We give an explicit description of the homology group $H_n(p)$ of a strong type $p$ in any stable theory under the assumption that for every non-forking extension $q$ of $p$ the groups $H_i(q)$ are trivial for $2 \leq i < n$. The group $H_n(p)$ turns out to be isomorphic to the automorphism group of a certain piece of the algebraic closure of $n$ independent realizations of $p$; it was shown earlier by the authors that such a group must be abelian. We call this the “Hurewicz correspondence” in analogy with the Hurewicz Theorem in algebraic topology.

The present paper is a part of the project to study type amalgamation properties in first-order theories by means of homology groups of types. Roughly speaking (more precise definitions are recalled below in Section 1), a strong type $p$ is said to have $n$-amalgamation if commuting systems of elementary embeddings among algebraic closures of proper subsets of the set of $n$ independent realizations of $p$ can always be extended to the algebraic closure of all $n$ realizations. The type $p$ has $n$-uniqueness if this extension is essentially unique. Generalized amalgamation properties for systems of models were introduced by Shelah in [12] and played an important role in [13]. The type amalgamation properties were studied extensively by Hrushovski in [9] and applications were given. In fact, the type amalgamation properties have been used in model theory at least as far back as Hrushovski’s classification of trivial totally categorical theories in [8].

In the previous paper [5], we introduced a notion of homology groups for a complete strong type in any stable, or even rosy, first-order theory. The idea was that these homology groups should measure information about the amalgamation properties of the type $p$. We could prove that if $p$ has $n$-amalgamation for all $n$, then $H_n(p) = 0$ for every $n$, and that the failure of 4-amalgamation (equivalently, the failure of 3-uniqueness)
over \(\text{dom}(p)\) in a stable theory corresponds to non-triviality of the group \(H_2(p)\). Furthermore, we established in that paper that \(H_2(p)\) is isomorphic in stable theories to a certain automorphism group of closures of realizations of \(p\). In [7], the theory is developed in a more general context of amenable collection of functors.

The current paper generalizes the main results of [5] in the stable context: if the type \(p\) does not have \((n + 2)\)-amalgamation, then for some \(i\) with \(2 \leq i \leq n\) and some nonforking extension \(p'\) of \(p\), the group \(H_i(p')\) must be nonzero. Furthermore, at the first \(i\) for which such \(p'\) with \(H_i(p') \neq 0\) exists, we show that \(H_i(p')\) is isomorphic to a certain automorphism group \(\Gamma_i(p')\), which immediately implies that \(H_i(p')\) is a profinite group.

In the remainder of this introduction, we state the main result more precisely, which requires some technical definitions.

In this paper, we always work with a fixed complete stable theory \(T\) in a language \(\mathcal{L}\), its saturated model \(\mathcal{M} = \mathcal{M}^{eq}\), and a complete type \(p\) (of possibly infinite arity) over a small set \(B = \text{acl}(B)\). Throughout this paper, independence is nonforking independence. We use the usual notational conventions of stability theory, plus some notational conventions from [5] or [6]. Note in particular that given a tuple or a set \(c\), \(\bar{c}\) denotes \(\text{acl}(cB)\).

For sets \(A, C\), \(\text{Aut}(A/C)\) denotes the group of elementary maps over \(C\) (i.e., fixing \(C\) pointwise) from \(A \cup C\) onto \(A \cup C\). For a type \(q = \text{tp}(f/C)\) with the solution set \(F\), \(\text{Aut}(q) := \text{Aut}(F/C)\).

Given a \(B\)-independent collection of elements \(c_1, \ldots, c_{n+1}\) all having the same type over \(B\), we let \(\bar{c}_1\ldots\bar{c}_n := \overline{c_1\ldots c_n} \cap \text{dcl}(\bigcup_{i=1}^n \bar{c}_1\ldots\bar{c}_i\ldots\bar{c}_{n+1})\); and let

\[\partial(c_1\ldots c_n) := \text{dcl}(\bigcup_{i=1}^n \bar{c}_1\ldots\bar{c}_i\ldots\bar{c}_n).\]

We also define

\[\Gamma_n(p) := \text{Aut}(\overline{c_1\ldots c_n}/\partial(c_1\ldots c_n)),\]

where each \(c_i \models p\). Since \(p\) is stationary, it is routine to check that this definition does not depend on the choice of the realizations \(c_i\).

The main result of the paper is the following.

**Theorem 0.1.** Let \(n \geq 1\). Let \(T = T^{eq}\) be a stable theory and let \(p\) be a strong type in \(T\). Assume that \(p\) has \(k\)-amalgamation for each \(k \in \{2, \ldots, n + 1\}\). Then \(H_n(p) \cong \Gamma_n(p)\); the latter group is always a profinite abelian group.

