

TREE INDISCERNIBILITIES, REVISITED

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ABSTRACT. We give definitions that distinguish between two notions of indiscernibility for a sequence $(a_\eta \mid \eta \in \omega^{>\omega})$ that saw original use in [6], which we name *s-* and *ss-indiscernibility*. We restate proofs for [6, App. 2.6, 2.7], expanding on the details. Using these definitions and detailed proofs, we prove *s-* and *ss-indiscernible* modeling theorems and give some applications of these theorems. In particular, these clarified notions of indiscernibility supply a correct and complete proof of the fact that a theory has the tree property (TP) iff it has either TP_1 or TP_2 .

1. INTRODUCTION

Many classification-theoretic properties can be stated in terms of the existence of an infinite set of witnesses to some “forbidden” graph configuration, where the edge relation is some definable relation in the theory. The following properties are all such examples: being unstable, having the independence property, having the tree property (being non-simple), having TP_1 , TP_2 , or the SOP_n , for $n = 1, 2$. Being able to choose “very homogeneous” witnesses to the definable configuration, whenever witnesses exist, is a very powerful tool. We look at unstable theories as an example of this situation. A theory T is *unstable* just in case it has the *order property*, i.e. there exist some formula $\varphi(\bar{x}; \bar{y})$ in the language of T , and some infinite set of finite tuples $\{\langle \bar{a}_i, \bar{b}_i \rangle \mid i < \omega\}$ from the monster model \mathcal{M} such that $\models \varphi(\bar{a}_i; \bar{b}_j) \Leftrightarrow i < j$. The correct notion of a “very homogeneous” infinite set in this case is that of an *indiscernible sequence*, which is a sequence of parameters $(\bar{c}_i \mid i \in I)$, indexed by some linear order I , such that $\models \psi(\bar{c}_{i_1}, \dots, \bar{c}_{i_n}) \Leftrightarrow \psi(\bar{c}_{j_1}, \dots, \bar{c}_{j_n})$, whenever $i_1 < \dots < i_n$ and $j_1 < \dots < j_n$ are in I and $\psi(\bar{x}_1, \dots, \bar{x}_n)$ is any formula in the language. By Ramsey’s theorem, we may always choose an indiscernible sequence of witnesses to the order property in any unstable theory.

Suppose we consider the linear order I as a structure in its own right, a set with a binary relation, $(I, <)$. Then the sequence $(c_i : i \in I)$ of parameters from \mathcal{M} is indiscernible just in case for any n and $(i_1, \dots, i_n), (j_1, \dots, j_n)$ from I with the same quantifier-free type in I , we have that $(c_{i_1}, \dots, c_{i_n})$ and $(c_{j_1}, \dots, c_{j_n})$ share the same complete type in \mathcal{M} . Once viewed from this perspective, one may have as many notions of indiscernibility as there are useful index structures to serve in the place of I , as was first pointed out in [6]. If the indiscernible is indexed by a structure that is a tree under some expansion of the language for partial orders, $\{\preceq\}$, call it a

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tree-indexed indiscernible. Tree-indexed indiscernibles have been studied in several places, among them [3, 2]. A certain tree-indexed indiscernible was recently used in [4] to show that TP_1 is equivalent to $k\text{-TP}_1$. In addition, similar notions to the ones in this paper are currently being developed in [8].

In this paper we solidify definitions for two notions of tree-indexed indiscernibility that are used in [6], which we name s- and ss-indiscernibility. These notions are used in the proof of [6, Thm III.7.11], that $k\text{-TP}$ for some k is equivalent to 2-TP_1 or $k'\text{-TP}_2$, for some k' , however, the definitions for s- and ss-indiscernibles seem to be intermixed with one another in the discussion after [6, Def VII.3.1]. Say that the indiscernibles $(b_i \mid i \in I)$ are *based on* the parameters $(a_i \mid i \in I)$ if for any finite set of formulas Δ in the language of \mathcal{M} , any $(b_{j_1}, \dots, b_{j_n})$ where \bar{j} is closed under function symbols in the index language shares the same Δ -type with some $(a_{i_1}, \dots, a_{i_n})$ where the finite substructure of I indexing the a 's is isomorphic to that indexing the b 's by the isomorphism $i_k \mapsto j_k$ (see Definition 6.7 for a precise statement.) In this sense, the indiscernible that we obtain is “finitely modeled” on the existing set of parameters $(a_i \mid i \in I)$ as it is indexed by I . We say I -indexed indiscernibles have the *modeling property* if for any parameters $(a_i \mid i \in I)$ we may find indiscernible $(b_i \mid i \in I)$ based on the $(a_i \mid i \in I)$.

That s- and ss-indiscernibles have the modeling property is the content of Theorems 3.11 and 3.12, the s- (ss- resp.) modeling theorems. The combinatorial bedrock for these two theorems is given in [6, App. 2.6, 2.7], and these claims are implicit in [6, Thms III.7.11, VII.3.6], however explicit analogues of the proofs do not appear to be present, so we supply them here. Moreover, we define s- and ss-*type* properties, and prove in Theorem 6.3 that such properties may respectively be modeled by s- and ss-indiscernibles based on parameters witnessing those properties.

One property that cannot in general be modeled by ss-indiscernibles is the tree property. A step in the proof of [6, Thm III.7.11] claims that given a formula $\varphi(x; y)$ witnessing $k\text{-TP}$ with certain parameters, we may assume those parameters are ss-indiscernible by Ramsey’s theorem and compactness. In fact, the assumption that NTP_2 has essential use in obtaining such ss-indiscernibles, as we show in Remark 6.8. In Theorem 6.6 we patch this apparent gap in the original write-up, illustrating how the assumption of TP together with $k\text{-NTP}_2$ for all k , yields ss-indiscernible witnesses to TP . The original statement of Theorem III.7.11 states that the negation of $k'\text{-TP}_2$ for all k' yields 2-TP_1 , under the assumption of $k\text{-TP}$. A modified statement is given in [1]: that $k\text{-TP}$ is equivalent to 2-TP_1 or 2-TP_2 .

In obtaining the modified version of the theorem, crucial use is made of the fact that 2-TP_2 is equivalent to $k\text{-TP}_2$. This fact is argued for in [1] and we repeat the main argument in Prop 6.5 relying on a new proof of the supporting combinatorial result in Lemma 6.4.

We state our notation in section 2. The definitions of s- and ss-indiscernibles are given in section 3. In section 4 we restate proofs of key combinatorial lemmas from [6], providing further details. In section 5 we argue for Theorems 3.11 and 3.12. In section 6 we give formal definitions of s- and ss-type properties, and prove Theorem 6.3. Finally, we provide a patch-up for the proof of [6, Thm III.7.11] in Theorem 6.6.

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2. NOTATION AND CONVENTIONS

We use standard notation. We work in a saturated model \mathcal{M} of a complete theory T in a first-order language L , and x, y, a, b, \dots denote finite tuples. When there is little chance of confusion we also use T to refer a tree. For a tuple $\bar{\eta} = \langle \eta_0, \dots, \eta_{n-1} \rangle$, $l(\bar{\eta}) := n$. For a set X we write $\|X\|$ for the cardinality of X . For sets X, Y , by $X - Y$ we mean the set difference of X and Y . By ${}^Y X$ we mean the set of functions from Y into X ; for an ordinal α , by ${}^{\alpha} X$ we mean the set of functions from β into X for all ordinals $\beta < \alpha$. For an indexed set of parameters $(a_\eta | \eta \in {}^{\beta} \lambda)$ and a tuple $\bar{\eta}$ from ${}^{\beta} \lambda$, $\bar{a}_{\bar{\eta}} := \langle a_{\eta_0}, \dots, a_{\eta_{n-1}} \rangle$. As usual, $a \equiv b$ ($a \equiv_{\Delta} b$, resp.) means $\text{tp}(a) = \text{tp}(b)$ ($\text{tp}_{\Delta}(a) = \text{tp}_{\Delta}(b)$, resp.) as computed in \mathcal{M} .

2.1. Trees. Here we fix terminology for trees. In this paper, we will consider solely subtrees $T \subseteq {}^{\beta} \lambda$ for β, λ ordinals. Given trees S, T , an *embedding* $f : S \rightarrow T$ is an injection respecting the partial tree order \trianglelefteq , i.e. $\eta \triangleleft \nu$ (η is a proper initial segment of ν) in S iff $f(\eta) \triangleleft f(\nu)$ in T . The *meet* in the partial tree order (T, \trianglelefteq) of two elements η, ν is denoted by $\eta \wedge \nu$. A tuple $\bar{\eta}$ from a tree is said to be *meet-closed* if for each $i, j < l(\bar{\eta})$, there is $k < l(\bar{\eta})$ such that $\eta_k = \eta_i \wedge \eta_j$. Elements $\eta_0, \dots, \eta_{k-1} \in {}^{\beta} \lambda$ are called *siblings* if they are distinct elements sharing the same immediate predecessor. (i.e. there exist $\nu \in {}^{\beta} \lambda$ and distinct $t_0, \dots, t_{k-1} < \lambda$ such that $\nu \frown \langle t_i \rangle = \eta_i$ for each $i < k$.) Elements $\eta_0, \dots, \eta_{k-1} \in {}^{\beta} \lambda$ are called *distant siblings* if there exist $\nu \in {}^{\beta} \lambda$ and distinct $t_0, \dots, t_{k-1} < \lambda$ such that $\nu \frown \langle t_i \rangle \trianglelefteq \eta_i$ for each $i < k$. When distant siblings occur on the same level in a tree, we shall call them *same-level distant siblings*. (Recall that the *level* of an element $\eta \in {}^{\beta} \lambda$ means the (ordinal) length, i.e. the domain of η as a sequence.)

The following are useful subtrees:

Definition 2.1. Fix a tree T . Given $\eta \in T$, define $c(\eta, T) := \{\nu \in T \mid \eta \trianglelefteq \nu\}$, the *cone at η in T* ; define $T[\eta] := \{\nu \in T \mid \eta \trianglelefteq \nu \text{ or } \nu \trianglelefteq \eta\}$, the *neighborhood at η in T* .

We often consider a subtree of ${}^{\beta} \lambda$ with an additional linear ordering, the lexicographic ordering on sequences, using the natural well-ordering on λ . Say that an embedding of trees $f : S \rightarrow T$ is *orientation-preserving* if it respects this linear ordering.

