CHARACTERIZATION OF THE SECOND HOMOLOGY GROUP OF A STATIONARY TYPE IN A STABLE THEORY

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Abstract. Let T be a stable theory. It was shown in [5] that one can define the notions of homology groups attached to a stationary type of T. It was also shown that if T fails to have an amalgamation property called 3-uniqueness, then for some stationary type p the homology group $H_2(p)$ has to be a nontrivial abelian profinite group. The goal of this paper is to show that for any abelian profinite group G there is a stable (in fact, categorical) theory and a stationary type p such that $H_2(p) \cong G$.

1. Preliminaries

The paper [5] introduces the definitions of homology groups H_n, $n \geq 0$, for stable first-order theories. These groups measure the failure of generalized amalgamation properties for $n \geq 2$. It was shown that, for a stationary type p in a stable theory, the group $H_2(p)$ must be abelian profinite. In the present paper, for a given abelian profinite group G, we provide a construction of a stable theory T_G and a stationary type p in that theory such that $H_2(p) \cong G$.

We refer the reader to [5] for the definitions of homology groups and the generalized amalgamation properties. The presentation in this paper is self-contained, modulo the following key result.

Fact 1.1 (Theorem 2.1 in [5])**.** *If $T = T^{eq}$ is a stable theory and p is a stationary type, then the group $H_2(p)$ is isomorphic to the group $\operatorname{Aut}(\widetilde{ab}/\operatorname{acl}(a)\operatorname{acl}(b))$, where a, b are independent realizations of the type p and $\widetilde{ab} = \operatorname{acl}(ab) \cap \operatorname{dcl}(\operatorname{acl}(ac), \operatorname{acl}(bc))$ for some (equivalently, any) realization c of p such that c is independent from ab.*

Let $G = \varprojlim H_i$ be a given abelian profinite group. We construct a first-order theory T_G such that $\operatorname{Aut}(\widetilde{ab}/\operatorname{acl}(a)\operatorname{acl}(b)) \cong G$ for independent realizations of some stationary type. The needed theory T_G

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is a theory of a certain projective system of groupoids. We begin by presenting the definitions and facts about groupoids and projective systems of groupoids that will be needed to establish their model-theoretic properties.

1.1. **Groupoids.**

Definition 1.2. A *groupoid* $\mathcal{G}$ is a category in which every morphism is invertible. A groupoid carries the following structure:

(1) the universe of $\mathcal{G}$ is partitioned into disjoint sets O and M of objects and morphisms;
(2) the domain and range maps, which we will denote by d and r, from M to O;
(3) the ternary relation $\circ$ that defines the composition operation;
(4) the identity map from O to M which selects the identity element in every group $\mathrm{Mor}(a, a)$.

A groupoid is *connected* if there is a morphism between any two of its objects.

It is well known (see, for example, see [1]) that if $\mathcal{G}$ is a connected groupoid, then for any $a, b \in \mathrm{Ob}(\mathcal{G})$, the groups $\mathrm{Mor}(a, a)$ and $\mathrm{Mor}(b, b)$ are isomorphic. The isomorphism is given by a conjugation by a morphism from a to b. The group $\mathrm{Mor}(a, a)$ in this case is called the *vertex group of* $\mathcal{G}$. It is also well known that the isomorphism type of a connected groupoid is determined by the vertex group and by the cardinality of the set of objects. We will need a more detailed information about isomorphisms (and in particular, automorphisms) of groupoids, so we state the following facts.

Let $\mathcal{G}$ be a connected groupoid and let $a \in \mathrm{Ob}(\mathcal{G})$. A *star at* a is a function $s : \mathrm{Ob}(\mathcal{G}) \setminus \{a\} \to \mathrm{Mor}(\mathcal{G})$ such that $s(b) \in \mathrm{Mor}_{\mathcal{G}}(a, b)$.

Fact 1.3. (1) *Let $F : \mathcal{G} \to \mathcal{H}$ be an isomorphism of connected groupoids, let $a \in \mathrm{Ob}(\mathcal{G})$ be an arbitrary element and let s be a star at a in $\mathcal{G}$. Then F is uniquely determined by its restriction to the following three sets:* $\mathrm{Ob}(\mathcal{G})$, $\mathrm{Mor}_{\mathcal{G}}(a, a)$, *and the range of s.*

(2) *Conversely, given an arbitrary bijection σ between* $\mathrm{Ob}(\mathcal{G})$ *and* $\mathrm{Ob}(\mathcal{H})$, *a group isomorphism $\varphi : \mathrm{Mor}_{\mathcal{G}}(a, a) \to \mathrm{Mor}_{\mathcal{H}}(\sigma(a), \sigma(a))$, and arbitrary stars s in $\mathcal{G}$ at a and t in $\mathcal{H}$ at $\sigma(a)$, there is a unique isomorphism $F : \mathcal{G} \to \mathcal{H}$ that extends σ and φ and such that $F(s(b)) = t(F(b))$ for all $b \in \mathrm{Ob}(\mathcal{G}) \setminus \{a\}$.*

