[T_i]$ is a connected subgroup of G_j, so it must be contained in the connected component T_j of V_j. This gives the claim. □

Now we will define isomorphisms g_i from H_i to G_i such that, for $j < i$,

$$f_{i,j} g_i = g_j k_{i,j} \quad (*).$$

(Recall that $k_{i,j} : H_i \to H_j$ is the map induced by $f_{i,j}$.) Clearly, we can write $H_i = G_i/F_i$ as $T_i \oplus (S_i/\langle a_i \rangle)$. We define g_i to be the identity map on T_i, and to be equal to the map $\alpha_q : S_i/\langle a_i \rangle \to S_i$ induced by multiplication by q of representatives modulo $\langle a_i \rangle$ on $S_i/\langle a_i \rangle$, and extend additively to a map from $H_i = T_i \oplus (S_i/\langle a_i \rangle)$ to $T_i \oplus S_i = G_i$. To check $(*)$, notice that, for $x \in T_i$, both $f_{i,j} g_i(x)$ and $g_j k_{i,j}(x)$ are equal to $f_{i,j}(x)$, and for $y/\langle a_i \rangle \in S_i/\langle a_i \rangle$, we have that $f_{i,j} g_i(y/\langle a_i \rangle)$ and $g_j k_{i,j}(y/\langle a_i \rangle)$ are both equal to $q f_{i,j}(y)/\mathbb{Z}$, where $S_j = (\mathbb{R}, +)/\mathbb{Z}$.

Now, the system of isomorphisms g_i induces an isomorphism of topological groups G and G/F. □

Corollary 2.18. *Suppose T is G-compact, and $\mathrm{Gal}^1_{\mathrm{L}}(\bar{p})$ is abelian. Then, $\mathrm{Gal}^1_{\mathrm{L}}(\bar{p})$ and $\mathrm{Gal}^1_{\mathrm{L}}(p)$ are isomorphic as topological groups.*

Proof. By Proposition 2.11 or Theorem 2.5, K is finite, hence, by Lemma 2.16, the compact connected group $\mathrm{Gal}^1_{\mathrm{L}}(p)$ isomorphic to $\mathrm{Gal}^1_{\mathrm{L}}(\bar{p})/K$, must be isomorphic to $\mathrm{Gal}^1_{\mathrm{L}}(\bar{p})$ as well. □

Finally, we observe that Corollary 2.18 does not generalize to the non-abelian case.

Proposition 2.19. *Let G be a connected compact Lie group, and N its finite normal subgroup. Then, one can find types $p = \mathrm{tp}(a)$ and $\bar{p} = \mathrm{tp}(ac)$ (with $c \in \mathrm{acl}(a)$ finite) in some G-compact theory T with $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$ (in T^{eq}) so that $G = \mathrm{Gal}^1_{\mathrm{L}}(\bar{p})$ and $G/N = \mathrm{Gal}^1_{\mathrm{L}}(p)$. The same holds for the groups $\mathrm{Gal}^{\mathrm{res}}_{\mathrm{L}}$ and $\mathrm{Gal}^{\lambda}_{\mathrm{L}}$ for any λ.*

If we take $G = SO(4, \mathbb{R})$, $N = Z(G) = \{I, -I\}$, then $G/N = PSO(4, \mathbb{R})$ and $Z(G/N)$ is trivial, so G and G/N are not isomorphic.[2]

[2]This example is pointed out to us by Prof. Sang-hyun Kim from Seoul National University.

Proof. The Lie group G is definable in some o-minimal expansion $\mathcal{R}$ of the ordered field of real numbers, and (using elimination of imaginaries for o-minimal expansions of $\mathbb{R}$) we identify $\mathcal{R}$ with $\mathcal{R}^{eq}$. Put $H := G/N$. We can assume that $N \subseteq \mathrm{dcl}(\emptyset) = \mathrm{acl}(\emptyset)$, so H is 0-definable. We consider a structure $\mathcal{M} = (\mathcal{R}, X, Y)$, where $\mathcal{R}$ comes with its original structure, X and Y are sets equipped with a regular G-action and a regular G/N-action, respectively (both denoted by $\cdot$), and we add to the language the map $\pi : X \to Y$, defined as follows: fix any elements $x_0 \in X$ and $y_0 \in Y$, and, for any $g \in G$, we set $\pi(g \cdot x_0) := gN \cdot y_0$. Put $T = \mathrm{Th}(\mathcal{M})$.

Note that $\mathcal{M} = (\mathcal{R}, X, Y)$ is 0-interpretable in the structure $(\mathcal{R}, X)$, so every automorphism of $(\mathcal{R}, X)$ extends to an automorphism of $\mathcal{M}$. Moreover, this extension is unique, as $Y \subseteq \mathrm{dcl}(X)$. The same holds for the saturated extension $\mathcal{M}^* = (\mathcal{R}^*, X^*, Y^*)$, hence, by [9, Chapter 7], every automorphism of $\mathcal{M}^*$ is of the form $\bar{g}\bar{\phi}$, where $g \in G^*$ and $\phi \in \mathrm{Aut}(\mathcal{R}^*)$ (using the notation from [9], and identifying an automorphism of $(\mathcal{R}^*, X^*)$ with its unique extension to $\mathcal{M}^*$), and $\mathrm{Gal}_{\mathrm{L}}(T) = G$.

