Amalgamation functors and Homology groups in Model theory

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Outline

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- John Goodrick and Alexei Kolesnikov: Groupoids, covers, and 3-uniqueness in stable theories. To appear in *Journal of Symbolic Logic*.
- J. Goodrick, B. Kim, and A. Kolesnikov: Amalgamation functors and boundary properties in simple theories. To appear in *Israel Journal of Mathematics*.
- Tristram de Piro, B. Kim, and Jessica Millar: Constructing the type-definable group from the group configuration. *J. Math. Logic*, **6** (2006), 121–139.
- D. Evans: Higher amalgamation properties and splitting of finite covers. Preprint.

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- B. Kim and A. Pillay: Simple theories. *Annals of Pure and Applied Logic*, **88** (1997) 149–164.

Definition

Recall that by a *category* $\mathcal{C} = (\mathrm{Ob}(\mathcal{C}), \mathrm{Mor}(\mathcal{C}))$, we mean a class $\mathrm{Ob}(\mathcal{C})$ of members called *objects* of the category; equipped with a class $\mathrm{Mor}(\mathcal{C}) = \{\mathrm{Mor}(a, b) | \ a, b \in \mathrm{Ob}(\mathcal{C})\}$ where $\mathrm{Mor}(a, b) = \mathrm{Mor}_{\mathcal{C}}(a, b)$ is the class of *morphisms* between objects a, b (we write $f : a \to b$ to denote $f \in \mathrm{Mor}(a, b)$); and composition maps $\circ : \mathrm{Mor}(a, b) \times \mathrm{Mor}(b, c) \to \mathrm{Mor}(a, c)$ for each $a, b, c \in \mathrm{Ob}(\mathcal{C})$ such that

- (Associativity) if $f : a \to b$, $g : b \to c$ and $h : c \to d$ then $h \circ (g \circ f) = (h \circ g) \circ f$ holds, and
- (Identity) for each object c, there exists a morphism $1_c : c \to c$ called the identity morphism for c, such that for $f : a \to b$, we have $1_b \circ f = f = f \circ 1_a$.

A *groupoid* is a category where any morphism is invertible.

Note that any ordered set $(P, \leq)$ is a category where objects are members of P, and $\mathrm{Mor}(a, b) = \{(a, b)\}$ if $a \leq b$; $= \emptyset$ otherwise. Now we recall a functor F between two categories $\mathcal{C}, \mathcal{D}$.

Definition

The *functor* F sends an object $c \in \mathrm{Ob}(\mathcal{C})$ to $F(c) \in \mathrm{Ob}(\mathcal{D})$; and a morphism $f \in \mathrm{Mor}_{\mathcal{C}}(a, b)$ to $F(f) \in \mathrm{Mor}_{\mathcal{D}}(F(a), F(b))$ in such a way that

1. (Associativity) $F(g \circ f) = F(g) \circ F(f)$ for $f : a \to b$, $g : b \to c$;
2. (Identity) $F(1_c) = 1_{F(c)}$.

Throughout $\mathcal{C}$ is a fixed category, and s is a finite set of natural numbers.

Definition

Let $\mathcal{A}$ (or $\mathcal{A}_{\mathcal{C}}$) be a non-empty collection of functors $f : X \to \mathcal{C}$ for various downward-closed $X(\subseteq \mathcal{P}(s))$. We say that $\mathcal{A}$ is *amenable* if it satisfies all of the following properties:

