NON-COMMUTATIVE GROUPOIDS OBTAINED FROM THE FAILURE OF 3-UNIQUENESS IN STABLE THEORIES

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Abstract. Given an arbitrary connected groupoid $\mathcal{G}$ with its vertex group $\mathcal{G}_a$, if $\mathcal{G}_a$ is a central subgroup of a group F, then there is a canonical extension $\mathcal{F} = \mathcal{G} \otimes F$ of $\mathcal{G}$ in the sense that $\mathrm{Ob}(G) = \mathrm{Ob}(F)$, $\mathrm{Mor}(\mathcal{G}) \subseteq \mathrm{Mor}(\mathcal{F})$, and F is isomorphic to all the vertex groups of $\mathcal{F}$.

Meanwhile, from the failure of 3-uniqueness of a strong type p over $A = \mathrm{acl}^{\mathrm{eq}}(A)$ in a stable theory T, a canonical finitary connected commutative groupoid $\mathcal{G}$ with the binding group G is A-type definably constructed by John Goodrick and Alexei Kolesnikov [2]. In this paper we take a certain (possibly non-commutative) automorphism group F where G is embedded centrally (so inducing $\iota_a : \mathcal{G}_a \to Z(F)$), and show that the abstract groupoid $\mathcal{G} \otimes F$ lives A-invariantly in models of T. More precisely, we A-invariantly construct a connected groupoid $\mathcal{F}$, isomorphic to $\mathcal{G} \otimes F$ as abstract groupoids, satisfying the following:

1. $\mathrm{Ob}(\mathcal{F}) = \mathrm{Ob}(\mathcal{G})$, and $\mathrm{Mor}(\mathcal{F})$ and composition maps are A-invariant (i.e., described by infinite disjunctions of conjunctions of formulas over A), so that an A-automorphism of a model of T induces a groupoid automorphism of $\mathcal{F}$;
2. There is an A-invariant faithful functor $I : \mathcal{G} \to \mathcal{F}$ which is the identity on the objects, and $I(\mathcal{G}_a) = i_a \circ \iota_a$, where i_a is a canonical group isomorphism from F onto a vertex group $\mathcal{F}_a$ of $\mathcal{F}$.

An automorphism group approximated by the vertex groups of the non-commutative groupoids is suggested as a "fundamental group" of the strong type p.

1. Introduction

In singular homology theory, one of the differences between the fundamental group and the first homology group H_1 is that the former is not necessarily commutative while the latter is. In the earlier papers [2],[3],[4] by Goodrick, Kolesnikov (and the first author), an analogue of

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homotopy/homology theory is developed in the context of model theory but where the "fundamental group" (more precisely "fundamental groupoid") introduced is always commutative. Namely, in a stable theory, given a strong type $p(x)$ of finite x over say $\emptyset = \operatorname{acl}(\emptyset)$ without 3-uniqueness whose realizations form a symmetric witness, a generic (abelian) groupoid $\mathcal{G}$ (Fact 4.4) is constructed. We may and can call this "a fundamental groupoid" in the context since the construction method (recalled after Claim 4.13) is exactly the same as the usual fundamental groupoid construction in homotopy theory (see for example [1]). That is, given two realizations c, d of p (analogously given two points x, y in a topological space X), consider paths, i.e. concatenated 'fibers' (or 'arrows') in a symmetric witness in p, from c to d (analogously consider paths, i.e. continuous images of a real closed interval, from x to y) where both cases two such paths are considered to be equivalent if they are "homotopic," and again both cases path composition is path concatenation. Here one of main points in model theory context is that, we consider a fiber f_{01} in a symmetric witness (in Definition 4.1) as a 'unit' path from a_0 to a_1.

But as said the constructed generic groupoid $\mathcal{G}$ is always abelian contrary to the case in algebraic topology. Indeed the vertex group of $\mathcal{G}$ in p corresponds to the 1st homology group of the topological space X. This correspondence is established in [4] (Fact 1.5 in this paper. But there is a numeric discrepancy. In general, $(n+1)$th homology group of p corresponds to nth homology group of a topological space). In conclusion, in model theory context, the fundamental group of a type and its homology group are isomorphic, while in usual topology, only the abelianization of the fundamental group of a space is its homology group.

Hence we think that some pieces of information are missing while constructing $\mathcal{G}$. In this paper, by finding more fibers or unit paths from one realization of a type to other, we suggest how to construct a different fundamental group in a non-commutative manner. More precisely, from the same symmetric witness to the failure of 3-uniqueness in a stable theory, by considering more conjugates of fibers of the witness (so adding more paths from one to the other), we construct a new groupoid $\mathcal{F}$ whose "vertex groups" $\operatorname{Mor}_{\mathcal{F}}(a, a)$ need not be abelian. In fact, we will show that $\operatorname{Mor}_{\mathcal{G}}(a, a) \leq Z(\operatorname{Mor}_{\mathcal{F}}(a, a))$. We may call $\mathcal{F}$ a *non-commutative groupoid* constructed from the symmetric witness. But unlike the groupoid $\mathcal{G}$, this new groupoid $\mathcal{F}$ living in a monster model is definable only in certain cases (e.g. under ω-categoricity); in general, it merely is invariant. More precisely, $\operatorname{Ob}(\mathcal{G}) = \operatorname{Ob}(\mathcal{F})$ =the solution set of $p(x)$, and for each $c, d \models p$, $\operatorname{Mor}_{\mathcal{F}}(c, d)$ is a cd-invariant set contained in $\mathrm{acl}(cd)$, so $\mathrm{Mor}(\mathcal{F})$ is an $\emptyset$-invariant set. The composition of morphisms is described $\emptyset$-invariantly. Moreover there is a faithful functor from $\mathcal{G}$ to $\mathcal{F}$, which is the identity map on $\mathrm{Ob}(\mathcal{G})$, and the injection from $\mathrm{Mor}_{\mathcal{G}}(c,d)$ to $\mathrm{Mor}_{\mathcal{F}}(c,d)$ is relatively $\emptyset$-definable.

As an abstract groupoid, $\mathcal{F}$ is isomorphic to $\mathcal{G} \otimes F$ (see Section 3) where F is a certain finite automorphism group isomorphic to all the vertex group of $\mathcal{F}$, and the binding group of $\mathcal{G}$ is centrally embedded into F.

We work in a complete *stable* theory T with a fixed monster model $\mathcal{M} = \mathcal{M}^{\mathrm{eq}}$. Unless said otherwise, tuples are from $\mathcal{M}$ and sets $A, B, \dots$ are small subsets of $\mathcal{M}$, and there is an independence notion among sets, induced by nonforking; an automorphism refers to that of $\mathcal{M}$, but occasionally we also mention an automorphism of a certain algebraic structure (e.g. a group or a groupoid) with clear distinction in the context. For tuples $a_0, a_1, \dots$, we write a_{01} to denote a_0a_1 and so on. Throughout this paper unless said otherwise **we also fix an algebraically closed set A and a complete type p of possibly infinite arity over A.** For a tuple c, $\overline{c}$ denotes $\mathrm{acl}(cA)$. If $\{a,b,c\}$ is an A-independent set of realizations of p, then we let

$$\widetilde{ab} := \mathrm{dcl}(\overline{ac}\,\overline{bc}) \cap \overline{ab}.$$

Due to stationarity, this set only depends on a and b. The rest notational convention we take is standard. For example, $a \equiv_B b$ means $\mathrm{tp}(a/B) = \mathrm{tp}(b/B)$; and $\mathrm{Aut}(C/B)$ is the group of elementary maps from C *onto* C fixing B pointwise. In addition, $\mathrm{Aut}(\mathrm{tp}(f/B))$ means $\mathrm{Aut}(Y/B)$ where Y is the solution set of $\mathrm{tp}(f/B)$. As usual, in this paper, a set or an algebraic object in $\mathcal{M}$ is said to be *B-invariant* if it is described by infinite disjunctions of conjunctions of formulas over B (equivalently, it is B-automorphism invariant), and we may allow an element of the algebraic object can be a B-invariant equivalence class in a B-invariant subset of the model.

Now we recall definitions of notions which we will use throughout. A group homomorphism is said to be *central* if its image is in the center of the target group. A *groupoid* $\mathcal{G}$ is a category where every morphism is invertible. Hence in $\mathcal{G}$, for each object a, $\mathcal{G}_a := \mathrm{Mor}_{\mathcal{G}}(a,a)$ forms a group called a *vertex group*. If all the vertex groups are abelian we call $\mathcal{G}$ *abelian* or *commutative*. We say the groupoid $\mathcal{G}$ is *connected* (*finitary*, resp.) if $\mathrm{Mor}_{\mathcal{G}}(a,b)$ is non-empty (finite, resp.) for any two objects a, b.

Remark 1.1. Assume $\mathcal{G}$ is connected. Then any two vertex groups are isomorphic via a map $v^a_b : \mathcal{G}_a \to \mathcal{G}_b$ sending $u \in \mathcal{G}_a$ to $v.u.v^{-1}$, for each $v \in \mathrm{Mor}_{\mathcal{G}}(a,b)$. In general for distinct $v, w \in \mathrm{Mor}_{\mathcal{G}}(a,b)$, v^a_b and w^a_b can be distinct, but if we restrict v^a_b and w^a_b to the centers of the vertex groups then they are the same. Hence the *central binding group*

$$(\bigcup\{Z(\mathcal{G}_a) : \ a \in \mathrm{Ob}(\mathcal{G})\})/\sim$$

canonically isomorphic to each $Z(\mathcal{G}_a)$ is associated with $\mathcal{G}$, where $\sim$ is an equivalence relation such that $u_1 \sim u_2$ for $u_i \in Z(\mathcal{G}_{a_i})$ if $v^{a_1}_{a_2}(u_1) = u_2$ for some (any) $v \in \mathrm{Mor}_{\mathcal{G}}(a_1, a_2)$. If $\mathcal{G}$ is abelian then the central binding group is simply called the *binding group*.

Originally, 3-uniqueness is defined functorially in [6],[3], but as we will not use amalgamation notion the following equivalent definition would suffice in this note.

Definition 1.2. [3] We say the fixed complete type p has 3-*uniqueness over* A if whenever $\{a_0, a_1, a_2\}$ is an A-independent set of realizations of p, and for $0 \leq i < j \leq 2$, $\sigma_{ij} \in \mathrm{Aut}(\overline{a_{ij}}/\overline{a_i}\ \overline{a_j})$, then $\sigma_{01} \cup \sigma_{02} \cup \sigma_{12}$ is also an elementary map.

Fact 1.3. [6],[3] *Let $a, b, c \models p$ be independent over A. Then p has 3-uniqueness over A iff $\widetilde{ab} = \mathrm{dcl}(\overline{a}\overline{b})$.*

We now recall a certain automorphism group which plays a role of the (abelian) fundamental group of p in the homotopy theory of model theory introduced in [2],[4].

Definition 1.4. [3],[4] Let $\{a, b, c\}$ be A-independent set of realizations of p. We let

$$\Gamma_2(p) := \mathrm{Aut}(\widetilde{ab}/\overline{a}\overline{b}).$$

Since the homology groups of p will not be dealt with, we do not recall the definition of those, but only point out the following proved in [4].

Fact 1.5. *The group $\Gamma_2(p)$ is profinite abelian and isomorphic to the type's 2nd homology group $H_2(p)$.*

There indeed is a mismatch in numbering. The group $\Gamma_2(p)$ corresponds to the fundamental group π_1 (or its abelianization), and so should do $H_2(p)$ to the first homology group in algebraic topology. An higher dimensional version of Fact 1.5 is proved in [5].