Indeed in [5], the theorem for \(n = 1, 2\) is already shown, and the whole generalization above was conjectured. We call this the Hurewicz
correspondence since the result connects the homology groups of $p$ to $\Gamma_n(p)$, which are similar to homotopy groups, as in algebraic topology (see [1]). In particular, it is shown in [4] that $\Gamma_2(p)$ is the profinite limit of the vertex groups of relatively definable groupoids obtained from the failure of 4-amalgamation of the type. Since the definable vertex groups consist of something like “homotopy equivalence of paths,” as described in [3], the limit $\Gamma_2(p)$ of definable vertex groups is analogous to the fundamental groupoid $\pi_1$ of the type, as in singular homology theory [2]. (Note that there is a mismatch in the numbering: our group $\Gamma_n$ corresponds to the group $\pi_{n-1}$ in algebraic topology.)

In trying to prove Theorem 0.1, it was natural to seek a generalization of the notion of a definable groupoid to higher dimensions, with the idea that the profinite limit of the groups acting regularly on the fibers of the higher groupoids should be isomorphic to the groups $\Gamma_n(p)$ defined above. Although there are various notions of “$n$-groupoids” in the literature, we could not find one that precisely fits our needs, so in [6] we defined a new kind of structure called $n$-ary polygroupoids.\footnote{The closest thing which existed previously in the literature would be the Kan complexes discussed by Lurie in [11].}

In the case where $n = 2$, a “binary polygroupoid” is essentially equivalent to an ordinary connected groupoid. In an $n$-ary polygroupoid, there is an $n$-ary composition rule which satisfies a kind of associativity rule, generalizing associativity for ordinary groupoids. However, in polygroupoids, since the “morphisms” only live in fibers above ordered $n$-tuples of distinct objects, there is no notion of vertex groups contrary to usual groupoids. Instead we have a notion of binding groups acting regularly on fibers of connected polygroupoids, which must always be abelian.

In [6], we showed that if $p$ has $(n+1)$-CA but fails $(n+2)$-amalgamation over $B$, then there exists a $B$-type-definable $n$-ary polygroupoid canonically witnessing the failure, and a $B$-definable finite abelian binding group of the groupoid isomorphic to an automorphism group, a component of the profinite abelian group $\Gamma_n(p)$. In this paper, in contrast with the proof of $n = 2$ case in [5], we directly build a homomorphism from $H_n(p)$ to the binding group of elementary maps, and show that its profinite limit is the desired group isomorphism between $H_n(p)$ and $\Gamma_n(p)$. We use almost all of the results from [6] on $n$-ary polygroupoids, but the proof here is more straightforward than that in [5] for $n = 2$.

In sections 1 and 2, we summarize basic definitions and results from [4],[5] and [6]. Several new results are shown in section 2 as well. Among them, the uniformization of $Q$-relations (Lemma 2.17) is the
key result. All of this is used in section 3, where we prove Theorem 0.1. Finally, in section 4, we describe examples of stable theories in which $H^n(p)$ is congruent to any given finite (or even profinite) abelian group.

1. Type amalgamation and homology groups

In this section we recall from [4] and [5] the definition of homology groups of the type $p$, and the basic notions of the amalgamation properties related to the computation of the homology groups.

If $X$ is a family of sets ordered by inclusion, then we consider it to be a category with a single inclusion map $i_{u,v} : u \to v$ between any two sets $u, v \in X$ with $u \subseteq v$. The set $X$ is called downward-closed if whenever $u \subseteq v \in X$, then $u \in X$.

We let $C_B$ denote the category of all small algebraically closed subsets of $\mathcal{M}$ containing $B$, where morphisms are elementary maps over $B$ (i.e., fixing $B$ pointwise). For a downward closed $X$ and a functor $f : X \to C_B$ and $u \subseteq v \in X$, we write $f_u := f(i_{u,v})$ and $f_u(u) := f_u(f(u)) \subseteq f(v)$.

**Definition 1.1.** A (closed independent) $p$-functor is a functor $f : X \to C_B$ such that:

1. For some finite $s \subseteq \omega$, $X$ is a downward-closed subset of $\mathcal{P}(s)$;
2. $f(\emptyset) \supseteq B$; and for $i \in s$, $f(\{i\})$ (if it is defined) is of the form $acl(Cb)$ where $b(=p)$ is independent with $C = f(\{i\})$ over $B$.
3. For all non-empty $u \in X$, we have that $f(u) = acl(B \cup \bigcup_{i \in u} f_i(i))$ and the set $\{f_i(i) : i \in u\}$ is independent over $f_u(\emptyset)$.

If $f(\emptyset) = B$ (so for any $u \in X$, $f_u(\emptyset) = B$) then we say $f$ is over $B$.

**Definition 1.2.** Let $n \geq 0$ be a natural number. An $n$-simplex in $p$ is a $p$-functor $f : \mathcal{P}(s) \to C_B$ for some set $s \subseteq \omega$ with $|s| = n + 1$. The set $s$ is called the support of $f$, or $supp(f)$.

Let $S_n(p)$ denote the collection of all $n$-simplices over $B$ in $p$; and let $C_n(p)$ denote the free abelian group generated by $S_n(p)$; its elements are called $n$-chains in $p$. Similarly, we define $S(p) := \bigcup_n S_n(p)$, and $C(p) := \bigcup_n C_n(p)$. The support of a chain $c$ is the union of the supports of all the simplices that appear in $c$ with a nonzero coefficient.

