

Amalgamation functors and Homology groups in Model theory

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Outline

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- E. Hrushovski: Groupoids, imaginaries and internal covers. Preprint. arXiv:math.LO/0603413.

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- John Goodrick and Alexei Kolesnikov: Groupoids, covers, and 3-uniqueness in stable theories. To appear in *Journal of Symbolic Logic*.
- J. Goodrick, B. Kim, and A. Kolesnikov: Amalgamation functors and boundary properties in simple theories. To appear in *Israel Journal of Mathematics*.
- Tristram de Piro, B. Kim, and Jessica Millar: Constructing the type-definable group from the group configuration. *J. Math. Logic*, **6** (2006), 121–139.
- D. Evans: Higher amalgamation properties and splitting of finite covers. Preprint.

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- B. Kim and A. Pillay: Simple theories. *Annals of Pure and Applied Logic*, **88** (1997) 149–164.

Definition

Recall that by a *category* $\mathcal{C} = (\text{Ob}(\mathcal{C}), \text{Mor}(\mathcal{C}))$, we mean a class $\text{Ob}(\mathcal{C})$ of members called *objects* of the category; equipped with a class $\text{Mor}(\mathcal{C}) = \{\text{Mor}(a, b) \mid a, b \in \text{Ob}(\mathcal{C})\}$ where $\text{Mor}(a, b) = \text{Mor}_{\mathcal{C}}(a, b)$ is the class of *morphisms* between objects a, b (we write $f : a \rightarrow b$ to denote $f \in \text{Mor}(a, b)$); and composition maps $\circ : \text{Mor}(a, b) \times \text{Mor}(b, c) \rightarrow \text{Mor}(a, c)$ for each $a, b, c \in \text{Ob}(\mathcal{C})$ such that

- (Associativity) if $f : a \rightarrow b$, $g : b \rightarrow c$ and $h : c \rightarrow d$ then $h \circ (g \circ f) = (h \circ g) \circ f$ holds, and
- (Identity) for each object c , there exists a morphism $1_c : c \rightarrow c$ called the identity morphism for c , such that for $f : a \rightarrow b$, we have $1_b \circ f = f = f \circ 1_a$.

A *groupoid* is a category where any morphism is invertible.

Note that any ordered set (P, \leq) is a category where objects are members of P , and $\text{Mor}(a, b) = \{(a, b)\}$ if $a \leq b$; $= \emptyset$ otherwise. Now we recall a functor F between two categories \mathcal{C}, \mathcal{D} .

Definition

The *functor* F sends an object $c \in \text{Ob}(\mathcal{C})$ to $F(c) \in \text{Ob}(\mathcal{D})$; and a morphism $f \in \text{Mor}_{\mathcal{C}}(a, b)$ to $F(f) \in \text{Mor}_{\mathcal{D}}(F(a), F(b))$ in such a way that

- 1 (Associativity) $F(g \circ f) = F(g) \circ F(f)$ for $f : a \rightarrow b$,
 $g : b \rightarrow c$;
- 2 (Identity) $F(1_c) = 1_{F(c)}$.

Throughout \mathcal{C} is a fixed category, and s is a finite set of natural numbers.

Definition

Let \mathcal{A} (or $\mathcal{A}_{\mathcal{C}}$) be a non-empty collection of functors $f : X \rightarrow \mathcal{C}$ for various downward-closed $X (\subseteq \mathcal{P}(s))$. We say that \mathcal{A} is *amenable* if it satisfies all of the following properties:

