Amalgamation functors and homology groups in model theory

John Goodrick, Byunghan Kim[*] and Alexei Kolesnikov[†]

Abstract. We introduce the concept of an *amenable class of functors* and define homology groups for such classes. Amenable classes of functors arise naturally in model theory from considering types of independent systems of elements. Basic lemmas for computing these homology groups are established, and we discuss connections with type amalgamation properties.

Mathematics Subject Classification (2010). Primary 03C45; Secondary 55N35.

Keywords. Amalgamation functors, homology groups, model theory, Hurewicz correspondence, groupoids.

This paper abstracts the model theoretic results from [6] to a more general category-theoretic context. Namely, we introduce the concept of an *amenable class of functors.* It is a class of functors from a family of finite sets which is closed under subsets into a fixed image category satisfying a short list of axioms. We show that most of the general results proved in [6] hold in the broader *amenable* context. In addition we give some fundamental lemmas and examples which supplement the results of [6].

In Section 1, we introduce the notion of an amenable class of functors into a fixed category and we define the homology groups $H_n(\mathcal{A}, B)$ for an amenable class $\mathcal{A}$ and an object B in the image category. Model theory provides the best examples of amenable classes of functors, as described in [6].

In Section 2, given a rosy structure, we introduce the notion of the *type* homology groups, in contrast to the *set* homology groups defined in [6]. We show that the two homology groups are isomorphic.

Section 3 supplies some basic sufficient conditions for triviality of the homology groups and gives some additional examples of homology groups from model theory.

In Section 4 we outline some ongoing investigations related to our homology theory.

[*]The 2nd author was supported by NRF of Korea grant 2013R1A1A2073702, and Samsung Science Technology Foundation grant BA1301-03.

[†]The third author was partially supported by NSF grant DMS-0901315.

1. Simplicial homology in a category

In this section, we generalize the homology groups for rosy theories defined in [6] to a more general category-theoretic setting. Then we aim to provide a general framework for homology group computations. This section uses model theory only as a source of examples.

1.1. Basic definitions and facts.

Throughout this section, $\mathcal{C}$ denotes a category. If s is a set, then we consider the power set $\mathcal{P}(s)$ of s to be a category with a single inclusion map $\iota_{u,v} : u \to v$ between any pair of subsets u and v with $u \subseteq v$. A subset $X \subseteq \mathcal{P}(s)$ is called *downward-closed* if whenever $u \subseteq v \in X$, then $u \in X$. In this case we consider X to be a full subcategory of $\mathcal{P}(s)$. An example of a downward-closed collection that we will use often below is $\mathcal{P}^-(s) := \mathcal{P}(s) \setminus \{s\}$. We use ω for the set of natural numbers.

We are interested in a family of functors $f : X \to \mathcal{C}$ for downward-closed subsets $X \subseteq \mathcal{P}(s)$ for various finite subset sets s of the set of natural numbers. For $u \subseteq v \in X$, we shall write $f^u_v := f(\iota_{u,v}) \in \mathrm{Mor}_{\mathcal{C}}(f(u), f(v))$. Before specifying the desirable closure properties of the collection $\mathcal{A}$ of such functors, we need some auxiliary definitions.

Definition 1.1. (1) Let X be a downward closed subset of $\mathcal{P}(s)$ and let $t \in X$. The symbol $X|_t$ denotes the set $\{u \in \mathcal{P}(s \setminus t) \mid t \cup u \in X\} \subseteq X$.

(2) For s, t, and X as above, let $f : X \to \mathcal{C}$ be a functor. Then *the localization of f at t* is the functor $f|_t : X|_t \to \mathcal{C}$ such that

$$f|_t(u) = f(t \cup u)$$

and whenever $u \subseteq v \in X|_t$,

$$(f|_t)^u_v = f^{u\cup t}_{v\cup t}.$$

(3) Let $X \subset \mathcal{P}(s)$ and $Y \subset \mathcal{P}(t)$ be downward closed subsets, where s and t are finite sets of natural numbers. Let $f : X \to \mathcal{C}$ and $g : Y \to \mathcal{C}$ be functors.

We say that f and g are *isomorphic* if there is an order-preserving bijection $\sigma : s \to t$ with $Y = \{\sigma(u) : u \in X\}$ and a family of isomorphisms $\langle h_u : f(u) \to g(\sigma(u)) : u \in X\rangle$ in $\mathcal{C}$ such that for any $u \subseteq v \in X$, the following diagram commutes:

$$\begin{tikzcd}
f(u) \arrow[r, "h_u"] \arrow[d, "f^u_v"] & g(\sigma(u)) \arrow[d, "g^{\sigma(u)}_{\sigma(v)}"] \\
f(v) \arrow[r, "h_v"] & g(\sigma(v))
\end{tikzcd}$$

In the definition if σ is an arbitrary bijection, then f and g are said to be *weakly isomorphic.*

Remark 1.2. If X is a downward closed subset of $\mathcal{P}(s)$ and $t \in X$, then $X|_t$ is a downward closed subset of $\mathcal{P}(s \setminus t)$. Moreover $X|_t$ does not depend on the choice of s.

Definition 1.3. Let $\mathcal{A}$ be a non-empty set of functors $f : X \to \mathcal{C}$ such that $X \subseteq \mathcal{P}(\omega)$ is finite and downward closed and $\mathcal{C}$ is a fixed category (called the *image category of* $\mathcal{A}$). We say that $\mathcal{A}$ is *amenable* if it satisfies all of the following properties:

1. (Invariance under weak isomorphisms) If $f : X \to \mathcal{C}$ is in $\mathcal{A}$ and $g : Y \to \mathcal{C}$ is weakly isomorphic to f, then $g \in \mathcal{A}$.

2. (Closure under restrictions and unions) If $X \subseteq \mathcal{P}(s)$ is downward-closed and $f : X \to \mathcal{C}$ is a functor, then $f \in \mathcal{A}$ if and only if for every $u \in X$, we have that $f \restriction \mathcal{P}(u) \in \mathcal{A}$.

3. (Closure under localizations) Suppose that $f : X \to \mathcal{C}$ is in $\mathcal{A}$ for some $X \subseteq \mathcal{P}(s)$ and $t \in X$. Then $f|_t : X|_t \to \mathcal{C}$ is also in $\mathcal{A}$.

4. (Extensions of localizations are localizations of extensions) Suppose that $f : X \to \mathcal{C}$ is in $\mathcal{A}$ and $t \in X \subseteq \mathcal{P}(s)$ is such that $X|_t = X \cap \mathcal{P}(s \setminus t)$. Suppose that the localization $f|_t : X \cap \mathcal{P}(s \setminus t) \to \mathcal{C}$ has an extension $g : Z \to \mathcal{C}$ in $\mathcal{A}$ for some $Z \subseteq \mathcal{P}(s \setminus t)$. Then there is a map $g_0 : Z_0 \to \mathcal{C}$ in $\mathcal{A}$ such that $Z_0 = \{u \cup v : u \in Z, v \subseteq t\}$, $f \subseteq g_0$, and $g_0|_t = g$.

Remark 1.4. Model theory supplies the best examples of the amenable collection of functors. For example, as in [6] we could take $\mathcal{C}$ to be all boundedly (or algebraically) closed subsets of the monster model of a first-order theory, and let $\mathcal{A}$ be all functors which are "independence-preserving" (in Hrushovski's terminology [11]) and such that every face $f(u)$ is the bounded (or algebraic) closure of its vertices; then $\mathcal{A}$ is amenable. We may further restrict $\mathcal{A}$ by requiring, for instance, that all the "vertices" $f(\{i\})$ of functors $f \in \mathcal{A}$ realize the same type, or by placing further restrictions on "edges" $f(\{i,j\})$ and other higher-dimensional faces. We can also take $\mathcal{C}$ to be a category of *types* of the closed subsets of the model. These examples will be explained more precisely in Section 2.

Definition 1.5. Suppose that $f : X \to \mathcal{C}$ is a functor from a downward-closed collection X of sets and $B \in \mathrm{Ob}(\mathcal{C})$. If $f(\emptyset) = B$ then we say that f *is over* B. Let $\mathcal{A}_B$ denote the set of all functors $f \in \mathcal{A}$ that are over B.