Our goal in this paper is to introduce a possibly non-commutative "fundamental group" Π_2 of p, in which $\Gamma_2(p)$ places in the center. In Section 2, we give a motivational example. Namely a model of a connected groupoid having a given finite vertex group F. In the example, it turns out that $\Pi_2(p) = F$ and $\Gamma_2(p) = Z(F)$ if we take p as the 1-type of any object.

In Section 3, a general theory of groupoid extensions is developed. Namely given an arbitrary connected groupoid $\mathcal{G}$ with $a \in \mathrm{Ob}(\mathcal{G})$, its vertex group $\mathcal{G}_a$, and a group homomorphism $\iota : \mathcal{G}_a \to F$ into another group F, we canonically extends $\mathcal{G}$ to $\mathcal{F} = \mathcal{G} \otimes_\iota F$ such that all the vertex groups of $\mathcal{F}$ are isomorphic to F, and ι induces a canonical functor $I : \mathcal{G} \to \mathcal{F}$ with the identity map on objects. The results in this section can hold with any abstract groupoids.

In Section 4, we construct our desired non-commutative connected finitary groupoid from a symmetric witness to the non-3-uniqueness of p. The constructed groupoid is, as an abstract groupoid, isomorphic to an extension (in the sense of Section 3) of the abelian generic groupoid obtained from p in [2].

In Section 5, we show that $\Pi_2(p)$, a certain automorphism group partially approximated by the vertex groups of non-commutative groupoids constructed from the failure of 3-uniqueness of p, is a normal subgroup of $\mathrm{Aut}(\widetilde{ab}/\overline{a})$ where $a, b \models p$ are A-independent, and $\Gamma_2(p)$ is central in $\Pi_2(p)$.

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2. Finitary groupoid examples

Let F be an arbitrary finite group. Now let T_F be the complete totally categorical theory of the connected finitary groupoid $\mathcal{F} = (O, M, ., \mathrm{init}, \mathrm{ter})$ with the standard setting. Namely the sorts O, M represent the *infinite* sets of all objects and morphisms, respectively; . is the composition map between morphisms; $\mathrm{init}, \mathrm{ter} : M \to O$ are maps indicating initial and terminal objects, respectively, of a morphism; and $\mathcal{F}_a = \mathrm{Mor}_{\mathcal{F}}(a, a)$ is isomorphic to F for any $a \in O$. We let $G_a := Z(\mathcal{F}_a)$.

Remark 2.1. (1) Notice that $\widehat{G} := \bigcup_{a \in O} G_a$ is $\emptyset$-definable. As described in Remark 1.1, the central binding group $G = \widehat{G}/\sim$ ($\sim$ is $\emptyset$-definable here) isomorphic to each G_a lives in $\mathrm{acl}(\emptyset)(= \mathrm{acl}^{\mathrm{eq}}(\emptyset))$, and there is an a-definable bijection between G_a and G, so for distinct $u, v \in G_a$, $u \not\equiv_{\mathrm{acl}(\emptyset)} v$. Hence if F is abelian, then for any $a \in O$ and $b \in \mathcal{F}_a$, it follows $b \in \mathrm{dcl}(a\,\mathrm{acl}(\emptyset)) = \mathrm{acl}(a) = \mathrm{acl}(\mathcal{F}_a)$. However if F is not abelian the story is skewed, and in general $\mathrm{acl}(a) = \mathrm{acl}(\mathcal{F}_a) \neq \mathrm{dcl}(a\,\mathrm{acl}(\emptyset))$.

(2) For each $x \in O$, choose $g_x \in \mathcal{F}_x$. It is easy to see that the following map σ is an automorphism of the groupoid as a model:

(a) σ is the identity map on O;

(b) for $c, d \in O$ and $f \in \mathrm{Mor}_{\mathcal{F}}(c, d)$, we have $\sigma(f) = g_d.f.g_c^{-1}$.
If $g_a \in G_a$, then σ pointwisely fixes $\mathcal{F}_a$. Hence it follows that if $u \in \mathcal{F}_a$ is a conjugate of $v \in \mathcal{F}_a$, then there is an automorphism of $\mathcal{F}$ (as a model) such that it pointwisely fixes some small model of $\mathcal{F}$ while sending u to v, so in particular $u \equiv_{\mathrm{acl}(\emptyset)} v$ $(*)$. Therefore in general we can not expect a binding group for $\mathcal{F}_a$'s exists in $\mathrm{acl}(\emptyset)$ in exactly the same manner as the central binding group described in (1). But due to the construction method developed in Section 4, there *is* a finite group F_0 in $\mathrm{acl}(\emptyset)$ isomorphic to all $\mathcal{F}_a$'s. We explain this next.

(3) Let $\mathcal{F}'$ be some B-definable groupoid (in any theory) and $\mathcal{G}'$ is a B-definable central subgroupoid of $\mathcal{F}'$ such that $\mathrm{Ob}(\mathcal{F}') = \mathrm{Ob}(\mathcal{G}')$, $\mathrm{Mor}_{\mathcal{G}'}(a, b) \subseteq \mathrm{Mor}_{\mathcal{F}'}(a, b)$, and $\mathcal{G}'_a = Z(\mathcal{F}'_a)$. Then we can take v in Remark 1.1 from $\mathrm{Mor}_{\mathcal{G}'}(a, b)$, and find a B-definable equivalence relation $\sim$ so that $F' := (\bigcup\{\mathcal{F}'_a : a \in \mathrm{Ob}(\mathcal{F}')\})/\sim$ ($\subseteq \mathrm{acl}(B)$ if $\mathcal{F}'$ is finitary) as a group is isomorphic to each $\mathcal{F}'_a$. These facts altogether imply that *there is no* $\mathrm{acl}(\emptyset)$-definable central subgroupoid of our example $\mathcal{F}$. But these do not conflict with the generic abelian groupoid construction ([2] and Fact 4.4), and its extension in Section 4. Indeed for distinct $a, b \in \mathrm{Ob}(\mathcal{F})$, and $f \in \mathrm{Mor}_{\mathcal{F}}(a, b)$, in general (a, b, afb) does not form a subsequence of a full symmetric witness (defined in Definition 4.1), but $(\mathcal{F}_a, \mathcal{F}_b, \mathcal{F}_a f \mathcal{F}_b)$ does (where we identify $\mathcal{F}_a, \mathcal{F}_b$ as some enumeration of elements of those). This is because $\mathrm{tp}(f/ab)$ does not isolate $\mathrm{tp}(f/\mathcal{F}_a\mathcal{F}_b)$ (see Claim 2.2 and the remark after Claim 2.3). Hence the $\mathrm{acl}(\emptyset)$-definable generic abelian groupoid say $\mathcal{G}_0$ is constructed on $q(\bar{x}) := \mathrm{tp}(\mathcal{F}_a/\mathrm{acl}(\emptyset))$, so that $\mathrm{Ob}(\mathcal{G}_0)$ is the set of realizations of $q(\bar{x})$ and its binding group G_0 is in $\mathrm{acl}(\emptyset)$. Then by the main construction method in Section 4, we can find an $\mathrm{acl}(\emptyset)$-definable groupoid $\mathcal{F}_0$ such that $\mathcal{G}_0$ now is a central subgroupoid of $\mathcal{F}_0$. Therefore, as said there lives $(G_0 \leq) F_0 := (\bigcup\{(\mathcal{F}_0)_{\bar{c}} : \bar{c} \in \mathrm{Ob}(\mathcal{F}_0)\})/\sim$ in $\mathrm{acl}(\emptyset)$, which as a group is isomorphic to each $(\mathcal{F}_0)_{\bar{c}}$. The reader should notice that $\mathcal{F}_0$ will be isomorphic to $\mathcal{F}$ as abstract groupoids (so in fact the group F_0 is isomorphic to each $\mathcal{F}_a$, so to F), but in general this groupoid isomorphism is *not* $\mathrm{acl}(\emptyset)$-definable and there is no such (since if so then $(*)$ in (2) should not hold for distinct u, v). On the other hand, there *is* an $\emptyset$-definable binding group isomorphism between G and G_0.

Now we assume $\emptyset = \mathrm{acl}(\emptyset)$ by naming $\mathrm{acl}(\emptyset)$, then it follows T_F has weak elimination of imaginaries. Throughout this section, we let the $p(x)$ be the unique 1-type over $\emptyset$ of any object in O.

We further fix distinct $a, b \in O$. Due to weak elimination of imaginaries it follows that $\bar{a}(:= \mathrm{acl}(a)) = \mathrm{dcl}(\mathcal{F}_a)$; $\widetilde{ab}$, $\overline{ab}$, and $\mathrm{Mor}(a,b)\mathcal{F}_a\mathcal{F}_b$ are all interdefinable; and

$$\Gamma_2(p) = \mathrm{Aut}(\widetilde{ab}/\bar{a}, \bar{b}) = \mathrm{Aut}(\mathrm{Mor}(a,b)/\mathcal{F}_a\mathcal{F}_b).$$

Hence if we *fix* a morphism $f_0 \in \mathrm{Mor}(a,b)$, indeed (note that $\mathrm{Mor}(a,b) \subseteq \mathrm{dcl}(f_0\mathcal{F}_a)$)

$$\Gamma_2(p) = \mathrm{Aut}(X/\mathcal{F}_a\mathcal{F}_b),$$

where X is the finite solution set of $\mathrm{tp}(f_0/\mathcal{F}_a\mathcal{F}_b)$.

Now for any $f \in X$ there is unique $x \in \mathcal{F}_a$ such that $f = f_0.x$, and we claim that this x must be in $G_a = Z(\mathcal{F}_a)$.

Claim 2.2. *For $x \in \mathcal{F}_a$, we have $g := f_0.x \in X$ iff $x \in G_a$.*

Proof. ($\Rightarrow$) Since $g \in X$, $f_0 \equiv_{\mathcal{F}_a\mathcal{F}_b} g$. Then for any $y \in \mathcal{F}_a$, we have

$$f_0.y.f_0^{-1}(\in \mathcal{F}_b) = g.y.g^{-1} = f_0.x.y.x^{-1}.f_0^{-1}.$$

Hence $x \in G_a$.

($\Leftarrow$) There is $z \in \mathcal{F}_b$ such that $f_0 = z.g$. Now since $x \in G_a$, for any $y \in \mathcal{F}_b$ we have

$$g^{-1}.y.g.x^{-1} = f_0^{-1}.z.y.z^{-1}.f_0.x^{-1} = x^{-1}.f_0^{-1}.z.y.z^{-1}.f_0 = g^{-1}.z.y.z^{-1}.g.x^{-1}.$$

Hence $y = z.y.z^{-1}$, i.e. $z \in G_b$. Hence then by Remark 2.1(2) there is an automorphism fixing $\mathcal{F}_a\mathcal{F}_b$ pointwise while sending f_0 to g. Hence $g \in X$. $\square$

Claim 2.3. $\Gamma_2(p) = Z(F)$.

Proof. The proof will be similar to that of Proposition 2.22 in [3]. Note firstly that due to Claim 2.2, G_a acts on X as an obvious manner. This action is clearly regular. Secondly $\mathrm{Aut}(X/\mathcal{F}_a\mathcal{F}_b)$ also regularly acts on X. Moreover since each $\sigma \in \mathrm{Aut}(X/\mathcal{F}_a\mathcal{F}_b)$ fixes $\mathcal{F}_a$ pointwise, it clearly follows that the two actions commute. Hence they are the same group. $\square$

Due to Remark 2.1(2), we have $f \equiv_{\mathcal{F}_a b} f_0$ for any $f \in \mathrm{Mor}_{\mathcal{F}}(a,b)$, i.e., $\mathrm{Mor}(a,b)$ is the solution set of $\mathrm{tp}(f_0/\mathcal{F}_a b)$ or $\mathrm{tp}(f_0/\bar{a}b)$. Moreover for $f \in \mathrm{Mor}(a,b)$, it follows

$$\mathrm{dcl}(f\bar{a}) = \mathrm{dcl}(f_0\bar{a}) = \mathrm{dcl}(\mathrm{Mor}(a,b), \bar{a}) = \mathrm{dcl}(\mathrm{Mor}(a,b), \bar{a}\bar{b}) = \overline{ab}.$$

Hence,

$$\mathrm{Aut}(\widetilde{ab}/\bar{a}) = \mathrm{Aut}(\mathrm{Mor}(a,b)/\bar{a}) = \mathrm{Aut}(\mathrm{tp}(f_0/\bar{a}b)).$$

We further claim the following.