- 1 (Invariance under isomorphisms) Suppose that $f : X \rightarrow \mathcal{C}$ is in \mathcal{A} and $g : Y \rightarrow \mathcal{C}$ is isomorphic to f . Then $g \in \mathcal{A}$.
- 2 (Closure under restrictions and unions) If $X \subseteq \mathcal{P}(s)$ is downward-closed and $f : X \rightarrow \mathcal{C}$ is a functor, then $f \in \mathcal{A}$ if and only if for every $u \in X$, we have that $f \upharpoonright \mathcal{P}(u) \in \mathcal{A}$.
- 3 (Closure under localizations) Suppose that $f : X \rightarrow \mathcal{C}$ is in \mathcal{A} for some $X \subseteq \mathcal{P}(s)$ and $t \in X$. Then $f|_t : X|_t \rightarrow \mathcal{C}$ is also in \mathcal{A} ; where $X|_t := \{u \in \mathcal{P}(s \setminus t) \mid t \cup u \in X\} \subseteq X$, and $f|_t : X|_t \rightarrow \mathcal{C}$ is the functor such that $f|_t(u) = f(t \cup u)$ and whenever $u \subseteq v \in X|_t$, $(f|_t)_v^u = f_{v \cup t}^{u \cup t}$.
- 4 Extensions of localizations are localizations of extensions.

For $u \subseteq v$, we write $f_v^u(u) := f((u, v))(f(u))$. Given a model \mathcal{M} , $\mathcal{C}_{\mathcal{M}}$ is its canonical category (i.e. small subsets of \mathcal{M} together with their partial embeddings). Two examples have in mind.

Example

Let $\mathcal{A}_{tet.free} := \{f : X \rightarrow \mathcal{C}_{tet.free} \mid \text{downward closed } X \subseteq \mathcal{P}(s) \text{ for some } s; f_u^{\{i\}}(\{i\}) \neq f_u^{\{j\}}(\{j\}) \text{ are singletons for } i \neq j \in u \in X; f(u) = \{f_u^{\{i\}}(\{i\}) \mid i \in u\} \}$.

Example

Let G be a fixed finite group. $\mathcal{G}_G :=$ An infinite *connected* groupoid with the vertex group (= $\text{Mor}(a, a)$) G .

Let $\mathcal{A}_G := \{f : X \rightarrow \mathcal{C}_{\mathcal{G}_G} \mid \text{downward closed } X \subseteq \mathcal{P}(s) \text{ for some finite } s; f_u^{\{i\}}(\{i\}) \neq f_u^{\{j\}}(\{j\}) \text{ are single objects for } i \neq j \in u \in X; f(u) = \{f_u^{\{i\}}(\{i\}) \mid i \in u\} \}$.

Above two examples as 1st order structures have *simple* theories. In particular the theory of the 2nd example is *stable*.

For the rest fix $B \in \text{Ob}(\mathcal{C})$, and fix an amenable $\mathcal{A} = \mathcal{A}_{\mathcal{C}}$. Now $\mathcal{A}_B := \{f \in \mathcal{A} \mid f(\emptyset) = B\}$.

Definition

Let $n \geq 0$ be a natural number. An n -simplex in \mathcal{C} (over B) is a functor $f : \mathcal{P}(s) \rightarrow \mathcal{C}$ for some set s with $|s| = n + 1$ (such that $f \in \mathcal{A}_B$). The set s is called the *support* of f , or $\text{supp}(f)$.

Let $S_n(\mathcal{A}; B) = S_n(\mathcal{A}_B)$ denote the collection of all n -simplices in \mathcal{A} over B .

Let $C_n(\mathcal{A}; B)$ denote the free abelian group generated by $S_n(\mathcal{A}; B)$; its elements are called n -chains in \mathcal{A}_B , or n -chains over B . The *support* of a chain $c = \sum_i k_i f_i$ (nonzero $k_i \in \mathbb{Z}$) is the union of the supports of all simplices f_i .

Definition

If $n \geq 1$ and $0 \leq i \leq n$, then the *ith boundary map* $\partial_n^i : C_n(\mathcal{A}_B) \rightarrow C_{n-1}(\mathcal{A}_B)$ is defined so that if $f \in S(\mathcal{A}_B)$ is an n -simplex with domain $\mathcal{P}(s)$, where $s = \{s_0 < \dots < s_n\}$, then

$$\partial_n^i(f) = f \upharpoonright \mathcal{P}(s \setminus \{s_i\})$$

and extended linearly to a group map on all of $C_n(\mathcal{A}_B)$.