We also want notation for tree embeddings that preserve information about the levels of a tree. For $\eta \in T$, we let $|\eta|_T$ denote the level of η in T , which will be the (ordinal) length of η as a sequence. We drop the notation T and write $|\eta|$ when T is clear from context. Given a tree T , by the γ -th level of T , we mean the set $T(\gamma) := \{\eta \in T \mid |\eta|_T = \gamma\}$. We say that T is κ -branching at level γ if for all $\eta \in T(\gamma)$, η has κ immediate successors in T . Let $\text{Lev}(T)$ denote the set of all ordinals realized as levels by elements in T .

Definition 2.2. Fix trees S and T . By a *level function* we mean an injective function $f : \text{Lev}(S) \rightarrow \text{Lev}(T)$.

A subtree $S \subseteq T$ is a *graded subtree* if there is some level function f such that $|\eta|_S = \alpha$ if and only if $|\eta|_T = f(\alpha)$.

We say $f : S \rightarrow T$ is a *graded embedding* if it is an embedding and $f(S)$ is a graded subtree of T .

We say that a subtree (embedding) is *level-preserving* if it is graded witnessed by the identity function as level function.

2.2. k -TP, k -TP₁, weak k -TP₁ and k -TP₂. Here we recall some definitions. A theory is said to have k -TP if there exist a formula $\varphi(x, y)$ and a set $\{a_\eta \mid \eta \in {}^\omega > \omega\}$ of parameters such that $\{\varphi(x, a_{\mu \upharpoonright n}) \mid n < \omega\}$ is consistent for every $\mu \in {}^\omega \omega$, while for any siblings $\eta_0, \dots, \eta_{k-1} \in {}^\omega > \omega$, $\{\varphi(x, a_{\eta_j}) \mid j < k\}$ is inconsistent. The definitions of k -TP₁ and *weak* k -TP₁ are obtained by replacing the word ‘siblings’ by ‘pairwise incomparable elements’ and ‘distant siblings’, respectively. A theory is said to have k -TP₂ if there exist a formula $\varphi(x, y)$ and a set $\{a_j^i \mid i, j < \omega\}$ of parameters such that $\{\varphi(x, a_{f(i)}^i) \mid i < \omega\}$ is consistent for every function $f : \omega \rightarrow \omega$, while $\{\varphi(x, a_j^i) \mid j < \omega\}$ is k -inconsistent for every $i < \omega$. TP means k -TP for some $k \geq 2$. By TP₁ and TP₂, we shall mean 2-TP₁ and 2-TP₂, respectively.

3. S-INDISCERNIBLES AND SS-INDISCERNIBLES

Here we make a distinction between two kinds of indiscernibility for trees presented in [6]. We want two notions of similarity:

Definition 3.1. For $\bar{\eta}, \bar{\nu}$ meet-closed tuples from ${}^{\beta >} \lambda$, we say

- (1) $\bar{\eta} \sim_s \bar{\nu}$ ($\bar{\eta}$ is *s-similar* to $\bar{\nu}$) if
 - (a) For some n , $l(\bar{\eta}) = l(\bar{\nu}) = n$
 - (b) $(\eta_i \wedge \eta_j) \trianglelefteq (\eta_k \wedge \eta_l) \Leftrightarrow (\nu_i \wedge \nu_j) \trianglelefteq (\nu_k \wedge \nu_l)$, for all $i, j, k, l < n$
 - (c) $|\eta_i \wedge \eta_j| = |\nu_i \wedge \nu_j|$, for all $i, j < n$
 - (d) $(\eta_i \wedge \eta_j) <_{\text{lex}} (\eta_k \wedge \eta_l) \Leftrightarrow (\nu_i \wedge \nu_j) <_{\text{lex}} (\nu_k \wedge \nu_l)$, for all $i, j, k, l < n$;
- (2) $\bar{\eta} \sim_{ss} \bar{\nu}$ ($\bar{\eta}$ is *ss-similar* to $\bar{\nu}$) if
 - (a) For some n , $l(\bar{\eta}) = l(\bar{\nu}) = n$
 - (b) $(\eta_i \wedge \eta_j) \trianglelefteq (\eta_k \wedge \eta_l) \Leftrightarrow (\nu_i \wedge \nu_j) \trianglelefteq (\nu_k \wedge \nu_l)$, for all $i, j, k, l < n$
 - (c) $|\eta_i \wedge \eta_j| < |\eta_k \wedge \eta_l| \Leftrightarrow |\nu_i \wedge \nu_j| < |\nu_k \wedge \nu_l|$, for all $i, j, k, l < n$
 - (d) $(\eta_i \wedge \eta_j) <_{\text{lex}} (\eta_k \wedge \eta_l) \Leftrightarrow (\nu_i \wedge \nu_j) <_{\text{lex}} (\nu_k \wedge \nu_l)$, for all $i, j, k, l < n$.

Notation 3.2. We write s' when we mean to refer to *either* s- or ss-similarity.

Had we not required s' -similar tuples to be meet-closed, two tuples would be s' -similar iff their meet-closures were s' -similar. So in general there is no harm in defining s' -similarity between arbitrary tuples. In this paper however, s' -similarity is defined only between meet-closed tuples for convenience.

Definition 3.3. Let s' denote either s or ss. By the *s' -similarity class* of $\bar{\eta}$ we mean

$$[\bar{\eta}]_{s'} = \{\bar{\nu} \mid \bar{\nu} \sim_{s'} \bar{\eta}\}.$$

Remark 3.4. An s' -similarity class of a meet-closed tuple $\bar{\eta}$ from ${}^{\beta >} \lambda$ is exactly the set of realizations of the complete quantifier-free type of $\bar{\eta}$ in the language $L_{s'}(\lambda)$ defined below.

- $s' = s : L_s(\lambda)$ consists of
- $f(t, u)$, interpreted as $t \wedge u$
 - $t \trianglelefteq u$, interpreted as the partial tree order, \trianglelefteq
 - $P_n(t)$, interpreted to hold if and only if $|t|_{(\beta > \lambda)} = n$
 - $x < y$, interpreted as the lexicographic tree order, $<_{\text{lex}}$.
- $s' = \text{ss} : L_{\text{ss}}(\lambda)$ consists of
- $f(t, u)$, interpreted as $t \wedge u$
 - $t \trianglelefteq u$, interpreted as the partial tree order, \trianglelefteq
 - $L(t, u)$, interpreted to hold if and only if $|t|_{(\beta > \lambda)} < |u|_{(\beta > \lambda)}$
 - $x < y$, interpreted as the lexicographic tree order, $<_{\text{lex}}$.

Remark 3.5. Note that orientation-preserving embeddings from the ordered tree $(\beta > \lambda, \trianglelefteq, <_{\text{lex}})$ into itself are exactly model theoretic embeddings in the language $L_s(\lambda) \cap L_{\text{ss}}(\lambda)$.

Similarly the class of orientation- and level-preserving embeddings from $\beta > \lambda$ into itself is that of $L_s(\lambda)$ -embeddings, and for such embeddings f we have $\bar{\eta} \sim_s f(\bar{\eta})$; the class of orientation-preserving graded embeddings from $\beta > \lambda$ into itself is that of $L_{\text{ss}}(\lambda)$ -embeddings, and for such embeddings h , we have $\bar{\eta} \sim_{\text{ss}} h(\bar{\eta})$. Moreover, for such embeddings h , though we may have $\bar{\eta} \approx_s h(\bar{\eta})$, we still have that for any $\bar{\eta}, \bar{\nu}$, $\bar{\eta} \sim_s \bar{\nu}$ iff $h(\bar{\eta}) \sim_s h(\bar{\nu})$.

For the next lemma, we use the following notation: For any tuple $\bar{\eta} = (\eta_l)_{l < m}$ in $\omega > \lambda$ and any $n < \omega$, $\bar{\eta} \upharpoonright n := (\eta'_l)_{l < m}$ where

$$\eta'_l := \begin{cases} \eta_l & \text{if } |\eta_l| \leq n \\ \eta_l \upharpoonright n & \text{if } |\eta_l| > n \end{cases}$$

Lemma 3.6. *For any meet-closed tuples $\bar{\eta}, \bar{\nu} \in \omega > \lambda$ and any $n < \omega$, if $\bar{\eta} \sim_s \bar{\nu}$ then $\bar{\eta} \upharpoonright n \sim_s \bar{\nu} \upharpoonright n$.*

Proof. Let $k = \max\{|\eta_l| : l < m\} = \max\{|\nu_l| : l < m\}$. By induction it suffices to prove the case $n = k - 1$. Note that since $\bar{\eta}, \bar{\nu}$ are meet-closed, they are $L_s(\lambda)$ -structures. Moreover as they are s-similar, the mapping $\eta_l \mapsto \nu_l$ is an $L_s(\lambda)$ -isomorphism between the two tuples. Now clearly both $\bar{\eta} \upharpoonright k - 1 = (\eta'_l)_l$ and $\bar{\nu} \upharpoonright k - 1 = (\nu'_l)_l$ are meet-closed, so they are finite $L_s(\lambda)$ -structures as well. Now it can be routinely checked that the mapping $\eta'_l \mapsto \nu'_l$ is also an $L_s(\lambda)$ -isomorphism, i.e. the map preserves the interpretations of $L_s(\lambda)$ -symbols. Therefore $\bar{\eta} \upharpoonright k - 1 \sim_s \bar{\nu} \upharpoonright k - 1$. \square

Definition 3.7. Given a sequence $(a_\eta \mid \eta \in \beta > \lambda)$ such that $l(a_\eta) = l(a_\nu)$ for all η, ν we say it is

- (1) *Shelah-indiscernible* (s-indiscernible) if

$$\bar{\eta} \sim_s \bar{\nu} \Rightarrow \text{tp}(\bar{a}_{\bar{\eta}}) = \text{tp}(\bar{a}_{\bar{\nu}}).$$
- (2) *strongly Shelah-indiscernible* (ss-indiscernible) if

$$\bar{\eta} \sim_{\text{ss}} \bar{\nu} \Rightarrow \text{tp}(\bar{a}_{\bar{\eta}}) = \text{tp}(\bar{a}_{\bar{\nu}}).$$
- (3) *s'-indiscernible with respect to Γ and Δ* if

$$\bar{\eta} \sim_{s'} \bar{\nu} \text{ and } \bar{\nu} \in \Gamma \Rightarrow \text{tp}_\Delta(\bar{a}_{\bar{\eta}}) = \text{tp}_\Delta(\bar{a}_{\bar{\nu}})$$

where $s' = s$ or ss and Γ is a set of meet-closed tuples from ${}^{\beta>}\lambda$.