**Definition 1.3.** If $n \geq 1$ and $0 \leq i \leq n$, then the $i$th boundary operator $\partial_n^i : C_n(p) \to C_{n-1}(p)$ is defined so that if $f$ is an $n$-simplex in $p$ with domain $\mathcal{P}(s)$, where $s = \{s_0, \ldots, s_n\}$ with $s_0 < \ldots < s_n$, then

$$\partial_n^i(f) = f \upharpoonright \mathcal{P}(s \setminus \{s_i\})$$
and we extend $\partial'_n$ linearly to a group map on all of $C_n(p)$.

If $n \geq 1$ and $0 \leq i \leq n$, then the boundary map $\partial_n : C_n(p) \to C_{n-1}(p)$ is defined by the rule

$$\partial_n(c) = \sum_{0 \leq i \leq n} (-1)^i \partial'_n(c).$$

We write $\partial^i$ and $\partial$ for $\partial'_n$ and $\partial_n$, respectively, if $n$ is clear from context.

The kernel of $\partial_n$ is denoted $Z_n(p)$, and its elements are called $(n)$-cycles. The image of $\partial_{n+1}$ in $C_n(p)$ is denoted $B_n(p)$. The elements of $B_n(p)$ are called $(n)$-boundaries.

It can be shown (by the usual combinatorial argument) that $B_n(p) \subseteq Z_n(p)$, or more briefly, “$\partial_n \circ \partial_{n+1} = 0$.” Therefore we can define simplicial homology groups in the type $p$:

**Definition 1.4.** The $n$th (simplicial) homology group of the type $p \in S(B)$ is

$$H_n(p) := Z_n(p)/B_n(p).$$

Finally, we define the amalgamation properties of the type $p$. As usual, $n = \{0, \ldots, n-1\}$.

**Definition 1.5.** Let $n \geq 1$.

1. We say $p$ has $n$-amalgamation (or $n$-existence) if for any $p$-functor $f : P^{-}(n)(:= P(n) \setminus \{n\}) \to C_B$, there is an $(n-1)$-simplex $g$ in $p$ such that $g \supseteq f$. When $f$ above ranges only over $p$-functors over $B$, we say $p$ has $n$-amalgamation over $B$.

   If $p$ has $k$-amalgamation for every $k$ with $1 \leq k \leq n$, then we say $p$ has $n$-complete amalgamation (or $n$-CA, or $(\leq n)$-existence).

2. $p \in S(B)$ has $n$-uniqueness if for any closed independent $p$-functor $f : P^{-}(n) \to C_B$ and any two $(n-1)$-simplices $g_1$ and $g_2$ in $p$ extending $f$, there is a natural isomorphism $F : g_1 \to g_2$ such that $F \upharpoonright \text{dom}(f)$ is the identity. Similarly we say $p$ has $n$-uniqueness over $B$ when $f$ above ranges over $p$-functors over $B$.

   If $p$ has $k$-uniqueness for every $k$ with $2 \leq k \leq n$, then we say $p$ has $(\leq n)$-uniqueness.

It follows directly from the definitions above that the properties of $n$-uniqueness and $n$-existence of $p$ are preserved under nonforking extensions. Trivially, 1-amalgamation holds in $p$. The extension axiom and stationarity of nonforking imply 2-amalgamation, and 1- and 2-uniqueness of $p$, respectively. In general, the following holds.
Fact 1.6. [4] Let \( n \geq 1 \). Then \( p \) has \((\leq n)\)-uniqueness if and only if it has \((n+1)\)-CA. More precisely, assume \( p \) has \((\leq n)\)-uniqueness. Then \( p \) has \((n+1)\)-uniqueness over \( B \) iff it has \((n+2)\)-amalgamation over \( B \) iff \( \Gamma_n(p) = 0 \).

The following fact is proved in [10]:

Fact 1.7. For any \( k, n \geq 1 \), \( p \) has \( n \)-uniqueness if and only if \( p^{(k)} \) has \( n \)-uniqueness (where \( p^{(k)} \) is the complete type of \( k \) independent realizations of \( p \) over the base set \( B \)).

We now introduce a particular class of \( n \)-cycles representing all the members of \( H_n(p) \) under the assumption of \((n+1)\)-CA.

Definition 1.8. If \( n \geq 1 \), an \( n \)-shell is an \( n \)-chain \( c \) of the form

\[
\pm \sum_{0 \leq i \leq n+1} (-1)^i f_i,
\]

where \( f_0, \ldots, f_{n+1} \) are \( n \)-simplices such that whenever \( 0 \leq i < j \leq n + 1 \), we have \( \partial^i f_j = \partial^{j-1} f_i \).

Notice that any \( n \)-shell is an \( n \)-cycle; and, if \( f \) is any \((n+1)\)-simplex, then \( \partial f \) is an \( n \)-shell.

Theorem 1.9. [5] If \( p \) has \((\leq n)\)-uniqueness for some \( n \geq 1 \), then

\[
H_n(p) = \{ [c] : c \text{ is an } n \text{-shell in } p \text{ with support } n+2 \}.
\]

In particular due to 3-amalgamation, \( H_1(p) = 0 \).