If $n \geq 1$ and $0 \leq i \leq n$, then the *boundary map* $\partial_n : C_n(\mathcal{A}_B) \rightarrow C_{n-1}(\mathcal{A}_B)$ is defined by the rule

$$\partial_n(c) = \sum_{0 \leq i \leq n} (-1)^i \partial_n^i(c).$$

Definition

The kernel of ∂_n is denoted $Z_n(\mathcal{A}_B)$, and its elements are called *(n-)cycles*. The image of ∂_{n+1} in $C_n(\mathcal{A}_B)$ is denoted $B_n(\mathcal{A}_B)$, and its elements are called *(n-)boundaries*.

It can be shown (by the usual combinatorial argument) that $B_n(\mathcal{A}) \subseteq Z_n(\mathcal{A})$, or more briefly, " $\partial_n \circ \partial_{n+1} = 0$." Therefore we can define simplicial homology groups relative to \mathcal{A} :

Definition

The *n*th (simplicial) homology group of \mathcal{A} (over B) is

$$H_n(\mathcal{A}_B) = Z_n(\mathcal{A}_B)/B_n(\mathcal{A}_B).$$

Caution: \mathcal{A} and \mathcal{A}_\emptyset are distinct !!

Definition

Let $n \geq 1$. Recall that $n = \{0, \dots, n-1\}$ and $\mathcal{P}^-(n) := \mathcal{P}(n) \setminus \{n\}$.

- 1 \mathcal{A} has *n-amalgamation* (or *n-existence*) if for any functor $f : \mathcal{P}^-(n) \rightarrow \mathcal{C}$ in \mathcal{A} , there is an $(n-1)$ -simplex $g \supseteq f$ such that $g \in \mathcal{A}$.
- 2 \mathcal{A} has *n-complete amalgamation* or *n-CA* if \mathcal{A} has k -amalgamation for every k with $1 \leq k \leq n$.
- 3 \mathcal{A} has *strong 2-amalgamation* if whenever $f : X \rightarrow \mathcal{C}$ and $g : Y \rightarrow \mathcal{C}$ are *simplices* in \mathcal{A} , $f \upharpoonright (X \cap Y) = g \upharpoonright (X \cap Y)$, and $X, Y \subseteq \mathcal{P}(s)$ for some finite s , then $f \cup g$ can be extended to a functor $h : \mathcal{P}(s) \rightarrow \mathcal{C}$ in \mathcal{A} .
- 4 \mathcal{A} has *n-uniqueness* if for any functor $f : \mathcal{P}^-(n) \rightarrow \mathcal{A}$ and any two $(n-1)$ -simplices g_1 and g_2 in \mathcal{A} extending f , there is a natural isomorphism $F : g_1 \rightarrow g_2$ such that $F \upharpoonright \text{dom}(f)$ is the identity.

$\mathcal{A}_{\text{tet.free}}$ does not have 4-amalgamation. \mathcal{A}_G has 3-uniqueness iff 4-amalgamation iff $Z(G) = 0$.

For the rest we assume \mathcal{A} is *non-trivial* (i.e. has 1-amalgamation and strong 2-amalgamation).

Definition

If $n \geq 1$, an *n-shell* is an *n-chain* c of the form

$$\pm \sum_{0 \leq i \leq n+1} (-1)^i f_i,$$

where f_0, \dots, f_{n+1} are *n-simplices* such that whenever $0 \leq i < j \leq n+1$, we have $\partial^i f_j = \partial^{j-1} f_i$.

For example, if f is any $(n+1)$ -simplex, then ∂f is an *n-shell*.

Theorem

If \mathcal{A} has strong 2-amalgamation and $(n + 1)$ -CA (for some $n \geq 1$), then

$$H_n(\mathcal{A}_B) = \{[c] : c \text{ is an } n\text{-shell (over } B) \text{ with support } n + 2\}.$$

Corollary

If \mathcal{A} has $(n + 2)$ -CA, then $H_n(\mathcal{A}_B) = 0$.