(3) *If $\mathcal{G}$ and $\mathcal{H}$ are connected groupoids such that $|\mathrm{Ob}(\mathcal{G})| = |\mathrm{Ob}(\mathcal{H})|$ and the vertex groups of $\mathcal{G}$ and $\mathcal{H}$ are isomorphic, then the groupoids $\mathcal{G}$ and $\mathcal{H}$ are isomorphic.*

Proof. If $c, d \in \mathrm{Ob}(\mathcal{G}) \setminus \{a\}$, then $f \in \mathrm{Mor}_{\mathcal{G}}(c, d)$ can be written as $f = s(d) \circ h \circ s(c)^{-1}$ for a unique $h \in \mathrm{Mor}_{\mathcal{G}}(a, a)$. If $f \in \mathrm{Mor}_{\mathcal{G}}(a, b)$ ($f \in \mathrm{Mor}_{\mathcal{G}}(b, a)$) , $b \neq a$, then $f = s(b) \circ h$ ($f = h \circ s(b)^{-1}$, respectively) for a unique $h \in \mathrm{Mor}_{\mathcal{G}}(a, a)$. Thus, the for all $f \in \mathrm{Mor}(\mathcal{G})$, the value $F(f)$ is uniquely determined by $F \restriction O$, $F \restriction \mathrm{Mor}_{\mathcal{G}}(a, a)$ and by the restriction of F to the range of s.

The construction in (1) provides a way to define F extending σ, φ, and the map $s(b) \in \mathrm{Mor}_{\mathcal{G}}(a, b) \mapsto t(\sigma(b)) \in \mathrm{Mor}_{\mathcal{H}}(\sigma(a), \sigma(b))$. Associativity of groupoids ensures that the resulting map F preserves the composition. This establishes (2).

The third statement follows since we can choose at least one star in a connected groupoid (with at least two objects). $\dashv$

1.2. **Directed systems of groupoids.**

Definition 1.4. Let $(I, <)$ be a directed partially ordered set and let O be a non-empty set. For each $i \in I$, let $\mathcal{G}_i$ be a connected groupoid with $\mathrm{Ob}(\mathcal{G}_i) = O$ and suppose that we are given a system $\{\chi_{j,i} \mid i \leq j \in I\}$ of functors $\chi_{j,i} : \mathcal{G}_j \to \mathcal{G}_i$ such that

(1) the functor $\chi_{j,i}$ is the identity map on objects and is full on morphisms;
(2) the system commutes: for all $i < j < k \in I$ we have $\chi_{k,i} = \chi_{j,i} \circ \chi_{k,j}$.

We call the system $\boldsymbol{\mathcal{G}} := \{\mathcal{G}_i, \chi_{j,i} \mid i < j \in I\}$ a *projective system of groupoids*. We denote the common set of objects by the symbol $\mathrm{Ob}(\boldsymbol{\mathcal{G}})$. The symbol $\mathrm{Mor}(\boldsymbol{\mathcal{G}})$ will denote the disjoint union $\bigcup_{i \in I} \{i\} \times \mathrm{Mor}(\mathcal{G}_i)$ and $\mathrm{Mor}_{\boldsymbol{\mathcal{G}}}(a, b)$ is the set $\bigcup_{i \in I} \{i\} \times \mathrm{Mor}_{\mathcal{G}_i}(a, b)$.

The assumption that for all $i, j \in I$, $\mathrm{Ob}(\mathcal{G}_i) = \mathrm{Ob}(\mathcal{G}_j)$ is made only to simplify the notation; it can be replaced by the requirement that $\chi_{j,i}$ is bijective on objects.

An alternative way to describe the projective system of groupoids is by saying that $\boldsymbol{\mathcal{G}}$ is a (contravariant) functor from $(I, <)$ to the subcategory of the category of all connected groupoids in which the only morphisms are functors that are identity maps on objects and are full on morphisms. (The partially ordered set $(I, <)$ is viewed as a category in the natural way.)

We will need a description of isomorphisms for the projective systems of groupoids.

Definition 1.5. Let $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{\mathcal{H}}$ be projective systems of groupoids both indexed by a directed set I. We say that a family of functions $\{F_i \mid i \in I\}$ is an *isomorphism of the projective systems* if F_i is an isomorphism of groupoids $\mathcal{G}_i$ and $\mathcal{H}_i$ for each $i \in I$ and the isomorphisms F_i commute with the projection maps. That is, for all $i < j \in I$ we have $F_i \circ \chi^{\boldsymbol{\mathcal{G}}}_{j,i} = \chi^{\boldsymbol{\mathcal{H}}}_{j,i} \circ F_j$.