Claim 2.4. *F is isomorphic to* $\mathrm{Aut}(\mathrm{Mor}(a,b)/\overline{a}) = \mathrm{Aut}(\mathrm{Mor}(a,b)/\mathcal{F}_a)$. *Hence* $\Gamma_2(p) = \mathrm{Aut}(\mathrm{Mor}(a,b)/\mathcal{F}_a\mathcal{F}_b) = Z(\mathrm{Aut}(\widetilde{ab}/\overline{a}))$.

Proof. We know F and $\mathcal{F}_b$ are isomorphic. Now given $\sigma \in \mathcal{F}_b$ we assign an automorphism $\boldsymbol{\sigma} \in \mathrm{Aut}(\mathrm{Mor}(a,b)/\overline{a})$ sending $f(\in \mathrm{Mor}(a,b)) \mapsto \sigma.f$. This mapping is well-defined, since if $g \in \mathrm{Mor}(a,b)$ so that $g = f.\mu$ for some $\mu \in \mathcal{F}_a$, then $\boldsymbol{\sigma}(g) = \boldsymbol{\sigma}(f).\mu = \sigma.f.\mu = \sigma.g$. Now this correspondence is clearly 1-1 and onto. It is obvious that the correspondence is a group isomorphism. □

In the following sections we try to search this phenomenon in the general stable theory context. Namely given the abelian groupoid built from a *symmetric witness* introduced in [2], we construct an extended groupoid possibly non-abelian but the abelian groupoid places in the center of the new groupoid. In the case of above T_F, as pointed out in Remark 2.1(3), every vertex group of the abelian groupoid will be isomorphic to $Z(F)$. On the other hand the extended one in Section 4 is, as an abstract groupoid, isomorphic to the model $\mathcal{F}$ of T_F itself, in particular all vertex groups are isomorphic to the original F. The reader should have in mind that the example in this section is a motivational toy model for the rest of the development.

3. Groupoid extensions

In this section, we introduce a method of transforming a connected groupoid to another connected groupoid via a vertex group homomorphism. It is a groupoid extension if the homomorphism is an embedding. The basic idea of Definition 3.1 below is introduced by an anonymous referee, for which we express our thanks.

Let $\mathcal{G}$ be an abstract connected groupoid. For $a \in \mathrm{Ob}(\mathcal{G})$, let a group homomorphism $\iota : \mathcal{G}_a(= \mathrm{Mor}_{\mathcal{G}}(a,a)) \to F$ be given. From the triple $(\mathcal{G}, a, \iota : \mathcal{G}_a \to F)$, we construct a new connected groupoid $\mathcal{G}\otimes_\iota F$, whose vertex groups are isomorphic to F and there is a canonical functor from $\mathcal{G}$ to $\mathcal{G} \otimes_\iota F$ induced by ι.

Remark/Definition 3.1. (1) For $b, c \in \mathrm{Ob}(\mathcal{G})$, define a relation $\approx$ on $\mathrm{Mor}_{\mathcal{G}}(a,c) \times F \times \mathrm{Mor}_{\mathcal{G}}(b,a)$ as follows: For

$$(g,f,h),(g',f',h') \in \mathrm{Mor}_{\mathcal{G}}(a,c) \times F \times \mathrm{Mor}_{\mathcal{G}}(b,a),$$

we let

$$(g,f,h) \approx (g',f',h') \Leftrightarrow f = \iota(g^{-1}.g').f'.\iota(h'.h^{-1}).$$

It is easy to check that the relation $\approx$ is an equivalence relation. For $(g, f, h) \in \mathrm{Mor}_{\mathcal{G}}(a, c) \times F \times \mathrm{Mor}_{\mathcal{G}}(b, a)$, we write $\langle g, f, h\rangle$ to denote the $\approx$-class of (g, f, h).

Now from the data $(\mathcal{G}, a, \iota : \mathcal{G}_a \to F)$, we define a new connected groupoid denoted by $\mathcal{G} \otimes_\iota F$ as follows:

- $\mathrm{Ob}(\mathcal{G} \otimes_\iota F) := \mathrm{Ob}(\mathcal{G})$;
- For $b, c \in \mathrm{Ob}(\mathcal{G})$,

$$\mathrm{Mor}_{\mathcal{G}\otimes_\iota F}(b, c) := \mathrm{Mor}_{\mathcal{G}}(a, c) \times F \times \mathrm{Mor}_{\mathcal{G}}(b, a)/ \approx;$$

- Define a composition

$$\circ : \mathrm{Mor}_{\mathcal{G}\otimes_\iota F}(b, c) \times \mathrm{Mor}_{\mathcal{G}\otimes_\iota F}(c, d) \to \mathrm{Mor}_{\mathcal{G}\otimes_\iota F}(b, d)$$

  for $b, c, d \in \mathrm{Ob}(\mathcal{G})$, mapping

$$(\langle g, f, h\rangle, \langle g', f', h'\rangle) \mapsto \langle g', f', h'\rangle \circ \langle g, f, h\rangle := \langle g', f'.\iota(h'.g).f, h\rangle.$$

  It is easy to see that the composition is well-defined and associative;
- For the identity id_F of F, and $b \in \mathrm{Ob}(\mathcal{G} \otimes_\iota F)$, put $\mathrm{id}_b := \langle h^{-1}, \mathrm{id}_F, h\rangle$ for some (any) $h \in \mathrm{Mor}_{\mathcal{G}}(b, a)$. Then id_b is the left and right identity, that is, for $c \in \mathrm{Ob}(\mathcal{G} \otimes_\iota F)$ and for $\alpha \in \mathrm{Mor}_{\mathcal{G}\otimes_\iota F}(b, c)$, $\beta \in \mathrm{Mor}_{\mathcal{G}\otimes_\iota F}(c, b)$, we have $\alpha \circ \mathrm{id}_b = \alpha$ and $\mathrm{id}_b \circ \beta = \beta$. Moreover for $\alpha = \langle g, f, h\rangle$, put $\alpha^{-1} := \langle h^{-1}, f^{-1}, g^{-1}\rangle \in \mathrm{Mor}_{\mathcal{G}\otimes_\iota F}(c, b)$. Then it follows that $\alpha^{-1} \circ \alpha = \mathrm{id}_b$ and $\alpha \circ \alpha^{-1} = \mathrm{id}_c$.

(2) Moreover there is a canonical functor $I(= I_\iota)$ from $\mathcal{G}$ to $\mathcal{G} \otimes_\iota F$:
- On $\mathrm{Ob}(\mathcal{G}) = \mathrm{Ob}(\mathcal{G} \otimes_\iota F)$, I is the identity map;
- For $b, c \in \mathrm{Ob}(\mathcal{G})$ and $u \in \mathrm{Mor}_{\mathcal{G}}(b, c)$, put

$$I(u) := \langle u_2, \mathrm{id}_F, u_1\rangle \in \mathrm{Mor}_{\mathcal{G}\otimes_\iota F}(b, c)$$

  for some (any) $u_1 \in \mathrm{Mor}_{\mathcal{G}}(b, a)$, $u_2 \in \mathrm{Mor}_{\mathcal{G}}(a, c)$ with $u_2.u_1 = u$.

That I satisfies the functor axioms can be easily checked. Recall that for the connected groupoid $\mathcal{G}$, and for $v \in \mathrm{Mor}_{\mathcal{G}}(b, c)$, the mapping $\mathrm{Mor}_{\mathcal{G}}(b, b) \to \mathrm{Mor}_{\mathcal{G}}(c, c) : u \mapsto v.u.v^{-1}$ is a vertex group isomorphism. We let v^b_c denote this isomorphism. In particular for $u \in \mathrm{Mor}_{\mathcal{G}}(b, c)$, $I(u)^b_c = (I(u))^b_c$ is a vertex group isomorphism in $\mathcal{G} \otimes_\iota F$.

(3) For $u \in \mathcal{G}_a$, once can easily check that $i_u : F \to \mathrm{Mor}_{\mathcal{G}\otimes_\iota F}(a, a)$ sending f to $\langle u^{-1}, f, u\rangle$ is a group isomorphism. Hence F is isomorphic to every vertex group of $\mathcal{G} \otimes_\iota F$. It also follows $(i_u \circ \iota)(u) = I(u)$. Since I is a functor, for $v \in \mathrm{Mor}_{\mathcal{G}}(a, b)$ we have $I(v^a_b(u)) = I(v)^a_b(I(u))$. If ι is central, then $I_\iota(\mathcal{G})$ is

a central subgroupoid of $\mathcal{G} \otimes_\iota F$, thus as well known, for any $v, w \in \mathrm{Mor}_\mathcal{G}(b, c)$, it follows $I(v)^b_c = I(w)^b_c$.

Proposition 3.2. *Let $b, c \in \mathrm{Ob}(\mathcal{G} \otimes_\iota F)$. Then ι is injective iff so is $I : \mathrm{Mor}_\mathcal{G}(b, c) \to \mathrm{Mor}_{\mathcal{G}\otimes_\iota F}(b, c)$.*

Proof. Assume that ι is injective. Take $u \neq v \in \mathrm{Mor}_\mathcal{G}(b, c)$ and take $u_1, v_1 \in \mathrm{Mor}_\mathcal{G}(b, a)$, $u_2, v_2 \in \mathrm{Mor}_\mathcal{G}(a, c)$ such that $u = u_2.u_1$ and $v = v_2.v_1$. Since $u_2.u_1 = u \neq v = v_2.v_1$, we have $v_2^{-1}.u_2 \neq v_1.u_1^{-1} \in \mathcal{G}_a$. Thus $\iota(v_2^{-1}.u_2) \neq \iota(v_1.u_1^{-1})$. So $I(u) = \langle u_2, \mathrm{id}_F, u_1\rangle \neq \langle v_2, \mathrm{id}_F, v_1\rangle = I(v)$. The converse can be similarly proved. □

Remark 3.3. We can consider any group as a groupoid with a single object. Now consider two embeddings of additive groups $\iota_1 : 2\mathbb{Z} \to \mathbb{Z}$ and $\iota_2 : 2\mathbb{Z} \to \mathbb{Z}$ such that ι_1 is the inclusion map, and ι_2 is the group isomorphism mapping $2n \mapsto n$. Note that both $2\mathbb{Z} \otimes_{\iota_1} \mathbb{Z}$ and $2\mathbb{Z} \otimes_{\iota_2} \mathbb{Z}$ are isomorphic to $\mathbb{Z}$, but certainly there is no isomorphism $J : 2\mathbb{Z} \otimes_{\iota_1} \mathbb{Z} \to 2\mathbb{Z} \otimes_{\iota_2} \mathbb{Z}$ such that $J \circ I_{\iota_1} = I_{\iota_2}$.

If $\mathcal{G}$ is commutative and ι is central then the construction of $\mathcal{G} \otimes_\iota F$ above can be simplified as follows.

Remark/Definition 3.4. (1) Assume a groupoid $\mathcal{G}$ is connected and commutative. Then as noted in Section 1 there is the abelian binding group G and for each $a \in \mathrm{Ob}(\mathcal{G})$, there is the canonical group isomorphism $i_a : \mathcal{G}_a \to G$. Let F be a group with a central group homomorphism $\iota : G \to F$, and let $\iota_a = \iota \circ i_a$.