Remark 1.6. It is easy to see that condition (2) in Definition 1.3 is equivalent to the conjunction of the following two conditions:

(Closure under restrictions) If $f : X \to \mathcal{C}$ is in $\mathcal{A}$ and $Y \subseteq X$ with Y downward-closed, then $f \restriction Y$ is also in $\mathcal{A}$.

(Closure under unions) Suppose that $f : X \to \mathcal{C}$ and $g : Y \to \mathcal{C}$ are both in $\mathcal{A}$ and that $f \restriction X \cap Y = g \restriction X \cap Y$. Then the union $f \cup g : X \cup Y \to \mathcal{C}$ is also in $\mathcal{A}$.

For instance, if these two conditions are true and $f : X \to \mathcal{C}$ is a functor from a downward-closed set X such that $f \restriction \mathcal{P}(u) \in \mathcal{A}$ for every $u \in X$, then if $u_1, \ldots, u_n$ are maximal sets in X, we can use closure under unions $(n-1)$ times to see that $f \in \mathcal{A}$ (since it is the union of the functors $f \restriction \mathcal{P}(u_i)$).

For the remainder of this section, we fix a category $\mathcal{C}$ and a non-empty amenable collection $\mathcal{A}$ of functors mapping into $\mathcal{C}$. As we mentioned in the above remark, every functor in $\mathcal{A}$ can be described as the union of functors whose domains are $\mathcal{P}(s)$ for various finite sets s. Such functors will play a central role in this paper.

Definition 1.7. Let $n \geq 0$ be a natural number. A *(regular) n-simplex in $\mathcal{C}$* is a functor $f : \mathcal{P}(s) \to \mathcal{C}$ for some set $s \subseteq \omega$ with $|s| = n + 1$. The set s is called the *support of f*, or $\operatorname{supp}(f)$.

Let $S_n(\mathcal{A}; B)$ denote the collection of all regular n-simplices in $\mathcal{A}_B$. Then let $S(\mathcal{A}; B) := \bigcup_n S_n(\mathcal{A}; B)$ and $S(\mathcal{A}) := \bigcup_{B \in \mathrm{Ob}(\mathcal{C})} S(\mathcal{A}; B)$.

Let $C_n(\mathcal{A}; B)$ denote the free abelian group generated by $S_n(\mathcal{A}; B)$; its elements are called *n-chains in $\mathcal{A}_B$*, or *n-chains over B*. Similarly, we define $C(\mathcal{A}; B) := \bigcup_n C_n(\mathcal{A}; B)$ and $C(\mathcal{A}) := \bigcup_{B \in \mathrm{Ob}(\mathcal{C})} C(\mathcal{A}; B)$. The *support of a chain* c is the union of the supports of all the simplices that appear in c with a non-zero coefficient.

The adjective "regular" in the definition above is to emphasize that none of our simplices are "degenerate:" their domains must be *strictly* linearly ordered. It is more usual to allow for degenerate simplices, but for our purposes, this extra generality does not seem to be useful. Since all of our simplices will be regular, we will omit the word "regular" in what follows.

Now the rest of the development of the homology theory in this section will be exactly the same as the particular case of model theory described in the first section of [6]. The proofs are exactly the same and the reader will notice that the list of axioms for amenable family of functors singles out basic technical properties which enable the arguments in section 1 of [6] work. For the sake of completeness, we list here the all the definitions, lemmas, and theorems we will need for later sections but without giving proofs.

We begin with the notion of boundary operators used to define homology groups in our context.

Definition 1.8. If $n \geq 1$ and $0 \leq i \leq n$, then the *ith boundary operator* $\partial_n^i : C_n(\mathcal{A}; B) \to C_{n-1}(\mathcal{A}; B)$ is defined so that if f is an n-simplex with support $s = \{s_0 < \cdots < s_n\}$, then

$$\partial_n^i(f) = f \restriction \mathcal{P}(s \setminus \{s_i\})$$

and extended linearly to a group map on all of $C_n(\mathcal{A}; B)$.

If $n \geq 1$ and $0 \leq i \leq n$, then the *boundary map* $\partial_n : C_n(\mathcal{A}; B) \to C_{n-1}(\mathcal{A}; B)$ is defined by the rule

$$\partial_n(c) = \sum_{0 \leq i \leq n} (-1)^i \partial_n^i(c).$$

We write ∂^i and ∂ for ∂_n^i and ∂_n, respectively, if n is clear from context.

Definition 1.9. The kernel of ∂_n is denoted $Z_n(\mathcal{A}; B)$, and its elements are called *(n-)cycles*. The image of ∂_{n+1} in $C_n(\mathcal{A}; B)$ is denoted $B_n(\mathcal{A}; B)$. The elements of $B_n(\mathcal{A}; B)$ are called *(n-)boundaries*.

It can be shown (by the usual combinatorial argument) that $B_n(\mathcal{A}; B) \subseteq Z_n(\mathcal{A}; B)$, or more briefly, "$\partial_n \circ \partial_{n+1} = 0$." Therefore we can define simplicial homology groups relative to $\mathcal{A}$:

Definition 1.10. The *nth (simplicial) homology group of $\mathcal{A}$ over B* is

$$H_n(\mathcal{A}; B) = Z_n(\mathcal{A}; B)/B_n(\mathcal{A}; B).$$

There are two natural candidates for the definition of the boundary of a 0-simplex. One possibility is to define $\partial_0(f) = 0$ for all $f \in S_0(\mathcal{A}; B)$. Another possibility is to extend the definition of an n-simplex to $n = -1$; namely a (-1)-simplex f is an object $f(\emptyset)$ in $\mathcal{C}$. Then the definition of a boundary operator extends naturally to the operator $\partial_0 : f \in S_0(\mathcal{A}; B) \mapsto B$.

As we show in Lemma 3.1, computing the group H_0 in a specific context using the first definition gives $H_0 \cong \mathbb{Z}$ while using the second definition we get $H_0 = 0$. Thus, the difference between the approaches is parallel to that between the homology and reduced homology groups in algebraic topology [1].

Next we define the amalgamation properties. We use the convention that n denotes the set $\{0, 1, \ldots, n-1\}$.

Definition 1.11. Let $\mathcal{A}$ be an amenable family of functors into a category $\mathcal{C}$ and let $n \geq 1$.

1. $\mathcal{A}$ has *n-amalgamation* if for any functor $f : \mathcal{P}^-(n) \to \mathcal{C}$, $f \in \mathcal{A}$, there is an $(n-1)$-simplex $g \supseteq f$ such that $g \in \mathcal{A}$.

2. $\mathcal{A}$ has *n-complete amalgamation* or *n-CA* if $\mathcal{A}$ has k-amalgamation for every k with $1 \leq k \leq n$.

3. $\mathcal{A}$ has *strong 2-amalgamation* if whenever $f : \mathcal{P}(s) \to \mathcal{C}$, $g : \mathcal{P}(t) \to \mathcal{C}$ are simplices in $\mathcal{A}$ and $f \restriction \mathcal{P}(s \cap t) = g \restriction \mathcal{P}(s \cap t)$, then $f \cup g$ can be extended to a simplex $h : \mathcal{P}(s \cup t) \to \mathcal{C}$ in $\mathcal{A}$.

4. $\mathcal{A}$ has *n-uniqueness* if for any functor $f : \mathcal{P}^-(n) \to \mathcal{C}$ in $\mathcal{A}$ and any two $(n-1)$-simplices g_1 and g_2 in $\mathcal{A}$ extending f, there is a natural isomorphism $F : g_1 \to g_2$ such that $F \restriction \mathrm{dom}(f)$ is the identity.

Remark 1.12.
1. There is a mismatch that n-amalgamation refers to the existence of $(n-1)$-simplex extending its boundary. But this numbering is coherent with historical developments of amalgamation theory in model theory and homology theory in algebraic topology.

2. The definition of n-amalgamation can be naturally extended to $n = 0$: $\mathcal{A}$ has 0-amalgamation if it contains a functor $f : \{\emptyset\} \to \mathcal{C}$. This holds in any amenable family of functors.

Definition 1.13. We say that an amenable family of functors $\mathcal{A}$ is *non-trivial* if $\mathcal{A}$ has 1-amalgamation, and satisfies the strong 2-amalgamation property.

The following remark is immediate from the definitions.