1. (Invariance under isomorphisms) Suppose that $f : X \to \mathcal{C}$ is in $\mathcal{A}$ and $g : Y \to \mathcal{C}$ is isomorphic to f. Then $g \in \mathcal{A}$.
2. (Closure under restrictions and unions) If $X \subseteq \mathcal{P}(s)$ is downward-closed and $f : X \to \mathcal{C}$ is a functor, then $f \in \mathcal{A}$ if and only if for every $u \in X$, we have that $f \restriction \mathcal{P}(u) \in \mathcal{A}$.
3. (Closure under localizations) Suppose that $f : X \to \mathcal{C}$ is in $\mathcal{A}$ for some $X \subseteq \mathcal{P}(s)$ and $t \in X$. Then $f|_t : X|_t \to \mathcal{C}$ is also in $\mathcal{A}$; where $X|_t := \{u \in \mathcal{P}(s \setminus t) \mid t \cup u \in X\} \subseteq X$, and $f|_t : X|_t \to \mathcal{C}$ is the functor such that $f|_t(u) = f(t \cup u)$ and whenever $u \subseteq v \in X|_t$, $(f|_t)^u_v = f^{u \cup t}_{v \cup t}$.
4. Extensions of localizations are localizations of extensions.

For $u \subseteq v$, we write $f_v^u(u) := f((u,v))(f(u))$. Given a model $\mathcal{M}$, $\mathcal{C}_{\mathcal{M}}$ is its canonical category (i.e. small subsets of $\mathcal{M}$ together with their partial embeddings). Two examples have in mind.

Example

Let $\mathcal{A}_{tet.free} := \{f : X \to \mathcal{C}_{tet.free} |$ downward closed $X \subseteq \mathcal{P}(s)$ for some s; $f_u^{\{i\}}(\{i\}) \neq f_u^{\{j\}}(\{j\})$ are singletons for $i \neq j \in u \in X$; $f(u) = \{f_u^{\{i\}}(\{i\}) | \ i \in u\} \}$.

Example

Let G be a fixed finite group. $\mathcal{G}_G$:=An infinite *connected* groupoid with the vertex group ($=$ Mor(a,a)) G.
Let $\mathcal{A}_G := \{f : X \to \mathcal{C}_{\mathcal{G}_G} |$ downward closed $X \subseteq \mathcal{P}(s)$ for some finite s; $f_u^{\{i\}}(\{i\}) \neq f_u^{\{j\}}(\{j\})$ are single objects for $i \neq j \in u \in X\}$; $f(u) = \{f_u^{\{i\}}(\{i\}) | \ i \in u\} \}$.

Above two examples as 1st order structures have *simple* theories. In particular the theory of the 2nd example is *stable*.

For the rest fix $B \in \mathrm{Ob}(\mathcal{C})$, and fix an amenable $\mathcal{A} = \mathcal{A}_{\mathcal{C}}$. Now $\mathcal{A}_B := \{f \in \mathcal{A} | \ f(\emptyset) = B\}$.

Definition

Let $n \geq 0$ be a natural number. An *n-simplex in $\mathcal{C}$ (over B)* is a functor $f : \mathcal{P}(s) \to \mathcal{C}$ for some set s with $|s| = n+1$ (such that $f \in \mathcal{A}_B$). The set s is called the *support of f*, or $\mathrm{supp}(f)$.

Let $S_n(\mathcal{A}; B) = S_n(\mathcal{A}_B)$ denote the collection of all n-simplices in $\mathcal{A}$ over B.

Let $C_n(\mathcal{A}; B)$ denote the free abelian group generated by $S_n(\mathcal{A}; B)$; its elements are called *n-chains in $\mathcal{A}_B$*, or *n-chains over B*. The *support* of a chain $c = \sum_i k_i f_i$ (nonzero $k_i \in \mathbb{Z}$) is the union of the supports of all simplices f_i.

Definition

If $n \geq 1$ and $0 \leq i \leq n$, then the *ith boundary map* $\partial_n^i : C_n(\mathcal{A}_B) \to C_{n-1}(\mathcal{A}_B)$ is defined so that if $f \in S(\mathcal{A}_B)$ is an n-simplex with domain $\mathcal{P}(s)$, where $s = \{s_0 < \ldots < s_n\}$, then

$$\partial_n^i(f) = f \restriction \mathcal{P}(s \setminus \{s_i\})$$

and extended linearly to a group map on all of $C_n(\mathcal{A}_B)$.
If $n \geq 1$ and $0 \leq i \leq n$, then the *boundary map* $\partial_n : C_n(\mathcal{A}_B) \to C_{n-1}(\mathcal{A}_B)$ is defined by the rule

$$\partial_n(c) = \sum_{0 \leq i \leq n} (-1)^i \partial_n^i(c).$$

Definition

The kernel of ∂_n is denoted $Z_n(\mathcal{A}_B)$, and its elements are called *(n-)cycles*. The image of ∂_{n+1} in $C_n(\mathcal{A}_B)$ is denoted $B_n(\mathcal{A}_B)$, and its elements are called *(n-)boundaries*.