In category theory language, projective systems of groupoids are isomorphic if they are naturally isomorphic as functors. We "unwrap" this definition mostly to fix the notation for the component isomorphisms.

Definition 1.6. Let $\boldsymbol{\mathcal{G}} = \{\mathcal{G}_i, \chi_{j,i} \mid i < j \in I\}$ be a projective system of connected groupoids and fix $a \in \mathrm{Ob}(\boldsymbol{\mathcal{G}})$. A *star system at a* is a function $s : I \times (\mathrm{Ob}(\boldsymbol{\mathcal{G}}) \setminus \{a\}) \to \mathrm{Mor}(\boldsymbol{\mathcal{G}})$ such that $s(i,b) \in \mathrm{Mor}_{\mathcal{G}_i}(a,b)$ and $s(i,b) = \chi_{j,i}(s(j,b))$ for all $i < j \in I$.

The following is an easy generalization of Fact 1.3.

Proposition 1.7. *Let $\boldsymbol{\mathcal{G}}$ and $\boldsymbol{\mathcal{H}}$ be projective systems of groupoids both indexed by a directed set I.*

(1) *Let $\boldsymbol{F} = \{F_i \mid i \in I\}$ be an isomorphism of the projective systems. Let $a \in \mathrm{Ob}(\boldsymbol{\mathcal{G}})$ be an arbitrary element and let s be a star system at a in $\boldsymbol{\mathcal{G}}$. Then $\boldsymbol{F}$ is uniquely determined by the restrictions of F_i, $i \in I$, to the sets $\mathrm{Ob}(\boldsymbol{\mathcal{G}})$, $\mathrm{Mor}_{\mathcal{G}_i}(a,a)$, and the range of s.*

(2) *Given an arbitrary bijection σ between $\mathrm{Ob}(\boldsymbol{\mathcal{G}})$ and $\mathrm{Ob}(\boldsymbol{\mathcal{H}})$; a family $\{\varphi_i \mid i \in I\}$ of group isomorphisms $\varphi_i : \mathrm{Mor}_{\mathcal{G}_i}(a,a) \to \mathrm{Mor}_{\mathcal{H}_i}(\sigma(a),\sigma(a))$ that commute with projections; and arbitrary star systems s in $\boldsymbol{\mathcal{G}}$ at a and t in $\boldsymbol{\mathcal{H}}$ at $\sigma(a)$, there is a unique isomorphism $\{F_i \mid i \in I\}$ of projective systems such that for each $i \in I$ we have: $F_i : \mathcal{G}_i \to \mathcal{H}_i$ extends σ and φ_i; and $F_i(s(i,b)) = t(i,(F_i(b))$ for all $b \in \mathrm{Ob}(\boldsymbol{\mathcal{G}}) \setminus \{a\}$.*

(3) *Suppose that $\boldsymbol{\mathcal{G}} = \{\mathcal{G}_i \mid i \in I\}$ and $\boldsymbol{\mathcal{H}} = \{\mathcal{H}_i \mid i \in I\}$ are projective systems of groupoids and that there is a system of isomorphisms $\{\varphi_i \mid i \in I\}$ between the vertex groups of $\mathcal{G}_i$ and $\mathcal{H}_i$ that commutes with the projection maps. If the object sets $\mathrm{Ob}(\boldsymbol{\mathcal{G}})$ and $\mathrm{Ob}(\boldsymbol{\mathcal{H}})$ have the same cardinality, then there is a system of isomorphisms $\{F_i : \mathcal{G}_i \to \mathcal{H}_i \mid i \in I\}$ that commutes with the projection maps in the system.*

Proof. The first statement follows from Fact 1.3(2) "level-by-level." The existence of the functions F_i follows from Fact 1.3(3); it remains to verify that the system $\{F_i \mid i \in I\}$ commutes with the projections. For $i < j \in I$, take $(j,f) \in \mathrm{Mor}_{\boldsymbol{\mathcal{G}}}(c,d)$. We consider the case when $c, d \neq a$, the remaining cases are similar. Then $f = s(j,d) \circ h_j \circ s(j,c)^{-1}$ for a unique $h_j \in \mathrm{Mor}_{\mathcal{G}_j}(a,a)$. Let $h_i := \chi^{\boldsymbol{\mathcal{G}}}_{j,i}(h_j)$. Then we have

$$\begin{aligned} F_i(\chi^{\boldsymbol{\mathcal{G}}}_{j,i}(j,f) &= F_i(s(i,d) \circ h_i \circ s(i,c)^{-1}) = t(i,F_i(d)) \circ \varphi_i(h_i) \circ t(i,F_i(c))^{-1} \\ &= \chi^{\boldsymbol{\mathcal{H}}}_{j,i}(t(j,F_j(d)) \circ \chi^{\boldsymbol{\mathcal{H}}}_{j,i}(\varphi_j(h_j)) \circ \chi^{\boldsymbol{\mathcal{H}}}_{j,i}(t(j,F_j(c))^{-1}) = \chi^{\boldsymbol{\mathcal{H}}}_{j,i}(F(j,f)). \end{aligned}$$