Now from the data $(\mathcal{G}, \iota : G \to F)$, we similarly define a new connected groupoid denoted by $\mathcal{G} \otimes_\iota F$ as follows. (Notice that although we use the same symbol, the domain of ι in Definition 3.1 is different.) The reader can easily verify each of the following more in detail:

- $\mathrm{Ob}(\mathcal{G}\otimes_\iota F) := \mathrm{Ob}(\mathcal{G})$, and for $a, c \in \mathrm{Ob}(\mathcal{G})$, $\mathrm{Mor}_{\mathcal{G}\otimes_\iota F}(a, c) := (\mathrm{Mor}_\mathcal{G}(a, c) \times F)/\approx$, where $\approx$ is an equivalence relation on $\mathrm{Mor}_\mathcal{G}(a, c)\times F$ such that, for $(g, f), (g', f') \in \mathrm{Mor}_\mathcal{G}(a, c)\times F$, we let $(g, f) \approx (g', f')$ if $f = \iota_a(g^{-1}.g').f'$. For such (g, f), we write $\langle g, f\rangle$ to denote its $\approx$-class;
- Define a composition $\circ :\ \mathrm{Mor}_{\mathcal{G}\otimes_\iota F}(a, c) \times \mathrm{Mor}_{\mathcal{G}\otimes_\iota F}(c, d) \to \mathrm{Mor}_{\mathcal{G}\otimes_\iota F}(a, d)$ for $a, c, d \in \mathrm{Ob}(\mathcal{G})$, mapping $(\langle g, f\rangle, \langle g', f'\rangle) \mapsto \langle g', f'\rangle \circ \langle g, f\rangle := \langle g'.g, f'.f\rangle$. Since $\mathcal{G}$ is commutative and ι is central, this associative composition is well-defined;
- For the identity id_F of F, and $b \in \mathrm{Ob}(\mathcal{G} \otimes_\iota F)$, put $\mathrm{id}_b := \langle 1_b, \mathrm{id}_F\rangle$ where 1_b is the identity of $\mathcal{G}_b$. Then id_b is the left

and right identity in $\mathcal{G}\otimes_\iota F$. For $\alpha = \langle g, f\rangle \in \mathrm{Mor}_{\mathcal{G}\otimes_\iota F}(a,b)$, put $\alpha^{-1} := \langle g^{-1}, f^{-1}\rangle \in \mathrm{Mor}_{\mathcal{G}\otimes_\iota F}(b,a)$, so that $\alpha^{-1}\circ\alpha = \mathrm{id}_a$ and $\alpha\circ\alpha^{-1} = \mathrm{id}_b$.

(2) There is a canonical functor $I(= I_\iota)$ from $\mathcal{G}$ to $\mathcal{G}\otimes_\iota F$: On $\mathrm{Ob}(\mathcal{G}) = \mathrm{Ob}(\mathcal{G}\otimes_\iota F)$, I is the identity map, and for $u \in \mathrm{Mor}_{\mathcal{G}}(b,c)$, let $I(u) = \langle u, \mathrm{id}_F\rangle \in \mathrm{Mor}_{\mathcal{G}\otimes_\iota F}(b,c)$.

(3) For $a \in \mathrm{Ob}(\mathcal{G})$, $j_a : F \to \mathrm{Mor}_{\mathcal{G}\otimes_\iota F}(a,a)$ sending f to $\langle \mathrm{id}_a, f\rangle$ is a group isomorphism. Hence F is isomorphic to every vertex group of $\mathcal{G}\otimes_\iota F$. It also follows $(j_a\circ\iota_a)(u) = I(u)$, for $u \in \mathcal{G}_a$. $I(\mathcal{G})$ is a central subgroupoid of $\mathcal{G}\otimes_\iota F$, and I is faithful iff ι is injective. Moreover $\mathcal{G}\otimes_\iota F$, and $\mathcal{G}\otimes_{\iota_a} F$ as described in Definition 3.1, are isomorphic via a functor J which is the identity map on objects and for $\langle w, f\rangle \in \mathrm{Mor}_{\mathcal{G}\otimes_\iota F}(b,c)$, $J(\langle w, f\rangle) = \langle w_2, f, w_1\rangle$ for some (any) $w_1 \in \mathrm{Mor}_{\mathcal{G}}(b,a), w_2 \in \mathrm{Mor}_{\mathcal{G}}(a,c)$ with $w = w_2.w_1$. Notice that $J\circ I_\iota = I_{\iota_a}$.

As described in [2], from a symmetric witness in a strong type over A without 3-uniqueness, an A-type-definable generic commutative groupoid $\mathcal{G}$ with its finite binding group G is constructed. In the following section we will consider a certain automorphism group F, possibly non-commutative, and a central group embedding $\iota : G \to F$. Then we will construct an A-invariant groupoid $\mathcal{F}$ isomorphic to $\mathcal{G}\otimes_\iota F$ as abstract groupoids.

4. The non-commutative groupoid $\mathcal{F}$

We recall from [2] or [4], the notion of symmetric witnesses which is a building block in constructing an *abelian* (generic) groupoid (Fact 4.4).

Definition 4.1. A *(full) symmetric witness to non-3-uniqueness* (over the algebraically closed set A) is a tuple $(a_0, a_1, a_2, f_{01}, f_{12}, f_{02}, \theta(x,y,z))$ such that a_0, a_1, a_2 and f_{01}, f_{12}, f_{02} are *finite* tuples, $\{a_0, a_1, a_2\}$ is independent over A, $\theta(x,y,z)$ is a formula over A, and:

(1) $f_{ij} \in \overline{a_{ij}} \smallsetminus \mathrm{dcl}(\overline{a_i}, \overline{a_j})$, while $a_{ij} \subset f_{ij}$;
(2) $a_{01}f_{01} \equiv_A a_{12}f_{12} \equiv_A a_{02}f_{02}$;
(3) f_{01} is the unique realization of $\theta(x, f_{12}, f_{02})$, and so are f_{12}, f_{02} of $\theta(f_{01}, y, f_{02})$, $\theta(f_{01}, f_{12}, z)$, respectively; and
(4) each $\mathrm{tp}(f_{ij}/\overline{a_i}\ \overline{a_j})$ is isolated by $\mathrm{tp}(f_{ij}/a_{ij}A)$.

The following (proved in [2]) is the key technical point saying that we have "enough" symmetric witnesses:

Proposition 4.2. *If (a_0', a_1', a_2') is the beginning of a Morley sequence of finite tuples over A and f' is a finite tuple in $\widetilde{a_{01}'} \smallsetminus \mathrm{dcl}(\overline{a_0'}, \overline{a_1'})$, then* *there is some full symmetric witness* $(a_0, a_1, a_2, f, g, h, \theta)$ *such that* $f' \in \operatorname{dcl}(fA)$ *and* $a_i' \in \operatorname{dcl}(a_i A) \subseteq \overline{a_i'}$ *for* $i = 0, 1, 2$.

Hence if the complete type p *does not have* 3*-uniqueness over* A*, then there is a symmetric witness* $(a_0, a_1, a_2, \dots)$ *over* A *such that* $a_i \in \overline{c_i}$ *for some* A*-independent realizations* c_0, c_1, c_2 *of* p.

From now on for notational simplicity, we suppress A to $\emptyset$ (by naming the set), so $\emptyset = \operatorname{acl}(\emptyset)$. *Throughout this Section 4*, we also fix a *finite* tuple b_0 and let $p(x) = \operatorname{tp}(b_0)$.

Definition 4.3. By a *generic abelian groupoid in* p, we mean an $\emptyset$-type-definable connected finitary abelian groupoid $\mathcal{G}$ such that

(1) $\operatorname{Ob}(\mathcal{G}) = p(\mathcal{M})$, $\operatorname{Mor}(\mathcal{G})$ is $\emptyset$-type-definable, and maps

$$\text{init},\ \text{ter} : \operatorname{Mor}(\mathcal{G}) \to \operatorname{Ob}(\mathcal{G})$$

indicating initial and terminal objects of a morphism respectively are relatively $\emptyset$-definable, and so is the composition map between morphisms; and

(2) for $f \in \operatorname{Mor}_{\mathcal{G}}(b_0, b_1)$ with independent b_0, b_1,

$$\operatorname{Mor}_{\mathcal{G}}(b_0, b_1) = \{g |\ g \equiv_{b_{01}} f\} = \{g |\ g \equiv_{\overline{b_0}\ \overline{b_1}} f\}.$$

We fix some more notations that we will refer to throughout the rest. Fix a symmetric witness $W = (b_0, b_1, b_2, f_{01}', f_{12}', f_{02}', \theta)$ to the failure of 3-uniqueness over $\emptyset$. The following facts are shown in [2],[3] (see also [4],[5]).

Fact 4.4. *From the witness* W*, we can construct an* $\emptyset$*-type-definable generic abelian groupoid* $\mathcal{G}$ *in* p *such that*

(1) there exist a formula $\pi(x, y; z_0 z_1)$ *over* $\emptyset$ *such that for any independent* $a, c \models p$ *and* f' *with* $f'ac \equiv f_{01}' b_0 b_1$, $\pi(x, y; ac)$ *defines a bijection*

$$\pi_{ac} : \{f |\ f \equiv_{ac} f'\} = \{f |\ f \equiv_{\bar{a}\bar{c}} f'\} \to \operatorname{Mor}_{\mathcal{G}}(a, c), \text{ and}$$

(2) θ *represents the composition, i.e., for* $f_{ij} := \pi_{b_{ij}}(f_{ij}') \in \operatorname{Mor}_{\mathcal{G}}(b_i, b_j)$, *we have* $f_{12}.f_{01} = f_{02}$.

Remark 4.5. We also fix such $\mathcal{G}$, the *abelian groupoid obtained from the witness* W. For $a, b \models p$, for convenience, X_{ab} denotes $\operatorname{Mor}_{\mathcal{G}}(a, b)$, and $X_a = X_{aa}$ denotes the vertex group $\mathcal{G}_a$. Hence if a, b are independent then X_{ab} is the solution set of $\operatorname{tp}(f/ab)$ (so of $\operatorname{tp}(f/\overline{a}\overline{b})$) where $f \in X_{ab}$.

As pointed out in Section 1, the finite binding group $G = (\bigcup\{X_a \mid a \in \operatorname{Ob}(\mathcal{G})\})/\sim$ of $\mathcal{G}$ lives in $\operatorname{acl}(\emptyset)$, which is canonically isomorphic to each group X_a. The group $(G, .)$ naturally acts on the set $\operatorname{Mor}(\mathcal{G})$.

For $f \in X_{ab}$ and $\boldsymbol{\sigma} \in G$, the ($\emptyset$-definable) left action $\boldsymbol{\sigma}.f$ is given by the composition $\sigma.f$, where σ is the unique element in $\boldsymbol{\sigma} \cap X_b$; the right action $f.\boldsymbol{\sigma}$ is given similarly. But the two actions are equal. Namely, for all $f \in X_{ab}$, $g \in X_{bc}$, for all $\boldsymbol{\sigma}, \boldsymbol{\tau} \in G$ we have $\boldsymbol{\sigma}.f = f.\boldsymbol{\sigma}$; $(g.f).\boldsymbol{\sigma} = g.(f.\boldsymbol{\sigma}) = (\boldsymbol{\sigma}.g).f$; and $f.(\boldsymbol{\sigma}.\boldsymbol{\tau}) = (f.\boldsymbol{\sigma}).\boldsymbol{\tau}$. Clearly this action on X_{ab} is regular, and $|G| = |X_{ab}|$.