Notation 3.8. From now on, parameters $(a_\eta \mid \eta \in {}^{\beta>}\lambda)$ will always denote same-length tuples, i.e. $l(a_\eta) = l(a_\nu)$ for all $\eta, \nu \in {}^{\beta>}\lambda$.

Definition 3.9. Fix parameters $(a_\eta : \eta \in {}^{\beta>}\lambda)$ and a meet-closed tuple $\bar{\eta} \in {}^n({}^{\beta>}\lambda)$, for some n . We say a set of formulas $\Gamma(\bar{x})$ is *realized by* $[\bar{a}_{\bar{\eta}}]_s$ (or $[\bar{a}_{\bar{\eta}}]_{ss}$) if

- $\bar{x} = (x_1, \dots, x_n)$ is a tuple of variables such that $l(x_i) = l(a_\eta)$, for $\eta \in {}^{\beta>}\lambda$, and
- for any $\bar{\nu} \sim_s \bar{\eta}$ (\sim_{ss} resp.) we have $\models \Gamma(\bar{a}_{\bar{\nu}})$.

Remark 3.10. Every ss -indiscernible sequence is s -indiscernible since \sim_s implies \sim_{ss} .

Now the following two theorems are our main tree indiscernibility results.

Theorem 3.11. (*s-indiscernible modeling theorem*) *Let $(a_\eta \mid \eta \in {}^{\beta>}\lambda)$ be any sequence of parameters, where λ is infinite. Then, for any finite meet-closed tuples $\bar{\eta}_i$ from ${}^{\beta>}\lambda$ ($i \in I$) and sets of formulas $\Gamma_i(\bar{x}_i)$ realized by $[\bar{a}_{\bar{\eta}_i}]_s$, there exists an s -indiscernible sequence $(b_\eta \mid \eta \in {}^{\beta>}\lambda)$ such that $[\bar{b}_{\bar{\eta}_i}]_s \models \Gamma_i(\bar{x}_i)$ for each $i \in I$.*

Theorem 3.12. (*ss-indiscernible modeling theorem*) *The same holds if in Theorem 3.11, we replace s by ss , and assume that both β, λ are infinite ordinals.*

Definition 3.13. Following [9, Def 15.2], we name $\Gamma = \Gamma(x_\eta \mid \eta \in {}^{\beta>}\lambda)$ the s' -EM-type of a sequence of parameters $(a_\eta \mid \eta \in {}^{\beta>}\lambda)$ if $\varphi(x_{\eta_1}, \dots, x_{\eta_n}) \in \Gamma$ just in case $\varphi(a_{\nu_1}, \dots, a_{\nu_n})$ holds for all $(\nu_1, \dots, \nu_n) \sim_{s'} (\eta_1, \dots, \eta_n)$. For a sequence of parameters \mathcal{I} , we denote the s' -EM-type of \mathcal{I} by $\text{EM}_{s'}(\mathcal{I})$.

The above definition is not in direct analogy in the sense that we do not restrict the variables in the EM-type to $\bar{x}_{\bar{\eta}}$ where $\bar{\eta}$ are canonical representatives of each $\sim_{s'}$ -similarity class.

Remark 3.14. Theorems 3.11, 3.12 imply that for any parameters $\mathcal{I} := (a_\eta \mid \eta \in {}^{\beta>}\lambda)$ (for appropriate β, λ) we may find s' -indiscernible $(b_\eta \mid \eta \in {}^{\beta>}\lambda)$ realizing $\text{EM}_{s'}(\mathcal{I})$.

To prove these theorems, we need preliminary tree homogeneity results which we describe in the next section.

4. TREE HOMOGENEITY

We repeat the proofs of two combinatorial results from [6], providing additional clarification in the steps.

First we fix some notation. For the following definitions, we fix an infinite cardinal κ , an integer $m \geq 1$, a disjoint family $\{X_\alpha \mid \alpha < \kappa\}$ of well-ordered sets X_α , where each X_α has cardinality $\beth_{m(m+1)}(\kappa)^+$, $X := \bigcup_{\alpha < \kappa} X_\alpha$ and a function $f : X^m \rightarrow \kappa$. We easily think of $f(\bar{x}) \in \kappa$ as a ‘‘color’’ assigned by f to \bar{x} .

Notation 4.1. For any m -tuple $\bar{s} := (s_0, \dots, s_{m-1}) \in X^m$, we define

- (1) $i(s_j) :=$ the index α such that $s_j \in X_\alpha$.

- (2) $L(\bar{s}) := \{i(s_j) \mid j < m\}$.
- (3) For $\beta < \kappa$, $L_\beta(\bar{s}) := \{\alpha \in L(\bar{s}) \mid \alpha \geq \beta\}$.

Definition 4.2. For m -tuples $\bar{s} := (s_0, \dots, s_{m-1})$ and $\bar{t} := (t_0, \dots, t_{m-1})$ of elements in X , we write

- (1) $\bar{s} \parallel \bar{t}$ if
 - (a) $\forall j < m, i(s_j) = i(t_j)$,
 - (b) $\forall j, k < m$, if $i(s_j) = i(s_k)$ then $(s_j < s_k \Leftrightarrow t_j < t_k)$.
- (2) $\bar{s} \sim_{(\beta, n)} \bar{t}$ (for $\beta < \kappa$ and $n < \omega$) if
 - (a) $\bar{s} \parallel \bar{t}$,
 - (b) $\beta \in L(\bar{s})$,
 - (c) $\|L_\beta(\bar{s})\| \leq n$,
 - (d) $\forall j < m, i(s_j) < \beta \Rightarrow s_j = t_j$.
- (3) $\bar{s} \approx_\beta \bar{t}$ (for $\beta < \kappa$) if
 - (a) $\bar{s} \parallel \bar{t}$,
 - (b) $\beta \in L(\bar{s})$,
 - (c) $\forall j < m, i(s_j) \neq \beta \Rightarrow s_j = t_j$.

Remark 4.3. It is important in (2) and (3) that we ask β to be in $L(\bar{s})$. For $\bar{s} \approx_\beta \bar{t}$ or $\bar{s} \sim_{(\beta, n)} \bar{t}$, we refer to the sets X_α for $\alpha \in L_\beta(\bar{s})$ as the *exceptional neighborhoods of \bar{s} (and \bar{t})*, of which X_β is the least, in the linear ordering.

Definition 4.4. For a family $\mathcal{F} = \{Y_\alpha \mid \alpha < \kappa\}$ of subsets $Y_\alpha \subseteq X_\alpha$, we say \mathcal{F} is (β, n) -*indiscernible* if any $\sim_{(\beta, n)}$ -equivalent tuples in $\bigcup_{\alpha < \kappa} Y_\alpha$ are mapped to the same image by f . We say \mathcal{F} is \approx_β -*indiscernible* if any \approx_β -equivalent tuples are mapped to the same image by f . We say \mathcal{F} is n -*indiscernible* if it is (β, n) -indiscernible for all $\beta < \kappa$.

Remark 4.5. For a family \mathcal{F} as described above,

- (1) \mathcal{F} is vacuously 0-indiscernible, since no tuples satisfy $\bar{s} \sim_{(\beta, 0)} \bar{t}$, for any $\beta < \kappa$.
- (2) \mathcal{F} is $(n + 1)$ -indiscernible $\Rightarrow \mathcal{F}$ is n -indiscernible.
- (3) \mathcal{F} is m -indiscernible \Leftrightarrow any \parallel -equivalent m -tuples are mapped to the same image by f .

Theorem 4.6 ([6, App. 2.7]). *Fix an infinite cardinal κ , an integer $m \geq 1$ and pairwise disjoint sets $(X_\alpha \mid \alpha < \kappa)$ such that $X := \bigcup_{i < \kappa} X_i$ is well ordered by $<$, and for $x \in X_i, y \in X_j$ and $i < j$, we have that $x < y$. Suppose further that for each i , $\|X_i\| = \beth_{m(m+1)}(\kappa)^+$. Suppose we are given $f : X^m \rightarrow \kappa$.*

Then there exists an m -indiscernible family $\{Y_\alpha \mid \alpha < \kappa\}$ of subsets $Y_\alpha \subseteq X_\alpha$, where each Y_α has size κ^+ .¹

Proof. We will build families $\mathcal{F}_j = \{X_\alpha^j \mid \alpha < \kappa\}$ for $j = 0, \dots, m$, satisfying

First induction hypothesis:

¹For completeness we allow $m = 1$ where of course $\|X_i\| \geq \kappa^+$ would suffice.

- (1) $X_\alpha^{j+1} \subseteq X_\alpha^j, \forall \alpha < \kappa,$
- (2) $\|X_\alpha^j\| = \beth_{(m-j)(m+1)}(\kappa)^+, \forall \alpha < \kappa,$ ²
- (3) \mathcal{F}_j is j -indiscernible.

By Remark 4.5, \mathcal{F}_m will be the desired family of sets. We shall build the \mathcal{F}_j by induction on j . Again, by Remark 4.5, clearly if we set $X_\alpha^0 := X_\alpha, \forall \alpha < \kappa,$ then $\mathcal{F}_0 := \{X_\alpha^0 \mid \alpha < \kappa\}$ satisfies the First induction hypothesis.

Remark 4.7. We give some guidance on the proof. The induction has two parts. At the n -th stage of the First induction, we have found refinements X_i^n of our original neighborhoods, $X_i \supseteq X_i^n$, from which $\mathbb{1}$ -similar m -tuples must have the same color provided they satisfy $s_i \neq t_i$ in (at most) n exceptional neighborhoods. In going from n to $n+1$ we allow an additional neighborhood to be exceptional, where in general an m -tuple intersects at most m neighborhoods.

In the Second induction, at stage $i < \kappa,$ we focus on tuples \bar{s} such that the i -th neighborhood is the first exceptional neighborhood of \bar{s} . Having defined $(X_j^{n+1})_{j < i}$ we are looking for a refinement $X_i^{n+1} \subseteq X_i^n$ such that all $\mathbb{1}$ -similar m -tuples from $(\bigcup_{\alpha < i} X_\alpha^{n+1}) \cup X_i^{n+1} \cup (\bigcup_{i < \alpha < \kappa} X_\alpha^n)$ whose first exceptional neighborhood is at $i,$ and who have $n+1$ exceptional neighborhoods in total, get mapped by f to the same color. The resulting $\{X_\alpha^{n+1} \mid \alpha < \kappa\}$ satisfy the First induction hypothesis.