Let q and p be the unique (strong) types of the sorts X^* and Y^*, respectively. It follows from [9] that an automorphism $\bar{g}\bar{\phi}$ is Lascar-strong iff $g \in G^*_{\mathrm{inf}}$ (the group of infinitesimals of G^*). It follows that two elements $x, y \in X^*$ have the same Lascar type iff there is $g \in G^*_{\mathrm{inf}}$ such that $g \cdot x = y$, and two elements $w, u \in Y$ have the same Lascar type iff there is $g \in G^*_{\mathrm{inf}}$ and $n \in N$ such that $(gn) \cdot w = u$. As all automorphism of the form $\bar{\phi}$ move any $x \in X^*$ to its translate by some element of G^*_{inf}, we conclude easily that

$$\mathrm{Autf}^1(q) = \mathrm{Autf}^{\mathrm{res}}(q) = \{\bar{g}\bar{\phi} : g \in G^*_{\mathrm{inf}}, \phi \in \mathrm{Aut}(R^*)\}$$

and

$$\mathrm{Autf}^1(p) = \mathrm{Autf}^{\mathrm{res}}(p) = \{\bar{g}\bar{\phi} : g \in G^*_{\mathrm{inf}}N, \phi \in \mathrm{Aut}(R^*)\}.$$

Hence, the map

$$g \mapsto [\bar{g}_{|X^*}]$$

is an isomorphism $G \to \mathrm{Gal}^1_{\mathrm{L}}(q) = \mathrm{Gal}^{\mathrm{res}}_{\mathrm{L}}(q)$, and the map

$$g \mapsto [\bar{g}_{|Y^*}]$$

is an epimorphism $G \to \mathrm{Gal}^1_{\mathrm{L}}(p) = \mathrm{Gal}^{\mathrm{res}}_{\mathrm{L}}(p)$ with kernel N, so $\mathrm{Gal}^1_{\mathrm{L}}(p) = \mathrm{Gal}^{\mathrm{res}}_{\mathrm{L}}(p) = G/N = H$.

Now, if we take $c \models q$ and $a = \pi(c) \models p$ and $\bar{p} = \mathrm{tp}(ac)$, then $c \in \mathrm{acl}(a)$ (as the fibers of π are finite), but the Lascar-Galois groups of p are isomorphic to H, and we see (as ac is interdefinable with c) that those of $\bar{p}$ are isomorphic to G.

Finally, since $\mathrm{Gal}_{\mathrm{L}}(T) = G$ is connected, its connected component $\mathrm{Gal}^0_{\mathrm{L}}(T)$ is the same as G, so we get by Theorem 21 from [9] that all algebraic imaginaries are fixed by all automorphisms over $\emptyset$, hence $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$. □

We will see in Theorem 3.7 that, in a non-G-compact T, some $\mathrm{tp}(\mathrm{acl}(a)/\mathrm{acl}^{\mathrm{eq}}(\emptyset))$ fails to be a Lascar type while $\mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(\emptyset))$ is a Lascar type. We will also see in Proposition 3.10 that, even in a G-compact theory, $\mathrm{Gal}^1_{\mathrm{L}}(\mathrm{tp}(\mathrm{acl}(a)/\mathrm{acl}^{\mathrm{eq}}(\emptyset))$ may be abelian but not isomorphic to $\mathrm{Gal}^1_{\mathrm{L}}(\mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(\emptyset))$.

3. Examples

As pointed out in the beginning of Section 2, in [2], a motivating result was observed that in G-compact T if $\mathrm{stp}(a)$ is a Lascar type, then so is $\mathrm{stp}(\mathrm{acl}(a))$. In this section we give an example showing that T being G-compact is essential in the result. Moreover, in regards to Corollary 2.18, it is natural to ask whether the premise conditions are essential. In Example 2.19, we have already seen that the abelianness assumption cannot be removed, and we present in this section an example showing that the assumption that c is finite cannot be removed either. Lastly, we find a Lie group structure example answering a question raised in [7], which asks the existence of an RN-pattern minimal 2-chain not equivalent to a Lascar pattern 2-chain having the same boundary.

We now explain preliminary examples. Throughout this section we will use $\mathcal{M}, \mathcal{N}, \dots$ to denote some models which need not be saturated. For a positive integer n, consider a structure $\mathcal{M}_{1,n} = (M; S, g_n)$, where M is a unit circle; S is a ternary relation on M such that $S(a, b, c)$ holds iff a, b, c are distinct and b comes before c going around the circle clockwise starting at a; and $g_n = \sigma_{1/n}$, where σ_r is the clockwise rotation by $2\pi r$-radians.

For $p = \mathrm{tp}(a)$, we will denote $\mathrm{tp}(\mathrm{acl}(a))$ by $\bar{\bar{p}}$.