Remark 1.14. Any non-trivial amenable collection of functors $\mathcal{A}$ contains an n-simplex for each $n \geq 1$.

Everywhere below, we only deal with non-trivial amenable families of functors.

1.2. Computing homology groups.

As in [6] we introduce special kinds of n-chains which are useful for computing homology groups.

Definition 1.15. If $n \geq 1$, an *n-shell* is an n-chain c of the form

$$\pm \sum_{0 \leq i \leq n+1} (-1)^i f_i,$$

where $f_0, \ldots, f_{n+1}$ are n-simplices such that whenever $0 \leq i < j \leq n+1$, we have $\partial^i f_j = \partial^{j-1} f_i$.

Definition 1.16. If $n \geq 1$, and *n-fan* is an n-chain of the form

$$\pm \sum_{i \in \{0, \ldots, \widehat{k}, \ldots, n+1\}} (-1)^i f_i$$

for some $k \leq n+1$, where the f_i are n-simplices such that whenever $0 \leq i < j \leq n$, we have $\partial^i f_j = \partial^{j-1} f_i$. In other words, an n-fan is an n-shell missing one term.

If c is an n-fan, then ∂c is an $(n-1)$-shell; and $\mathcal{A}$ has n-amalgamation if and only if every $(n-2)$-shell in $\mathcal{A}$ is the boundary of an $(n-1)$-simplex in $\mathcal{A}$. And $\mathcal{A}$ has n-uniqueness if and only if every $(n-2)$-shell in $\mathcal{A}$ is the boundary of at most one $(n-1)$-simplex in $\mathcal{A}$ up to isomorphism.

As mentioned earlier, we now state without proofs a series of lemmas and theorems analogous to those in [6], Section 1. In particular, we state two "prism lemmas" (1.25 and 1.27) and a result that every element of a homology group is the equivalence class of a shell (Theorem 1.28).

Lemma 1.17. *If $n \geq 2$ and $\mathcal{A}$ has n-CA, then every $(n-1)$-cycle is a sum of $(n-1)$-shells. Namely, for each $c \in Z_{n-1}(\mathcal{A}; B)$, $c = \sum_i \alpha_i f_i$, there is a finite collection of $(n-1)$-shells $c_i \in Z_{n-1}(\mathcal{A}; B)$ such that $c = \sum_i (-1)^n \alpha_i c_i$.*

Moreover, if S is the support of the chain c and m is any element not in S, then we can choose $\sum_i \alpha_i c_i$ so that its support is $S \cup \{m\}$.

Corollary 1.18. *Assume $\mathcal{A}$ has n-CA for some $n \geq 2$. Then $H_{n-1}(\mathcal{A}; B)$ is generated by*

$$\{[c] \,:\, c \text{ is an } (n-1)\text{-shell over } B\}.$$

In particular, if any $(n-1)$-shell over B is a boundary, then so is any $(n-1)$-cycle.

Corollary 1.19. *If $\mathcal{A}$ has n-CA for some $n \geq 3$, then $H_{n-2}(\mathcal{A}; B) = 0$.*

Corollary 1.18 will be strengthened to Theorem 1.28.

Definition 1.20. If $n \geq 1$, an *n-pocket* is an n-cycle of the form $f - g$, where f and g are n-simplices with support S (where S is an $(n+1)$-element set).

Lemma 1.21. *Suppose that $f, g \in S_n(\mathcal{A})$ are* isomorphic *functors such that $\partial_n f = \partial_n g$. Then the n-pocket $f - g$ is a boundary.*

Lemma 1.22. *Suppose that $n \geq 1$ and $\mathcal{A}$ has $(n+1)$-amalgamation. Then for any n-fan*

$$g = \pm \sum_{i \in \{0,\ldots,\widehat{k},\ldots,n+1\}} (-1)^i f_i$$

there is some n-simplex f_k and some $(n+1)$-simplex f such that $g + (-1)^k f_k = \partial f$.

The next lemma says that n-pockets are equal to n-shells, "up to a boundary."

Lemma 1.23. *Assume that $\mathcal{A}$ has the $(n+1)$-amalgamation property for some $n \geq 1$. For any $B \in \mathcal{C}$, any n-shell in $\mathcal{A}_B$ with support $n+2$ is equivalent, up to a boundary in $B_n(\mathcal{A}; B)$, to an n-pocket in $\mathcal{A}_B$ with support $n+1$. Conversely, any n-pocket with support $n+1$ is equivalent, up to a boundary, to an n-shell with support $n+2$.*

From Corollary 1.18 and Lemma 1.23 we derive the following:

Corollary 1.24. *If $\mathcal{A}$ has 3-amalgamation, then $H_2(\mathcal{A}; B)$ is generated by equivalence classes of 2-pockets.*

Lemma 1.25 (Prism lemma). *Let $n \geq 1$. Suppose that $\mathcal{A}$ has $(n+1)$-amalgamation. Let $f - f'$ be an n-pocket in $\mathcal{A}_B$ with support s, where $|s| = n+1$. Let t be an $(n+1)$-element set disjoint from s. Then given n-simplex g in $\mathcal{A}_B$ with the domain $\mathcal{P}(t)$, there is an n-simplex g' such that $g - g'$ forms an n-pocket in $\mathcal{A}_B$ and is equivalent, modulo $B_n(\mathcal{A}; B)$, to the pocket $f - f'$. We may choose g' first and then find g to have the same conclusion.*

Corollary 1.26. *Let $n \geq 1$. Suppose $\mathcal{A}$ has $(n+1)$-CA. The group $H_n(\mathcal{A}; B)$ is generated by equivalence classes n-shells with support $n+2$.*

We have a shell version of the prism lemma as well:

Lemma 1.27 (Prism lemma, shell version). *Let $\mathcal{A}$ satisfy $(n+1)$-amalgamation for some $n \geq 1$. Suppose that an n-shell $f := \sum_{0 \leq i \leq n+1} (-1)^i f_i$ and an n-fan $g^- := \sum_{i \in \{0,\ldots,\hat{k},\ldots,n+1\}} (-1)^i g_i$ are given, where f_i, g_i are n-simplices over B,* $\mathrm{supp}(f) = s$ *with $|s| = n+2$, and* $\mathrm{supp}(g^-) = t = \{t_0, \ldots, t_{n+1}\}$, *where $t_0 < \ldots < t_{n+1}$ and $s \cap t = \emptyset$. Then there is an n-simplex g_k over B with support $\partial_k t := t \setminus \{t_k\}$ such that $g := g^- + (-1)^k g_k$ is an n-shell over B and $f - g \in B_n(\mathcal{A}; B)$.*

The next theorem gives an even simpler standard form for elements of $H_n(\mathcal{A}; B)$.

Theorem 1.28. *If $\mathcal{A}$ has $(n+1)$-CA for some $n \geq 1$, then*

$$H_n(\mathcal{A}; B) = \{[c] : c \text{ is an } n\text{-shell over } B \text{ with support } n+2\}.$$

Now using Theorem 1.28 and Lemma 1.23, we obtain the following:

Corollary 1.29. *If $(n+1)$-CA (for some $n \geq 1$) holds in $\mathcal{A}$, then*

$$H_n(\mathcal{A}; B) = \{[c] : c \text{ is an } n\text{-pocket in } \mathcal{A} \text{ over } B \text{ with support } n+1\}.$$

2. Type versus set homology groups in model theory

In this section, we define some amenable classes of functors that arise in model theory. Namely given either a complete rosy theory T or a complete type $p \in S(A)$ in a rosy theory, we will define both the "type homology groups" $H^t_n(T)$ (or $H^t_n(p)$) and the "set homology groups" $H^{set}_n(T)$ (or $H^{set}_n(p)$). As noted, $H^{set}_n(p)$ and the classes of p-set-functors were already introduced in [6] and the properties of those were the motivation for Definition 1.3. As we show below, these definitions will lead to isomorphic homology groups (Proposition 2.12).