Hence above theorem and Fact 1.6 imply Theorem 0.1 holds for \( n = 1 \).

2. \( n \)-ary polygroupoids and symmetric witnesses

In this section we continue recalling definitions and facts from [6], and add some new observations, all of which will be used in Section 3. We use the same terminology as in [6]. In particular, \([n] := \{1, \ldots, n\}\); and for a set \( A \), \( A^{(n)} \) denotes the set of \( n \)-tuples of distinct elements from \( A \), and \( A^{(\leq n)} := A^{(1)} \cup \cdots \cup A^{(n)} \).

Definition 2.1. Let \( M = (P_1, \ldots, P_n, \pi^2, \ldots, \pi^n) \) be a structure with sorts \( P_i, i = 1, \ldots, n \) and functions \( \pi^k : P_k \to (P_{k-1})^k, k = 2, \ldots, n \). We use the symbol \( \pi^k_i(f) \) to refer to the \( i \)th element of the tuple \( \pi^k(f) \).

1. We say that a tuple \( (f_1, \ldots, f_{k+1}) \in (P_k)^{k+1} \) is compatible if
   a) \( k = 1 \) and \( f_1 \neq f_2 \), or
   b) \( k \geq 2 \) and \( \pi^k_i(f_j) = \pi^k_{j-1}(f_i) \) for all \( 1 \leq i < j \leq k + 1 \).
We say that a $k$-tuple

$$(f_1, \ldots, f_{\ell-1}, \widehat{f}_\ell, f_{\ell+1}, \ldots, f_{k+1}) \in (P_k)^k$$

with the deleted term number $\ell (\leq k+1)$, is compatible if above
(a) (b) hold as far as the values of the terms are defined, i.e.,
either $k = 1$, or $k \geq 2$ and $\pi^k_i(f_j) = \pi^k_{j-1}(f_i)$ for all $1 \leq i(\neq 
\ell) < j(\neq \ell) \leq k + 1$.

(2) Assume that for every $f \in P_k$, the image $\pi^k(f)$ forms a compatible tuple: Then for any $i \in \{2, \ldots, n\}$ and any $f \in P_i$, we iteratively say that $f$ is over $(a_1, \ldots, a_i) \in (P_i)^{(i)}$ if,
(a) $i > 2$ and $(a_2, a_1) = \pi^2(f)$, or
(b) $i > 2$ and for every $j \in \{1, \ldots, i\}$, $\pi^i_j(f)$ is over $(a_1, \ldots, a_{j-1}, a_j, a_i)$;
for any $(a_1, \ldots, a_i) \in I^{(i)}$, we denote by $P_i(a_1, \ldots, a_i)$ the set of all $f \in P_i$ which are over $(a_1, \ldots, a_i)$. If $f \in P_i(a_1, \ldots, a_i)$, we also write “$\pi(f) = (a_1, \ldots, a_i)$”; Note that then the sort $P_i$ is the disjoint union of all the “fibers” $P_i(a_1, \ldots, a_i)$ where $(a_1, \ldots, a_i) \in (P_i)^{(i)}$.

Given $f \in \pi(a_1, \ldots, a_i)$, and a $j$-tuple $s (0 < j \leq i)$ of increasing numbers from $[i]$, we write $\pi_s(f)$ to denote the unique element in $P_{[s]}(a_j \mid j \in s)$ which is an image of $u$ under a composition of $\pi^2, \ldots, \pi^i$. For example, $\pi_{[1]}(f) = f$, $\pi_{[1]}(f) = a_1$, and $\pi_{[1,2,3,\ldots,n]}(f) = \pi^{i-1}(\pi^2(f))$.

**Remark 2.2.** In Definition 2.1(1), we warn the reader not to confuse a tuple

$$(f_1, \ldots, f_{\ell-1}, \widehat{f}_\ell, f_{\ell+1}, \ldots, f_{k+1}) \in (P_k)^k$$

being compatible, with that

$$(g_1, \ldots, \widehat{g}_\ell, \ldots, g_{k+1}) = (g_1, \ldots, g_{\ell-1}, g_{\ell+1}, \ldots, g_{k+1}) \in (P_{k-1})^k$$

is compatible.

Notice also that in the context of a polygroupoid which is connected (see Definition 2.4 below), the tuple $(f_1, \ldots, f_{\ell-1}, \widehat{f}_\ell, f_{\ell+1}, \ldots, f_{k+1})$ is compatible if and only if there is some element $f_\ell \in P_k$ such that $(f_1, \ldots, f_\ell, \ldots, f_{k+1})$ is compatible.

**Definition 2.3.** If $n \geq 2$, an $n$-ary quasigroupoid is a structure $\mathcal{H} = (I, P_2, \ldots, P_{n-1}, P, Q)$ with $n$ disjoint sorts $I = P_1, P_2, \ldots, P_n = P$
equipped with an $(n + 1)$-ary relation $Q \subseteq P^{n+1}$ and a system of maps

$$\langle \pi^k : 2 \leq k \leq n \rangle$$
satisfying the following axioms:

1. (Coherence) For each $k \in \{2, \ldots, n\}$, the function $\pi^k$ maps an element $f \in P_k$ to a compatible $k$-tuple $(\pi^k_1(f), \ldots, \pi^k_k(f)) \in (P_{k-1})^k$. 