Fact 3.1. [1]

(1) $\mathrm{Th}(\mathcal{M}_{1,n})$ *has the unique 1-complete type* $p_n(x)$ *over* $\emptyset$*, which is isolated by the formula* $x = x$*.*

(2) $\mathrm{Th}(\mathcal{M}_{1,n})$ *is* $\aleph_0$*-categorical and has quantifier-elimination.*

(3) For any subset $A \subseteq M_n$*,* $\mathrm{acl}(A) = \mathrm{dcl}(A) = \bigcup_{0 \le i < n} g^i_n(A)$ *(in the home-sort), where* $g^i_n = \underbrace{g_n \circ \cdots \circ g_n}_{i \text{ times}}$*.*

(4) The unique 1-complete type p_n *is also a Lascar type.*

3.1. **Non G-compact theory with p Lascar type but $\bar{\bar{p}}$ not Lascar type.** We give an example of non G-compact theory which has a Lascar strong type $\mathrm{tp}(a)$ but $\mathrm{tp}(\mathrm{acl}(a))$ is not a Lascar strong type. Let $\mathcal{M} = (M_i, S_i, g_i, \pi_i)_{i\geq 1}$ be a multi-sorted structure where $\mathcal{M}_{1,i} = (M_i, S_i, g_i)$ (as introduced in Fact 3.1, but of course M_i and M_j are disjoint for $i \neq j$) and $\pi_i : M_i \to M_1$ sending $x \mapsto x^i$ for each $i \geq 1$ (if we identify each M_i with the unit circle in the complex plane). For each $n \geq 1$, let $\mathcal{M}_{\leq n} = (M_i, S_i, g_i, \pi_i)_{1\leq i\leq n}$. Let $T = \mathrm{Th}(\mathcal{M})$, $T_i = \mathrm{Th}(\mathcal{M}_{1,i})$, and $T_{\leq n} = \mathrm{Th}(\mathcal{M}_{\leq n})$.

Remark 3.2. Both T and $T_{\leq n}$'s are $\aleph_0$-categorical.

Remark 3.3. Let $\mathcal{N} = (N_i, \ldots)$ be a model of T. For $A \subseteq \mathcal{N}$, define $A_1 := \bigcup_{i\geq 1} \pi_i(\bigcup_{j\leq i} g_i^j(A \cap N_i))$, and $\mathrm{cl}(A) = \bigcup_{i\geq 1} \pi_i^{-1}[A_1]$.

(1) $\mathrm{cl}(A)$ is the smallest substructure containing A.
(2) For $B, C \subseteq \mathcal{N}$ algebraically closed, if $B_1 = C_1$, then $B = C$.

Fact 3.4. [2, Theorem 5.5] *Let T be $\aleph_0$-categorical and let $\mathcal{M} = (M, \ldots)$ be a saturated model of T. Suppose that if $X \subseteq M^1$ is definable over each of two algebraically closed sets A_0 and A_1, then X is definable over $B := A_0 \cap A_1$.*

Then, for any subset Y of M^n, if Y is both A_0-definable and A_1-definable, then it is B-definable. Furthermore, in this case, T has weak elimination of imaginaries.

Proposition 3.5. *(1) Both T and $T_{\leq n}$, $n \geq 1$, have quantifier elimination.*
(2) Both T and $T_{\leq n}$, $n \geq 1$, weakly eliminate imaginaries.

Proof. (1) Here we prove that T has quantifier elimination. By a similar reason, each $T_{\leq n}$ has also quantifier elimination. Let $\mathcal{N}_1 = (N_i^1, \ldots)$ and $\mathcal{N}_2 = (N_i^2, \ldots)$ be a model of T. Take finite $A \subseteq \mathcal{N}_1$, and let $f : \mathrm{cl}(A) \to \mathcal{N}_2$ be a partial embedding. Take $a \in \mathcal{N}_1$ arbitrary. It is enough to show that there is a partial embedding $g : \mathrm{cl}(Aa) \to \mathcal{N}_2$ extending f. By Remark 3.3(1), we may assume that $A = A_1 \subset N_1^1$. Suppose $a \in N_i^1$. Let $a_1 = \pi_i(a)$. If $a_1 \in A$, then $a \in \mathrm{acl}(A)$, and we are done. Without loss of generality, we may assume that $a_1 \notin A$. Since T_1 has quantifier elimination, we can pick $b_1 \in N_1^2$ such that $b_1 \models \mathrm{tp}_{T_1}(a_1/A)$. Let $A_i = \pi_i^{-1}[A] \subseteq N_i^1$, which is finite, and let $\bar{a}_i = (a_i^1, \ldots, a_i^i)$ be an enumeration of $\pi_i^{-1}(a_1)$. Since each T_i has quantifier elimination, we can pick an enumeration $\bar{b}_i = (b_i^1, \ldots, b_i^i)$ of $\pi_i^{-1}(b_1)$ such that $\bar{b}_i \models \mathrm{tp}_{T_i}(\bar{a}_i/A_i)$. Note that $\mathrm{cl}(Aa) = \mathrm{cl}(A) \cup \bigcup_{i\geq 1} \pi_i^{-1}(a_1)$. Consider a map $g = f \cup \{(a_i^j, b_i^j)|\ 1 \leq i, 1 \leq j \leq i\}$. For each i, $g \restriction_{N_i^1}$ is a partial embedding to N_i^2, and for each $j \leq i$, $b_1 = \pi_i(b_i^j) = \pi_i(g(a_i^j))$. Therefore $g : \mathrm{cl}(Aa) \to \mathcal{N}_2$ is a partial embedding extending f, and we are done. As a consequence, $\mathrm{acl}(A) = \mathrm{cl}(A)$ for each A.