We fix more notations. Given independent $a, b \models p$ and $f_{ab} \in X_{ab}$ (so $b_{01}f_{01} \equiv abf_{ab}$), we write G_{ab} to denote $\mathrm{Aut}(\mathrm{tp}(f_{ab}/\overline{a}\overline{b})) = \mathrm{Aut}(\mathrm{tp}(f_{ab}/ab))$, which is isomorphic to G [3, 2.22] and the canonical isomorphism is given below Fact 4.6(3). Note that $G_{ab} = \mathrm{Aut}(\mathrm{tp}(f'_{ab}/\overline{a}\overline{b})) = \mathrm{Aut}(\mathrm{tp}(f'_{ab}/ab))$ too, where $b_{01}f'_{01} \equiv abf'_{ab}$, since $\mathrm{dcl}(f_{ab}) = \mathrm{dcl}(f'_{ab}) \ni a, b$.

Fact 4.6. *Let $a, b \models p$ be independent.*

(1) If we let $(f_0, g_0) \sim (f_1, g_1) \in X^2_{ab}$ when there is $\boldsymbol{\sigma} \in G$ such that $g_j = \boldsymbol{\sigma}.f_j$ ($j = 0, 1$), then $\sim$ is an equivalence relation on X^2_{ab} and the map $[\ ] : X^2_{ab}/\sim \to G$ sending $[(f_j, g_j)]$ to $\boldsymbol{\sigma}$ is the unique bijection such that for $f, g, h \in X_{ab}$,

$$[(f, g)].[(g, h)] = [(f, h)].$$

(2) For any $f, g \in X_{ab}$ and $\boldsymbol{\sigma} \in G$, we have $\mathrm{dcl}(f) = \mathrm{dcl}(g)$ and $\mathrm{tp}(f, \boldsymbol{\sigma}.f) = \mathrm{tp}(g, \boldsymbol{\sigma}.g)$.

(3) There exists the canonical isomorphism $\rho_{ab} : G \to G_{ab}$ sending $\boldsymbol{\mu} \in G$ to $\mu \in G_{ab}$ such that $\boldsymbol{\mu}.f = \mu(f)$ for some (any) $f \in X_{ab}$. In other words G again uniformly and canonically binds all the groups of the form G_{ab} with independent $a, b \models p$.

Proof. (1) This easily follows from the regularity of the action of G on X_{ab}.

(2) comes from that $G \subseteq \mathrm{acl}(\emptyset)$.

(3) Here the abelianity of G is used. It needs to show that $\boldsymbol{\mu}.f = \mu(f)$ implies $\boldsymbol{\mu}.g = \mu(g)$, for any $f, g \in X_{ab}$ and $\boldsymbol{\mu} \in G$, $\mu \in G_{ab}$. Now there is $\sigma \in G_{ab}$ such that $g = \sigma(f)$. Then from (2), or directly, $\sigma(\boldsymbol{\mu}.f) = \boldsymbol{\mu}.\sigma(f) = \boldsymbol{\mu}.g$. Thus $\mu(g) = \mu \circ \sigma(f) = \sigma \circ \mu(f) = \sigma(\boldsymbol{\mu}.f) = \boldsymbol{\mu}.g$. The rest can easily be checked. □

Now we are ready to extend the construction method given in [2] to find another groupoid $\mathcal{F}$ (which in general is not $\emptyset$-type-definable but $\emptyset$-invariant) from the fixed symmetric witness W. The class $\mathrm{Ob}(\mathcal{F})$ of objects will be the same as $\mathrm{Ob}(\mathcal{G})$. But $\mathcal{F}$ need not be abelian as the vertex group $\mathrm{Mor}_{\mathcal{F}}(b_i, b_i)$ will be isomorphic to $\mathrm{Aut}(Y_{01}/\overline{b}_0)$, where Y_{01} is the *possibly infinite* set

$$Y_{b_{01}} = Y_{01} := \{f |\ f \equiv_{\overline{b_0}} f_{01} \text{ and } \mathrm{dcl}(f\overline{b_0}) = \mathrm{dcl}(f_{01}\overline{b_0})\}.$$

Note that $\mathrm{dcl}(f_{01}, \overline{b_0}) = \mathrm{dcl}(f_{01} b_1 \overline{b_0})$ since $b_1 \in \mathrm{dcl}(f_{01})$. Moreover Y_{01} and Y'_{01}, the set defined the same way as Y_{01} but substituting f'_{01} for f_{01}, are interdefinable. Furthermore, we shall see that $\mathrm{Mor}_{\mathcal{G}}(b_i, b_i) \leq Z(\mathrm{Mor}_{\mathcal{F}}(b_i, b_i))$ (Claim 4.9). We will call $\mathcal{F}$ a *non-commutative groupoid* constructed from the symmetric witness W.

Remark 4.7. The set Y_{01} defined above depends only on b_0 and b_1 and not on the choice of $f_{01} \in X_{b_{01}}$.

Proof. Due to Facts 4.4 and 4.6, even if we replace f_{01} by any $g \in X_{b_{01}}$, we obtain the same Y_{01}. ☐

Lemma 4.8. *A set $C = \{c_i\}_i$ of realizations of p with $b_0 \downarrow C$, and $g_i \in X_{b_0 c_i}$ are given. Then for $\sigma \in X_{b_0}$, there is an automorphism $\mu = \mu_\sigma$ of $\mathcal{M}$ fixing each $\overline{c_i}$ and $\overline{b_0}$ pointwise and $\mu(g_i) = g_i.\sigma$. Similarly, if $D = \{d_i\}_i (\downarrow b_0)$ is a set of realizations of p and $h_i \in X_{d_i b_0}$, then there is an automorphism τ fixing $\overline{d_i}$ and $\overline{b_0}$ such that $\tau(h_i) = \sigma.h_i$.*

Proof. Take $d \models p$ independent from $b_0 C$; and take $h \in X_{b_0 d}$. For each i, there is $h_i \in X_{d c_i}$ such that $g_i = h_i.h$. Now by stationarity we have $g_0 \equiv_{\overline{b_0}, \overline{Cd}} g_0.\sigma$ witnessed by some automorphism μ sending g_0 to $g_0.\sigma$ and fixing $\overline{b_0}, \overline{Cd}$ pointwise. Then $\mu(g_i) = \mu(h_i.h) = h_i.\mu(h)$ since $h_i \in \overline{Cd}$. Now there is unique $\tau \in X_{b_0}$ such that $\mu(h) = h.\tau$. Thus $\mu(g_0) = g_0.\sigma = h_0.h.\tau$. Hence $\sigma = \tau$. Similarly there is $\tau_i \in X_{b_0}$ such that $\mu(g_i) = g_i.\tau_i$, and then $\mu(g_i) = g_i.\tau_i = h_i.h.\sigma$. Hence $\tau_i = \sigma$, so $\mu(g_i) = g_i.\sigma$ as desired. The second clause can be proved similarly. ☐

Now consider $F_{b_{01}} = F_{01} := \mathrm{Aut}(Y_{01}/\overline{b_0})$.

Claim 4.9. *(1) $X_{b_{01}} \subseteq Y_{01} \subseteq \overline{b_{01}}$, and Y_{01} is b_{01}-invariant.*

(2) The action of F_{01} on Y_{01} (obviously by $\sigma(g)$ for $\sigma \in F_{01}$ and $g \in Y_{01}$) is regular (so $|F_{01}| = |Y_{01}|$ but can be infinite). Hence any element of $G_{01} := G_{b_{01}}$ has a unique extension in F_{01} (we may identify those two).

(3) $G_{01} \leq Z(F_{01})$. Hence for $f, k \in Y_{01}$ and $\sigma \in G_{01}$, we have $f, \sigma(f) \equiv_{\overline{b_0}} k, \sigma(k)$; and for any $\mu \in F_{01}$, b_1 and $b' = \mu(b_1)$ are interdefinable over $\overline{b_0}$.

(4) Let $f, g \in X_{b_{01}}$, so $g = \sigma(f) = f.e$ for unique $e \in X_{b_0}$ and $\sigma \in G_{01}$. Then for any $\tau \in F_{01}$, we have $f, \tau(f) \equiv_{\overline{b_0}} g, \tau(g)$; $\tau(g) = \tau(f.e) = \tau(f).e$; and $\sigma(f, \tau(f)) = (f.e, \tau(f).e)$.

(5) If we let $(f_0, g_0) \sim (f_1, g_1) \in X_{01} \times Y_{01}$ when there is (unique) $\sigma \in F_{01}$ such that $g_j = \sigma(f_j)$ ($j = 0, 1$), then $\sim$ is an equivalence relation on $X_{01} \times Y_{01}$, and the map $[\ ] : X_{01} \times Y_{01}/\sim \to F_{01}$ sending $[(f_j, g_j)]$ to σ is the unique bijection such that

$$[(f, h)] \circ [(f, g)] = [(f, k)]$$

holds for any $f \in X_{01}, g, h \in Y_{01}$ *with* $fg \equiv_{\overline{b_0}} hk$. *Since* $\sim$ *is* $\overline{b_0}b_1$*-invariant, this bijection endows* $X_{01} \times Y_{01}/\sim$ *with a* $\overline{b_0}b_1$*-invariant group structure isomorphic to* F_{01}.

Proof. (1) follows easily.

(2) comes from the fact that for any $g, g' \in Y_{01}$, they are interdefinable over $\overline{b_0}$, and $Y_{01} \subseteq \mathrm{dcl}(g\overline{b_0}) = \mathrm{dcl}(g'\overline{b_0}) = \mathrm{dcl}(f_{01}\overline{b_0})$. In particular if $(g_0, g_1, \dots)$ is an enumeration of Y_{01} then any $\sigma \in F_{01}$ is completely determined by $(g_0, g_i = \sigma(g_0))$. Hence from (1), it follows G_{01} is a subgroup of F_{01}. The rest clearly follows.

(3) Suppose $\sigma \in G_{01}, \tau \in F_{01}$ are given. Let $g = \sigma(f_{01}) = f_{01}.\sigma_0$ for some $\sigma_0 \in X_{b_0}$, and let $h = \tau(f_{01})$. Then $\tau(g) = \tau(f_{01}.\sigma_0)$, and since τ fixes $\overline{b_0}$, $\tau(f_{01}.\sigma_0) = \tau(f_{01}).\sigma_0 = h.\sigma_0$. Now by Lemma 4.8, there is an automorphism fixing $\overline{b}_0$ and sending (f_{01}, h) to $(f_{01}.\sigma_0, h.\sigma_0)$, so $(f_{01}, h) \equiv_{\overline{b_0}} (f_{01}.\sigma_0, h.\sigma_0) = (g, \tau(g))$. But since $h \in \mathrm{dcl}(f_{01}, \overline{b_0})$ and $g = \sigma(f_{01})$, we must have that $\sigma(h) = \tau(g)$, so $= \sigma\circ\tau(f_{01}) = \tau\circ\sigma(f_{01})$. Then due to regularity, we conclude $\sigma \in Z(F_{01})$.

Hence if $k = \mu(f)$ for some $\mu \in F_{01}$, then $\mu(f, \sigma(f)) = (k, \sigma(k))$, in particular $f, \sigma(f) \equiv_{\overline{b_0}} k, \sigma(k)$ (*). Now if an automorphism σ' fixes $b_1\overline{b_0}$ then clearly we can assume $\sigma' \in G_{01}$. Then by (*), σ' fixes b' too. Hence $b' \in \mathrm{dcl}(b_1\overline{b_0})$. Similarly it follows $b_1 \in \mathrm{dcl}(b'\overline{b_0})$.

(4) Due to (3), $\sigma(f, \tau(f)) = (g, \tau(g))$. Hence $f, \tau(f) \equiv_{\overline{b_0}} g, \tau(g)$. Now since τ fixes $\overline{b_0} \supseteq X_{b_0}$, particularly it fixes e. Hence $\tau(f.e) = \tau(f).\tau(e) = \tau(f).e$, and the last one follows too.