First induction step: ($0 \leq n < m$)

Assume that we have defined an n -indiscernible family $\mathcal{F}_n = \{X_\alpha^n \mid \alpha < \kappa\}$ where $X_\alpha^n \subseteq X_\alpha$ and $\|X_\alpha^n\| = \beth_{(m-n)(m+1)}(\kappa)^+$ for each $\alpha < \kappa.$ We now build an $(n+1)$ -indiscernible family $\mathcal{F}_{n+1} = \{X_\alpha^{n+1} \mid \alpha < \kappa\}$ where $X_\alpha^{n+1} \subseteq X_\alpha^n$ and $\|X_\alpha^{n+1}\| = \beth_{(m-n-1)(m+1)}(\kappa)^+$ for each $\alpha < \kappa.$

We build such sets X_α^{n+1} by induction on $i < \kappa.$ For all $\gamma < \kappa$ we wish to define a sequence $(X_\alpha^{n+1} \mid \alpha \leq \gamma)$ satisfying

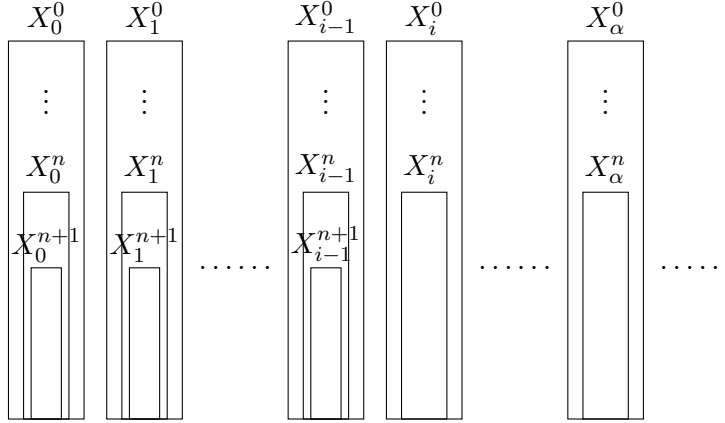
Second induction hypothesis:

- (1) $X_\alpha^{n+1} \subseteq X_\alpha^n,$
- (2) $\|X_\alpha^{n+1}\| = \beth_{(m-n-1)(m+1)}(\kappa)^+,$ for each $\alpha \leq \gamma,$ and
- (3) $\{X_\alpha^{n+1} \mid \alpha \leq \gamma\} \cup \{X_\alpha^n \mid \gamma < \alpha < \kappa\}$ is $(\beta, n+1)$ -indiscernible for all $\beta \leq \gamma.$

Second induction step: ($i < \kappa$)

Having found $(X_\alpha^{n+1} \mid \alpha < i)$ satisfying (1)-(3) above for γ equal to any $j < i,$ now we define X_i^{n+1} so that $(X_\alpha^{n+1} \mid \alpha \leq i)$ satisfies the Second induction hypothesis, for $\gamma = i.$

²Here we diverge from [6] where we believe there is a typo in setting up the induction.



It suffices to find a subset $X_i^{n+1} \subseteq X_i^n$ of size $\beth_{(m-n-1)(m+1)}(\kappa)^+$ such that the family $\{X_\alpha^{n+1} \mid \alpha \leq i\} \cup \{X_\alpha^n \mid i < \alpha < \kappa\}$ is $(i, n+1)$ -indiscernible.

Claim 4.8. *We may find X_i^{n+1} as described above.*

Proof. For each $\alpha > i$, choose an arbitrary finite subset $S_\alpha \subseteq X_\alpha^n$ of size $m-1$. We will eliminate the use of these sets in time.

Claim 4.9. *There exists a subset $\hat{X}_i^{n+1} \subseteq X_i^n$ of size $\beth_{(m-n-1)(m+1)}(\kappa)^+$ such that the family $\{X_\alpha^{n+1} \mid \alpha < i\} \cup \{\hat{X}_i^{n+1}\} \cup \{S_\alpha \mid i < \alpha < \kappa\}$ is \approx_i -indiscernible.*

Proof. Let $\Omega := (\bigcup_{\alpha < i} X_\alpha^{n+1}) \cup (\bigcup_{i < \alpha < \kappa} S_\alpha)$. To any subset $B \subseteq m$, we can associate a map $G_B : {}^B(X_i^n) \times {}^{m-B}\Omega \rightarrow \kappa$ defined by $(\mu, \nu) \mapsto f(\gamma)$, where $\gamma \in {}^m X$ satisfies $\gamma(j) = \mu(j)$ for $j \in B$, and $\gamma(j) = \nu(j)$ for $j \in m-B$. We can also view G_B as a map $G_B : {}^B(X_i^n) \rightarrow \text{Func}({}^{m-B}\Omega, \kappa)$, where $\text{Func}({}^{m-B}\Omega, \kappa)$ denotes the set of functions from ${}^{m-B}\Omega$ to κ . Now, for each $\eta \in {}^m(X_i^n)$, let $F(\eta)$ be the finite sequence $\langle G_B(\eta \upharpoonright_B) \mid B \subseteq m \rangle$. The idea is that there are at most $\beth_{((m-n-1)(m+1)+2)}(\kappa)$ many sequences in such a form since

$$\|\Omega\| \leq \kappa^{\beth_{(m-n-1)(m+1)}(\kappa)^+} \leq \beth_{((m-n-1)(m+1)+2)}(\kappa)$$

Hence, F can be viewed as a coloring map, coloring m -tuples in X_i^n using

$\beth_{((m-n-1)(m+1)+2)}(\kappa)$ many colors. Moreover, if we set $\lambda := \beth_{((m-n-1)(m+1)+2)}(\kappa)$, then:

$$\begin{aligned} \|X_i^n\| &= \beth_{(m-n)(m+1)}(\kappa)^+ \\ &= \beth_{(m-1)}(\beth_{((m-n-1)(m+1)+2)}(\kappa))^+ \\ &= \beth_{(m-1)}(\lambda)^+ \end{aligned}$$

By Erdős-Rado, $\beth_{(m-1)}(\lambda)^+ \rightarrow (\lambda^+)_\lambda^m$. Thus there exists a subset $\hat{X}_i^{n+1} \subseteq X_i^n$ of size $\lambda^+ = \beth_{((m-n-1)(m+1)+2)}(\kappa)^+$ such that $F(\mu)$ for $\mu \in {}^m(\hat{X}_i^{n+1})$ depends

only on the order type of μ : i.e., $F(\mu) = F(\tau)$ whenever $\mu, \tau \in {}^m(\hat{X}_i^{n+1})$ such that $(\mu)_s < (\mu)_t$ iff $(\tau)_s < (\tau)_t$ for all $s, t < m$. This means that the family $\{X_\alpha^{n+1} \mid \alpha < i\} \cup \{\hat{X}_i^{n+1}\} \cup \{S_\alpha \mid i < \alpha < \kappa\}$ is \approx_i -indiscernible. Clearly, we can choose such \hat{X}_i^{n+1} to have a smaller size $(\beth_{(m-n-1)(m+1)}(\kappa))^+$. This completes the proof of Claim 4.9. \square

We continue the proof of Claim 4.8, by arguing that \hat{X}_i^{n+1} yields an X_i^{n+1} as desired. In fact, by the Second induction hypothesis, we argue that we may take X_i^{n+1} to be \hat{X}_i^{n+1} .

So, set $X_i^{n+1} := \hat{X}_i^{n+1}$. To see that $\{X_\alpha^{n+1} \mid \alpha < i\} \cup \{X_i^{n+1}\} \cup \{X_\alpha^n \mid i < \alpha < \kappa\}$ is $\sim_{(i, n+1)}$ -indiscernible, suppose $\bar{s} = (s_0, \dots, s_{m-1})$ and $\bar{t} = (t_0, \dots, t_{m-1})$ are $\sim_{(i, n+1)}$ -equivalent tuples in

$$\left(\bigcup_{\alpha < i} X_\alpha^{n+1} \right) \cup X_i^{n+1} \cup \left(\bigcup_{i < \alpha < \kappa} X_\alpha^n \right)$$

We need to show $f(\bar{s}) = f(\bar{t})$. Clearly it suffices to assume $\|L_i(\bar{s})\| = n+1$, as the case for $\|L_i(\bar{s})\| \leq n$ has already been dealt with. Let i^* be the least index α such that $i < \alpha < \kappa$ and $\alpha = i(s_j)$ for some $j < m$. First, it is clear that we can choose an m -tuple \bar{s}^* from

$$\left(\bigcup_{\alpha < i} X_\alpha^{n+1} \right) \cup X_i^{n+1} \cup \left(\bigcup_{i < \alpha < \kappa} S_\alpha \right)$$

such that $\bar{s}^* \sim_{(i^*, n)} \bar{s}$, by having \bar{s}^* coincide with \bar{s} inside the j -th neighborhoods for $j \leq i$, while having them obey the same ordering inside the j -th neighborhoods for $j > i$. Next, for each $j < m$, let $t_j^* := t_j$ if $i(t_j) \leq i$, and let $t_j^* := s_j^*$ if $i(t_j) > i$. Because $\bar{s}^* \sim_{(i^*, n)} \bar{s}$, this is enough to know that $\bar{t}^* \sim_{(i^*, n)} \bar{t}$. Also clearly $\bar{s}^* \approx_i \bar{t}^*$. Summarizing:

$$\bar{s} \sim_{(i^*, n)} \bar{s}^* \approx_i \bar{t}^* \sim_{(i^*, n)} \bar{t}.$$

Then $f(\bar{s}) = f(\bar{s}^*)$ and $f(\bar{t}) = f(\bar{t}^*)$ by the n -indiscernibility of $\{X_\alpha^n \mid \alpha < \kappa\}$, while $f(\bar{s}^*) = f(\bar{t}^*)$ by the \approx_i -indiscernibility of $\{X_\alpha^{n+1} : \alpha \leq i\} \cup \{S_\alpha \mid i < \alpha < \kappa\}$. Hence $f(\bar{s}) = f(\bar{t})$. This completes the proof of Claim 4.8, and thus, the Second induction. \square

This completes the First induction, and Theorem 4.6 is proved. \square

Theorem 4.10 ([6, App. 2.6]). *For every $n, m < \omega$ there is some $k = k(n, m) < \omega$ such that for any infinite cardinal χ , the following is true of $\lambda := \beth_k(\chi)^+$: for every $f : ({}^{n \geq \lambda})^m \rightarrow \chi$ there is a level-preserving, orientation-preserving subtree $I \subseteq {}^{n \geq \lambda}$ such that*

- (i) $\langle \rangle \in I$ and whenever $\eta \in I \cap {}^{n > \lambda}$, $\|\{\alpha < \lambda \mid \eta \frown \langle \alpha \rangle \in I\}\| \geq \chi^+$.
- (ii)_f If $\bar{\eta}, \bar{\nu} \in I$ are such that $\bar{\eta} \sim_s \bar{\nu}$ then $f(\eta_0, \dots, \eta_{m-1}) = f(\nu_0, \dots, \nu_{m-1})$.