The third statement follows from the existence of the star systems. ⊣

2. Any profinite abelian group can occur as $H_2(p)$

In this section, we prove the main result of the paper.

Theorem 2.1. *Let $(I,<)$ be a directed partially ordered set and let $\{H_i, \varphi_{j,i} \mid i \leq j \in I\}$ be an inverse system of non-trivial finite abelian groups, where each of the group homomorphisms $\varphi_{j,i}$ is surjective. Let $G = \varprojlim H_i$. There is a stable theory T_G and a stationary type p of T_G such that the group $H_2(p)$ is isomorphic to G.*

The main idea is to axiomatize, in first-order logic, the class of projective systems of groupoids indexed by the partially ordered set I and such that the vertex group of the groupoid $\mathcal{G}_i$ is isomorphic to H_i. We need to be particularly careful to axiomatize in a way that fixes both the set I and the groups H_i. This is accomplished by coding I and $\{H_i \mid i \in I\}$ into the language of the structure.

2.1. **Description of T_G. Language.** The language L_G is a multi-sorted language with sorts O and $\{M_i \mid i \in I\}$ and containing the following:

(1) for each $i \in I$, the function symbols d_i and r_i, both from the sort M_i to the sort O;
(2) for each $i \in I$, the ternary relation $\circ_i$ on M_i;
(3) for each pair $i < j \in I$, function symbols $\chi_{j,i}$ from the sort M_j to the sort M_i;
(4) for each $i \in I$, a finite set $\{P^i_1, \ldots, P^i_{k_i}\}$ of unary predicates on the sort M_i, where $k_i = |H_i|$.

Standard structure. We describe a "standard" L_G-structure and then give a list of axioms T_G satisfied by this structure. We will then show that T_G is categorical in every infinite cardinal greater than $|I|$ (so we may assume that all the structures we are dealing with are standard).

Let O be an infinite set. Let $M_i = O^2 \times H_i$, and define the functions d_i, r_i, and the relation $\circ_i$ so that $(O, M_i, d_i, r_i, \circ_i)$ is a definable groupoid with the vertex group H_i. For each $i \in I$, fix an enumeration $\{h^i_1, \ldots, h^i_{k_i}\}$ of the group H_i. Define $P^i_\ell := \{(a, a, h^i_\ell) \mid a \in O\}$ for $\ell = 1, \ldots, k_i$. Define functions $\chi_{j,i} : M_j \to M_i$ as follows: $\chi_{j,i}(a, b, h) = (a, b, \varphi_{j,i}(h))$.

Axiomatization. Let T_G be the following list of $\forall\exists$-sentences:

(1) the sorts O, M_i, $i \in I$, are pairwise disjoint;

(2) O is an infinite set;
(3) for each $i \in I$, the structure $\mathcal{G}_i := (O, M_i, d_i, r_i, \circ_i)$ is a definable connected groupoid with the set O of objects and the set M_i of morphisms. For $a, b \in O$, we use the symbol $M_i(a, b)$ to denote the set $\{f \in M_i \mid d_i(f) = a,\ r_i(f) = b\}$;
(4) for each $i \in I$, for each $a \in O$ and $\ell = 1, \ldots, k_i$, there is a unique element $f^i_{a,\ell} \in M_i(a, a) \cap P^i_\ell$ and $P^i_\ell = \{f^i_{a,\ell} \mid a \in O\}$;
(5) for each $i \in I$, a sentence expressing that for all $a \in O$, the map $h^i_\ell \mapsto f^i_{a,\ell}$ is a group isomorphism between H_i and $M_i(a, a)$ (note however that the elements h^i_ℓ and the map are not a part of the structure);
(6) the system $\{\mathcal{G}_i, \chi_{j,i} \mid i < j \in I\}$ is a projective system of groupoids and $\chi_{j,i} \upharpoonright M_j(a, a)$ agrees with $\varphi_{j,i}$.

It is straightforward to check that the following holds.