(2) We first show that each $T_{\leq n}$ weakly eliminates imaginaries. We consider $\mathcal{M}_{\leq n}$ as $\mathcal{M}'_{\leq n} = (M', S', g', \pi')$, where $M' = M_1 \cup \cdots \cup M_n$, $S' = S_1 \cup \cdots \cup S_n$, $g' = g_1 \cup \cdots \cup g_n$, and $\pi' = \pi_1 \cup \cdots \cup \pi_n$. Let $T'_{\leq n} = \mathrm{Th}(\mathcal{M}'_{\leq n})$. It is enough to show that $T'_{\leq n}$ weakly eliminates imaginaries. Note that $T'_{\leq n}$ is $\aleph_0$-categorical. Let $\mathcal{N} = (N(= N_1 \cup \cdots \cup N_n), \ldots)$ be a saturated model of $T'_{\leq n}$. By Fact 3.4, it is enough to show that a subset X of N^1 which is both A-definable and B-definable, where A and B are algebraically closed, is also definable over $C = A \cap B$. Let A_i, B_i and C_i denote the intersections of A, B, and C, respectively, with N_i, $1 \leq i \leq n$. Note that $C_i = A_i \cap B_i$ for each $1 \leq i \leq n$ and $C = \mathrm{acl}(C_1)$. We may assume that $X \subseteq N_i$ for some $1 \leq i \leq n$ by taking $X \cap N_i$. By quantifier elimination, X is definable over A_i and B_i. Then from the proof of [6, Theorem 4.3(1)], X is definable over $C_i = A_i \cap B_i$ and clearly over $C = \mathrm{acl}(C_i)$. Thus $T'_{\leq n}$ weakly eliminates imaginaries and so does $T_{\leq n}$.

Let $\mathcal{M}'$ be a saturated model of T and let $\mathcal{M}'_{\leq n}$ be the saturated structure corresponding to $\mathcal{M}_{\leq n}$. By Remark 3.3(1) and quantifier elimination, there is no strictly decreasing chain of algebraically closures of finite sets in $\mathcal{M}'$. Since $\mathcal{M}' = \bigcup_n \mathcal{M}_{\leq n}$, by the same reasoning as in the proof of [2, Theorem 5.9], we conclude that T weakly eliminates imaginaries. □

Let $\mathcal{N} = (N_i, \ldots)$ be a saturated model of T. By Proposition 3.5, $\mathrm{acl}^{\mathrm{eq}}(\emptyset) = \emptyset$. Let $a \in N_1^1$, and consider strong types $p := \mathrm{tp}(a)$ (isolated by $x = x$) and $\bar{\bar{p}} := \mathrm{tp}(\mathrm{acl}(a))$.

Fact 3.6. [1] *For $a \neq b \in N_n$, a and b have the same type over some elementary substructure of N_n if and only if $N_n \models S_n(a, b, g_n(a)) \wedge S_n(g_n^{-1}(a), b, a)$. So there are $a, b \in N_n$ whose Lascar distance is at least $n/2$.*

Theorem 3.7. *The type p is a Lascar strong type but $\bar{\bar{p}}$ is not a Lascar strong type.*

Proof. By Fact 3.6, p is a Lascar strong type. Take $a, b \in N_1$ which are Lascar equivalent. Let $\bar{a}_i = (a_i^1, \ldots, a_i^i)$ and $\bar{b}_i = (b_i^1, \ldots, b_i^i)$ be enumerations of $\pi_i^{-1}(a)$ and $\pi_i^{-1}(b)$. Denote $\sigma_i(a_i) = (a_i^{\sigma(1)}, \ldots, a_i^{\sigma(i)})$ for each permutation $\sigma \in S_i$. Let τ_i be the permutation of $\{1, 2, \ldots, i\}$ sending j to $j+1$ for $j < i$ and i to 1. Then $(a_1, \tau_2^{k_2}(\bar{a}_2), \tau_3^{k_3}(\bar{a}_3), \ldots) \models \bar{p}$ for arbitrary $k_2, k_3, \ldots$ (∗). For each i, there are $a_i \in \pi_i^{-1}(a_1)$ and

$b_i \in \pi_i^{-1}(b_1)$ whose Lascar distance is at least $n/2$. So, by $(*)$, $\bar{\bar{p}}$ is not a Lascar strong type. $\square$

3.2. **A G-compact theory with $\mathrm{Gal}^1_{\mathrm{L}}(\bar{\bar{p}})$ abelian and not isomorphic to $\mathrm{Gal}^1_{\mathrm{L}}(p)$.** We will use the symbols σ_r, g_n and (M_i, S_i) defined in the previous subsection. Put $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}_1 = (M_1, S_1, g_{1,n})_{n\geq 1}$, where $g_{1,n}$ is the same rotation as g_n on the unit circle $M_1 = S^1$. We define a multi-sorted structure

$$\mathcal{M}' = ((\tilde{\mathcal{M}}_k), \delta_k)_{k\geq 1},$$

where each $\tilde{\mathcal{M}}_k := (M_k = \{x_{(k)} : x \in S^1\}, S_k, g_{k,n})_{n\geq 1}$ is a copy of $\tilde{\mathcal{M}}_1$ (again $g_{k,n}$ is the same rotation as g_n on M_k, and we may omit k in S_k and $g_{k,n}$), and $\delta_k : \tilde{\mathcal{M}}_{k+1} \to \tilde{\mathcal{M}}_k$ is given by $\delta_k(x_{(k+1)}) = x^2_{(k)}$. Put $T' = \mathrm{Th}(\mathcal{M}')$.