Notation 4.11. Refer to a subtree I with property (ii)_f as \sim_s -homogeneous for the coloring f .

Proof. We may assume $n, m > 0$, since otherwise, f is already assumed to be a constant function, and so is easily constant on m -tuples.

Case 1. ($m = 1$) Fix n . Choose $k(n, 1) = 0$. f is a unary function in this case. Fix any infinite cardinal χ . Then $\lambda := \chi^+$. Let f be any function as in the claim.

$$f : {}^{n \geq}(\chi^+) \rightarrow \chi$$

Define by induction on $j \leq n$

$$\langle (I_\eta; \alpha_{n-j}(\eta), \dots, \alpha_n(\eta)) \mid \eta \in {}^{n-j}\lambda \rangle$$

such that $I_\eta \subseteq c(\eta, {}^{n \geq}\lambda)$ and $\alpha_{n-j}(\eta), \dots, \alpha_n(\eta)$ are ordinals $< \chi$ with the properties:

- (a) $\eta \in I_\eta$ and whenever $\nu \in I_\eta \cap {}^{n >}\lambda$, then $|\{\alpha < \lambda \mid \nu \hat{\ } \langle \alpha \rangle \in I_\eta\}| = \chi^+$
- (b) For any $\nu \in I_\eta \cap {}^i\lambda$, for $n - j \leq i \leq n$, $f(\nu) = \alpha_i(\eta)$.

I_η is a thinned-out, χ^+ -branching subtree of the cone at η . At $\eta = \langle \rangle$ we will obtain a level- and orientation-preserving subtree of the original tree satisfying (i). Property (ii) will be satisfied by virtue of the fact that any $\eta \in {}^i\lambda \cap I_\langle \rangle$ will have color $\alpha_i(\langle \rangle)$. So in fact, the color of a node depends only on its quantifier-free 1-type in the L_s language, as desired.

Here we build the sequence $(I_\eta; \alpha_{n-j}(\eta), \dots, \alpha_n(\eta))$.

($j = 0$) Then $I_\eta = \{\eta\}$, (a) is trivially satisfied, and (b) is easily satisfied if we let $\alpha_n(\eta) = f(\eta)$.

($j > 0$) Suppose I_η has been defined for all $\eta \in {}^{n-j+1}\lambda$. Fix $\eta \in {}^{n-j}\lambda$, by induction, for all $\beta < \lambda$, we have defined both $I_{\eta \hat{\ } \langle \beta \rangle}$ and sequences

$$\langle \alpha_n(\eta \hat{\ } \langle \beta \rangle), \dots, \alpha_{n-j+1}(\eta \hat{\ } \langle \beta \rangle) \rangle.$$

These are $(n - j)$ -tuples of elements from χ , so there are at most χ of these tuples. By the pigeonhole principle, there is a size- $(\chi)^+$ subset $Y \subseteq \lambda$ on which all these strings agree and equal some $\langle \alpha_n, \dots, \alpha_{n-j+1} \rangle$. Set $\langle \alpha_n(\eta), \dots, \alpha_{n-j+1}(\eta) \rangle := \langle \alpha_n, \dots, \alpha_{n-j+1} \rangle$ and $\alpha_{n-j}(\eta) := f(\eta)$. Then we may take I_η to be $= \{\eta\} \cup \{I_{\eta \hat{\ } \langle \beta \rangle} \mid \beta \in Y\}$.

Case 2. ($n = 1, m > 1$) Let $k(n, m) := m - 1$. We are working with a height-1 tree with λ leaves. The lexicographic ordering $<$ between the leaves coincides in this case to the usual ordering on λ . Observe that, for any meet-closed m -tuples $\bar{\eta}, \bar{\nu} \in {}^{1 \geq}\lambda$, $\bar{\eta} \sim_s \bar{\nu}$ if and only if there exists a subset $B \subseteq m$ such that

- (1) $B = \{j < m \mid |\eta_j| = 1\} = \{j < m \mid |\nu_j| = 1\}$
- (2) $\forall j, k \in B, \eta_j < \eta_k$ iff $\nu_j < \nu_k$.

Hence, to prove Case 2, it suffices to find a subset $A \subseteq \lambda$ of size χ^+ such that, whenever $\bar{\eta}, \bar{\nu}$ from ${}^{1 \geq}A$ and $B \subseteq m$ satisfy the conditions (1) and

(2) above, then $f(\bar{\eta}) = f(\bar{\nu})$. (Then the subtree ${}^{1\geq}A$ would be a desired χ^+ -branching, \sim_s -homogeneous subtree of ${}^{1\geq}\lambda$.)

Let $B \subseteq m$ be any subset. To any $\nu \in {}^B\lambda$, we can associate the value $G_B(\nu) := f(\gamma)$, where $\gamma \in {}^m({}^{1\geq}\lambda)$ satisfies $\gamma(j) = \nu(j)$ for $j \in B$ and $\gamma(j) = \langle \rangle$ for $j \in m-B$. Now, for each $\eta \in {}^m\lambda$, let $F(\eta)$ be the finite sequence $\langle G_B(\eta \upharpoonright_B) \mid B \subseteq m \rangle$. Since there are at most χ many finite sequences in such a form, F can be viewed as a coloring map, coloring m -tuples in λ , using χ many colors. By Erdős-Rado,

$$\beth_{(m-1)}(\chi)^+ \rightarrow (\chi^+)_\chi^m$$

Thus, there exists a subset $A \subseteq \lambda$ of size χ^+ such that $F(\eta)$ depends only on the order type of $\langle (\eta)_0, \dots, (\eta)_{m-1} \rangle$ for $\eta \in {}^m A$. Then the subtree $I := {}^{1\geq}A$ is a desired χ^+ -branching, \sim_s -homogeneous subtree of ${}^{1\geq}\lambda$.

Case 3. ($n > 1, m > 1$) Assume we have proved the claim for n and any $m > 1$ (and any χ). We shall prove the case for $n+1$ for all $m > 1$. Fix $m > 1$ and infinite χ . Define $k(n+1, m) := k(n, m) + m^2 + m + 4$. Define

$$\kappa := \beth_{m^2+m+3}(\chi).$$

We are given a function f as in the claim,

$$f : ({}^{n+1\geq}\lambda)^m \rightarrow \chi.$$

Consider the auxiliary function g defined on $({}^{n\geq}\lambda)^m$

$$g(\eta_0, \dots, \eta_{m-1}) = \{ \langle w, \beta_0, \dots, \beta_{m-1}, \alpha \rangle \mid w \subseteq m \text{ is finite,} \\ \text{each } \beta_l < \kappa, \text{ and, for } (\nu_l)_{l < m} \text{ such that} \\ \nu_l := \eta_l \text{ for } l \notin w \text{ and } \nu_l := \eta_l \hat{\ } \langle \beta_l \rangle \text{ for } l \in w, \\ \text{we have } \alpha = f(\nu_0, \dots, \nu_{m-1}) \}.$$

There are κ -many possible tuples $\langle w, \beta_0, \dots, \beta_{m-1}, \alpha \rangle$, and every $\langle \eta_0, \dots, \eta_{m-1} \rangle$ maps to some subset of this size- κ set. Thus the range of g is no larger than the number of subsets, 2^κ .

$$\lambda = \beth_{k(n,m)+m^2+m+4}(\chi)^+.$$

$$\beth_{k(n,m)+m^2+m+4}(\chi) = \beth_{k(n,m)}(\beth_{m^2+m+4}(\chi)) = \beth_{k(n,m)}(2^\kappa).$$

$$\text{Thus, } \lambda = \beth_{k(n,m)}(2^\kappa)^+.$$

By the induction hypothesis, we have a level- and orientation-preserving subtree $I_0 \subseteq {}^{n\geq}\lambda$, \sim_s -homogeneous for the coloring g , containing the root and $(2^\kappa)^+$ -branching at every level below the n -th level. We may as well assume that I_0 is only χ^+ -branching at every level below the n -th level, by simply thinning out the tree from the bottom up.

Claim 4.12. *There is $I_1 \subseteq {}^{n+1\geq}\lambda$ a level- and orientation-preserving subtree containing the root, χ^+ -branching at each level below the $(n+1)$ -st level, and \sim_s -homogeneous for the coloring f .*

Proof. Consider the following subtree of $^{n+1}\geq\lambda$,

$$I^* := I_0 \cup \bigcup \{ \eta \hat{\ } \langle \alpha \rangle \mid \eta \in (I_0 \cap {}^n\lambda), \text{ and } \alpha < \kappa \}.$$

This is the tree that agrees with I_0 on all levels $\leq n$, but nodes at level n have κ successors.

Since I_0 is \sim_s -homogeneous for g , by considering when $\emptyset = w$, the first component of the tuples in the value of g , we see that I_0 is \sim_s -homogeneous for f too. Hence we merely need to worry about those tuples $\bar{\eta}, \bar{\nu}$ some of whose entries are from the top level, $I^* \cap {}^{n+1}\lambda$. We work on finding a subtree $I_1 \subseteq I^*$ that is level- and orientation-preserving, such that I_1 agrees with I^* (and therefore I_0) at all levels $\leq n$, and at level $n+1$, I_1 chooses χ^+ successors of each node at level n .

For $\eta \in I_0 \cap {}^n\lambda$, let X_η be the subset $\{ \eta \hat{\ } \langle \alpha \rangle : \alpha < \kappa \} \subseteq I^*$. Since I_0 is χ^+ -branching of height n , $I_0 \cap {}^n\lambda$ is indexed by the ordinal $(\chi^+)^n$ in such a way that $i < j \Leftrightarrow \hat{\eta}_i <_{\text{lex}} \hat{\eta}_j$. We write $I_0 \cap {}^n\lambda = \langle \hat{\eta}_i : i < (\chi^+)^n \rangle$.