It can be shown (by the usual combinatorial argument) that $B_n(\mathcal{A}) \subseteq Z_n(\mathcal{A})$, or more briefly, "$\partial_n \circ \partial_{n+1} = 0$." Therefore we can define simplicial homology groups relative to $\mathcal{A}$:

Definition

The *nth (simplicial) homology group of $\mathcal{A}$ (over B)* is

$$H_n(\mathcal{A}_B) = Z_n(\mathcal{A}_B)/B_n(\mathcal{A}_B).$$

Caution: $\mathcal{A}$ and $\mathcal{A}_\emptyset$ are distinct !!

Definition

Let $n \geq 1$. Recall that $n = \{0, ..., n-1\}$ and $\mathcal{P}^-(n) := \mathcal{P}(n) \setminus \{n\}$.

1. $\mathcal{A}$ has *n-amalgamation* (or *n-existence*) if for any functor $f : \mathcal{P}^-(n) \to \mathcal{C}$ in $\mathcal{A}$, there is an $(n-1)$-simplex $g \supseteq f$ such that $g \in \mathcal{A}$.
2. $\mathcal{A}$ has *n-complete amalgamation* or *n-CA* if $\mathcal{A}$ has k-amalgamation for every k with $1 \leq k \leq n$.
3. $\mathcal{A}$ has *strong 2-amalgamation* if whenever $f : X \to \mathcal{C}$ and $g : Y \to \mathcal{C}$ are *simplices* in $\mathcal{A}$, $f \restriction (X \cap Y) = g \restriction (X \cap Y)$, and $X, Y \subseteq \mathcal{P}(s)$ for some finite s, then $f \cup g$ can be extended to a functor $h : \mathcal{P}(s) \to \mathcal{C}$ in $\mathcal{A}$.
4. $\mathcal{A}$ has *n-uniqueness* if for any functor $f : \mathcal{P}^-(n) \to \mathcal{A}$ and any two $(n-1)$-simplices g_1 and g_2 in $\mathcal{A}$ extending f, there is a natural isomorphism $F : g_1 \to g_2$ such that $F \restriction \mathrm{dom}(f)$ is the identity.

$\mathcal{A}_{tet.free}$ *does not have* 4-amalgamation. $\mathcal{A}_G$ has 3-uniqueness iff 4-amalgamation iff $Z(G) = 0$.

For the rest we assume $\mathcal{A}$ is *non-trivial* (i.e. has 1-amalgamation and strong 2-amalgamation).

Definition

If $n \geq 1$, an *n-shell* is an n-chain c of the form

$$\pm \sum_{0 \leq i \leq n+1} (-1)^i f_i,$$

where $f_0, \ldots, f_{n+1}$ are n-simplices such that whenever $0 \leq i < j \leq n+1$, we have $\partial^i f_j = \partial^{j-1} f_i$.

For example, if f is any $(n+1)$-simplex, then ∂f is an n-shell.

Theorem

If $\mathcal{A}$ has strong 2-amalgamation and $(n+1)$-CA (for some $n \geq 1$), then

$$H_n(\mathcal{A}_B) = \{[c] : c \text{ is an } n\text{-shell (over } B\text{) with support } n+2\}\,.$$

Corollary

If $\mathcal{A}$ has $(n+2)$-CA, then $H_n(\mathcal{A}_B) = 0$.