Claim 2.2. *Let $G = \varprojlim H_i$ be an abelian profinite group. Then for any infinite set O, the standard L_G-structure on O is a model of T_G.*

2.2. **Model-theoretic properties of T_G.** We establish that T_G is a complete and categorical theory. The latter fact allows us to work with the standard models of T_G we described above. We describe the algebraically closed sets in models of T_G and show that the theory has weak elimination of imaginaries.

Lemma 2.3. *The theory T_G is complete, categorical in every $\lambda > |I|$ and is model-complete.*

Proof. If M, N are models of T_G of cardinality $\lambda > |I|$, then $|O(M)| = |O(N)| = \lambda$.

The axioms of T_G guarantee that for any $M, N \models T_G$, for any $a \in \mathrm{Ob}(M)$ and $b \in \mathrm{Ob}(N)$, the groups $\mathrm{Mor}_i(a, a) \subset M$ and $\mathrm{Mor}_i(b, b) \subset N$ are isomorphic and that there is a family of isomorphisms of these groups that commutes with the maps $\chi^M_{j,i}$ and $\chi^N_{j,i}$. Thus, by Proposition 1.7(3), there is an isomorphism between the projective systems M and N.

Since T_G is categorical, it is complete. Since T_G is categorical, has no finite models, and is $\forall\exists$-axiomatizable, it is model-complete by Lindström's test. $\dashv$

Definition 2.4. Let $\mathfrak{C}$ be a large model of T_G. If A is a small subset of $\mathfrak{C}$, then we say that the set of all $b \in O(\mathfrak{C})$ such that either $b \in A$ or b is the domain or the range of a morphism in A is *the support of* A and denoted $\mathrm{supp}(A)$.

Let $\mathfrak{C}$ be a large model of T_G. By Lemma 2.3, we may assume that $\mathfrak{C}$ is the standard model of T_G. Since T_G completely determines the vertex groups, by Proposition 1.7, every automorphism of $\mathfrak{C}$ is uniquely determined by a permutation of $O(\mathfrak{C})$ and by the image of a star system under the automorphism. We fix a specific star system.

Definition 2.5. Fix $a \in O(\mathfrak{C})$, where $\mathfrak{C}$ is a large standard model of T_G, and consider the star system s_0 at a that picks out the zeros: $s_0(i, b) = (a, b, 0_{H_i})$. We call this function the *zero star system at a*.

Lemma 2.6. *Let $\mathfrak{C}$ be a large model of T_G and let $O = O(\mathfrak{C})$.*

(1) *If A is a small subset of $\mathfrak{C}$ and $A' = \mathrm{supp}(A)$, then for any $i \in I$ and any $f \in M_i$ if $\mathrm{supp}(f) \not\subseteq A'$, then there is an automorphism σ of $\mathfrak{C}$ that fixes pointwise $A \cup O$ and moves f.*

(2) *If A is a small subset of $\mathfrak{C}$ and $A' = \mathrm{supp}(A)$, then any permutation of O that fixes A' can be extended to an automorphism of $\mathfrak{C}$ that fixes A.*

Proof. As we pointed out above, the theory T_G completely determines the structure of the vertex groups and the commuting system of maps between the vertex groups. Therefore, by Proposition 1.7, to specify an automorphism σ of $\mathfrak{C}$ we need to describe the permutation $\sigma \restriction O$ and the star system $\sigma \circ s_0$, where s_0 is the zero star system at some $a \in O$.

For the first statement, $\sigma \restriction O$ is the identity map. Choose an arbitrary element $a \in O$ and let s_0 be the zero star system at a. Let $f \in M_i(b, c)$, where at least one of b, c is not in A'. We may assume that $b \notin A'$; the argument in the case $c \notin A'$ is similar.

Let $h \in H_i$ be a non-zero element (it exists since we assume that all the groups H_i are non-trivial) and define the second star system s at a as follows. Let $s(j, d) = (a, d, 0)$ if $d \neq b$ or if $d = b$ but j is not $<_I$-comparable with i; let $s(j, b) = \chi_{i,j}(h_i)$ if $j \leq i$; and otherwise let $s(j, b) = (a, b, h_j)$, where $h_j \in H_j$ is an element such that $\chi_{j,i}(h_j) = h_i$ and such that $\chi_{k,j}(h_k) = h_j$ for all $i < j < k$. It is clear that the resulting automorphism will move f. It remains to show that for any $j \in I$ and any $f \in M_j \cap A$, we have $\sigma(f) = f$. This is immediate since neither $d_j(f)$ nor $r_j(f)$ are equal to b.