We make the same assumptions on our underlying theory T as in [6]: **in what follows, we assume that $T = T^{\mathrm{eq}}$ is a complete rosy theory** (e.g. stable, simple, or o-minimal) and we work in its fixed large saturated model $\mathfrak{C} = \mathfrak{C}^{\mathrm{eq}}$. The reason for this is so that we have a nice independence notion [3]. Throughout, "$\mathop{\smile\!\!\!\!\!\!\!\;|}$", "independence" or "non-forking" will mean thorn-independence. So if T is simple then we assume it has elimination of hyperimaginaries in order for non-forking independence to be equal to thorn-independence [3]. But the assumptions are for convenience not for full generality. For example if T is simple, then one may assume the independence is usual non-forking in $\mathfrak{C}^{\mathrm{heq}}$ while replacing acl by bdd and so on. Moreover there are non-rosy examples having suitable independence notions that fit in our amenable category context (see [10] and [13]).

We refer the reader to [12], [20] and to [3], [18] for general background on simple and rosy theories, respectively.

2.1. Type homology.

We will work with $*$-types – that is, types with possibly infinite sets of variables – and to avoid some technical issues, we will place an absolute bound on the cardinality of the variable sets of the types we consider. Fix some infinite cardinal $\kappa_0 \geq |T|$. We will assume that every $*$-type has no more than κ_0 free variables. We also fix a set $\mathcal{V}$ of variables such that $|\mathcal{V}| > \kappa_0$ and assume that all variables in $*$-types come from the set $\mathcal{V}$ (which is a "master set of variables.") We work in a monster model $\mathfrak{C} = \mathfrak{C}^{\mathrm{eq}}$ which is saturated in some cardinality greater than $2^{|\mathcal{V}|}$. We let $\bar{\kappa} = |\,\mathfrak{C}\,|$. As we will see in the next section, the precise values of κ_0 and $|\mathcal{V}|$ will not affect the homology groups.

Given a set A, strictly speaking we should write "**a** complete $*$-type of A" instead of "**the** complete $*$-type of A" – there are different such types corresponding to different choices for associating each element of A with a variable from $\mathcal{V}$, and this distinction is crucial for our purposes.

If X is any subset of the variable set $\mathcal{V}$, $\sigma : X \to \mathcal{V}$ is any injective function, and $p(\overline{x})$ is any $*$-type such that $\overline{x}$ is contained in X, then we let

$$\sigma_* p = \{\varphi(\sigma(\overline{x})) : \varphi(\overline{x}) \in p\}.$$

Definition 2.1. If A is a small subset of the monster model, then $\mathcal{T}_A$ is the category such that

1. The objects of $\mathcal{T}_A$ are are all the complete $*$-types in T over A, including (for convenience) a single distinguished type $p_\emptyset$ with no free variables;
2. $\mathrm{Mor}_{\mathcal{T}_A}(p(\overline{x}), q(\overline{y}))$ is the set of all injective maps $\sigma : \overline{x} \to \overline{y}$ such that $\sigma_*(p) \subseteq q$.

Note that this definition gives a notion of two types $p(\overline{x})$ and $q(\overline{y})$ being "isomorphic:" namely, that q can be obtained from p by relabeling variables.

Definition 2.2. If $A = \mathrm{acl}(A)$ is a small subset of the monster model, a *closed independent type-functor based on* A is a functor $f : X \to \mathcal{T}_A$ such that:

1. X is a downward-closed subset of $\mathcal{P}(s)$ for some finite $s \subseteq \omega$.
2. Suppose $w \in X$ and $u, v \subseteq w$. Recall our notational convention $f^u_w := f(\iota_{u,w})$. Let us write $\overline{x}_w$ to be the variable set of $f(w)$. Then whenever $\overline{a}$ realizes the type $f(w)$ and $\overline{a}_u$, $\overline{a}_v$, and $\overline{a}_{u\cap v}$ denote subtuples corresponding to the variable sets $f^u_w(\overline{x}_u)$, $f^v_w(\overline{x}_v)$, and $f^{u\cap v}_w(\overline{x}_{u\cap v})$, then
$$\overline{a}_u \underset{A \cup \overline{a}_{u\cap v}}{\bigcup\!\!\!\!\smile} \overline{a}_v.$$
3. For all non-empty $u \in X$ and any $\overline{a}$ realizing $f(u)$, we have (using the notation above) $\overline{a} = \mathrm{acl}\left(A \cup \bigcup_{i\in u} \overline{a}_{\{i\}}\right)$.

(The adjective "closed" in the definition refers to the fact that, by (3), all the types $f(u)$ are $*$-types of algebraically closed tuples.)

Let $\mathcal{A}^t(T; A)$ denote all closed independent type-functors based on A.

Remark 2.3. It follows from the definition above and the basic properties of nonforking that if f is a closed independent type-functor based on A and $u \in \mathrm{dom}(f)$ is a non-empty set of size k, then any realization $\overline{a}$ of $f(u)$ is the algebraic closure of an AB-independent set $\{\overline{a}_1, \ldots, \overline{a}_k\}$, where B is the subtuple of $\overline{a}$ corresponding to the variables $f^\emptyset_u(\overline{x}_\emptyset)$ and each $\overline{a}_i$ is the subtuple corresponding to the variables in $f^{\{i\}}_u(\overline{x}_{\{i\}})$.

Definition 2.4. If $A = \mathrm{acl}(A)$ is a small subset of the monster model and $p \in S(A)$, then a *closed independent type-functor in* p is a closed independent type-functor $f : X \to \mathcal{T}_A$ based on A such that if $X \subseteq \mathcal{P}(s)$ and $i \in s$, then $f(\{i\})$ is the complete $*$-type of $\mathrm{acl}(AC \cup \{b\})$ over A, where C is some realization of $f(\emptyset)$ and b is some realization of a nonforking extension of p to AC.

Let $\mathcal{A}^t(p)$ denote all closed independent type-functors in p.

Now using the basic independence properties of rosy theories, it is not hard to verify amenability of the above families of functors. In particular one may consult the proof of [6, 1.19]

Proposition 2.5. *The sets $\mathcal{A}^t(T;A)$ and $\mathcal{A}^t(p)$ are non-trivial amenable families of functors.*

Definition 2.6. If A is a small algebraically closed subset of $\mathfrak{C}$, then we write $S_n\mathcal{T}_A$ as an abbreviation for $S_n(\mathcal{A}^t(T;A);p_\emptyset)$ (the collection of closed n-simplices in $\mathcal{A}^t(T;A)$ over the distinguished type $p_\emptyset$), $B_n\mathcal{T}_A$ and $Z_n\mathcal{T}_A$ for the boundary and cycle groups, and $H^t_n(T;A)$ for the homology group $H_n(\mathcal{A}^t(T;A);p_\emptyset)$.

Similarly if $p \in S(A)$, then we use the abbreviations $S_n\mathcal{T}(p)$ for $S_n(\mathcal{A}^t(p);p_\emptyset)$; and $B_n\mathcal{T}(p)$, $Z_n\mathcal{T}(p)$, and $H^t_n(p)$.

2.2. Set homology.

Definition 2.7. Let A be a small subset of $\mathfrak{C}$. By $\mathcal{C}_A$ we denote the category of all subsets *containing* A of $\mathfrak{C}$ of size no more that κ_0, where morphisms are partial elementary maps over A (that is, fixing A pointwise).

For a functor $f : X \to \mathcal{C}_A$ and $u \subseteq v \in X$, we write $f^u_v(u) := f^u_v(f(u)) \subseteq f(v)$.

Definition 2.8. A *closed independent set-functor based on* $A = \operatorname{acl}(A)$ is a functor $f : X \to \mathcal{C}_A$ such that:

1. X is a downward-closed subset of $\mathcal{P}(s)$ for some finite $s \subseteq \omega$.
2. For all non-empty $u \in X$, we have that $f(u) = \operatorname{acl}(A \cup \bigcup_{i\in u} f^{\{i\}}_u(\{i\}))$ and the set $\{f^{\{i\}}_u(\{i\}) : i \in u\}$ is independent over $f^{\emptyset}_u(\emptyset)$.

Let $\mathcal{A}^{set}(T;A)$ denote all closed independent set-functors based on A.

Now we recall the following in [6].

Definition 2.9. If $A = \operatorname{acl}(A)$ is a small subset of the monster model and $p \in S(A)$, then a *closed independent set-functor in* p is a closed independent set-functor $f : X \to \mathcal{C}_A$ based on A such that if $X \subseteq \mathcal{P}(s)$ and $i \in s$, then $f(\{i\})$ is a set of the form $\operatorname{acl}(C \cup \{b\})$ where $C = f^{\emptyset}_{\{i\}}(\emptyset) \supseteq A$ and b realizes some non-forking extension of p to C.