Lemma 3.8. *T' admits q.e.*

Proof. It is enough to prove q.e. for the restriction of $\mathcal{M}'$ to finitely many sorts $\tilde{\mathcal{M}}_{\leq k}$ for each k, but such a restriction is quantifier-free interpretable in the structure $\mathcal{M}'$ for some n, which is known to admit q.e. ([2, Theorem 5.8]). $\square$

Using q.e, it is easy to check that two (possibly infinite) tuples have the same Lascar type iff their corresponding coordinates are infinitesimally close (in the same sort), and S induces corresponding partial orders on the tuples. As this is a type-definable condition, we get that T' is G-compact.

By Theorem 5.9 from [2], $\tilde{\mathcal{M}}$, and hence also any $\tilde{\mathcal{M}}_{\leq k}$, weakly eliminates imaginaries. Hence, we get:

Remark 3.9. T' weakly eliminates imaginaries.

By Lemma 3.8, one easily gets that $\mathrm{acl}_{T'}(\emptyset) = \emptyset$ in the home-sort, hence, by Remark 3.9, $\mathrm{acl}^{\mathrm{eq}}_{T'}(\emptyset) = \emptyset$. Now, let p be the (unique) type of an element $a \in \tilde{\mathcal{M}}$. Then $\mathrm{Gal}^1_{\mathrm{L}}(p) \cong \mathrm{Gal}_{\mathrm{L}}(\mathrm{Th}(\tilde{\mathcal{M}})) \cong S^1$. On the other hand, for $\bar{\bar{p}} = \mathrm{tp}(\mathrm{acl}(a))$, we have:

Proposition 3.10. *$\mathrm{Gal}^1_{\mathrm{L}}(\bar{\bar{p}})$ is isomorphic to the 2-solenoid $G := \varprojlim_i \mathbb{R}/2^i\mathbb{Z}$ (the projections in the inverse limit are the natural quotient maps). Hence, $\mathrm{Gal}^1_{\mathrm{L}}(\bar{\bar{p}})$ is abelian and non-isomorphic to $\mathrm{Gal}^1_{\mathrm{L}}(p)$.*

Proof. Consider the homomorphism $\phi : G = \varprojlim_i \mathbb{R}/2^i\mathbb{Z} \to \mathrm{Gal}^1_{\mathrm{L}}(\bar{\bar{p}})$ sending a sequence $(r_i/2^i\mathbb{Z})_i$ to the class induced by the automorphism $f_{(x_i)_i}$ (defined on a monster model of T') equal to $\sigma_{r_i/2^i}$ on $\tilde{\mathcal{M}}_i$ for each i. If $r_i \notin 2^i\mathbb{Z}$, then $f_{(r_i)_i}$ does not preserve the Lascar types of elements of $\tilde{\mathcal{M}}_i$, hence $\mathrm{Ker}(\phi) = \{0\}$. It remains to check that ϕ is surjective. Consider a class in $\mathrm{Gal}^1_{\mathrm{L}}(\bar{\bar{p}})$ induced by an automorhpism f. As all elements of $\tilde{\mathcal{M}}_1$ have the same Lascar type, we may assume that $f(a) = a$ for some $a \in \tilde{\mathcal{M}}_1$ (by composing f with a strong automorphism sending $f(a)$ to a). Now, for each i, as f commutes with $\delta_1\delta_2\ldots\delta_{i-1}$, it must preserve $\mathrm{acl}(a) \cap \tilde{\mathcal{M}}_i$ setwise. Hence, there is r_i such that on $\mathrm{acl}(a) \cap \tilde{\mathcal{M}}_i$, f is equal to $\sigma_{r_i/2^i}$. Then f and $f_{(r_i)_i}$ have infinitesimally close values on every element of the monster model, so $f^{-1}f_{(r_i)_i}$ is Lascar strong and $\phi(f_{(r_i)_i})$ is equal to the class induced by f. □

3.3. **RN but not a Lascar pattern 2-chain.** Let p be a strong type of an algebraically closed set. In [6], it was shown that any 2-chain in p with 1-shell boundary can be reduced to a 2-chain having an *RN-pattern* or an *NR-pattern*.[3] Also for $a, b \models p$, if $a \equiv^L b$, then (a, b) is an endpoint pair of a 1-shell which is the boundary of a 2-chain having a *Lascar pattern*, a kind of an RN-pattern. In [7, Question 4.5], it was asked whether all minimal 2-chains having an RN-pattern with 1-shell boundary should be equivalent to that having a Lascar pattern. It was also noticed that if there is a counterexample, then its length should be at least 7.

In this subsection, we assume the reader has some familiarity with the notions described in [6, 7], and we work with trivial independence to define independent functors in a strong type (see Definition 0.6), and corresponding 1-shells and 2-chains. We now give descriptions of 2-chains having an RN- or a Lascar pattern in terms of balanced walks, rather than original definitions. For more combinatorial descriptions for RN- or Lascar pattern 2-chains, see [7].