(5) Clearly $\sim$ is an equivalence relation on $X_{01} \times Y_{01}$, which is, by (4), $\overline{b_0}b_1$-invariant. Now if $g = \mu(f)$ and $h = \tau(f)$ for $\mu, \tau \in F_{01}$, then since $fg \equiv_{\overline{b_0}} hk$ and f, g are interdefinable over $\overline{b_0}$, actually $\tau(fg) = hk$. Hence $k = \tau(g) = \tau \circ \mu(f)$, and due to (2), particularly the regularity of the action, the mapping clearly is a bijection satisfying the property. Moreover the group operation on $X_{01} \times Y_{01}$ given by the isomorphic mapping is clearly $\overline{b_0}b_1$-invariant. □

One may let $(f_0, g_0) \approx (f_1, g_1) \in Y_{01}^2$ when there is (unique) $\sigma \in F_{01}$ such that $g_j = \sigma(f_j)$ $(j = 0, 1)$, then $\approx$ is an equivalence relation on Y_{01}^2 and the map $[\ ] : Y_{01}^2/\approx \to F_{01}$ sending $[(f_j, g_j)]$ to σ is the unique bijection such that $[(g, h)] \circ [(f, g)] = [(f, h)]$ for $f, g, h \in Y_{01}$. But this equivalence relation $\approx$ need not be $\overline{b_0}b_1$-invariant. Namely, given $\tau \in F_{01}$ and $f, g \in Y_{01}$, in general $\mathrm{tp}(f, \tau(f)/\overline{b_0}) \neq \mathrm{tp}(g, \tau(g)/\overline{b_0})$, so $(f, f') \equiv_{\overline{b_0}} (g, g')$ need not imply $(f, f') \approx (g, g')$ (in contrast to Claim 4.9(3),(4),(5) and Fact 4.6). Neither needs G_{01} be equal to $Z(F_{01})$ (see Example 4.18).

For the rest of the paper we fix independent $a, b \models p$ and $f_{ab} \in X_{ab}$. We define Y_{ab} just like Y_{01} but with b_{01}, f_{01} being replaced by ab, f_{ab}. Now we start to construct the new groupoid mentioned. Our first approximation of $\mathrm{Mor}_{\mathcal{F}}(a,b)$ is Y_{ab}. Beware that $Y_{ab}(\supseteq X_{ab})$ need not be definable nor type-definable. It is just an ab-invariant set. So our groupoid $\mathcal{F}$ will only be invariant, and it will be definable only under additional hypotheses (e.g. ω-categoricity).

We recall the binding group G acting on $\mathcal{G}$ as described in Remark 4.5. The action need *not* be a structure automorphism, since for $\boldsymbol{\sigma} \in G$, in general $\mathrm{id}_a \not\equiv \boldsymbol{\sigma}.\,\mathrm{id}_a \in X_a$. But $f \equiv \boldsymbol{\sigma}.f$ for $f \in X_{ab}$, and more generally Lemma 4.8 holds. As pointed out in Fact 4.6(3), there is the group isomorphism $\rho_{ab} : G \to G_{ab}$ such that $\rho_{ab}(\boldsymbol{\sigma})(f) = \boldsymbol{\sigma}.f$ for any $f \in X_{ab}$. We write σ_{ab} for $\rho_{ab}(\boldsymbol{\sigma})$. But when there is no chance of confusion, we use σ for both $\boldsymbol{\sigma} \in G$ and $\sigma_{ab} \in G_{ab}$. Moreover, σ_a denotes the unique element in $\boldsymbol{\sigma} \cap X_a$. Hence for $f \in X_{ab}$, $\sigma(f) = \sigma.f = \sigma_b.f = f.\sigma_a$

We now let $F_{ab} := \mathrm{Aut}(Y_{ab}/\bar{a})$. Then as in Claim 4.9(2), $G_{ab} \leq F_{ab}$. As just said for any $cd \equiv ab$, there is the canonical isomorphism between $\rho_{cd} \circ \rho_{ab}^{-1} : G_{ab} \to G_{cd}$. We somehow try to find extended isomorphisms between F_{ab} and F_{cd}'s in a uniform manner as follow. Fix an enumeration of $Y_{ab} = \{g_i\}_i \cup \{g'_j\}_j$ such that $X_{ab} = \{g_i\}_i$, (and *the rest construction depends on this*). Let $Y_{cd} = \{h_i\}_i \cup \{h'_j\}_j$ such that $X_{cd} = \{h_i\}_i$ and $\langle g_i\rangle^\frown\langle g'_j\rangle\bar{a}b \equiv \langle h_i\rangle^\frown\langle h'_j\rangle\bar{c}d$. Now due to regularity of the action, for each i or j there is unique μ_i^{ab} or $\mu_j^{ab} \in F_{ab}$ such that $\mu_i(g_0) = g_i$ or $\mu_j(g_0) = g'_j$. Similarly we have μ_i^{cd} or $\mu_j^{cd} \in F_{cd}$.

Claim 4.10. *The choices of μ_i^{cd}, μ_j^{cd} only depend on $\mathrm{tp}(\langle h_i\rangle\langle h'_j\rangle/\bar{c}d)$, and the correspondence $\mu_i^{ab} \mapsto \mu_i^{cd}$ or $\mu_j^{ab} \mapsto \mu_j^{cd}$ is an isomorphism from F_{ab} to F_{cd} extending $\rho_{cd} \circ \rho_{ab}^{-1}$.*

Proof. Assume $\{k_i\}_i \cup \{k'_j\}_j$ is another arrangement of Y_{cd} such that $\langle k_i\rangle^\frown\langle k'_j\rangle \equiv_{\bar{c}d} \langle h_i\rangle^\frown\langle h'_j\rangle$. Then $k_0 = \sigma(h_0)$ for some $\sigma \in G_{cd}$. Thus by Claim 4.9, we have $\sigma(h_0, \mu_i^{cd}(h_0)) = (k_0, \mu_i^{cd}(k_0))$ and so $h_0, \mu_i^{cd}(h_0) \equiv_{\bar{c}} k_0, \mu_i^{cd}(k_0)$. Then due to interdefinability, we must have $\mu_i^{cd}(k_0) = k_i$. Similarly $\mu_j^{cd}(k_0) = k'_j$, and the first clause is proved. It easily follows that the correspondence is an isomorphism. Moreover due to 4.6(2) we see that it extends $\rho_{cd} \circ \rho_{ab}^{-1}$. □

Remark 4.11. We now fix an invariant group $\mathbb{F}$ with an invariant isomorphism $\rho_{ab}^{\mathbb{F}} : \mathbb{F} \to F_{ab}$ as follows: For $uv \equiv ab$, let $X'_{uv} = \{f\bar{u}v \mid f \in X_{uv}\}$, and $Y'_{uv} = \{h\bar{u}v \mid h \in Y_{uv}\}$. Consider the following $\emptyset$-invariant set

$$\mathcal{U} := \bigcup \{X'_{uv} \times Y'_{uv} |\ uv \equiv ab\}.$$

Then we let $\mathbb{F} = \mathcal{U}/\sim$, where $\sim$ is an induced $\emptyset$-invariant equivalent relation on $\mathcal{U}$ as in Claim 4.9(5). Namely, $(f\overline{u}v, h\overline{u}v)\sim(g\overline{s}t, k\overline{s}t)$ if the two tuples realize the same type. Again as in Claim 4.9(5), $\mathbb{F}$ is endowed with an $\emptyset$-invariant group structure. Now define $\rho^{\mathbb{F}}_{cd} : \mathbb{F} \to F_{cd}$ as: $[(f\overline{c}d, h\overline{c}d)] \mapsto \mu^{cd}_i$ such that $\mu^{cd}_i(f) = h$. Due to Claim 4.10, this mapping is a well-defined group isomorphism such that $\rho^{\mathbb{F}}_{cd} \circ (\rho^{\mathbb{F}}_{ab})^{-1}$ is the correspondence defined in 4.10. Obviously there is a central group embedding $\rho : G \to \mathbb{F}$ such that $\rho^{\mathbb{F}}_{cd} \circ \rho$ extends ρ_{cd}. Due to Fact 4.6, we can regard ρ as an $\emptyset$-invariant embedding.

Now for $\boldsymbol{\mu} \in \mathbb{F}$, we use μ_{cd} or simply μ to denote $\rho^{\mathbb{F}}_{cd}(\boldsymbol{\mu})$. Clearly $(\boldsymbol{\mu} \in)\mathbb{F}$ regularly acts on $(h \in)Y_{cd}$ (via $\mu_{cd}(h)$) extending the action of G on X_{cd} described in Remark 4.5 and Fact 4.6(3).

Claim 4.12. *Assume $cd \mathop{\smile\!\!\!\!\!\!\mid}\, a$, $f_1, f_2 \in X_{cd}, g \in X_{ac}$ and $\boldsymbol{\sigma} \in G$. Then $\sigma(f_i.g) = \sigma(f_i).g = f_i.\sigma(g)$. For $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathbb{F}$ if $\mu_1(f_1) = \mu_2(f_2) \in Y_{cd}$, then $\mu_1(f_1.g) = \mu_2(f_2.g) \in Y_{ad}$.*

Proof. The first assertion follows from Remark 4.5. Now $f_1 = \sigma'(f_2)$ for unique $\sigma' \in G$. Hence $\boldsymbol{\mu}_2 = \boldsymbol{\mu}_1 \circ \sigma'$, and $\mu_2(f_2.g) = \mu_1(\sigma'(f_2.g)) = \mu_1(f_1.g)$. □

Assume now $c(\models p) \mathop{\smile\!\!\!\!\!\!\mid}\, ab$, and $g \in Y_{ab}, h \in Y_{bc}$ are given. We want to define a composition $h.g \in Y_{ac}$ extending that for $\mathcal{G}$. Note now $g = \tau_0(g_0)$ and $h = \sigma_0(h_0)$ for some $\boldsymbol{\tau_0}, \boldsymbol{\sigma_0} \in \mathbb{F}$ and $g_0 \in X_{ab}, h_0 \in X_{bc}$. We define $h.g := (\sigma_0 \circ \tau_0)(h_0.g_0)$ (where obviously $\sigma_0 \circ \tau_0 = \rho^{\mathbb{F}}_{ac}(\boldsymbol{\sigma_0} \circ \boldsymbol{\tau_0})$). One has to be cautious here that even if we choose $g \in X_{ab}$, *$h.g$ defined here need not be equal to $h.g$ defined in $\mathcal{G}$* unless $h \in X_{bc}$, since the latter may not be in Y_{ac}.