We consider the category \mathcal{C} in the context of model theory. Let T be rosy (having e.h.i, and e.i.) So T has a good notion of independence between subsets from a model of T , satisfying basic independence axioms. We work in a fixed large saturated model $\mathcal{M} \models T$. Fix a (small) set $B \subseteq \mathcal{M}$ such that $B = \text{acl}(B)$. Let \mathcal{C}_B be the category of all (small) subsets of \mathcal{M} containing B , with partial elementary maps over B , i.e. $\mathcal{C}_B = \mathcal{C}_{\mathcal{M}_B}$. Fix a complete type p over B .

Definition

A *closed independent functor in p* is a functor $f : X \rightarrow \mathcal{C}_B$ such that:

- 1 X is a downward-closed subset of $\mathcal{P}(s)$ for some finite $s \subseteq \omega$; $f(\emptyset) \supseteq B$; and for $i \in s$, $f(\{i\})$ is of the form $\text{acl}(Cb)$ where $b(\models p)$ is independent with $C = f_{\{i\}}^\emptyset(\emptyset)$ over B .
- 2 For all non-empty $u \in X$, we have $f(u) = \text{acl}(B \cup \bigcup_{i \in u} f_u^{\{i\}}(\{i\}))$; and $\{f_u^{\{i\}}(\{i\}) \mid i \in u\}$ is independent over $f_u^\emptyset(\emptyset)$.

Let \mathcal{A}_p denote all closed independent functors in p .

Now \mathcal{A} is amenable. Due to the extension axiom of independence, \mathcal{A}_p is non-trivial. $H_n(p) := H_n(\mathcal{A}_p; B)$. Similarly $S_n(p)$, $C_n(p)$, $Z_n(p)$, $B_n(p)$ are defined.

If T is simple, then we know that \mathcal{A}_p has 3-amalgamation.

Corollary

If \mathcal{A}_p has $(n + 2)$ -CA, then $H_n(p) = 0$.

If T is simple, then $H_1(p) = 0$.

Indeed if T is o-minimal, still $H_1(p) = 0$.

Example

- $H_n(\mathcal{A}_{tet.free}) = 0$ for all n , although $\mathcal{A}_{tet.free}$ does not have 4-amalgamation.
- $H_2(\mathcal{A}_G) = Z(G)$. So if G has non-trivial center then \mathcal{A}_G does not have 4-amalgamation.

If T is stable, then we have the following theorem which is analogous to Hurewicz's theorem in algebraic topology connecting homotopy groups and homology groups.

Suppress now $B = \emptyset$.

For a tuple c , we write $\bar{c} := \text{acl}(cB) = \text{acl}(c)$.

Theorem

T stable. Then $H_2(p) = \text{Aut}(\widetilde{a_0 a_1} / \overline{a_0}, \overline{a_1})$ where $\{a_0, a_1, a_2\}$ is independent, $a_i \models p$, and

$$\widetilde{a_0 a_1} := \overline{a_0 a_1} \cap \text{dcl}(\overline{a_0 a_2}, \overline{a_1 a_2}).$$

Moreover $H_2(p)$ is always an abelian profinite group. Conversely any abelian profinite group can occur as $H_2(p)$.

Conjecture

T stable having $(n + 1)$ -CA. Then

$$H_n(p) = \text{Aut}(\widetilde{a_0 \dots a_{n-1}} / \bigcup_{i=0}^{n-1} \overline{\{a_0 \dots a_{n-1}\} \setminus \{a_i\}})$$

where $\{a_0, \dots, a_n\}$ is independent, $a_i \models p$, and

$$\widetilde{a_0 \dots a_{n-1}} := \overline{a_0 \dots a_{n-1}} \cap \text{dcl}\left(\bigcup_{i=0}^{n-1} \overline{\{a_0 \dots a_n\} \setminus \{a_i\}}\right).$$

Lemma

If $n \geq 1$ and \mathcal{A} has $(n + 1)$ -CA, then every n -cycle is a sum of n -shells. More precisely, for each $c \in Z_n(\mathcal{A}; B)$, $c = \sum_i k_i f_i$, there corresponds n -shells $c_i \in Z_n(\mathcal{A}; B)$ such that $c = (-1)^n \sum_i k_i c_i$. Moreover, if s is the support of the chain c and m is any element not in s , then we can choose $\text{supp}(\sum_i k_i c_i) = s \cup \{m\}$.