The statement (2) is clear when $A = \emptyset$. Otherwise, we can take $a \in A'$ and obtain an automorphism of $\mathfrak{C}$ from the following data: the given permutation of O that fixes A' and two zero star systems at a. $\dashv$

The next step is to describe the definable closure and the algebraic closure of a subset of a large saturated model $\mathfrak{C}$ of T_G (for background on such models, called *monster models* in model theory, see, for example Section 10.1 of [6]). Recall that the definable closure of a set A is the set of all elements that are fixed by $\mathrm{Aut}(\mathfrak{C}/A)$ (the group of automorphisms of $\mathfrak{C}$ that fix A pointwise); and the algebraic closure of A is the set of all elements that have a finite orbit under the action of $\mathrm{Aut}(\mathfrak{C}/A)$.

The general statement about the algebraic and definable closures is given in Proposition 2.7 below. Let us illustrate these model-theoretic notions in our context on the following simple example. Let $\boldsymbol{\mathcal{G}}$ be a model of T_G and take $f \in \mathrm{Mor}_{\mathcal{G}_k}(a, b)$ for some $a \neq b \in \mathrm{Ob}(\mathcal{G})$. Then the definable closure of the set $\{f\}$ contains: a, b, the vertex groups $\mathrm{Mor}_{\mathcal{G}_i}(a, a)$ and $\mathrm{Mor}_{\mathcal{G}_i}(b, b)$ for all $i \in I$, as well as the hom-sets $\mathrm{Mor}_{\mathcal{G}_j}(a, b)$ for all $j \leq_I k$. The algebraic closure will also contain the sets $\mathrm{Mor}_{\mathcal{G}_i}(a, b)$ for all $i \in I$.

Proposition 2.7. *Let A be a subset of the monster model of T_G. Then* $\mathrm{dcl}(A) \cap O = \mathrm{supp}(A)$ *and* $\mathrm{acl}(A)$ *is the set* $\bigcup_{i\in I}\{M_i(a, b) \mid a, b \in \mathrm{supp}(A)\}$.

Proof. It is clear that every element of the support of A is definable from an element of A. For the reverse inclusion, take arbitrary elements $c \neq d \in O$ which are not in the support of A and define an automorphism σ of $\mathfrak{C}$ by the following data: the permutation of O that transposes c and d and fixes all other elements; and two zero star systems at a point $a \in O$, $a \neq c, d$. It is clear that σ fixes A pointwise. Thus, for any element $c \notin \mathrm{supp}(A)$, there is an automorphism σ such that $\sigma(c) \neq c$.

The argument for the algebraic closure of A is similar, using Lemma 2.6. $\dashv$

Next we show that the theory T_G has *weak elimination of imaginaries* in the sense of [9, Section 16.5]: for every formula $\varphi(\overline{x}, \overline{a})$ defined over a model M, there is a smallest algebraically closed set $A \subseteq M$ such that $\varphi(\overline{x}, \overline{a})$ is equivalent to a formula with parameters in A.

This is a technical step; it is needed because the results of [5] hold for structures that have sorts (called *imaginary sorts*) for quotient spaces modulo all the definable equivalence relations (this is the role of the assumption $T = T^{eq}$). There is a general procedure of expanding any first order theory T so that its expansion T^{eq} has all the imaginary sorts (yet whose models have the same automorphism groups as the models of T). We are proving in the following two lemmas that the algebraic closure in T_G^{eq} is controlled by subsets of models of T_G.

Lemma 2.8. *The theory T_G has weak elimination of imaginaries.*

Proof. By Lemma 16.17 of [9], it suffices to prove the following two statements:

1. There is no strictly decreasing sequence $A_0 \supsetneq A_1 \supsetneq \ldots$, where every A_i is the algebraic closure of a finite set of parameters; and

2. If A and B are algebraic closures of finite sets of parameters in the monster model $\mathfrak{C}$, then $\mathrm{Aut}(\mathfrak{C}/A \cap B)$ is generated by $\mathrm{Aut}(\mathfrak{C}/A)$ and $\mathrm{Aut}(\mathfrak{C}/B)$.

Statement 1 follows immediately from the characterization of algebraically closed sets in Proposition 2.7 (that is, algebraic closures of finite sets are equal to algebraic closures of finite subsets of O).

To check statement 2, suppose that $\sigma \in \mathrm{Aut}(\mathfrak{C}/A \cap B)$, and assume that $A = \mathrm{acl}(A_0)$ and $B = \mathrm{acl}(B_0)$ where $A_0, B_0 \subseteq O(\mathfrak{C})$. By Lemma 2.6(2), any permutation of $O(\mathfrak{C})$ which fixes A_0 can be extended to an automorphism of $\mathrm{Aut}(\mathfrak{C}/A)$, and likewise for B_0 and B.

So as a first step, we can use the fact that $\mathrm{Sym}(O/A_0 \cap B_0)$ is generated by $\mathrm{Sym}(O/A_0)$ and $\mathrm{Sym}(O/B_0)$ to find an automorphism $\tau \in \mathrm{Aut}(\mathfrak{C})$ such that τ is in the subgroup generated by $\mathrm{Aut}(\mathfrak{C}/A)$ and $\mathrm{Aut}(\mathfrak{C}/B)$ and $\widetilde{\sigma} := \sigma \circ \tau^{-1}$ fixes O pointwise.