We consider the category $\mathcal{C}$ in the context of model theory.
Let T be rosy (having e.h.i, and e.i.) So T has a good notion of independence between subsets from a model of T, satisfying basic independence axioms. We work in a fixed large saturated model $\mathcal{M} \models T$. Fix a (small) set $B \subseteq \mathcal{M}$ such that $B = \operatorname{acl}(B)$. Let $\mathcal{C}_B$ be the category of all (small) subsets of $\mathcal{M}$ containing B, with partial elementary maps over B, i.e. $\mathcal{C}_B = \mathcal{C}_{\mathcal{M}_B}$. Fix a complete type p over B.

Definition

A *closed independent functor in p* is a functor $f : X \to \mathcal{C}_B$ such that:

1. X is a downward-closed subset of $\mathcal{P}(s)$ for some finite $s \subseteq \omega$; $f(\emptyset) \supseteq B$; and for $i \in s$, $f(\{i\})$ is of the form $\mathrm{acl}(Cb)$ where $b(\models p)$ is independent with $C = f^{\emptyset}_{\{i\}}(\emptyset)$ over B.
2. For all non-empty $u \in X$, we have $f(u) = \mathrm{acl}(B \cup \bigcup_{i\in u} f_u^{\{i\}}(\{i\}))$; and $\{f_u^{\{i\}}(\{i\}) | i \in u\}$ is independent over $f_u^{\emptyset}(\emptyset)$.

Let $\mathcal{A}_p$ denote all closed independent functors in p.

Now $\mathcal{A}$ is amenable. Due to the extension axiom of independence, $\mathcal{A}_p$ is non-trivial. $H_n(p) := H_n(\mathcal{A}_p; B)$. Similarly $S_n(p)$, $C_n(p)$, $Z_n(p)$, $B_n(p)$ are defined.

If T is simple, then we know that $\mathcal{A}_p$ has 3-amalgamation.

Corollary

If $\mathcal{A}_p$ has $(n+2)$-CA, then $H_n(p) = 0$.
If T is simple, then $H_1(p) = 0$.
Indeed if T is o-minimal, still $H_1(p) = 0$.

Example

- $H_n(\mathcal{A}_{tet.free}) = 0$ for all n, although $\mathcal{A}_{tet.free}$ does not have 4-amalgamation.
- $H_2(\mathcal{A}_G) = \mathrm{Z}(G)$. So if G has non-trivial center then $\mathcal{A}_G$ does not have 4-amalgamation.

If T is stable, then we have the following theorem which is analogous to Hurewicz's theorem in algebraic topology connecting homotopy groups and homology groups.

Suppress now $B = \emptyset$.
For a tuple c, we write $\overline{c} := \mathrm{acl}(cB) = \mathrm{acl}(c)$.

Theorem

T stable. Then $H_2(p) = \mathrm{Aut}(\widetilde{a_0 a_1}/\overline{a_0}, \overline{a_1})$ where $\{a_0, a_1, a_2\}$ is independent, $a_i \models p$, and

$$\widetilde{a_0 a_1} := \overline{a_0 a_1} \cap \mathrm{dcl}(\overline{a_0 a_2}, \overline{a_1 a_2}).$$

Moreover $H_2(p)$ is always an abelian profinite group. Conversely any abelian profinite group can occur as $H_2(p)$.

Conjecture

T stable having $(n+1)$-CA. Then

$$H_n(p) = \mathrm{Aut}(\widetilde{a_0 \ldots a_{n-1}} / \bigcup_{i=0}^{n-1} \overline{\{a_0 \ldots a_{n-1}\} \setminus \{a_i\}})$$

where $\{a_0, \ldots, a_n\}$ is independent, $a_i \models p$, and

$$\widetilde{a_0 \ldots a_{n-1}} := \overline{a_0 \ldots a_{n-1}} \cap \mathrm{dcl}(\bigcup_{i=0}^{n-1} \overline{\{a_0 \ldots a_n\} \setminus \{a_i\}}).$$

Lemma

If $n \geq 1$ and $\mathcal{A}$ has $(n+1)$-CA, then every n-cycle is a sum of n-shells. More precisely, for each $c \in Z_n(\mathcal{A}; B)$, $c = \sum_i k_i f_i$, there corresponds n-shells $c_i \in Z_n(\mathcal{A}; B)$ such that $c = (-1)^n \sum_i k_i c_i$. Moreover, if s is the support of the chain c and m is any element not in s, then we can choose $\operatorname{supp}(\sum_i k_i c_i) = s \cup \{m\}$.