Note:

$$I^* \cap {}^{n+1}\lambda = \bigcup_{i < (\chi^+)^n} X_{\hat{\eta}_i}.$$

Set up a new function \bar{g} on m -tuples from $I^* \cap {}^{n+1}\lambda$:

$$\begin{aligned} \bar{g}(\langle \sigma_l \rangle_{l < m}) &= \{ \langle w, \rho_0, \dots, \rho_{m-1}, \alpha \rangle \mid w \subseteq m; \text{ each } \rho_l \in I_0, \\ &\quad l \in w \leftrightarrow |\rho_l| = n; \text{ and } \alpha = f(\nu_0, \dots, \nu_{m-1}), \text{ where} \\ &\quad \nu_l := \rho_l \text{ for } l \notin w, \text{ and } \nu_l := \sigma_l \text{ for } l \in w \}. \end{aligned}$$

The range of \bar{g} is composed of subsets of a set of size χ^+ , so

$$\bar{g} : \left(\bigcup_{i < (\chi^+)^n} X_{\hat{\eta}_i} \right)^m \rightarrow K, \text{ with } \|K\| \leq \mu := 2^{(\chi^+)}.$$

$$\begin{aligned} \text{Calculate } \kappa &= \beth_{m^2+m+3}(\chi) \geq \beth_{m^2+m+2}(\chi^+) = \beth_{m^2+m+1}(\mu) \\ &= \beth(\beth_{m^2+m}(\mu)) \geq \beth_{m^2+m}(\mu)^+. \end{aligned}$$

Now we use Theorem 4.6 on the $(X_{\hat{\eta}_i} \mid i < (\chi^+)^n)$, understanding the ordering in \vDash -equivalence as the $<_{\text{lex}}$ -ordering³. The theorem gives subsets $Y_{\hat{\eta}_i} \subseteq X_{\hat{\eta}_i}$ of size μ^+ such that for any $(\sigma_l)_l, (\tau_l)_l$ that are \vDash -equivalent, $\bar{g}(\langle \sigma_l \rangle_l) = \bar{g}(\langle \tau_l \rangle_l)$. Now consider the tree

$$I_1 := I_0 \cup \bigcup_{i < (\chi^+)^n} Y_{\hat{\eta}_i} = I_0 \cup \{ \hat{\eta}_i \hat{\ } \langle \beta \rangle \in Y_{\hat{\eta}_i} \mid \beta < \kappa, i < (\chi^+)^n \}.$$

I_1 is a level- and orientation-preserving subtree of I^* , thus by transitivity, of the original tree.

Subclaim 4.13. I_1 is homogeneous for the coloring f .

³We have fewer than κ actual sets X_η , so the tail sequence may be supplied by copies of κ .

Proof. We introduce some notation for m -tuples $\bar{\eta} = (\eta_l)_{l < m}$, $\bar{\beta} = (\beta_l)_{l < m}$ from I_1 and κ respectively (we may write $\eta_l = (\bar{\eta})_l$ as well), with $w = w_{\bar{\eta}} \subseteq m$ such that $|\eta_l| = n + 1 \leftrightarrow l \in w$:

- $\text{short}_w(\bar{\eta}) := \bar{\eta} \upharpoonright n$ as in Lemma 3.6, i.e. $= (\eta'_l)_l$, where $\eta'_l := \eta_l$ if $l \notin w$ and $\eta'_l := \eta \upharpoonright n$ if $l \in w$.
- $\text{complete}_w(\bar{\eta}, \bar{\beta}) := (\epsilon_l)_l$ defined by $\epsilon_l = (\text{short}(\bar{\eta}))_l$ if $l \notin w$, and $\epsilon_l := (\text{short}(\bar{\eta}))_l \hat{\wedge} \langle \beta_l \rangle$, if $l \in w$.
- For $\sigma \in I_1$ with $|\sigma| = n + 1$, $\text{topseq}_w(\bar{\eta}, \sigma) := (\nu_l)_l$ defined by $\nu_l := \eta_l$ if $l \in w$ and $\nu_l := \sigma$ if $l \notin w$.
- $\text{topval}_w(\bar{\eta}) := (\beta_l)_l$ defined by $\beta_l := \eta_l(n)$ if $l \in w$ and $\beta_l := 0$ if $l \notin w$.

Now we show that for $\bar{\eta} \sim_s \bar{\nu}$ from I_1 , $f(\bar{\eta}) = f(\bar{\nu})$.

By similarity, $w_{\bar{\eta}} = w_{\bar{\nu}}$, which we fix as w . Now as in Lemma 3.6,

$$(1) \text{short}_w(\bar{\eta}) \sim_s \text{short}_w(\bar{\nu}).$$

Equation (1) guarantees that $g(\text{short}_w(\bar{\eta})) = g(\text{short}_w(\bar{\nu}))$. Thus, by the definition of g , given any m -tuple $\bar{\beta}$ from κ ,

$$(2) f(\text{complete}_w(\bar{\eta}, \bar{\beta})) = f(\text{complete}_w(\bar{\nu}, \bar{\beta})).$$

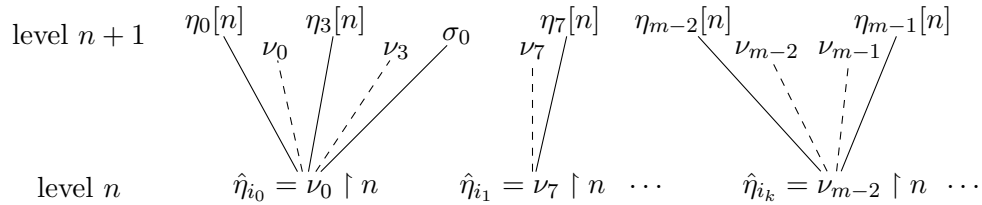
We apply (2) in the case that $\bar{\beta} := \text{topval}_w(\bar{\eta})$:

$$(3) f(\text{complete}_w(\bar{\eta}, \text{topval}_w(\bar{\eta}))) = f(\text{complete}_w(\bar{\nu}, \text{topval}_w(\bar{\eta}))).$$

Since, $\bar{\eta} \sim_s \bar{\nu}$, clearly for some $\sigma_0 \in I_1$ with $|\sigma_0| = n + 1$,

$$(4) \text{topseq}_w(\text{complete}_w(\bar{\nu}, \text{topval}_w(\bar{\eta})), \sigma_0) \parallel$$

$$\text{topseq}_w(\text{complete}_w(\bar{\nu}, \text{topval}_w(\bar{\nu})), \sigma_0),$$



Remark 4.14. This picture is for $k \leq m$, m -tuples $\bar{\eta}$, $\bar{\nu}$, such that $\bar{\eta} \sim_s \bar{\nu}$ and $\hat{\eta}_{i_0} = \nu_0 \upharpoonright n$. In this example, the m -tuples are indexed in $<$ -increasing order. The set w happens to be $\{0, 3, 7, \dots, m-2, m-1\}$. The i -th coordinate of $\bar{\eta}$ is sequence $\eta_i \in {}^{n+1}I_1$, taking value $\eta_i[n]$ at n . Notice how \sim_s -similarity yields \sim_s -similarity of $\text{complete}_w(\bar{\nu}, \text{topval}_w(\bar{\eta}))$ with ν , and thereby \parallel -similarity of the top sequence, as in (4).

In fact, we may choose σ_0 to be right of all nodes in the left-most neighborhood. Thus, by homogeneity of \bar{g} on $I_1 \cap {}^{n+1}\lambda$, by (4) we have

$$(5) \quad \bar{g}(\text{topseq}_w(\text{complete}_w(\bar{\nu}, \text{topval}_w(\bar{\eta})), \sigma_0)) = \\ \bar{g}(\text{topseq}_w(\text{complete}_w(\bar{\nu}, \text{topval}_w(\bar{\nu})), \sigma_0)),$$

and by the definition of \bar{g} , (5) implies

$$(6) \quad f(\text{complete}_w(\bar{\nu}, \text{topval}_w(\bar{\eta}))) = f(\text{complete}_w(\bar{\nu}, \text{topval}_w(\bar{\nu}))).$$

By Equations (3) and (6),

$$(7) \quad f(\text{complete}_w(\bar{\eta}, \text{topval}_w(\bar{\eta}))) = f(\text{complete}_w(\bar{\nu}, \text{topval}_w(\bar{\nu}))).$$

However, $\text{complete}_w(\bar{\nu}, \text{topval}_w(\bar{\eta})) = \bar{\eta}$, and $\text{complete}_w(\bar{\nu}, \text{topval}_w(\bar{\nu})) = \bar{\nu}$, so we are done. □

At this point the n -th level of I_1 is κ -branching. To obtain the desired tree, we thin out the top level of the tree to obtain a χ^+ -branching tree, at all levels. □

This completes the proof of all three cases. □

In the proof above, the Case 3 proof may cover Case 1, but in separating the cases, one gets better bounds k and λ in Case 1.

5. PROOFS OF THEOREM 3.11 AND 3.12

Now we are ready to prove our main theorems.

Theorem 3.11. (*s-indiscernible modeling theorem*) *Let $(a_\eta \mid \eta \in {}^{\beta>\lambda})$ be any sequence of parameters, where λ is infinite. Then, for any finite meet-closed tuples $\bar{\eta}_i$ from ${}^{\beta>\lambda}$ ($i \in I$) and sets of formulas $\Gamma_i(\bar{x}_i)$ realized by $[\bar{a}_{\bar{\eta}_i}]_s$, there exists an s-indiscernible sequence $(b_\eta \mid \eta \in {}^{\beta>\lambda})$ such that $[\bar{b}_{\bar{\eta}_i}]_s \models \Gamma_i(\bar{x}_i)$ for each $i \in I$.*

Proof. For convenience, we assume that $l(a_\eta) = 1$. Note that there is a set $\Phi(x_\eta \mid \eta \in {}^{\beta>\lambda})$ of formulas describing the existence of the desired sequence $(b_\eta \mid \eta \in {}^{\beta>\lambda})$. Hence by compactness it suffices to satisfy a finite subset of Φ . For this aim, fix $X = \{\bar{\eta}_i \mid i \in I_0\}$ where I_0 is an arbitrary finite subset of I , and let m be the maximum arity of the tuples in X . Indeed there is no harm in assuming that all tuples in X have arity m . Now fix $n < \omega$ and some orientation-preserving graded embedding $h : {}^{n \geq \omega} \rightarrow {}^{\beta>\lambda}$ such that $X \subseteq (\text{Im}(h))^m$. It remains to show the existence of $(b_\eta \mid \eta \in \text{Im}(h))$ which is s-indiscernible for any m -tuples (computed in ${}^{\beta>\lambda}$), and $[\bar{b}_{\bar{\eta}_i}]_s \models \Gamma_i(\bar{x}_i)$ for $i \in I_0$.