Remark/Definition 3.11. *Let $a, b \models p$. For $n \geq 1$, a* balanced edge-walk of length $2n$ from a to b *is a finite sequence $(d_i)_{0\leq i\leq 2n}$ of realizations of p satisfying the following conditions:*

(1) $d_0 = a$, and $d_{2n} = b$; and
(2) there is a bijection

$$\sigma : \{0, 1, \ldots, n-1\} \to \{0, 1, \ldots, n-1\}$$

such that $d_{2i}d_{2i+1} \equiv d_{2\sigma(i)+2}d_{2\sigma(i)+1}$ for $i < n$),

A balanced edge-walk of length $2n$ from a to b induces a 2-chain of length $2n+1$ (having the 1-shell boundary whose end point pair is (a, b)), which is a chain-walk (c.f. the proof of [2, Theorem 4.2]*). Such*

[3] In this paper, we change terminology "...-type" to "...-pattern" in order to avoid confusion with the existing notion of a type - a set of formulas.

an induced 2-chain need not be unique but it is unique up to the first homology class.

In particular, given a minimal 2-chain α of length $2n+1$ with the 1-shell boundary s, α is equivalent to a Lascar pattern 2-chain iff α is equivalent to a chain-walk induced by a balanced edge-walk of length $2n$ from a to b with $\sigma = \mathrm{id}$ such that $[a,b] = [s] \in H_1(p)$.

Fact 3.12. [2, 7] *Let α be a* minimal *2-chain of length $2n+1$ ($n \geq 1$) in p with the boundary $s = g_{12} - g_{02} + g_{01}$ such that $\mathrm{supp}(g_{ij}) = \{i,j\}$. Then the following are equivalent.*

1. *The 2-chain α has an RN-pattern.*
2. *The 2-chain α is equivalent to a 2-chain*
$$\alpha' = \sum_{i=0}^{2n} (-1)^i f_i$$
which is a chain-walk of 2-simplices from g_{01} to $-g_{02}$ such that $\partial^2 f_0 = g_{01}$, $\partial^0 f_{2n} = g_{12}$, $\partial^1(f_{2n}) = -g_{02}$ and $\mathrm{supp}(\alpha') = 3 = \{0,1,2\}$.
3. *There are an endpoint pair (a,b) of s, and a balanced-edge-walk of length $2n$ from a to b which induces a 2-chain equivalent to α.*

In this subsection we will give a minimal 2-chain example (with 1-shell boundary) having an RN-pattern but not equivalent to that having a Lascar pattern. The example will have length 7. We start with preparatory constructions. Let G be a connected compact Lie group definable in $\mathbb{R}$ in the ordered ring language. Let X be a set equipped with a regular G-action. Let $\mathcal{M}_2 = (\mathcal{R}, X, \cdot)$ be a two sorted structure, where $\mathcal{R}$ is the field of real numbers with a named finite set of elements over which G is definable. Let $\mathcal{M}_n = (\mathcal{M}_{1,n}, \mathcal{M}_2)$, and there is no interaction between $\mathcal{M}_{1,n}$ and $\mathcal{M}_2$.

Let $\mathcal{N}_n^* = (\mathcal{M}_{1,n}^*, \mathcal{M}_2^*)$ be a saturated model of $\mathrm{Th}(\mathcal{M}_n)$ such that $\mathcal{M}_{1,n}^* = (M^*, S, g_n)$ and $\mathcal{M}_2^* = (\mathcal{R}^*, X^*, \cdot)$ are saturated models of $\mathrm{Th}(\mathcal{M}_{1,n})$ and $\mathrm{Th}(\mathcal{M}_2)$ respectively. Since there are no other interactions between $\mathcal{M}_{1,n}^*$ and $\mathcal{M}_2^*$, we have that

- $\mathrm{Aut}(\mathcal{N}_n^*) = \mathrm{Aut}(\mathcal{M}_{1,n}^*) \times \mathrm{Aut}(\mathcal{M}_2^*)$.
- For $a \in \mathcal{M}_{1,n}^*$ and $b \in \mathcal{M}_2^*$, $\mathrm{acl}(ab) = \mathrm{acl}_{\mathcal{M}_{1,n}}(a) \cup \mathrm{acl}_{\mathcal{M}_2}(b)$, where $\mathrm{acl}_{\mathcal{M}_{1,n}}$ and $\mathrm{acl}_{\mathcal{M}_2}$ are the algebraic closures taken in $\mathcal{M}_{1,n}^*$ and $\mathcal{M}_2^*$ respectively.
- For $a \in \mathcal{M}_{1,n}^*$ and $b \in \mathcal{M}_2^*$, $\mathrm{tp}(ab) = \mathrm{tp}_{\mathcal{M}_{1,n}}(a) \cup \mathrm{tp}_{\mathcal{M}_2}(b)$ where $\mathrm{tp}_{\mathcal{M}_{1,n}}(a)$ and $\mathrm{tp}_{\mathcal{M}_2}(b)$ are complete types in $\mathcal{M}_{1,n}^*$ and $\mathcal{M}_2^*$ respectively.

First, let us recall a fact on minimal lengths of 1-shells in p_n from [6].

Definition 3.13. Let $\mathcal{A}$ be a non-trivial amenable collection and let s be a 1-shell. Define

$$B(s) := \min\{\ |\tau| : \tau \text{ is a (minimal) 2-chain and } \partial(\tau) = s\ \}.$$

If s is not the boundary of any 2-chain, define $B(s) := -\infty$.