Claim 4.13. *The composition map is well-defined, invariant under any (A-)automorphism of $\mathcal{M}$, and extends that on* $\mathrm{Mor}(\mathcal{G})$*. For any $f \in Y_{ac}$, there is unique $h' \in Y_{bc}$ ($g' \in Y_{ab}$, resp.) such that $f = h'.g$ ($f = h.g'$ resp.).*

Proof. Let $g = \tau_1(g_1)$ and $h = \sigma_1(h_1)$ for some $\boldsymbol{\tau_1}, \boldsymbol{\sigma_1} \in \mathbb{F}$ and $g_1 \in X_{ab}, h_1 \in X_{bc}$. Then since $\sigma_0^{-1} \circ \sigma_1(h_1) = h_0$ and $\tau_0^{-1} \circ \tau_1(g_1) = g_0$, due to uniqueness we have that both $\boldsymbol{\sigma_0^{-1}} \circ \boldsymbol{\sigma_1}$, $\boldsymbol{\tau_0^{-1}} \circ \boldsymbol{\tau_1}$ are in $\rho(G)$ so in the center of $\mathbb{F}$. Now due to Claim 4.12,

$$\begin{aligned}
\sigma_0 \circ \tau_0(h_0.g_0) &= \sigma_0 \circ \tau_0 \circ \sigma_0^{-1} \circ \sigma_0(h_0.g_0) &&= \sigma_0 \circ \tau_0 \circ (\sigma_0^{-1} \circ \sigma_1)(h_1.g_0) \\
&= \sigma_1 \circ \tau_0(h_1.g_0) &&= \sigma_1 \circ \tau_1 \circ (\tau_1^{-1} \circ \tau_0)(h_1.g_0) \\
&= \sigma_1 \circ \tau_1(h_1.(\tau_1^{-1} \circ \tau_0)(g_0)) &&= \sigma_1 \circ \tau_1(h_1.(\tau_1^{-1}(\tau_1(g_1)))) \\
&= \sigma_1 \circ \tau_1(h_1.g_1).
\end{aligned}$$

The automorphism invariance of the composition follows from Claim 4.10. Moreover by taking $\tau_0 = \sigma_0 = \mathrm{id}$, we see that the composition extends that for $\mathcal{G}$. Lastly $f = \tau(f_1)$ for some $f_1 \in X_{ac}$. Now there is $h_1' \in X_{bc}$ such that $f_1 = h_1'.g_1$. Put $h' = \tau \circ \tau_1^{-1}(h_1')$. Then by the definition, $f = (\tau \circ \tau_1^{-1}) \circ \tau_1(h_1'.g_1) = h'.g$. For any $h''(\neq h') \in Y_{bc}$ it easily follows that $f \neq h''.g$. Hence h' is unique such element. $\square$

The rest of the construction of $\mathcal{F}$ will be similar to that of $\mathcal{G}$ in [2]. $\mathrm{Ob}(\mathcal{F})$ will be the same as $\mathrm{Ob}(\mathcal{G}) = p(\mathcal{M})$. Now for arbitrary $c, d \models p$, an *n-step directed path* from c to d is a sequence $(c_0, g_1, c_1, g_2 ..., c_n)$ such that $c = c_0, d = c_n, c_{i-1}c_i \equiv ab$ and $g_i \in Y_{c_{i-1}c_i}$. Let $D^n(c,d)$ be the set of all n-step directed paths. For $q = (c_0, g_1, c_1, g_2 ..., c_n) \in D^n(c,d)$ and $r = (d_0, h_1, d_1, h_2 ..., d_m) \in D^m(c,d)$ we say they are equivalent (write $r \sim q$) if for some $c^*(\models p) \mathop{\smile\!\!\!\!\!\mid}\, qr$ and $g^* \in Y_{c^*c}$, we have $g_n^* = h_m^* \in Y_{c^*d}$ where $g_0^* = h_0^* = g^*$ and $g_{i+1}^* = g_{i+1}.g_i^*$ $(i = 0, ..., n-1)$ and $h_{j+1}^* = h_{j+1}.h_j^*$ $(j = 0, ..., m-1)$. Due to stationarity the relation is independent from the choices of c^* and g^*, and is an equivalence relation. Similarly to Lemma [2, 2.12], one can easily see using Claim 4.13 that for any $q \in D^n(c,d)$, there is $r \in D^2(c,d)$ such that $q \sim r$. Then $D^2(c,d)/\sim$ will be our $\mathrm{Mor}_{\mathcal{F}}(c,d)$, and composition will be concatenation of paths. The identity morphism in $\mathrm{Mor}_{\mathcal{F}}(c,c)$ can be defined just like in [2, 2.15]. Now our groupoid $\mathcal{F}$ is clearly connected, and it extends $\mathcal{G}$ (see Proposition 4.16). An argument similar to that in [2, 2.14] implies there is a canonical ab-invariant 1-1 correspondence between Y_{ab} and $\mathrm{Mor}_{\mathcal{F}}(a,b)$. Indeed the same argument shows that for any $c, d \models p$ (not necessarily independent), there too exists a canonical injection from X_{cd} to $\mathrm{Mor}_{\mathcal{F}}(c,d)$.

Now for $f \in Y_{ab}$ (or $\in X_{cd}$, resp.), in the rest *we let $\underline{f}$ denote the corresponding element in* $\mathrm{Mor}_{\mathcal{F}}(a,b)$ (or $\mathrm{Mor}_{\mathcal{F}}(c,d)$, resp.). Note that $\mathcal{F}$ need not be definable nor type-definable nor hyperdefinable. It is just an (A-)invariant groupoid. Now for notational simplicity, write $\underline{Y}_{cd}$ to denote $\mathrm{Mor}_{\mathcal{F}}(c,d)$, and write $\underline{Y}_c$ for $\underline{Y}_{cc}$. We state some observations regarding $\mathcal{F}$.

Remark 4.14. (1) Note that for $\mu \in F_{ab}$ and $f \in Y_{ab}$, we have $\mu(f) \in Y_{ab}$ and both $\underline{f}, \underline{\mu(f)} \in \underline{Y}_{ab}$. However in general we can not consider F_{ab} as a group of *elementary* maps on $\underline{Y}_{ab}$, since $\underline{f}$ is of the form $(a, g_1, c, g_2, b)/\sim$ and then $\mu(\underline{f}) \in \underline{Y}_{a\mu(b)}$. But still we can consider a group $\underline{F_{ab}} := \{\underline{\mu} : \mu \in F_{ab}\}$ of *permutations* on $\underline{Y}_{ab}$ by letting $\underline{\mu}(\underline{f}) := \underline{\mu(f)}$. Obviously, the mapping $\mu \mapsto \underline{\mu}$ is a group isomorphism between F_{ab} and $\underline{F_{ab}}$.

(2) Notice that for $\mu \in F_{ab}$ and $f, g \in Y_{ab}$, we have $\underline{\mu(g)}(\in \underline{Y}_{ab}) = k.\mu(\underline{g})$ for some $k \in \underline{Y}_{\mu(b)b}$. Then as the composition in $\mathcal{F}$ is concatenation of paths, it follows $\underline{\mu(f)} = k.\mu(\underline{f})$. Hence it too follows $\underline{\mu(g)} = \underline{\mu(f)}.x$ if $\underline{g} = \underline{f}.x$ for some $x \in \underline{Y}_a$.

(3) We know that $\underline{Y}_{ab} \subseteq \overline{ab}$. For any $c \models p$, we too have $\underline{Y}_{bc} \subseteq \overline{bc}$: We can assume $a \mathop{\smile\!\!\!\!\!\!|}\, bc$. Now let $f \in \underline{Y}_{bc}$. Suppose that $f \notin \overline{bc}$, and let $\{f_i\}$ be a set of infinitely many conjugates of f over $\bar{b}\bar{c}$. Now due to stationarity, we can then assume that all f_i's have the same type over $\bar{c} \cup \overline{ba}$. Hence given $x \in \underline{Y}_{ab}$, all $f_i.x \in \underline{Y}_{ac}$ have the same type over ac, contradicting $f_i.x \in \overline{ac}$.

We get now the following results for $\mathcal{F}$ similarly to those of $\mathcal{G}$.

Proposition 4.15. *The group F_{ab} is isomorphic to $\underline{Y}_a$. In fact for any $\sigma \in \underline{Y}_b$, there is $\sigma_b \in F_{ab}$ such that for any $f \in Y_{ab}$, $\underline{\sigma_b(f)} = \sigma.\underline{f}$. Hence $F_{ab} = \{\sigma_b |\ \sigma \in \underline{Y}_b\}$.*

Proof. The proof will be similar to that of Claim 2.4. Define a map $\eta : \underline{Y}_b \to F_{ab}$ such that for $\sigma \in \underline{Y}_b$ and any $f \in Y_{ab}$, we let $\eta(\sigma)(f) = g$ where $\underline{g} = \sigma.\underline{f}$. We claim now that η is well-defined: For $h \in Y_{ab}$, there is $x \in \underline{Y}_a$ such that $\underline{h} = \underline{f}.x$ Then due to Remark 4.14(2), we have $\underline{\eta(\sigma)(h)} = \underline{\eta(\sigma)(f)}.x = \underline{g}.x = \sigma.\underline{f}.x = \sigma.\underline{h}$, and thus $\eta(\sigma)(h) = k$ where $\underline{k} = \sigma.\underline{h}$.

Moreover clearly η is 1-1 and onto since any $\mu \in F_{ab}$ is determined by $(f_{ab}, \mu(f_{ab}))$. It is obvious η is in fact a group isomorphism. Now we take $\sigma_b = \eta(\sigma)$. □

Proposition 4.16. *For $c(\models p) \mathop{\smile\!\!\!\!\!\!|}\, ab$ and $f \in Y_{ab}, g \in Y_{bc}, h \in Y_{ac}$, we have $h = g.f$ (the composition map is defined before Claim 4.13) iff $\underline{h} = \underline{g}.\underline{f}$. Moreover, $\mathcal{F}$ extends the composition of $\mathcal{G}$.*

Proof. Since the composition relation defined in 4.13 is invariant relation, we can find an $\emptyset$-invariant relation $\theta(x, y, z)$ such that for any $a'b'c' \equiv abc$ and $f' \in Y_{a'b'}, g' \in Y_{b'c'}, h' \in Y_{a'c'}$, we have $h' = g'.f'$ iff $\theta(a'b'f', b'c'g', a'c'h')$ holds. Then the rest proof of the proposition will be exactly the same as that of [3, 2.19], hence we omit it.

We now step by step show that $\mathcal{F}$ extends the composition of $\mathcal{G}$. Let $c \models p$ be given such that $ac \mathop{\smile\!\!\!\!\!\!|}\, b$. Let $y \in X_{ab}, z \in X_{bc}$. Choose $a'(\models p) \mathop{\smile\!\!\!\!\!\!|}\, abc$. Then for each $x \in X_{a'a}$, we have $(z.y).x \in X_{a'c}$. Thus by the definitions of the concatenating composition and the injection from X_{ac} to $\underline{Y}_{ac}$, it follows $\underline{z}.\underline{y} = \underline{z.y}$ in $\mathcal{F}$.

Now more generally let $d \models p$ be given such that $acd \mathop{\smile\!\!\!\!\!\!|}\, b$. Let $s \in X_{ac}$, $t \in X_{cd}$. Choose $u \in X_{ab}$. Then there are $v \in X_{bc}$ and

$w \in X_{bd}$ such that $s = v.u \in X_{bc}$, and $w.v^{-1} = t$ (in $\mathcal{G}$). Then by the previous argument, in $\mathcal{F}$, we have $\underline{t}.\underline{s} = \underline{w.v^{-1}}.\underline{v.u} = \underline{w}.\underline{v^{-1}}.\underline{v}.\underline{u} = \underline{w}.\underline{v}^{-1}.\underline{v}.\underline{u} = \underline{w.u} = \underline{t.s}$. □

Remark 4.17. We have seen that $\mathcal{F}$ is an $\emptyset$-invariant groupoid, and there is an $\emptyset$-invariant faithful functor $I : \mathcal{G} \to \mathcal{F}$ such that it is the identity map on $\mathrm{Ob}(\mathcal{G}) = \mathrm{Ob}(\mathcal{F}) = p(\mathcal{M})$ and for $f \in \mathrm{Mor}(\mathcal{G})$, $I(f) = \underline{f}$. Indeed $\mathcal{G}$ is a central subgroupoid of $\mathcal{F}$, so as pointed out in Remark 2.1(3), there is an $\emptyset$-invariant *binding* group $\widehat{F}$ isomorphic to each of F_{ab}, $\underline{Y}_a$, and $\mathbb{F}$ (defined in Remark 4.11). But one should notice that in general there is no analogous isomorphism from $\widehat{F}$ to F_{ab} extending the property in Fact 4.6(3), since F_{ab} need not be a group of automorphisms of $\underline{Y}_{ab}$ as pointed out in Remark 4.14(1) (but there is an analogous isomorphism between $\widehat{F}$ and $\underline{F_{ab}}$).