Now let $\chi := |S_m(T)|$. Then there corresponds an infinite cardinal $\lambda(\chi) = \lambda'$ as in Theorem 4.10. Since λ above is infinite, by compactness there is a sequence $A_{\lambda'} = (a'_\eta \mid \eta \in {}^{\beta>} \lambda')$ such that for each $i \in I$, $\Gamma_i(\bar{x}_i)$ is still realized by $[\bar{a}'_{\eta_i}]_s$ (here $[\bar{\eta}_i]_s \in {}^{\beta>} \lambda' / \sim_s$). Consider a λ' -branching subtree Λ of ${}^{\beta>} \lambda'$ containing $\text{Im}(h)$ but $\text{Lev}(\text{Im}(h)) = \text{Lev}(\Lambda)$, and a function

$$f : \Lambda^m \rightarrow S_m(T)$$

defined by $f(\bar{\eta}) = \text{tp}(\bar{a}'_{\bar{\eta}})$. Then again by the conclusion of Theorem 4.10, we have the desired subsequence $(b_\eta \mid \eta \in \text{Im}(h))$ of $A_{\lambda'}$, after some suitable re-indexing if necessary. \square

Theorem 3.12. (*ss-indiscernible modeling theorem*) *Let $(a_\eta \mid \eta \in {}^{\beta>} \lambda)$ be any sequence of parameters, where both β, λ are infinite ordinals. Then, for any meet-closed finite tuples $\bar{\eta}_i$ from ${}^{\beta>} \lambda$ ($i \in I$) and sets of formulas $\Gamma_i(\bar{x}_i)$ realized by $[\bar{a}_{\bar{\eta}_i}]_{ss}$, there exists an ss-indiscernible sequence $(b_\eta \mid \eta \in {}^{\beta>} \lambda)$ such that $[\bar{b}_{\bar{\eta}_i}]_{ss} \models \Gamma_i(\bar{x}_i)$ for each $i \in I$.*

Proof. This can be proved in a straight-forward way by Theorem 3.11 and Ramsey's theorem. Note that \sim_s implies \sim_{ss} , so any \sim_{ss} -similarity class can be represented as a union of \sim_s -similarity classes. Hence by Theorem 3.11, there is an s-indiscernible $(c_\eta \mid \eta \in {}^{\beta>} \lambda)$ such that $[\bar{c}_{\bar{\eta}_i}]_{ss} \models \Gamma_i(\bar{x}_i)$ for each $i \in I$.

Now consider a set of formulas $\Psi(x_\eta \mid \eta \in {}^{\beta>} \lambda)$ describing the existence of the desired sequence $(b_\eta \mid \eta \in {}^{\beta>} \lambda)$. We apply compactness to Ψ . For this, fix finitely many meet-closed tuples $\bar{v}^0, \dots, \bar{v}^{k-1} \in {}^{\beta>} \lambda$. We can assume $l(\bar{v}^i) = m$ for all $i < k$ by extending the length of the tuple by repeating a component in the tuple if necessary. Let

$$r_i := |L(\bar{v}^i)| \text{ where } L(\bar{v}^i) = \{|\bar{v}_j^i (= (\bar{v}^i)_j)| : j < m\} \quad (i < k),$$

and let $r = \max\{r_i \mid i < k\}$. Also fix a set $\Delta(\bar{x}) = \{\varphi_0(\bar{x}), \dots, \varphi_{n-1}(\bar{x})\}$ of formulas where $|\bar{x}| = m$. Let E be an equivalence relation on \mathcal{M}^m defined by $\bigwedge_{i < n} (\varphi_i(\bar{x}) \leftrightarrow \varphi_i(\bar{y}))$. Consider a function

$$f : [\omega]^r \rightarrow (\mathcal{M}^m / E)^k$$

defined by

$$f(n_0, \dots, n_{r-1}) = (\bar{c}_{\bar{\mu}^0} / E, \dots, \bar{c}_{\bar{\mu}^{k-1}} / E)$$

where $\bar{\mu}^i = h_i(\bar{v}^i)$ with some orientation-preserving, graded-embedding

$$h_i : \{\nu_0^i, \dots, \nu_{m-1}^i\} \rightarrow {}^{\omega>} \omega$$

such that $L(\bar{\mu}^i)$ is the initial r_i -element subset of $\{n_0 < \dots < n_{r-1}\}$. Note that since each $\bar{\mu}^i$ is meet-closed, it has the \triangleleft -least element. Hence such h_i always exists, and $\bar{v}^i \sim_{ss} \bar{\mu}^i$. Moreover since $(c_\eta \mid \eta \in {}^{\omega>} \omega)$ is s-indiscernible, $c_{\bar{\mu}^i} / E$ does not depend on the choice of h_i . Therefore f is well-defined.

Now by Ramsey's theorem, there is some infinite subset $\tau \subseteq \omega$ such that f is constant on $[\tau]^r$. Consider an orientation-preserving graded-embedding $h : {}^{\omega>} \omega \rightarrow {}^{\omega>} \omega$ such that $\text{Lev}(\text{Im}(h)) = \tau$. Then by our setting, $(c_{h(\eta)} \mid \eta \in {}^{\omega>} \omega)$ is ss-indiscernible

with respect to $\{\bar{\nu}^0, \dots, \bar{\nu}^{k-1}\}$ and Δ . Moreover we know that for $i \in I$, $[\bar{c}_{h(\bar{\eta}_i)}]_{\text{ss}} \models \Gamma_i$, in particular $\models \Gamma_i \upharpoonright \Delta$. Then as $\{\bar{\nu}^0, \dots, \bar{\nu}^{k-1}\}$ may be chosen to contain any subset of $\{\bar{\eta}_i\}_{i \in I}$, this means that each finite fragment of Ψ is consistent. Therefore there is a realization of Ψ , which will be our desired ss-indiscernible. \square

6. APPLICATIONS

We define s' -type properties for $s' = s$ or ss . In the following $\omega^{>\omega}$ could easily be replaced by $\beta^{>\lambda}$ for arbitrary ordinals β and λ .

Recall the definition of $\text{EM}_{s'}(\mathcal{I})$ from Definition 3.13.

Definition 6.1. Say that a property P of theories is *pre- s' -type* if there exists a partial type $\Gamma(x_i \mid i \in \omega^{>\omega})$ such that

- (1) T has P just in case there exist witnesses $(a_i : i \in \omega^{>\omega})$ to Γ ,
- (2) For any $\mathcal{I} = (a_\eta \mid \eta \in \omega^{>\omega})$ witnessing Γ , $\text{EM}_{s'}(\mathcal{I}) \vdash \Gamma(x_\eta \mid \eta \in \omega^{>\omega})$.

By an *s' -type property*, we mean a (possibly infinite) disjunction of pre- s' -type properties.

By the definitions, we see that for a formula φ , the property “ φ witnesses k -TP” is a pre- s -type property, while “ φ witnesses (weak) k -TP₁” is a pre- ss -type property. Hence we have the following claim.

Claim 6.2. *k -TP is an s -type property; weak- k -TP₁, k -TP₁ are ss -type properties.*

Using these notions, by Remark 3.14,

Theorem 6.3. *If T has an s (ss resp.)-type property, then it can be witnessed by an s (ss resp.)-indiscernible sequence indexed by $\omega^{>\omega}$. In particular the following hold.*

- (1) *Assume T has (weak resp.) k -TP₁ witnessed by $\varphi(x, y)$. Then there must be an ss-indiscernible sequence $(a_\eta \mid \eta \in \omega^{>\omega})$ witnessing (weak resp.) k -TP₁ with the same $\varphi(x, y)$.*
- (2) *Assume T has k -TP witnessed by $\varphi(x, y)$. Then there must be an s -indiscernible sequence $(a_\eta \mid \eta \in \omega^{>\omega})$ witnessing k -TP with the same $\varphi(x, y)$.*

On the other hand, we have the following lemma, which is implicitly assumed in the proof of [1, Prop. 13], but we give an alternate proof.

Lemma 6.4. *Assume T has k -TP₂ witnessed by $\varphi(x, y)$. Then there is $\{a_j^i \mid i, j \in \omega\}$ witnessing k -TP₂ with $\varphi(x, y)$ such that*

$$(1) \quad a_0^0 \dots a_n^0, a_0^1 \dots a_n^1, \dots, a_0^n \dots a_n^n \equiv a_{j(0,0)}^{i_0} \dots a_{j(0,n)}^{i_0}, a_{j(1,0)}^{i_1} \dots a_{j(1,n)}^{i_1}, \dots, a_{j(n,0)}^{i_n} \dots a_{j(n,n)}^{i_n}$$

for any $i_0 < \dots < i_n$ and $j(i, 0) < \dots < j(i, n)$.

Proof. For this we use Theorem 4.6. Now given arbitrary m , we let $\kappa := |S_{m^2}(T)|$, $\lambda := \beth_{m^2(m^2+1)}(\kappa)^+$. By compactness there is $C = \{c_j^i \mid i < \kappa; j < \lambda\}$ witnessing k -TP₂ with φ . Then consider $f : C^{m^2} \rightarrow S_{m^2}(T)$ defined by $f(\bar{c}) = \text{tp}(\bar{c})$. By Theorem 4.6, there is (at least) countable $\tau_i \subseteq \lambda$, for $i < \kappa$, such that any \equiv -equivalent m^2 -tuples

from $C'' = \{c_{j_i}^i \mid i < \omega; j_i \in \tau_i\}$ realize the same type. Re-index $C''' = \{a_j^i \mid i, j < \omega\}$. Then due to Π -equivalence,

$$a_0^{i_0} \dots a_{m-1}^{i_0}, a_0^{i_1} \dots a_{m-1}^{i_1}, \dots, a_0^{i_{m-1}} \dots a_{m-1}^{i_{m-1}} \equiv \\ a_{j(0,0)}^{i_0} \dots a_{j(0,m-1)}^{i_0}, a_{j(1,0)}^{i_1} \dots a_{j(1,m-1)}^{i_1}, \dots, a_{j(m-1,0)}^{i_{m-1}} \dots a_{j(m-1,m-1)}^{i_{m-1}}$$

for any $i_0 < \dots < i_{m-1}$ and $j(i, 0) < \dots < j(i, m-1)$.