Now we present $\widetilde{\sigma}$ as the composition of two automorphisms $\sigma_A^0 \in \mathrm{Aut}(\mathfrak{C}/A)$ and $\sigma_B^0 \in \mathrm{Aut}(\mathfrak{C}/B)$. Take an arbitrary point $a \in A_0 \cap B_0$ (if the intersection is empty, choose $a \in A_0$). Let s_0 be the zero star system at a. Let $s = \widetilde{\sigma} \circ s_0$. Then s is a star system at a. Since $\widetilde{\sigma}$ fixes $A \cap B$ pointwise, for all $c \in A_0 \cap B_0$ and all $i \in I$ we have $s(i,c) = 0$. Define the star system s_A as follows: for all $c \in A_0$ and all $i \in I$, let $s_A(i,c) = 0$ and for $c \notin A_0$ let $s_A(i,c) = s(i,c)$ for all $i \in I$. Define a star system s_B by setting $s_B(i,c) = s(i,c)$ for all $c \in A_0$ and all $i \in I$ and otherwise let $s_B(i,c) = 0$. Note that for all $c \in B_0$ and all $i \in I$ we have $s_B(i,c) = 0$ (this is true by definition for $c \in B_0 \setminus (A_0 \cap B_0)$; and recall that $s(i,c) = 0$ for $c \in A_0 \cap B_0$).

Let $\sigma_A^0 \in \mathrm{Aut}(\mathfrak{C})$ be the automorphism determined by the identity permutation of O and the star system s_A and similarly let $\sigma_B^0 \in \mathrm{Aut}(\mathfrak{C})$ be the automorphism determined by s_B. Since the star systems $s_A(i,c) = 0$ for all $a \in A_0$ and $s_B(i,c) = 0$ for all $c \in B_0$, we have $\sigma_A^0 \in \mathrm{Aut}(\mathfrak{C}/A)$ and $\sigma_B^0 \in \mathrm{Aut}(\mathfrak{C}/B)$. Finally, since $s_A(i,c) + s_B(i,c) = s(i,c)$ for all $i \in I$ and $c \in O$, we have $\sigma_B^0 \circ \sigma_A^0 = \widetilde{\sigma}$. Finally, we get $\sigma_B^0 \circ \sigma_A^0 \circ \tau = \sigma$. ⊣

Lemma 2.9. *Let $\mathfrak{C}$ be a monster model of T_G, let $O = O(\mathfrak{C})$. If X is a small subset of O, then*

$$\mathrm{acl}^{eq}(X) = \mathrm{dcl}^{eq}\left(\bigcup_{i \in I} \{M_i(a,b) \mid a, b \in X\}\right).$$

Proof. Suppose $g \in \operatorname{acl}^{eq}(X)$. Then $g = c/E$ for some X-definable finite equivalence relation $E(x, y)$. By Lemma 2.8, there is a smallest algebraically closed set A such that $E(x, c)$ is equivalent to a formula with parameters from A. By Theorem 16.15 in [9], the set A is the algebraic closure of a finite set of tuples. If the set $\operatorname{supp}(A)$ contained objects $a'_1, \ldots, a'_k \notin X$, then (by the description of automorphisms) for any $b'_1, \ldots, b'_k \notin X$, the formula $E(x, c)$ would be equivalent to a formula with parameters from $\operatorname{acl}(X \cup \{b'_1, \ldots, b'_k\})$. In particular, we could choose $\{b'_i \mid i = 1, \ldots, k\}$ to be disjoint from $\{a'_i \mid i = 1, \ldots, k\}$, contradicting the minimality of A. ⊣

2.3. **Computing the automorhism group.** Let a_0, a_1, a_2 be distinct elements in $O(\mathfrak{C})$. Let $\Gamma_2 := \operatorname{Aut}(\widetilde{a_0 a_1}/ \operatorname{acl}^{eq}(a_0), \operatorname{acl}^{eq}(a_1))$, where $\widetilde{a_0 a_1} = \operatorname{acl}^{eq}(a_0 a_1) \cap \operatorname{dcl}^{eq}(\operatorname{acl}^{eq}(a_0 a_2), \operatorname{acl}^{eq}(a_1 a_2))$.