Let $\mathcal{A}^{set}(p)$ denote all closed independent set-functors in p.

Just as in the previous subsection, we have:

Proposition 2.10. *The sets $\mathcal{A}^{set}(T;A)$ and $\mathcal{A}^{set}(p)$ are non-trivial amenable families of functors.*

Definition 2.11. If A is a small subset of $\mathfrak{C}$, then we write "$S_n\mathcal{C}_A$" to denote $S_n(\mathcal{A}^{set}(T;A);A)$ (the collection of closed n-simplices in $\mathcal{A}^{set}(T;A)$ *over* A), and similarly we write $B_n\mathcal{C}_A$ and $Z_n\mathcal{C}_A$ for the boundary and cycle groups over A, and use the notation $H^{set}_n(T;A)$ for the homology group $H_n(\mathcal{A}^{set}(T;A);A)$.

If $A = \operatorname{acl}(A)$ and $p \in S(A)$, then we use similar abbreviations $S_n\mathcal{C}(p) := S_n(\mathcal{A}^{set}(p);A)$, $B_n\mathcal{C}(p)$, $Z_n\mathcal{C}(p)$, and $H^{set}_n(p)$.

Proposition 2.12. *1. For any n and any set A, $H_n^t(T;A) \cong H_n^{set}(T;A)$.*

2. For any n and any complete type $p \in S(A)$, $H_n^t(p) \cong H_n^{set}(p)$.

Proof. The idea is that we can build a correspondence $F : S\mathcal{C}_A \to S\mathcal{T}_A$ which maps each set-simplex f to its "complete $*$-type" $F(f)$. Note that this will involve some non-canonical choices: namely, which variables to use for $F(f)$, and in what order to enumerate the various sets in f (since our variable set $\mathcal{V}$ is indexed and thus implicitly ordered). We will write out a proof of part (1) of the proposition, and part (2) can be proved similarly by relativizing to p.

Let $S_{\leq n}\mathcal{C}_A$ and $S_{\leq n}\mathcal{T}_A$ denote, respectively, $\bigcup_{i\leq n} S_i\mathcal{C}_A$ and $\bigcup_{i\leq n} S_i\mathcal{T}_A$. We will build a sequence of maps $F_n : S_{\leq n}\mathcal{C}_A \to S_{\leq n}\mathcal{T}_A$ whose union will be F. Given such an F_n, let $\overline{F}_n : C_{\leq n}\mathcal{C}_A \to C_{\leq n}\mathcal{T}_A$ be its natural extension to the class of all set-k-chains over A for $k \leq n$.

Claim 2.13. *There are maps $F_n : S_{\leq n}\mathcal{C}_A \to S_{\leq n}\mathcal{T}_A$ such that:*

1. *F_{n+1} is an extension of F_n;*
2. *If $f \in S_{\leq n}\mathcal{C}_A$ and $\mathrm{dom}(f) = \mathcal{P}(s)$, then $\mathrm{dom}(F_n(f)) = \mathcal{P}(s)$ and $[F_n(f)](s)$ is a complete $*$-type of $f(s)$ over A;*
3. *For any $k \leq n$, any $f \in S_k\mathcal{C}_A$, and any $i \leq k$, $F_n(\partial^i f) = \partial^i [F_n(f)]$; and*
4. *F_n is surjective, and in fact for every $g \in S_k\mathcal{T}_A$ (where $0 \leq k \leq n$), there are **more** than $2^{|\mathcal{V}|}$ simplices $f \in S_k\mathcal{C}_A$ such that $F_n(f) = g$.*

Proof. We prove the claim by induction on n. The case where $n = 0$ is simple: only conditions (2) and (4) are relevant, and note that we can insure (4) because the monster model $\mathfrak{C}$ is $(2^{|\mathcal{V}|})^+$-saturated and there are at most $2^{|\mathcal{V}|}$ elements of $S_0\mathcal{T}_A$. So suppose that $n > 0$ and we have $F_0, \ldots, F_n$ satisfying all these properties, and we want to build F_{n+1}. We build F_{n+1} as a union of a chain of partial maps from $S_{\leq n+1}\mathcal{C}_A$ to $S_{\leq n+1}\mathcal{T}_A$ extending F_n (that is, functions whose domains are subsets of $S_{\leq n+1}\mathcal{C}_A$).

Subclaim 2.14. *Suppose that $F : X \to S_{\leq n+1}\mathcal{T}_A$ is a function on a set $X \subseteq S_{\leq n+1}\mathcal{C}_A$ of size at most $(2^{|\mathcal{V}|})^+$ and that F satisfies (1) through (3). Then for any simplex $g \in S_{n+1}\mathcal{T}_A$, there is an extension F_0 of F satisfying (1) through (3) such that $|\mathrm{dom}(F_0)| \leq (2^{|\mathcal{V}|})^+$ and:*

(*) There are $(2^{|\mathcal{V}|})^+$ distinct $f \in S_{n+1}\mathcal{C}_A$ such that $F'(f) = g$.

Proof. Let $\partial g = g_0 - g_1 + \ldots + (-1)^n g_n$ (where $g_i = \partial^i g$), and let $\mathcal{P}(s)$ be the domain of g. By induction, each g_i is the image under F_n of $(2^{|\mathcal{V}|})^+$ different n-simplices in $\mathcal{C}_A$; let $\langle f_i^j : j < (2^{|\mathcal{V}|})^+ \rangle$ be a sequence of distinct simplices such that for every $j < (2^{|\mathcal{V}|})^+$, $F_n(f_i^j) = g_i$. By saturation of the monster model, for each $j < (2^{|\mathcal{V}|})^+$ we can pick an $(n+1)$-simplex $f_j \in \mathcal{C}_A$ with domain $\mathcal{P}(s)$ such that $\partial f_j = f_0^j - f_1^j + \ldots + (-1)^n f_n^j$ and $\mathrm{tp}(f_j(s)) = g(s)$. Then the f_j are all distinct, so we can let $F_0 = F \cup \{(f_j, g) : j < (2^{|\mathcal{V}|})^+\}$. $\dashv$

Now by the subclaim, we can use transfinite induction to build a **partial** map $F' : S_{\leq n+1}\mathcal{C}_A \to S_{\leq n+1}\mathcal{T}_A$ satisfying (1) through (4) (also using the fact that there only (at most) $2^{|\mathcal{V}|}$ different simplices in $S_{n+1}\mathcal{T}_A$ and the fact that the union of a chain of partial maps from $S_{\leq n+1}\mathcal{C}_A$ to $S_{\leq n+1}\mathcal{T}_A$ satisfying conditions (2) and (3) will still satisfy these conditions).

Finally, we can extend F' to a function on all of $S_{\leq n+1}\mathcal{C}_A$ by a second transfinite induction, extending F' to each $f : \mathcal{P}(s) \to \mathcal{C}_A$ in $\mathcal{C}_A$ one at a time; to ensure that properties (2) and (3) hold, we just have to pick $F_{n+1}(f)$ to be some $(n+1)$-simplex with the same domain $\mathcal{P}(s)$ whose n-faces are as specified by F_n and such that $[F_{n+1}(f)]\,(s)$ is a complete $*$-type of $f(s)$ over A.

$\dashv$

Now let $F = \bigcup_{n<\omega} F_n$. By property (3) above, it follows that for any chain $c \in C\mathcal{C}_A$, we have $\overline{F}(\partial c) = \partial\left[\overline{F}(c)\right]$. Hence $\overline{F}$ maps $Z_n\mathcal{C}_A$ into $Z_n\mathcal{T}_A$ and $B_n\mathcal{C}_A$ into $B_n\mathcal{T}_A$, and so $\overline{F}$ induces group homomorphisms $\varphi_n : H_n^{set}(T;A) \to H_n^t(T;A)$. Verifying that φ_n is injective amounts to checking that whenever $\overline{F}(c) \in B_n\mathcal{T}_A$, the set-chain c is in $B_n\mathcal{C}_A$, but this is staightforward: if, say, $\overline{F}(c) = \partial c'$, then we can pick a set-simplex $\hat{c}$ "realizing" c' such that $\partial\hat{c} = c$. Finally, condition (4) implies that φ_n is surjective, so $H_n^{set}(T;A) \cong H_n^t(T;A)$.