Prism Lemma

Let $\mathcal{A}$ be a non-trivial amenable family of functors that satisfies $(n+1)$-amalgamation for some $n \geq 1$. Suppose that an n-shell $f := \sum_{0 \leq i \leq n+1} (-1)^i f_i$ and an n-fan $g^- := \sum_{i \in \{0,\ldots,\hat{k},\ldots,n+1\}} (-1)^i g_i$ are given, where f_i, g_i are n-simplices over B, $\operatorname{supp}(f) = s$ with $|s| = n+2$, and $\operatorname{supp}(g^-) = t = \{t_0, \ldots, t_{n+1}\}$, where $t_0 < \ldots < t_{n+1}$ and $s \cap t = \emptyset$. Then there is an n-simplex g_k over B with support $t \smallsetminus \{t_k\}$ such that $g := g^- + (-1)^k g_k$ is an n-shell over B and $f - g \in B_n(\mathcal{A}; B)$.

Skeleton of the proof of Hurewicz's Theorem for stable theory.

(1) The type p has 3-uniqueness iff p has 4-amalgamation iff $\mathrm{Aut}(\widetilde{a_0 a_1}/\overline{a_0}, \overline{a_1})$ is trivial iff $H_2(p)$ is trivial.

(2) (Hrushovski; Goodrick, Kolesnikov) p does not have 3-uniqueness iff $\widetilde{a_0 a_1}$ is non-empty.
Moreover for each finite $i \in \widetilde{a_0 a_1}$, there is a definable (in p) connected groupoid $\mathcal{G}_i$ whose vertex group G_i is finite non-trivial abelian and isomorphic to $\mathrm{Aut}(i/\overline{a_0}, \overline{a_1})$. For $j \in \widetilde{a_0 a_1}$, put $i \leq j$ if $i \in \mathrm{dcl}(j)$.

(3) $\mathrm{Aut}(\widetilde{a_0 a_1}/\overline{a_0}, \overline{a_1}) = \varprojlim \{\mathrm{Aut}(i/\overline{a_0}, \overline{a_1}) |\ i \in \widetilde{a_0 a_1}\} (\stackrel{\text{let}}{=} G)$ with restriction maps π_{ji}.

(4) For each such f, define suitably a map

$$\epsilon_i : S_2(p) \to G_i,$$

and extend it linearly to $C_2(p)$.

(5) Show that if a 2-chain c is a 2-boundary, then $\epsilon_i(c) = 0$. Thus the map ϵ_i induces a map $\widetilde{\epsilon_i} : H_2(p) \to G_i$, so induces a map

$$\epsilon : H_2(p) \to G$$

as well.

(6) Show that for a 2-cycle c, if $\widetilde{\epsilon_i}(c) = 0$ for every i, then c is 2-boundary. Therefore ϵ is injective.
Lastly show that ϵ is surjective.

More details for the steps (4),(5):
Choose an arbitrary selection function

$$\alpha_i : S_1(p) \to \mathrm{Mor}(G_i)$$

such that $\alpha_i(g) \in \mathrm{Mor}_{G_i}(b_0, b_1)$ where $\mathrm{supp}(g) = \{n_0 < n_1\}$ and $b_j := g_{\{n_0,n_1\}}^{\{n_j\}}(g(\{n_j\}))$.
Then define $\epsilon_i : S_2(p) \to G_i$, as

$$\epsilon_i(f) := [f_{02}^{-1} \circ f_{12} \circ f_{01}]_{G_i}$$

where for $\mathrm{supp}(f) = \{n_0 < n_1 < n_2\} = s$,

$$f_{jk} := f_s^{\{n_j,n_k\}}(\alpha_i(f \restriction \mathrm{dom}(\{n_j, n_k\}))).$$