(2) (Compatibility and \(Q\)) If \(Q(p_1, \ldots, p_{n+1})\) holds, then \((p_1, \ldots, p_{n+1})\) is a compatible \((n + 1)\)-tuple of elements of \(P\).

(3) (Uniqueness of horn-filling) Whenever \(Q(p_1, \ldots, p_{n+1})\) holds, then for any \(i \in \{1, \ldots, n + 1\}\), \(p_i\) is the unique element \(x \in P\) which satisfies
\[
Q(p_1, \ldots, p_{i-1}, x, p_{i+1}, \ldots, p_{n+1}).
\]

We say the \(n\)-ary quasigroupoid is locally finite if for every \(w \in I^{(n)}\), the set \(P(w)\) is finite.

**Definition 2.4.** Let \(H = (I, \ldots, P, Q)\) be an \(n\)-ary quasigroupoid. We call \(H\) connected if it satisfies both of the following conditions:

1. If \(i \in \{1, \ldots, n - 1\}\), then any compatible tuple \((p_1, \ldots, p_{i+1})\) from \(P_i\) is in the image of \(\pi^{i+1}\).
2. (Existence of horn fillers) Suppose that \(i \in \{1, \ldots, n + 1\}\) and \(\langle p_j : 1 \leq j \leq n + 1 \rangle\) is a tuple of elements from \(P\) which is compatible with respect to \(\pi^n\). Then there is an element \(p_i' \in P\) such that \(Q\) holds of \((p_1, \ldots, p_i', \ldots, p_{n+1})\).

Notice that (1) above, when \(i = 1\), simply says that every \((a_1, a_2) \in I^{(2)}\), \(P_2(a_1, a_2) \neq \emptyset\). But in general (1) is more stronger than the condition that \(P_n(a_1, \ldots, a_n) \neq \emptyset\) for any \((a_1, \ldots, a_n) \in I^{(n)}\).

**Definition 2.5.** If \(H = (I, \ldots, P, Q)\) is an \(n\)-ary quasigroupoid, we say that \(H\) is an \(n\)-ary polygroupoid if it satisfies the following condition:

(Associativity) Suppose that \(\{p_j^i : 1 \leq i \leq n + 2, 1 \leq j \leq n + 1\}\) is a collection of elements in \(P\) such that for each \(i = 1, \ldots, n + 2\) the elements \(\{p_j^i : 1 \leq j \leq n + 1\}\) are compatible and such that \(p_j^i = p_i^{j+1}\) for all \(1 \leq i \leq j \leq n + 1\).

For each \(\ell = 1, \ldots, n + 2\), if \(Q(p_1^\ell, \ldots, p_{n+1}^\ell)\) hold for all \(i \in \{1, \ldots, n + 2\} \setminus \{\ell\}\), then \(Q(p_1^\ell, \ldots, p_{n+1}^\ell)\) holds too.

If associativity holds for any compatible such tuples \(p_j^i\) with \(\pi(p_j^i) = (d_1^i, \ldots, d_j^i, \ldots, d_{n+1}^i) \in I^{(n)}\) where \(d_1^i \ldots d_{n+1}^i = c_1 \ldots c_i \ldots c_{n+2}\), then we say the associativity of \(Q\) holds on \((c_1, \ldots, c_{n+2})\).

**Definition 2.6.** A symmetric witness to the failure of \((n+1)\)-uniqueness (over \(B = acl(B)\)) is a Morley sequence \(I = \langle a_1, \ldots, a_{n+1} \rangle\) over \(B\) equipped with a locally finite, connected quasigroupoid \((I, P_2, \ldots, P_n; Q; \pi')\) such that, \(Q\) and \(\pi'\) are the restrictions of \(B\)-definable functions to the corresponding domains (so all elements in \(I\) or \(P_i\) are finite tuples), and for any any \(i \in \{2, \ldots, n\}\) and any \(f, g \in P_i\):

1. (Isolation of types)
The Hurewicz Correspondence

(a) \( \text{tp}(f/\pi^i(f)B) \vdash \text{tp}(f/\partial(\pi(f))) \), and if \( f' \equiv_{\pi(f)B} f \) then 
\( f' \in P_i \);
(b) \( \pi^i(f) = \pi^i(g) \) iff \( f \equiv_{\pi(f)B} g \); and
(2) (Algebraicity) \( f \in \text{acl}(\pi(f)B) \).

Note that in the definition above, since each \( \pi^i \) is \( B \)-definable, for \( f \in P_i(\pi(f)) \) and any \( B \)-automorphism \( \sigma \) permuting \( I \), it follows \( \sigma(\pi^i(f)) = \pi^i(\sigma(f)) \) and \( \sigma(f) \in P_i(\sigma(\pi(f))) \). Moreover if \( \text{tp}(a_1/B) \) has \( (\leq n) \)-uniqueness, then for any \( u, v \in P_i \) (\( i = 2, \ldots, n \)), \( u \pi^i(u) \equiv_B v \pi^i(v) \).