$\dashv$

Remark 2.15. Since Proposition 2.12 is true for any choices of κ_0, $\mathcal{V}$, and the monster model $\mathfrak{C}$ as long as $|T| \leq \kappa_0 < |\mathcal{V}|$ and $2^{|\mathcal{V}|} \leq |\,\mathfrak{C}\,|$, it follows that our homology groups (with the restriction of the set A) do not depend on the choices of κ_0, $|\mathcal{V}|$, or the monster model.

Without specifying a base set A, one could also define $C_n(T)$ to be the direct sum $\bigoplus_{i<\bar{\kappa}} C_n\mathcal{C}_{A_i}$ where $\{A_i | i < \bar{\kappa}\}$ is the collection of all small subsets of $\mathfrak{C}$, and similarly $Z_n(T)$, $B_n(T)$, and $H_n(T) := Z_n(T)/B_n(T)$. Then the boundary operator ∂ sends n-chains to $(n-1)$-chains componentwise. Hence it follows $H_n(T) = \bigoplus_{i<\bar{\kappa}} H_n(T;A_i)$. This means the homology groups defined without specifying a base set depends on the choice of monster model, and so this approach would not give invariants for the theory T.

2.3. An alternate definition of the set homology groups.

In our definition of the set homology groups $H_n^{set}(T;A)$ and $H_n^{set}(p)$ (where $p \in S(A)$), we have been assuming that the base set A is fixed pointwise by all of the elementary maps in a set-simplex – this is built into our definition of $\mathcal{C}_A$. It will turn out that we get an equivalent definition of the homology groups if we allow the base set to be "moved" by the images of the inclusion maps in a set-simplex, as we will show in this subsection.

Definition 2.16.
1. A *set-n-simplex weakly over A* is a set-n-simplex $f : \mathcal{P}(s) \to \mathcal{C}(= \mathcal{C}_\emptyset)$ such that $f(\emptyset) = A$.
2. If $p \in S(A)$, then a set-n-simplex $f : \mathcal{P}(s) \to \mathcal{C}$ is *weakly of type p* if $f(\emptyset) = A$,

and for every $i \in s$,

$$f(\{i\}) = \operatorname{acl}\left(f^{\emptyset}_{\{i\}}(A) \cup \{a_i\}\right)$$

for some a_i such that $\operatorname{tp}(a_i/f^{\emptyset}_{\{i\}}(A))$ is a conjugate of p.

Let $S'_n\mathcal{C}_A$ be the collection of all set-n-simplices weakly over A. Note that the boundary operator ∂ takes an n-simplex weakly over A to a chain of $(n-1)$-simplices weakly over A, and so we can define "weak set homology groups over A," which we denote $H'_n(T;A)$. Similarly, we can define $H'_n(p)$, the "weak set homology groups of p," from chains of set-simplices that are weakly of type p.

Proposition 2.17. *1. For any n and any $A \in \mathcal{C}$, $H'_n(T;A) \cong H^{set}_n(T;A)$.*

2. For any n and any complete type $p \in S(A)$, $H'_n(p) \cong H^{set}_n(p)$.

Proof. As usual, the two parts have identical proofs, and we only prove the second part.

We will identify $S'_0\mathcal{C}_A$ as a big single *complex* as follows. Due to our cardinality assumption, for each $n < \omega$, there are $\bar{\kappa}$-many 0-simplices in $S'_0\mathcal{C}_A$ having the common domain $\mathcal{P}(\{n\})$. Then we consider the following domain set $\mathcal{D}_0 = \{\emptyset\} \cup \{\{(n,i)\} |\ n < \omega, i < \bar{\kappa}\}$. Now as said we identify $S'_0\mathcal{C}_A$ as a *single* functor F'_0 from $\mathcal{D}_0$ to $\mathcal{C}$ such that $F'_0(\emptyset) = A$, and $F'_0(\{(n,i)\}) = (f')^n_i(\{n\})$ where $(f')^n_i \in S'_0\mathcal{C}_A$ is the corresponding 0-simplex with $((f')^n_i)^{\emptyset}_{\{n\}} = (F'_0)^{\emptyset}_{\{(n,i)\}}$. Similarly we consider $S_0\mathcal{C}_A$ as a functor F_0 from $\mathcal{D}_0$ to $\mathcal{C}_A$ such that $F_0(\emptyset) = A$, and $F_0(\{(n,i)\}) = f^n_i(\{n\}) \equiv (f')^n_i(\{n\})$ where $f^n_i \in S_0\mathcal{C}_A$ is the corresponding 0-simplex over A with $(f^n_i)^{\emptyset}_{\{n\}} = (F_0)^{\emptyset}_{\{(n,i)\}}$. Now F'_0 and F_0 are naturally isomorphic by η^0 with $\eta^0_{\emptyset}$ =the identity map of A, and suitable $\eta^0_{\{(n,i)\}}$ sending $(f')^n_i(\{n\})$ to $f^n_i(\{n\})$.

Now for $S'_1\mathcal{C}_A$, note that for each pair $(f')^{n_0}_{i_0}, (f')^{n_1}_{i_1}$ with $n_0 < n_1$, there are $\bar{\kappa}$-many 1-simplices f'_j in $S'_1\mathcal{C}_A$ having the common domain $\mathcal{P}(\{n_0, n_1\})$ with $\partial^0 f'_j = (f')^{n_1}_{i_1}$ and $\partial^1 f'_j = (f')^{n_0}_{i_0}$. Hence we now put the domain set $\mathcal{D}_1 = \mathcal{D}_0 \cup \{\{(n_0,i_0),(n_1,i_1),j\} |\ n_0 < n_1 < \omega; i_0, i_1, j < \bar{\kappa}\}$. Then we identify $S'_1\mathcal{C}_A$ as a functor F'_1 from $\mathcal{D}_1$ to $\mathcal{C}$ such that $F'_1 \restriction \mathcal{D}_0 = F'_{01}$, and $F'_1(\{(n_0,i_0),(n_1,i_1),j\})$ corresponds jth 1-simplex having $(f')^{n_0}_{i_0}, (f')^{n_1}_{i_1}$ as 0-faces. Similarly we try to identify $S'_1\mathcal{C}_A$ as a functor F_1 from $\mathcal{D}_1$ to $\mathcal{C}_A$, extending F_0. But to make F'_1 and F_1 isomorphic, we need extra care when defining F_1. For each $j < \bar{\kappa}$ and a set $a'_j = f'_j(\{n_0,n_1\})$ of corresponding 1-simplex f'_j, assign an embedding $\eta^1_j = \eta^1_{\{(n_0,i_0),(n_1,i_1),j\}}$ sending a'_j to a_j, extending the inverse of $(f'_j)^{\emptyset}_{\{n_0,n_1\}}$. Then we define $F_1(\{(n_0,i_0),(n_1,i_1),j\}) = a'_j$, and

$$(F_1)^{\{(n_k,i_k)\}}_{\{(n_0,i_0),(n_1,i_1),j\}} = \eta^1_j \circ (f'_j)^{\{n_k\}}_{\{n_0,n_1\}} \circ (\eta^0_{\{(n_k,i_k)\}})^{-1}.$$

Now then clearly η^1 with $\eta^1 \restriction \mathcal{D}_0 = \eta^0$ is an isomorphism between F'_1 and F_1.

By iterating this argument we can respectively identify $S'_n\mathcal{C}_A$ and $S_n\mathcal{C}_A$, as functors F'_n and F_n having the same domain $\mathcal{D}_n$ extending $\mathcal{D}_1$. Moreover we can also construct an isomorphism η^n, extending η^1, between F'_n and F_n. Note that each $x \in \mathcal{D}_n - \mathcal{D}_{n-1}$ corresponds an n-simplex $f' \in S'_n\mathcal{C}$, and η^n_x corresponds an n-simplex over A $f \in S_n\mathcal{C}$. This correspondence $f' \mapsto f$ induces a bijection from $C'_n\mathcal{C}_A$ to $C_n\mathcal{C}_A$, mapping $c' \mapsto c$, which indeed is an isomorphism of the two groups. Notice that by the construction, if an n-shell c' is the boundary of some $(n+1)$-simplex f', then c is the boundary of f. In general, it follows $(\partial d)' = \partial d'$ (*). Thus this correspondence also induces an isomorphism between $Z'_n(T;A)$ and $Z_n(T;A)$. Moreover it follows from (*) that the correspondence sends $B'_n(T;A)$ to $B_n(T;A)$: Let $c' = \partial d' \in B'_n(T;A)$. Then by (*), we have $c = \partial d \in B_n(T;A)$. Conversely for $c' \in Z'_n(T;A)$, assume $c = \partial e \in B_n(T;A)$. Now for e', again by (*), $\partial e' = c'$. Hence we have $c' \in B'_n(T;A)$. ⊣

3. Basic facts and examples

From now on, we will usually drop the superscripts t and *set* from $H^t_n(p)$ and $H^{set}_n(p)$ defined in Section 2, since these groups are isomorphic, and use "$H_n(p)$" to refer to the isomorphism class of these two groups. In computing the groups below, we generally use $H^{set}_n(p)$ rather than $H^t_n(p)$.