Prism Lemma

Let \mathcal{A} be a non-trivial amenable family of functors that satisfies $(n+1)$ -amalgamation for some $n \geq 1$. Suppose that an n -shell $f := \sum_{0 \leq i \leq n+1} (-1)^i f_i$ and an n -fan $g^- := \sum_{i \in \{0, \dots, \hat{k}, \dots, n+1\}} (-1)^i g_i$ are given, where f_i, g_i are n -simplices over B , $\text{supp}(f) = s$ with $|s| = n+2$, and $\text{supp}(g^-) = t = \{t_0, \dots, t_{n+1}\}$, where $t_0 < \dots < t_{n+1}$ and $s \cap t = \emptyset$. Then there is an n -simplex g_k over B with support $t \setminus \{t_k\}$ such that $g := g^- + (-1)^k g_k$ is an n -shell over B and $f - g \in B_n(\mathcal{A}; B)$.

Skeleton of the proof of Hurewicz's Theorem for stable theory.

- (1) The type p has 3-uniqueness iff p has 4-amalgamation iff $\text{Aut}(\widetilde{a_0 a_1}/\overline{a_0}, \overline{a_1})$ is trivial iff $H_2(p)$ is trivial.
- (2) (Hrushovski; Goodrick, Kolesnikov) p does not have 3-uniqueness iff $\widetilde{a_0 a_1}$ is non-empty.
Moreover for each finite $i \in \widetilde{a_0 a_1}$, there is a definable (in p) connected groupoid \mathcal{G}_i whose vertex group G_i is finite non-trivial abelian and isomorphic to $\text{Aut}(i/\overline{a_0}, \overline{a_1})$. For $j \in \widetilde{a_0 a_1}$, put $i \leq j$ if $i \in \text{dcl}(j)$.
- (3) $\text{Aut}(\widetilde{a_0 a_1}/\overline{a_0}, \overline{a_1}) = \varprojlim \{ \text{Aut}(i/\overline{a_0}, \overline{a_1}) \mid i \in \widetilde{a_0 a_1} \} \stackrel{\text{let}}{=} G$ with restriction maps π_{ji} .
- (4) For each such f , define suitably a map

$$\epsilon_i : S_2(p) \rightarrow G_i,$$

and extend it linearly to $C_2(p)$.

- (5) Show that if a 2-chain c is a 2-boundary, then $\epsilon_j(c) = 0$. Thus the map ϵ_j induces a map $\tilde{\epsilon}_j : H_2(p) \rightarrow G_j$, so induces a map

$$\epsilon : H_2(p) \rightarrow G$$

as well.

- (6) Show that for a 2-cycle c , if $\tilde{\epsilon}_i(c) = 0$ for every i , then c is 2-boundary. Therefore ϵ is injective. Lastly show that ϵ is surjective.

More details for the steps (4),(5):

Choose an arbitrary selection function

$$\alpha_i : S_1(p) \rightarrow \text{Mor}(G_i)$$

such that $\alpha_i(g) \in \text{Mor}_{G_i}(b_0, b_1)$ where $\text{supp}(g) = \{n_0 < n_1\}$ and $b_j := g_{\{n_0, n_1\}}^{\{n_j\}}(g(\{n_j\}))$.

Then define $\epsilon_i : S_2(p) \rightarrow G_i$, as

$$\epsilon_i(f) := [f_{02}^{-1} \circ f_{12} \circ f_{01}]_{G_i}$$

where for $\text{supp}(f) = \{n_0 < n_1 < n_2\} = s$,

$$f_{jk} := f_s^{\{n_j, n_k\}}(\alpha_i(f \upharpoonright \text{dom}(\{n_j, n_k\}))).$$