Proposition 2.10. *The group Γ_2 described above is isomorphic to the group $G = \varprojlim H_i$.*

Proof. By Lemma 2.9 and the fact that any morphism in $M_i(a_0, a_1)$ is a composition of morphisms in $M_i(a_0, a_2)$ and $M_i(a_2, a_1)$, it follows that the set $\widetilde{a_0 a_1}$ is interdefinable with $\bigcup_{i \in I} (M_i(a_0, a_1) \cup (M_i(a_0, a_0) \cup M_i(a_1, a_1))$. By construction, any automorphism of $\mathfrak{C}$ fixes $M_i(a, a)$ for all $a \in O$ and all $i \in I$. So to compute the group Γ_2, it is enough to describe all the automorphisms of $\operatorname{acl}(a_0, a_1)$ that fix a_0 and a_1.

We define an isomorphism $\Phi : G \to \Gamma_2$. Take $g \in G$. By Lemma 2.6, to define an automorphism of $\operatorname{acl}(a_0, a_1)$ over $a_0 a_1$, it is enough to specify a star system at a_0 over $\{a_0, a_1\}$. Let $s(i, a_1) = g(i)$. This induces a unique automorphism of $\operatorname{acl}(a_0, a_1)$ that sends the zero star system to s. If $s_1(i, a_1) = g_1(i)$ and $s_2(i, a_1) = g_2(i)$, the automorphism determined by s_2 maps $0 \in M_i(a, b)$ to $g_2(i)$ and therefore, it maps $g_1(i)$ to $g_2(i) + g_1(i)$. Thus, the function Φ is a injective homomorphism. Surjectivity is also clear: given an automorphism $\psi \in \Gamma_2$, the function $i \in I \mapsto \psi(s_0(i, a_1)) \in H_i$ is an element of G because an automorphic image of a star system is a star system. ⊣

2.4. **Proof of the main theorem.** To complete the proof of the main theorem, we need to check that the type of an element in $O(\mathfrak{C})$ is stationary and that distinct elements of $O(\mathfrak{C})$ form an independent set.

Proposition 2.11. *Let $\mathfrak{C}$ be a large model of T_G, let A be a small subset of $\mathfrak{C}$ and let $c \in O(\mathfrak{C})$. If $c \notin \operatorname{supp}(A)$, then $\operatorname{tp}(c/A)$ does not fork over $\emptyset$.*

Proof. By finite character of non-forking, it is enough to prove the statement for a finite set A. If $c, d \notin \text{supp}(A)$, then $\text{tp}(c/A) = \text{tp}(d/A)$ by Lemma 2.6(2). So the type of c over the finite set A is isolated by the formula expressing $c \notin \text{supp}(A)$. It is clear that this formula does not divide over $\emptyset$, so $\text{tp}(c/A)$ does not fork over $\emptyset$. ⊣

From Proposition 2.11 and Lemma 2.6(2), we immediately obtain the following.

Corollary 2.12. *Let* $c \in O(\mathfrak{C})$*. Then the type* $\text{tp}(c)$ *is stationary.*

Now we can complete the proof of the main result.

Proof of Theorem 2.1. Let $\mathfrak{C}$ be a large model of T_G, take $a \in O(\mathfrak{C})$. By Corollary 2.12, the type $p = \text{tp}(a)$ is a stationary type. By Proposition 2.11, any distinct realizations of p form an independent set over $\emptyset$. By Theorem 1.1, it is enough to show that G is isomorphic to the group $\Gamma_2(p)$, and this was done in Proposition 2.10. ⊣

References

[1] Ronald Brown. From groups to groupoids: a brief survey. *Bull. London Math. Soc.*, **19**, (1987), 113-134.

[2] Tristram de Piro, Byunghan Kim, and Jessica Millar. Constructing the type-definable group from the group configuration. *Journal of Mathematical Logic*, **6** (2006), 121–139.

[3] John Goodrick and Alexei Kolesnikov. Groupoids, covers, and 3-uniqueness in stable theories. *Journal of Symbolic Logic*, **75** (2010), 905–929.

[4] John Goodrick, Byunghan Kim, and Alexei Kolesnikov. Amalgamation functors and boundary properties in simple theories. *Israel Journal of Mathematics*, **193** (2013), 169–207.

[5] John Goodrick, Byunghan Kim, and Alexei Kolesnikov. Homology groups of types in model theory and the computation of $H_2(p)$. *Journal of Symbolic Logic*, **78** (2013), 1086–1114.

[6] Wilfrid Hodges. Model theory. Cambridge University Press, Cambridge. 1993.

[7] Ehud Hrushovski. Groupoids, imaginaries and internal covers. *Turkish Journal of Mathematics* **36** (2012), 173–198.

[8] Byunghan Kim and Hyeung-Joon Kim. Notions around tree property 1. *Annals of Pure and Applied Logic*, **162** (2011), 698–709.

[9] Bruno Poizat. A course in model theory: an introduction to contemporary mathematical logic. Springer, 2000.

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