3.1. Computing $\boldsymbol{H_0}$. In this subsection, we observe that H_0 does not give any information, since it is always isomorphic to $\mathbb{Z}$, if $\partial_0(f) = 0$ for any 0-simplex f; or is trivial if $\partial_0(f)$ is defined to be $f(\emptyset)$:

Lemma 3.1. *1. If $\partial_0(f) = 0$, then for any complete type p over an algebraically closed set A, $H_0(p) \cong \mathbb{Z}$ and for any small subset A of $\mathfrak{C}$, $H_0(T;A) \cong \mathbb{Z}$.*

2. If $\partial_0(f) = f(\emptyset)$, then both groups in (1) are trivial.

Proof. Both parts of the lemma can be proved by essentially the same argument, so we only write out the proof for the group $H_0(p)$ in (1).

For the proof we will define an augmentation map ϵ as in topology. Since we can add parameters to the language for A, we can assume that $A = \emptyset$.

Define $\epsilon : C_0\mathcal{C}(p) \to \mathbb{Z}$ by $\epsilon(c) = \sum_i n_i$ for a 0-chain $c = \sum_i n_i f_i$ of type p. Then ϵ is a homomorphism such that $\epsilon(b) = 0$ for any 0-boundary b (since $\epsilon(\partial f) = 0$ for any 1-simplex f). Thus ϵ induces a homomorphism $\epsilon_* : H_0(p) \to \mathbb{Z}$. Note that *any* 0-chain c is in $Z_0(p)$, so clearly ϵ_* is onto. We claim that ϵ_* is one-to-one, i.e. $\ker \epsilon_* = B_0(p)$. Given a 0-chain $c = \sum_{i\in I} n_i f_i$ such that $\epsilon_*(c) = \sum_{i\in I} n_i = 0$, we shall show c is a boundary. Pick some natural number m greater than every k_i where $\operatorname{dom} f_i = \mathcal{P}(\{k_i\})$. Let $a_i = \operatorname{acl}(a_i) = f_i(\{k_i\})$. Then choose a realizing p such that $a \mathop{\smile\!\!\!\!\!\!|}\ \{a_i : i \in I\}$. Now let g_i be a closed 1-simplex of p such that $\operatorname{dom} g_i = \mathcal{P}(\{k_i, m\})$, $g_i(\{k_i\}) = a_i$, and $g_i(\{m\}) = a$. Then $\partial g_i = c_m - f_i$, where c_m is the 0-simplex such that $c_m(\emptyset) = \emptyset$ and $c_m(\{m\}) = a$. Then $c + \partial(\sum_i n_i g_i) = \sum_i n_i f_i + \sum_i n_i(c_m - f_i) = (\sum_i n_i)c_m = 0$. Hence c is a 0-boundary, and $H_0(p) \cong \mathbb{Z}$.

⊣

3.2. Amalgamation properties. The amalgamation properties in Definition 1.11 can be specialized to the context of model theory, yielding the usual notion of n-amalgamation (as in [11]).

Definition 3.2. 1. If A is a small subset of $\mathfrak{C}$, then T has the *n-amalgamation property over* (*based on*, resp.) A if for every $(n-2)$-shell c *over* (*based on*, resp.) A, there is an $(n-1)$-simplex f such that $c = \partial f$.

2. A complete type p has the *n-amalgamation* if any closed functor $f : \mathcal{P}^-(n) \to \mathcal{C}_A$ in p can be extended to an $(n-1)$-simplex.

3. Similarly, "n-uniqueness" *over* A, *based on* A, or *of the type* p can be defined and so can be the notion of "n-CA".

Remark 3.3. 1. Amalgamation properties *based on* A is equivalent to amalgamation properties *over* all $B \supseteq A$, which implies n-amalgamation for any type p. A stable theory has 4-amalgamation over any model M, as noted in [2]. However, it need not have 4-amalgamation based on M. For suppose that T is a stable theory in which there is a definable groupoid $\mathcal{G}$ which has unboundedly many connected components, each of which is not almost retractable (see [4]). Then if $M \models T$, a is the name of a connected component of $\mathcal{G}$ which does not intersect M (noting that these are equivalence classes which live in T^{eq}), and $B = \mathrm{acl}(Ma)$, then T does not have 4-amalgamation over the set B.

2. Similarly, if p has n-amalgamation, then so does any non-forking extension, but the converse need not hold even in a stable theory; see Remark 1.8 of [6].

3. As is well known, if T is simple then T has 3-CA; and if T is stable, then T has 2-uniqueness by stationarity. A non-simple rosy theory cannot have 3-amalgamation [16] but it may have n-amalgamation for all $n \geq 4$ (e.g. the theory of dense linear ordering).

Now we can restate Corollary 1.28 as:

Fact 3.4. *Assume T has n-CA based on $A = \mathrm{acl}(A)$ for $n \geq 2$. Then*

$$H_{n-1}(T;A) = \{[c] |\ c \textit{ is an } (n-1)-\textit{shell over } A \textit{ with support } n+1\}.$$

and

$$H_{n-1}(p) = \{[c] |\ c \textit{ is an } (n-1)-\textit{shell of } p \textit{ with support } n+1\}.$$

So it follows:

Fact 3.5. *Suppose $n \geq 3$.*

1. *If T has n-CA based on $A = \mathrm{acl}(A)$, then $H_{n-2}(T;A) = 0$.*

2. *If $p \in S(A)$ (where $A = \mathrm{acl}(A)$) has n-CA, then $H_{n-2}(p) = 0$.*

However, the converse of the above fact is false in general: the theory of the random tetrahedron-free hypergraph does not have 4-amalgamation, but all of its homology groups are trivial ([6, 1.32]).

Fact 3.6. *If T is simple, then $H_1(T;A)=0$ and $H_1(p)=0$ for any strong type p in T.*

The fact above is extended to any rosy theory in [14].

3.3. More examples. Homology groups of some examples are already given in [6, Section 1.2]. There $H_2(p)$ of a strong type p in a stable theory is computed too. In this subsection, we compute some homology groups for o-minimal examples.

Example 3.7. Let p be the unique 1-type over $\emptyset$ in the theory T_{dlo} of dense linear ordering (without end points). Due to weak elimination of imaginaries it is a strong type. We show that $H_n(p)=0$ for every $n \geq 1$, even though it does not have 3-amalgamation. It is not hard to see that p has n-amalgamation for all $n \neq 3$. Now we claim that, just like in Claim 1.33 in [5], any n-cycle is a sum of n-shells. The proof will be similar, and we use the same notation. We want to construct the edges h_{ij}. The trick this time is to take a^* greater than all the points of the form $a' = g_{ij}(\{k\})$. Then given any edge $\{b,c\} = g_{ij}(\{k,\ell\})$, where either $b<c$ or $c<b$, pick $a>b,c$. Then since $\mathrm{tp}(a'a^*) = \mathrm{tp}(ba) = \mathrm{tp}(ca)$, the construction of h_{ij} on this level is compatible. For the rest of the construction, use n-amalgamation.

Due to the claim and $(n+2)$-amalgamation, all of the groups $H_n(p)$ are 0 for $n \neq 1$. Furthermore, $H_1(p)=0$ because any 1-shell is the boundary of a 2-fan (choose a point greater than all the vertices of all the terms in the 1-shell).