**Definition 2.7.** Let \( q(x) \in S(B) \) with finite variable \( x \). We call a quasigroupoid \( (P_1, P_2, \ldots, P_n; Q; \pi^i) \) an \( n \)-ary generic groupoid in \( q \) if it is \( B \)-type-definable and locally finite such that

1. \( Q \) and \( \pi^i \) (\( i = 2, \ldots, n \)) are \( B \)-definable;
2. \( P_1 \) is the solution set of \( q(x) \);
3. for \( b_i \models q \), \( P_1(b_1, \ldots, b_i) \not= \emptyset \) iff \( \{ b_1, \ldots, b_i \} \) is \( B \)-independent;
4. for any \( B \)-independent \( I = \{ b_1, \ldots, b_{n+1} \} \subset P_1 \), the subgroupoid of \( (P_1, \ldots, P_n, Q) \) on \( I \) with its all fibers forms an \( (n + 1) \)-symmetric witness over \( B \); conversely if \( Q(v_1, \ldots, v_{n+1}) \) holds with \( v_j \in P_1(c_1, \ldots, \widehat{c_j}, \ldots, c_{n+1}) \) then \( \langle c_1, \ldots, c_{n+1} \rangle \) is Morley in \( q \), and
5. associativity of \( Q \) holds on any Morley sequence of \( (n+2) \)-tuples in \( q \).

Notice that if we restrict a generic groupoid to the fibers over some \( I_0 \subseteq I \) where \( I_0 \) is an infinite Morley sequence, then the result will be a connected quasigroupoid. Moreover the \( Q \)-relation in a generic groupoid is almost associative.

**Lemma 2.8.** Assume \( (P_1, P_2, \ldots, P_n; Q; \pi^2, \ldots, \pi^n) \) is an \( n \)-ary generic groupoid in \( q \in S(B) \). Let \( J = \langle c_1, \ldots, c_{n+2} \rangle \) be given such that for each \( j \in [n+2] \), \( J_j := \langle c_1, \ldots, \widehat{c_j}, \ldots, c_{n+2} \rangle \) is Morley in \( q \). Then associativity of \( Q \) holds on \( J \).

**Proof.** Let \( U_i \) be the set of all \( i \)-subsets of \( [n + i] \) (\( i = 2, 3 \)), and let \( U'_i(\subset U_i) \) be the set of all \( i \)-subsets of \( [n + 1] \). Now for \( \{ j < k \} \in U_2 \), let \( J_{jk} = J_{kj} := \langle c_1, \ldots, \widehat{c_j}, \ldots, \widehat{c_k}, \ldots, c_{n+2} \rangle \). Assume there are \( v_{jk} = v_{kj} \in P_n(J_{jk}) \) such that 

\[ Q(v_{1,k}, \ldots, \widehat{v_{k,k}}, \ldots, v_{n+2,k}) \]

holds, for each \( k \in [n + 1] \). We want to show that it also holds when \( k = n + 2 \). (This is only the case \( \ell = n + 2 \) in Definition 2.5, but the other cases are similar.)

For this goal, let us take \( c_{n+3} \models q \), which is \( B \)-independent from \( J \). Then for each \( j \), \( J_j c_{n+3} \) is Morley in \( q \); so the associativity of \( Q \) holds
on \( J_n c_{n+3} \). Now for \( \{ j < k < m \} \in U_3 \), let \( J_{jkm} = J_{jnk} \) (and so on) := \( \langle c_1, \ldots, \tilde{c}_j, \ldots, \tilde{c}_k, \ldots, c_m, \ldots, c_{n+3} \rangle \). And we put \( v_{j,k,n+3} := v_{jk} \) for each \( \{ j, k \} \in U_2 \). Then for each \( \{ j, k, m \} \in U_3 \), we can choose \( v_{jkm} = v_{jnk} \) (and so on)\( \in P_n(J_{jkm}) \) such that

\[
(v_{j,k,1}, \ldots, v_{j,k,j-1}, v_{j,k,j+1}, \ldots, v_{j,k,k-1}, v_{j,k,k+1}, \ldots, v_{j,k,n+1}, \cdots, v_{j,k,n+3} = v_{jk})
\]

is a compatible \( n \)-tuple from \( P_n \) (see Definition 2.1(1)). This is possible mainly by the connectedness of witnesses. Notice that since elements \( v_{jk} \) are already compatible, in each \( P_i(c_{j_1}, \ldots, c_{j_l}) \) \( (1 \leq j_1 < \cdots < j_l \leq n + 2; 2 \leq i < n) \) there is a unique element such that the set of such elements is the image of \( v_{jk} \)'s under the compositions of \( \pi^{i+1} \)'s. Then by the connectedness of the witnesses, we can iteratively select an element of each remaining fiber in \( P_i(i < n) \) of an increasing \( i \)-tuple from \( J_{c_{n+3}} \), such that the element in an upper level fiber is chosen from the inverse image under \( \pi^i \) of selected elements in lower level fibers. Finally, we apply connectedness again to select such elements \( v_{jkm} \) in \( P_n(J_{jkm}) \).