Fact 3.14. *For $n \geq 7$, there is a 1-shell s in p_n such that $B(s) = 7$.*

Note that in [6], the authors considered 1-shells and 2-chains defined under acl-independence for Fact 3.14. By similar methods as in the proofs, it is not hard to see that the same Fact 3.14 holds for 1-shells and 2-chains defined under trivial independence as in this subsection.

Next, let us recall some facts about $\mathcal{M}_2^*$ from [9]. Let G^* be the extension of G in $\mathcal{R}^*$. Let μ be the normal subgroup of infinitesimals in G^*. We fix a base point $x_0 \in X^*$ so that for $x \in X^*$ there is a unique element $h \in G^*$ with $x = h \cdot x_0$.

Fact 3.15. *(1) Each $\phi \in \mathrm{Aut}(\mathcal{R}^*)$ is extended to $\mathrm{Aut}(\mathcal{M}^*)$ defined by $\bar{\phi}(h \cdot x_0) := \phi(h) \cdot x_0$. An automorphism which fixes $\mathcal{R}^*$ pointwise is of the form $\bar{g}$ for some $g \in G^*$, where $\bar{g}(h \cdot x_0) = (hg^{-1}) \cdot x_0$, and the commutation rule is given by $\bar{\phi}\bar{g} = \overline{\phi(g)}\bar{\phi}$. So we have $\mathrm{Aut}(\mathcal{M}^*) = \mathrm{Aut}(\mathcal{R}^*) \ltimes G^*$.*

(2) An automorphism $\Phi = \bar{\phi}\bar{g}$ is a strong automorphism if and only if g is an infinitesimal. So the Lascar Galois group of $\mathrm{Th}(\mathcal{M})$ is isomorphic to G.

(3) $\mu = \{h^{-1}\phi(h) |\ h \in G^*,\ \phi \in \mathrm{Aut}(\mathcal{R}^*)\}$.

We note that in X^*, there is a unique 1-type over $\emptyset$ and it is also a strong type over $\emptyset$.

Proposition 3.16. *For any $x \in X$, $x \equiv x_0$ and $\mathrm{tp}(x_0) = \mathrm{stp}(x_0)$.*

Proof. Suppose there are $x, x' \in X^*$ with $\mathrm{stp}(x) \neq \mathrm{stp}(x')$. Then there is a $\emptyset$-definable finite equivalence relation E on X^* such that $\neg E(x, x')$. Define $H := \{g \in G^* |\ E(x, g \cdot x)\}$. Since E is $\emptyset$-definable, H is $\emptyset$-definable subgroup of G^*, and it is proper. We claim that $[G^* : H]$ is finite. Note that $gg^{-1} \in H$ if and only if $E(g \cdot x, g' \cdot x)$. So the map sending $gH \in G^*/H \mapsto g(x)E \in X^*/E$ is a bijection because G^* acts on X^* transitively. Since X^*/E is finite, so is G^*/H, contradicting the connectedness of G. □

Let $\mathrm{acl}^{\mathrm{eq}}_{\mathcal{R}^*}(\emptyset)$ be the algebraic closure of $\emptyset$ in $(\mathcal{R}^*)^{\mathrm{eq}}$. For $x \in X^*$, $\mathrm{acl}^{\mathrm{eq}}(x) = \{x\} \cup \mathrm{acl}^{\mathrm{eq}}_{\mathcal{R}^*}(\emptyset)$. From Proposition 3.16, we have that $\mathrm{tp}(x) \models$

$\mathrm{stp}(\mathrm{acl}^{\mathrm{eq}}(x))$. So we say x_1 and x_2 are endpoints if $\mathrm{acl}^{\mathrm{eq}}(x_1)$ and $\mathrm{acl}^{\mathrm{eq}}(x_2)$ are endpoints of a 1-shell in $p := \mathrm{stp}(\mathrm{acl}^{\mathrm{eq}}(x))$.

Lemma 3.17. *For $x_1, x_2, x_3 \in X^*$, if $x_1x_2 \equiv x_3x_2$, then there is $h \in \mu$ such that $\bar{h}(x_1) = x_3$.*

Proof. Let $x_1, x_2, x_3 \in X^*$ be such that $x_1x_2 \equiv x_3x_2$. Let $h_1, h_2, h_3 \in G^*$ be such that $x_i = h_i \cdot x_0$. It is enough to show that $h_3h_1^{-1} \in \mu$.

Take $\phi \in \mathrm{Aut}(\mathcal{R}^*)$ and $g \in G^*$ such that $\bar{g}\bar{\phi}(x_1x_2) = x_3x_2$. Then we have

$$\begin{aligned}\bar{g}\bar{\phi}(x_1x_2) &= \bar{g}((\phi(h_1) \cdot x_0)(\phi(h_2) \cdot x_0)) \\ &= (\phi(h_1)g^{-1} \cdot x_0)(\phi(h_2)g^{-1} \cdot x_0) \\ &= (h_3 \cdot x_0)(h_2 \cdot x_0).\end{aligned}$$

By the regularity of the action of G^*, we have $\phi(h_1)g^{-1} = h_3$ and $\phi(h_2)g^{-1} = h_2$, and so $g = h_2^{-1}\phi(h_2) \in \mu$. And $h_3h_1^{-1} = \phi(h_1)g^{-1}h_1^{-1} = \phi(h_1)h_1^{-1} = \mathrm{id}$ modulo μ because μ is a normal subgroup of G^*. Thus $h_3h_1^{-1} \in \mu$. □

Theorem 3.18. *For $x_1, x_2 \in X^*$, the following are equivalent:*

(1) $x_1 \equiv^L x_2$.
(2) There is $h \in \mu$ such that $\bar{h}(x_1) = x_2$.
(3) x_1 and x_2 are endpoints of a 1-shell which is the boundary of a 2-chain in p having a Lascar pattern.