Example 3.8. In [14], it is shown that for any strong type p in $\mathfrak{C}^{\mathrm{eq}}$ of a rosy theory, if it is a Lascar type too then $H_1(p)=0$. But the reason for the triviality of $H_1(p)$ can be arbitrarily complicated. Here we argue that if p is a complete 1-type over $A=\mathrm{acl}(A)$ in the home sort of an o-minimal theory then $H_1(p)=0$ due to a rather simple reason. Now fix such a p in an o-minimal theory.

Lemma 3.9. *Assume p is non-algebraic. Then there is a type $q(x,y) \in S(A)$ such that:*

1. *whenever $(a,b) \models q(x,y)$, then a and b are A-independent, and each realizes p; and*
2. *for any pair (a,b) of A-independent realizations of p, there is a third realization c of p such that c is A-independent from ab and both (a,c) and (b,c) realize q.*

Proof. Recall that since T is o-minimal, any A-definable unary function $f(x)$ is either eventually increasing (that is, there is some point c such that if $c<x<y$ then $f(x)<f(y)$, eventually decreasing, or eventually constant. If f is eventually constant with eventual value d, then $d \in \mathrm{dcl}(A)$.

We say an A-definable function $f(x_1,\dots,x_n)$ *bounded within* p if for any realizations $c_1,\dots,c_n \models p$, there is d realizing p such that $d > f(c_1,\dots,c_n)$. We call a pair of realizations (a,b) of p an *extreme pair* if whenever $f(x)$ is bounded within p, then $b > f(a)$.

First note that by the compactness theorem, for any a realizing p, there is a b realizing p such that (a,b) is an extreme pair. Also, if $b \in \mathrm{dcl}(aA) = \mathrm{acl}(aA)$, then there is an A-definable function $f : p(\mathfrak{C}) \to p(\mathfrak{C})$ such that $b = f(a)$, so since there is no maximal realization c of p (because such a realization c would be in $\mathrm{dcl}(A)$ and we are assuming that p is non-algebraic), it follows that (a,b) is **not** an extreme pair. So any extreme pair is algebraically independent over A and hence thorn-independent (see [18]).

Claim 3.10. *Any two extreme pairs have the same type over A.*

Proof. It is enough to check that if (a,b) and (a,c) are two extreme pairs, then $\mathrm{tp}(b/Aa) = \mathrm{tp}(c/Aa)$. By o-minimality, any Aa-definable set X is a finite union of intervals, and the endpoints $\{d_1,\dots,d_n\}$ of these intervals lie in $\mathrm{dcl}(Aa)$. So $d_i = f(a)$ for some A-definable function f, and as we already observed $b, c \neq d_i$. Hence it suffices to see $b > d_i$ iff $c > d_i$. Now by the definition of an extreme pair,

$$\forall x \models p \ \exists y \models p\, [y > f(x)] \Rightarrow b > f(a) = d_i.$$

Also,

$$\exists x \models p \ \forall y \models p\, [y \leq f(x)] \Rightarrow \forall x \models p \ \forall y \models p\, [y \leq f(x)]$$

because any two realizations of p are conjugate under an automorphism in $\mathrm{Aut}(\mathfrak{C}/A)$ which permutes $p(\mathfrak{C})$, and so

$$\exists x \models p\ \forall y \models p\, [y \leq f(x)] \Rightarrow b \leq f(a) = d_i.$$

The same reasoning applies with c in place of b, so

$$b > d_i = f(a) \Leftrightarrow \forall x \models p \ \exists y \models p\, [y > f(x)]$$

$$\Leftrightarrow c > f(a) = d_i.$$

⊣

Let $q(x,y) = \mathrm{tp}(a',b'/A)$ for some extreme pair. Condition (2) of the definition of weak 3-amalgamation can be ensured by picking $c \models p$ so that $c > g(a,b)$ for any A-definable function $g(y,z)$ bounded within p, which is possible by the compactness theorem.

⊣

The two conditions in Lemma 3.9 clearly mean that p has weak 3-amalgamation defined in [14]. Because of this or direct observation it follows $H_1(p) = 0$.

4. Work in progress

Here we summarize some work in progress concerning our homology groups.

In [6] the following was conjectured: Let T be stable having $(n+1)$-CA (over any algebraically closed set), and $p \in S(A)$ with $A = \operatorname{acl}(A)$. Then for every $n \geq 1$,

$$H_n(p) \cong \Gamma_n(p) := \operatorname{Aut}(\widetilde{a_0 ... a_{n-1}} / \bigcup_{i=0}^{n-1} \overline{\{a_0 ... a_{n-1}\} \smallsetminus \{a_i\}}),$$

where $\overline{a}$ denotes $\operatorname{acl}(aA)$; $\operatorname{Aut}(C/B)$ denotes the group of elementary permutations of the set C fixing B pointwise; $\{a_0, ..., a_n\}$ is A-independent, $a_i \models p$; and

$$\widetilde{a_0 ... a_{n-1}} := \overline{a_0 ... a_{n-1}} \cap \operatorname{dcl}(\bigcup_{i=0}^{n-1} \overline{\{a_0 ... a_n\} \smallsetminus \{a_i\}}).$$

In [6], the conjecture is proved when $n = 1, 2$. We plan to publish a proof for all n in the forthcoming preprint [8]. We may call this the Hurewicz correspondence since the result connects the homology groups to something analogous to a homotopy group, as in algebraic topology. To accomplish this, we needed to generalize the notion of groupoids to higher dimensions, and the vertex groups of the higher groupoids should be isomorphic to the groups $\Gamma_n(p)$ defined above. We could not find suitable generalization in the literature fit in our needs, so in [7] we define *n-ary polygroupoids*. A 2-ary polygroupoid is just an ordinary groupoid. In an n-ary polgroupoid, the “morphisms” live in fibers above ordered n-tuples of objects, and there is a sort of n-ary composition rule on these morphisms. Composition is only possible under certain compatibility conditions, and there are axioms generalizing invertibility and associativity for ordinary groupoids. In [7], we show that in any stable first-order theory that has k-uniqueness for all $k \leq n$ but fails $(n+1)$-uniqueness, there is an n-ary polygroupoid (definable in a mild extension of the language) which witnesses the failure of $(n+1)$-uniqueness.

In [14], as mentioned above it is proved that $H_1(p) = 0$ for any strong type p in a rosy theory as long as p is a Lascar type too; so any 1-shell in p is the boundary of a 2-chain. However, in contrast to the case of simple theories, we construct a series of types in rosy examples showing that there is no uniform bound for the minimal lengths of the 2-chains in the types having 1-shell boundaries. For this and its own research interests, in [14] and [17], all the possible 2-chains having the same 1-shell boundary are classified in a non-trivial amenable collection of functors. In this classification, the following results are obtained, among others: Any 2-chain with a 1-shell boundary is equivalent (preserving the boundary) to either an NR-type or an RN-type 2-chain with a support of size 3. Combinatorial and algebraic criteria determining the two types are given. A planar 2-chain is equivalent to a Lascar 2-chain.

In [4] and [5], from the failure of 3-uniqueness of a strong type p in a stable theory, a way of constructing canonical relatively definable groupiods is introduced. The profinite limit of vertex groups of the groupoids will be the automorphism group $\Gamma_2(p)$, and this seems to play a role in our setting analogous to that of a fundamental group; however, unlike $\pi_1(X)$ in topology, $\Gamma_2(p)$ is always abelian, since $\Gamma_2(p) \cong H_2(p)$. But in [15], a different canonical "fundamental" group for the type p is constructed which seems to give more information: this new group need not be abelian, and the group $\Gamma_2(p)$ is in the center of the new group.

In [6, 2.29], given an arbitrary profinite group G, only a brief sketch is given how to build a type p_G in a stable theory T_G such that $H_2(p_G) \cong G$. In [9], a more detailed proof is supplied.

Sustretov has recently found connections between 4-amalgamation and Galois cohomology in the preprint [19]. It would be very interesting to know if his work could be related to the computation of the homology groups discussed in this article.

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Department of Mathematics
Universidad de los Andes
Bogotá, Colombia
E-mail: jr.goodrick427@uniandes.edu.co

Department of Mathematics
Yonsei University
Seoul, Korea
E-mail: bkim@yonsei.ac.kr

Department of Mathematics
Towson University, MD
USA
E-mail: akolesnikov@towson.edu