Thus by the above reasoning and compactness we can additionally assume

$$\langle a_j^i \mid i, j < \omega \rangle \equiv \langle a_{h_i(j)}^i \mid i, j < \omega \rangle$$

for any order-preserving injections $h_i : \omega \rightarrow \omega$. Then again by Ramsey argument with compactness, we can further assume that $\langle \bar{a}_i \mid i < \omega \rangle$ is indiscernible, where each \bar{a}_i is an infinite tuple $(a_j^i \mid j < \omega)$. Therefore now $\{a_j^i \mid i, j < \omega\}$ witnessing k -TP₂ with φ , satisfies (1). \square

Hence now we can confirm the proof of the following theorem which is stated in [1, Prop. 13]. For completeness we repeat the proof here.

Proposition 6.5. *T has TP₂ iff it has k -TP₂ for some $k \geq 2$.*

Proof. (\Rightarrow) Clear.

(\Leftarrow) We show it inductively. When $k = 2$, there is nothing to show. So assume that this holds for all $2, \dots, k-1$. Now suppose that $\{\varphi(x, a_j^i) \mid i, j \in \omega\}$ as in Lemma 6.4 witnesses k -TP₂.

Case I) Assume $\{\varphi(x, a_0^i) \wedge \varphi(x, a_1^i) \mid i \in \omega\}$ is consistent: Then due to Lemma 6.4, it follows that $\varphi(x, y_0) \wedge \varphi(x, y_1)$ and $b_j^i := a_{2j}^i a_{2j+1}^i$ witness $\lceil \frac{k+1}{2} \rceil$ -TP₂. Hence by the induction hypothesis, T has TP₂.

Case II) Assume $\{\varphi(x, a_0^i) \wedge \varphi(x, a_1^i) \mid i \in \omega\}$ is inconsistent: Then there must be some n such that $\{\varphi(x, a_0^i) \wedge \varphi(x, a_1^i) \mid i < n\}$ is inconsistent. Then again due to Lemma 6.4, $\varphi(x, y_0) \wedge \dots \wedge \varphi(x, y_{n-1})$ and $b_j^i := a_j^{ni} a_j^{ni+1} \dots a_j^{ni+n-1}$ witness 2-TP₂. \square

Lastly we can confirm the proof of the following as well.

Theorem 6.6. *T has TP iff it has TP₁ or TP₂. Indeed if a formula $\varphi(x, y)$ has k -TP then some finite conjunction of $\varphi(x, y)$ has 2-TP₁ or 2-TP₂.*

Proof. (\Leftarrow) Clear.

(\Rightarrow) The proof of this was first stated in [6, Thm III.7.11], but that proof is not quite correct although some basic ideas were suggested. A revised proof is stated in [1, Thm 14], which seems to be all clear, except again a tree-indiscernibility condition is used without verification. So here we only clarify that part using our main result Theorem 6.3. For the rest of the proof, we refer the readers to [1, Thm 14], where it is quite clearly written. Condition (3) in [1, Thm 14] states that we may find $(b_\nu)_{\nu \in \omega \geq \omega}$ ss-indiscernible witnesses to k -TP. But this is not obvious since TP is only an s-type property. So as in Theorem 6.3, we can only assume that the $(b_\nu \mid \nu \in \omega^{>\omega})$ witnessing k -TP are s-indiscernible. In order to induce TP₁, it was also assumed that T does not have TP₂ (equivalently any k -TP₂, by Proposition 6.5

above.) Now it suffices to argue why we can assume $(b_\nu | \nu \in \omega^{>\omega})$ ss-indiscernible using this assumption.

We let $\lambda := (2^\omega)^+$. By Theorem 3.11, there is s-indiscernible $(a_\nu | \nu \in \lambda^{>\omega})$ witnessing k -TP with $\varphi(x, y)$. Let $\{\nu_i | i < \omega\} \subseteq \lambda^{>\omega}$ be any countable set of same-level distant siblings where the level is a successor. We claim that $\{\varphi(x, a_{\nu_i}) | i < \omega\}$ must be inconsistent: suppose this is not the case. Then, in particular, ν_i can not be siblings. Now, consider $\{\nu'_i \frown \langle j \rangle | i, j < \omega\}$, where ν'_i is the immediate predecessor of ν_i . Because the ν_i are distant siblings but not siblings, the ν'_i must be distinct. Due to s-indiscernibility, it follows that $\{\varphi(x, a_j^i) | i, j < \omega\}$ witnesses k -TP₂, where $a_j^i = a_{\nu'_i \frown \langle j \rangle}$. This contradicts our assumption that T does not have n -TP₂ for any $n \geq 2$, and hence the claim is verified.

By the claim and s-indiscernibility, for each $\alpha < \beta < \lambda$, there is $k(\alpha, \beta) < \omega$ such that for any set $\{\nu_i | i < \omega\} \subseteq \lambda^{>\omega}$ of same-level distant siblings with $|\nu_0 \wedge \nu_1| = \alpha + 1$ and $|\nu_0| = \beta + 1$, the formulas $\{\varphi(x, a_{\nu_i}) | i < \omega\}$ are $k(\alpha, \beta)$ -inconsistent. Now by Erdős-Rado, there is a homogeneous ω_1 -subset $\tau \subseteq \lambda$. Therefore there is $k' < \omega$ such that for an orientation-preserving and graded-embedding $h : \omega^{>\omega} \rightarrow \lambda^{>\omega}$ with $\text{Lev}(\text{Im}(h)) \subseteq \{\alpha + 1 | \alpha \in \tau\}$, the images of any same-level distant siblings witness k' -inconsistency. Note that the latter is now an ss-type property. Hence by Theorem 3.12, there are desired ss-indiscernible $(b_\nu | \nu \in \omega^{>\omega})$ witnessing this property, a fortiori, k' -TP with φ . \square

Before we make a remark, we recall a notion from [5]:

Definition 6.7. Let $s' := s$ or ss . The s' -indiscernible $(b_\eta | \eta \in \omega^{>\omega})$ in \mathcal{M} is s' -based on the parameters $(a_\eta | \eta \in \omega^{>\omega})$ if for any finite set of formulas Δ from the language of M , and for any meet-closed tuple (η_1, \dots, η_n) , there exist (ν_1, \dots, ν_n) such that

- (1) $(\eta_1, \dots, \eta_n) \sim_{s'} (\nu_1, \dots, \nu_n)$,
- (2) $\text{tp}_\Delta(b_{\eta_1}, \dots, b_{\eta_n}) = \text{tp}_\Delta(a_{\nu_1}, \dots, a_{\nu_n})$.

Remark 6.8. There exist ss-indiscernible $(b_\eta | \eta \in \omega^{>\omega})$ that fail to have k -TP witnessed by $\varphi(x; y)$ but are ss-based on $(a_\eta | \eta \in \omega^{>\omega})$ witnessing 2-TP with $\varphi(x; y)$. Thus, the assumption that NTP₂ has essential use in Theorem 6.6.

Consider $(a_\eta | \eta \in \omega^{>\omega})$ that witness 2-TP with $\varphi(x; y)$ *strictly*, i.e. such that, for any $k \geq 1$, $\{\varphi(x; a_{\eta_1}), \dots, \varphi(x; a_{\eta_k})\}$ is consistent whenever no two members of $\{\eta_1, \dots, \eta_k\}$ are siblings. We will take as our example the theory T_{feq}^* of infinitely many independent parameterized equivalence relations $x \sim_z y$ (see [7].) A model of this theory is a two-sorted structure in sorts Q, P . Given $c \in Q$, $x \sim_c y$ gives an equivalence relation on $P \times P$ that has infinitely many classes, but independently, so that

(*) if $k < \omega$ and $c_1, \dots, c_k \in Q$ with no repetition, and $b_1, \dots, b_k \in P$, then there is $a \in P$ such that $a \sim_{c_i} b_i$ for each i .

We let $\varphi(x; y, z) := x \sim_z y$. Now choose $(c_\nu)_{\nu \in \omega^{>\omega}}$, c distinct members of the index sort Q , and define $(d_{\nu \frown \langle i \rangle})_{i < \omega}$ to be infinitely many \sim_{c_ν} -inequivalent members of P , d any element of P . Then set $a_{\nu \frown \langle i \rangle} := (c_\nu, d_{\nu \frown \langle i \rangle})$, $a_\langle \rangle = (c, d)$. Clearly

$\{\varphi(x; a_\eta) \mid \eta \in {}^\omega > \omega\}$ witness 2-TP, as siblings index 2-inconsistent instances of φ , and paths are consistent by property (*). Moreover, $\{\varphi(x; a_{\eta_1}), \dots, \varphi(x; a_{\eta_k})\}$ is consistent whenever no two members of $\{\eta_1, \dots, \eta_k\}$ are siblings, by property (*).

Now consider $(a'_\eta \mid \eta \in {}^\omega > \omega)$ obtained by taking a subtree of the original tree where we delete all the odd levels. In other words, let $h : {}^\omega > \omega \rightarrow {}^\omega > \omega$ be the function such that $h(\langle \rangle) = \langle \rangle$, and for all $\nu \in {}^\omega > \omega$ and $t < \omega$, $h(\nu \frown \langle t \rangle) = h(\nu) \frown \langle t \rangle \frown \langle 0 \rangle$, and define $a'_\eta := a_{h(\eta)}$.

Note that for any size- k set of parameters $\{a'_{i_1}, \dots, a'_{i_k}\}$, no two members happen to be siblings in the original tree, thus $\{\varphi(x; a'_{i_1}), \dots, \varphi(x; a'_{i_k})\}$ is consistent. The latter is an ss-type property, so it is inherited by any ss-indiscernible $(b_\eta \mid \eta \in {}^\omega > \omega)$ that are ss-based on the $(a'_\eta : \eta \in {}^\omega > \omega)$. Thus, such b_η do not witness k -TP by φ , for any k . Because of the fact that h is an orientation-preserving graded embedding, the $(b_\eta \mid \eta \in {}^\omega > \omega)$ must also be ss-based on the $(a_\eta \mid \eta \in {}^\omega > \omega)$.

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