Using the connectedness of witnesses once again, we conclude that there is \( v_{j,k,n+2} \in P_n(J_{j,k,n+2}) \) for each \( \{ j < k \} \in U_2 \), such that

\[
Q(v_{j,k,1}, \ldots, v_{j,k,j}, \cdots, v_{j,k,k}, \ldots, v_{j,k,n+2}, v_{j,k,n+3} = v_{jk})
\]

holds. Hence by the associativity of \( Q \) on \( J_{c_{n+3}} \), we have that

\[
\models Q(v_{1,k,n+2}, \ldots, v_{k,k,n+2}, \ldots, v_{n+1,k,n+2} = v_{n+1,k}),
\]

for each \( k \in [n + 1] \). We finally observe, due to the associativity on \( J_{n+2 c_{n+3}} \),

\[
\models Q(v_{1,n+2,n+3} = v_{1,n+2}, \ldots, v_{n+1,n+2,n+3} = v_{n+1,n+2})
\]

holds. \( \square \)

In the proof above, if \( J_{n+2} \) is not independent (while so are all \( J_j \) for \( j \in [n + 1] \)) then clearly, by Definition 2.7(4), \( Q \) is not associative on \( J \).

The next fact is a compilation of observations that essentially come from [6] (even though the term “generic polygroupoid” did not appear there).

**Fact 2.9.** Assume that \( q \in S(B) \) has \( (\leq n) \)-uniqueness. Let \( (P_1, P_2, \ldots, P_n, Q) \) be a generic \( n \)-ary groupoid in \( q \). Then there is a \( B \)-definable finite abelian group \( G_q \) (so \( G_q \subseteq acl(B) = B \)) satisfying the following:

1. For any \( B \)-independent \( b_1, \ldots, b_n \models q \) and \( u \in P_n(b_1, \ldots, b_n) \), \( G_q \) acts regularly (transitively and faithfully) on the solution set \( \Pi_u \) of \( q_u \), where

\[
q_u := tp(u/\partial(b_1, \ldots, b_n)),
\]

2. \( \models Q(v_{1,n+2,n+3} = v_{1,n+2}, \ldots, v_{n+1,n+2,n+3} = v_{n+1,n+2}) \)

holds.

3. In the proof above, if \( J_{n+2} \) is not independent (while so are all \( J_j \) for \( j \in [n + 1] \)) then clearly, by Definition 2.7(4), \( Q \) is not associative on \( J \).

4. The next fact is a compilation of observations that essentially come from [6] (even though the term “generic polygroupoid” did not appear there).

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1. For any \( B \)-independent \( b_1, \ldots, b_n \models q \) and \( u \in P_n(b_1, \ldots, b_n) \), \( G_q \) acts regularly (transitively and faithfully) on the solution set \( \Pi_u \) of \( q_u \), where

\[
q_u := tp(u/\partial(b_1, \ldots, b_n)),
\]

2. \( \models Q(v_{1,n+2,n+3} = v_{1,n+2}, \ldots, v_{n+1,n+2,n+3} = v_{n+1,n+2}) \)

holds.

3. In the proof above, if \( J_{n+2} \) is not independent (while so are all \( J_j \) for \( j \in [n + 1] \)) then clearly, by Definition 2.7(4), \( Q \) is not associative on \( J \).
and \(|\Pi_u| = |G_q|\). Moreover, for any \(v, w \in \Pi_u\) and \(\sigma \in G_q\), we have \(\text{dcl}(vB) = \text{dcl}(wB)\) and \(\text{tp}(v, \sigma.v/B) = \text{tp}(w, \sigma.w/B)\). Hence if we let \((v_0, w_0) \sim (v_1, w_1) \in \Pi_u^2\) if there is \(\sigma \in G_q\) such that \(w_j = \sigma.v_j\) (\(j = 0, 1\)), then \(\sim\) is an equivalence relation on \(\Pi_u^2\) and the map \([\ ] : \Pi_u^2/\sim \to G_q\) sending \([(v_j, w_j)]\) to \(\sigma\) is the unique bijection such that for \(v, w, x \in \Pi_u\),

\[
[(v, w)] + [(w, x)] = [(v, x)].
\]

This induces the canonical isomorphism between \(G_q\) and \(G_u := \text{Aut}(q_u)\), sending \(\sigma \in G_q\) to \(\sigma_u \in G_u\) such that \(\sigma.v = \sigma_u(v)\) for some (any) \(v \in \Pi_u\). In other words \(G_q\) uniformly and canonically binds all the groups \(G_w\) with \(u' \equiv_B u\).

(2) Assume \(Q(v_1, \ldots, v_{n+1})\) holds. Then for any \(\sigma \in G_q\) and any \(i \in [n]\),

\[
Q(v_1, \ldots, v_{i-1}, \sigma.v_i, \sigma.v_{i+1}, v_{i+2}, \ldots, v_{n+1})
\]

holds too. More generally, for any \((\sigma_1, \ldots, \sigma_{n+1}) \in G_q^{n+1}\), we have

\[
Q(\sigma_1.v_1, \sigma_2.v_2, \ldots, \sigma_{n+1}.v_{n+1}) \iff \sum_{i=1}^{n+1} (-1)^i \sigma_i = 0.
\]

We state a new observation which will be crucial for the proof of Lemma 2.17.