Proof. (1) ⇔ (2) was proved in [9] and (1) ⇒ (3) was proved in [6]. It is enough to show (3) ⇒ (2). By definition of Lascar pattern 2-chain (in [6]) and Lemma 3.17, (3) ⇒ (2) is also true. □

Now we give a promised example of a minimal 2-chain having an RN-pattern but not equivalent to one having a Lascar pattern. Denote $[g, h] = ghg^{-1}h^{-1}$. Consider $\mathcal{N}_n^*$ for some $n \geq 7$ and let $q = \mathrm{tp}(ab)$ for $a \models p$ and $b \models p_n$, which is a strong type.

Theorem 3.19. *Suppose G is not abelian. Then there is a minimal 2-chain in q which has an RN-pattern but not equivalent to any Lascar pattern 2-chain.*

Proof. Suppose G is not abelian. We can choose $g_1, g_2, g_3 \in G$ such that $[(g_3g_2)^{-1}, (g_2g_1))] \neq \mathrm{id}$. Let $h_1 = g_1$, $h_2 = g_2$, $h_3 = g_3$, $h_4 = h^{1-}[h_1, (h_3h_2)]$, $h_5 = h^{-2}[h_2, (h_4h_3)]$, $h_6 = h^{-3}[h_3, (h_5h_4)]$, and $f_i = h_ih_{i-1} \ldots h_1$ for $1 \leq i \leq 6$. Set $x_i = (h_ih_{i-1} \cdots h_1)^{-1} \cdot x_0$ for $1 \leq i \leq 6$. Consider a 1-shell $s = f_{01} + f_{12} - f_{02}$, where $f_{01}(\{01\}) \equiv [x_0x_0]$, $s = f_{12}(\{1, 2\}) = [x_0x_0]$, and $f_{02}(\{0, 2\}) \equiv [x_6x_0]$, and its endpoints are x_0 and x_6. Set $h_1 := g_1$, $h_2 := g_2$, $h_3 := g_3$, $h_4 := h_1^{-1}[h_1, (h_3h_2)]$, $h_5 :=$

$h_2^{-1}[h_2,(h_4h_3)]$, and $h_6 := h_3^{-1}[h_3,(h_5h_4)]$. Let $f_i = h_ih_{i-1}\cdots h_1 \in G \subseteq G^*$ for $1 \le i \le 6$. Then we have that

- $f_4 = f_3f_1^{-1}$;
- $f_5f_1^{-1} = f_4f_2^{-1}$; and
- $f_6f_2^{-1} = f_5f_3^{-1}$.

This implies $\bar{f}_4(x_0x_1) = x_4x_3$, $\overline{f_5f_1^{-1}}(x_1x_2) = x_5x_4$, and $\overline{f_6f_2^{-1}}(x_2x_3) = x_6x_5$. Then we have that $x_0x_1 \equiv x_4x_3$, $x_1x_2 \equiv x_6x_5$, and $x_2x_3 \equiv x_5x_4$.

$$x_0 \xrightarrow{\bar{h}_1} x_1 \xrightarrow{\bar{h}_2} x_2 \xrightarrow{\bar{h}_3} x_3 \xrightarrow{\bar{h}_4} x_4 \xrightarrow{\bar{h}_5} x_5 \xrightarrow{\bar{h}_6} x_6$$

This gives a 2-chain α of length 7 having the 1-shell boundary s. Note that

$$\begin{aligned}
f_6 &= (h_6h_5h_4h_3)h_2h_1\\
&= (h_5h_4)h_2h_1\\
&= (h_4h_3h_2^{-1}h_3^{-1})h_2h_1\\
&= (h_4h_3)(h_2^{-1}h_3^{-1})(h_2h_1)\\
&= (h_3h_2)(h_2h_1)^{-1}(h_2^{-1}h_3^{-1})(h_2h_1)\\
&= [(h_3h_2),(h_2h_1)^{-1}] \neq \mathrm{id},
\end{aligned}$$

and $f_6 \notin \mu$. By Theorem 3.18, s is not the boundary of 2-chain with a Lascar pattern and $B(s) \le 7$.

Let $b, b' \models p_n$ where (b,b') is an endpoint pair of a 1-shell s' of p_n such that $B(s') = 7$ in $\mathcal{M}_{1,n}$. Let α' be a minimal 2-chain with an RN-pattern having the 1-shell boundary s' of length 7. Note that we can take such a 2-chain because p_n is a Lascar type. Consider $a = (x_0, b), a' = (x_6, b') \models q$. Let s'' be a 1-shell induced from s and s' with an endpoint pair (a, a'). Consider a 2-chain α'' in q induced from α and α', which does not a Lascar pattern and has length 7. We claim that α'' is minimal. Since $B(s) \le 7$ and $B(s') = 7$, we have that $B(s'') = 7$ and so α'' is minimal. By the construction of α'', it has an RN-pattern. Therefore, the minimal 2-chain α'' has an RN-pattern but is not equivalent to a Lascar pattern 2-chain. □

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