We now give an example where X_a is a proper subgroup of $Z(\underline{Y_a})$.

Example 4.18. Consider the same example $\mathcal{H} = (O, M, ., \mathrm{init}, \mathrm{ter})$ as in Section 2, but where the underlying finite group is a non-trivial abelian group H. Namely $\mathcal{H}$ is a connected finitary abelian groupoid. We add one more sort I and an equivalence relation E on I such that each E-class has 2 elements, and there also is a projection function $\pi_E : I \to O$ (all are in the language) so that $O = I/E$. We let $\mathcal{H}_I$ be the resulting extended structure. Now choose $c \neq d \in O$, $f \in \mathrm{Mor}(c, d)$, and let $\{c_0, c_1\} = \pi^{-1}(c)$, and $\{d_0, d_1\} = \pi^{-1}(d)$. Note that since $\mathcal{H}$ is abelian, as explained in Section 2, $\mathrm{tp}(f/\overline{c}\overline{d})$ is isolated by $\mathrm{tp}(f/cd)$ both in $\mathcal{H}$ and $\mathcal{H}_I$. Hence $(c_0c_1c, d_0d_1d, \dots, c_0c_1cfd_0d_1d, \dots, .(\bar{u}x\bar{v}, \bar{v}y\bar{w}, \bar{u}z\bar{w}))$ where $.(\bar{u}x\bar{v}, \bar{v}y\bar{w}, \bar{u}z\bar{w})$ is a formula realized by $(c_0c_1cfd_0d_1d, \dots, \dots)$ indicating the composition $z = y.x$, is a symmetric witness over $\mathrm{acl}(\emptyset)$ in $\mathcal{H}_I$. Let $\mathcal{G}$ be the abelian groupoid obtained from the symmetric witness. Then clearly $X_{c_0c_1c}$ is isomorphic to H. Let $\mathcal{F}$ be the groupoid as we constructed in this section from $\mathcal{G}$. Then since an automorphism of $\mathcal{H}_I$ can swap d_0, d_1 while fixing $\bar{c} = \overline{c_0c_1c}$, we have that H is central in $F := \underline{Y}_{c_0c_1c}$, while H has only two conjugates in F. Then it easily follows that F is abelian too, so $H \lneq Z(F) = F$.

Before finishing this section, we sketch another (possibly more direct) way to construct an $\emptyset$-invariant groupoid $\mathcal{F}'$ isomorphic to $\mathcal{F}$, using the method in Section 3. (We thank an anonymous referee suggesting this argument.) We use the group $\mathbb{F}$ and the central group embedding $\rho : G \to \mathbb{F}$ both $\emptyset$-invariantly defined in Remark 4.11. Then clearly the groupoid $\mathcal{F}' := \mathcal{G} \otimes_\rho \mathbb{F}$ introduced in Remark 3.4 can be constructed in the model $\emptyset$-invariantly. Namely, the equivalence relation $\approx$ and the composition defined on $\mathrm{Mor}(\mathcal{F}')$ are $\emptyset$-invariant. Moreover the canonical functor from $\mathcal{G}$ to $\mathcal{F}'$ described in 3.4.(2) is also $\emptyset$-invariant. In addition we argue that $\mathcal{F}'$ and $\mathcal{F}$ are $\emptyset$-invariantly isomorphic: Define a functor $J : \mathcal{F}' \to \mathcal{F}$ as follows.

- $J : \mathrm{Ob}(\mathcal{F}') \to \mathrm{Ob}(\mathcal{F})$ is the identity map.
- For $c, d \in \mathrm{Ob}(\mathcal{F}')$, define $J : \mathrm{Mor}_{\mathcal{F}'}(c, d) \to \mathrm{Mor}_{\mathcal{F}}(c, d) = \underline{Y}_{cd}$ by sending $\langle g, f\rangle$ ($g \in X_{cd}$ so $\underline{g} \in \underline{Y}_{cd}$, and $f \in \mathbb{F}$) to $\underline{g}.\sigma$ where $\sigma \in \underline{Y}_c$ is the image of f under the canonical isomorphism $\mathbb{F} \to \widehat{F} \to \underline{Y}_c$ described in Remark 4.17.

It is not hard to check this functor is well-defined and gives an isomorphism between $\mathcal{F}'$ and $\mathcal{F}$.

5. Approximation of the non-commutative groups

In this last section we discuss a possible limit of the vertex groups of the non-commutative groupoids we have constructed in previous section. As before, we keep suppressing $A = \mathrm{acl}(A)$ to $\emptyset$. Fix a complete type q (of possibly infinite arity) over $\emptyset$. Choose independent $u, v, w \models q$. Recall that

$$\Gamma_2(q) := \mathrm{Aut}(\widetilde{uv}/\overline{u}, \overline{v}),$$

where $\widetilde{uv} := \overline{uv} \cap \mathrm{dcl}(\overline{uw}, \overline{vw})$.

The following fact is simply a restatement of Proposition 4.2 and Fact 4.4.

Fact 5.1. *Let a finite tuple $f \in \widetilde{uv} \smallsetminus \mathrm{dcl}(\overline{u}, \overline{v})$ be given. Then there are a generic abelian groupoid $\mathcal{G}'$ in $q' \in S(\emptyset)$ of finite arity; and independent $u', v' \models q'$ with $f' \in \mathrm{Mor}_{\mathcal{G}'}(u', v')$ such that $f \in \mathrm{dcl}(f')$, and $u' \subseteq \overline{u}$, $v' \subseteq \overline{v}$.*

We let

$I = I_q := \{f \in \widetilde{u_f, v_f} :$ $\mathcal{G}_f$ is a generic abelian groupoid in $\mathrm{tp}(u_f) = \mathrm{tp}(v_f)$ such that $f \in \mathrm{Mor}_{\mathcal{G}_f}(u_f, v_f)$ with independent finite tuples $u_f (\subseteq \overline{u}), v_f (\subseteq \overline{v})\}$.

On the other hand we let,

$J = J_q := \{f \in \widetilde{u_f, v_f} :$ $\mathcal{F}_f$ is a non-commutative groupoid obtained from the generic abelian groupoid $\mathcal{G}_f$ in $\mathrm{tp}(u_f) = \mathrm{tp}(v_f)$ such that $f \in \mathrm{Mor}_{\mathcal{F}_f}(u_f, v_f)$ with independent finite tuples $u_f (\subseteq \overline{u}), v_f (\subseteq \overline{v})\}$.

For $f \in I_q$, we write G_f to denote $G_{u_f v_f}$ as in Section 4. Now by Fact 5.1, $(I, \leq_I)$ with letting $f \leq_I f'$ iff $f \in \mathrm{dcl}(f')$, $\mathrm{init}(f) \in \mathrm{dcl}(\mathrm{init}(f'))$, and $\mathrm{ter}(f) \in \mathrm{dcl}(\mathrm{ter}(f'))$ is a direct system.

Now for $f \leq_I f' \in I$, any $\sigma' \in G_{f'}$ fixes $u_{f'} v_{f'}$ pointwise. Hence $\sigma'(f) \in X_{u_f v_f}$, and we write $(\sigma' \restriction f)$ to denote the unique $\sigma \in G_f$

such that $\sigma(f) = \sigma'(f)$; and $\chi_f^{f'} : G_{f'} \to G_f$ to denote the group homomorphism sending σ' to $(\sigma' \restriction f)$. Due to stationarity it indeed is an epimorphism. Clearly $\chi_f^f = \mathrm{id}$. Then

$$\mathcal{S}_I := (\{G_f |\ f \in I\}, \{\chi_f^{f'} |\ f \leq_I f' \in I\})$$

forms a directed system of finite abelian groups. As pointed out in [3, Theorem 2.25], the inverse limit of $\mathcal{S}_I$ is isomorphic to $\Gamma_2(q)$, so it is a profinite abelian group.

However for $\{F_f |\ f \in J\}$ where $F_f := F_{u_f v_f}$ as in Section 3, it is not clear how to give an order relation and transition maps to make this a directed system of groups. There are a couple of obstacles to do this. For example, in general given an elementary map σ of $\mathcal{M}$, and a tuple cd, even if cd and $\sigma(cd)$ are interdefinable, c and $\sigma(c)$ need not be so, and vice versa. But we can consider *partial* transition maps among F_f's and their limit as follows. We let

$$\Pi_2(q) := \{\sigma \in \mathrm{Aut}(\widetilde{uv}/\overline{u}) :\ \text{for any } f \in J,\ \mathrm{dcl}(f\overline{u}) = \mathrm{dcl}(\sigma(f)\overline{u})\}.$$

For $f \in J$ and $\sigma \in \Pi_2(q)$, we similarly write $(\sigma \restriction f)$ to denote the unique $\sigma' \in F_f$ such that $\sigma(f) = \sigma'(f)$. We let

$$\Pi_f := \{(\sigma \restriction f) |\ \sigma \in \Pi_2(q)\}.$$

Proposition 5.2. *The following hold.*

(1) $\Pi_2(q) := \{\sigma \in \mathrm{Aut}(\widetilde{uv}/\overline{u}) :$ *for any* $f \in I$, $\mathrm{dcl}(f\overline{u}) = \mathrm{dcl}(\sigma(f)\overline{u})\}$.

(2) For $f \in J$, *we have* $G_f \leq \Pi_f \leq F_f$. *Now*

$$\mathcal{S}_J := (\{\Pi_f |\ f \in J\}, \{\chi_f^{f'} |\ f \leq_J f' \in J\})$$

forms a directed system of groups, where $\leq_J$ *and* $\chi_f^{f'}$ *are similarly defined as in* $\mathcal{S}_I$. *Moreover* $\Pi_2(q)$ *is the inverse limit of* $\mathcal{S}_J$.

(3) $\Gamma_2(q) \leq Z(\Pi_2(q))$.

(4) Both $\Gamma_2(q)$ *and* $\Pi_2(q)$ *are normal subgroups of* $\mathrm{Aut}(\widetilde{uv}/\overline{u})$.

Proof. (1) Clear (see Claim 4.9(3)).

(2) That $\Pi_f \leq F_f$ is clear by definition, and that $G_f \leq \Pi_f$ is also clear since $\Gamma_2(q) \leq \Pi_2(q)$. Now since every automorphism in Π_f is the restriction of that in $\Pi_2(q)$, it follows that $\mathcal{S}_J$ forms a directed system of groups with the transition maps $\chi_f^{f'}$. The rest proof of (2) is standard. Let Π be the inverse limit of $\mathcal{S}_J$. We define a homomorphism $\varphi : \Pi_2(q) \to \Pi$ by sending $\sigma \in \Pi_2(q)$ to the element in Π represented by the function $f \in J \mapsto (\sigma \restriction f) \in \Pi_f$. This embedding is obviously one-to-one and due to compactness it is surjective too.

(3) comes from (2) and that $G_f \leq Z(\Pi_f)$.

(4) Let $\sigma \in \Gamma_2(q)$. Then any of its conjugates in $\mathrm{Aut}(\widetilde{uv}/\overline{u})$ fixes $\overline{v}$ pointwise. Hence $\Gamma_2(q) \trianglelefteq \mathrm{Aut}(\widetilde{uv}/\overline{u})$.

Now let $\sigma \in \Pi_2(q)$ and $\mu \in \mathrm{Aut}(\widetilde{uv}/\overline{u})$. Then for any $f \in I$, we have $g := \mu(f) \in I$ too, and g and $\sigma(g)$ are interdefinable over $\overline{u}$. Hence so are f and $\mu^{-1} \circ \sigma(g) = \mu^{-1} \circ \sigma \circ \mu(f)$ over $\overline{u}$. Therefore $\mu^{-1} \circ \sigma \circ \mu \in \Pi_2(q)$ and (4) is proved. □

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