**Lemma 2.10.** Assume that \(q \in S(B)\) has \((\leq n)\)-uniqueness, and that a generic \(n\)-ary groupoid \((I, P_2, \ldots, P_n; Q; \pi^i)\) in \(q\) is given. Let \(\Phi_q(x_1, \ldots, x_{n+1})\) be a type over \(B\) saying that for some \(x_j^k\) (\(k = 1, \ldots, n+2\), \(j = 1, \ldots, n+1\)) with \(x_j = x_j^{n+2}\), and some Morley sequence \((a_1, \ldots, a_{n+2})\) in \(q\),

(1) \((x_1, \ldots, x_{n+1})\) is compatible from \(P_n\), and \(x_1 \ldots x_{n+1} \equiv_B x_1^k \ldots x_{n+1}^k\);

(2) \(x_j^k = x_j^{k+1}\) for all \(1 \leq k < j \leq n+1\); and

(3) \(\pi(x_j^k) = (d_1, \ldots, \hat{a}_j, \ldots, d_{n+1})\) where \(d_1 \ldots d_{n+1} = a_1 \ldots \hat{a}_k \ldots a_{n+2}\).

Then \(\models \Phi_q(v_1, \ldots, v_{n+1})\) if and only if there is some \(L(B)\)-formula \(Q'(x_1, \ldots, x_{n+1})\) realized by compatible \((v_1, \ldots, v_{n+1})\) from \(P_n\), such that \((I, \ldots, P_n; Q'; \pi^i)\) forms a generic \(n\)-ary groupoid in \(q\) (where we have just replaced \(Q\) by \(Q'\)).

**Proof.** (\(\Rightarrow\)): This direction is clear due to the associativity of \(Q'\) and the \((\leq n)\)-uniqueness property for \(q\).

(\(\Leftarrow\)): Assume \(\Phi_q(v_1, \ldots, v_{n+1})\) holds. So there is some Morley sequence \((a_1, \ldots, a_{n+1})\) in \(q\) such that \(\pi(v_j) = (a_1, \ldots, \hat{a}_j, \ldots, a_{n+1})\) is
witnessed by a formula, say $\theta(x_j; y_1, \ldots, \hat{y}_j, \ldots, y_{n+1}) \in \mathcal{L}(B)$. Moreover notice that $v_j \in \text{dcl}(v_1 \ldots \hat{v}_j \ldots v_{n+1}; B)$ for each $j \in [n + 1]$, since due to $(\leq n)$-uniqueness there is $v_j'$ such that $v_1 \ldots v_j' \ldots v_{n+1} = B w_1 \ldots w_{n+1}$ where $(w_1, \ldots, w_{n+1}) = Q$, so $v_j' \in \text{dcl}(v_1 \ldots v_j \ldots v_{n+1}; B)$, and by Fact 2.9(1), $v_j$ and $v_j'$ are interdefinable over $B$. Hence that $v_j \in \text{dcl}(v_1 \ldots \hat{v}_j \ldots v_{n+1}; B)$ is witnessed by some $\mathcal{L}(B)$-formula

$$\alpha_j(x_j; x_1, \ldots, \hat{x}_j, \ldots, x_{n+1}).$$

We also choose a formula $\beta(x_1, \ldots, x_{n+1}) \in \mathcal{L}(B)$ which says “the $(n+1)$-tuple $(x_1, \ldots, x_{n+1})$ is compatible.” Now we let $Q'(x_1, \ldots, x_{n+1})$ be the formula

$$\beta(x_1, \ldots, x_{n+1}) \land \exists y_1 \ldots y_{n+1} \in \mathcal{L}(B) [\theta(x_j; y_1, \ldots, \hat{y}_j, \ldots, y_{n+1}) \land \alpha_j(x_j; x_1, \ldots, \hat{x}_j, \ldots, x_{n+1})],$$

realized by $(v_1, \ldots, v_{n+1})$. (In fact, $Q'(x_1, \ldots, x_{n+1})$ together with a type over $B$ saying that for some Morley $\langle a_1, \ldots, a_{n+1} \rangle$ in $q$, $\pi(x_j) = (a_1, \ldots, \hat{a}_j, \ldots, a_{n+1})$ for each $j \in [n+1]$, determines $\text{tp}(v_1, \ldots, v_{n+1}; B)$.) Now it suffices to check that $Q'$ satisfies the associativity on a Morley sequence of $(n+2)$-tuple in $q$.

Since $(v_1, \ldots, v_{n+1})$ satisfies (1),(2), and (3), there exist elements $v_j^k$ and a $B$-Morley sequence $\langle a_1, \ldots, a_{n+2} \rangle$ satisfying these conditions (where $v_j^k$ plays the role of $x_j^k$). In particular, $= Q'(v_1^k, \ldots, v_{n+1}^k)$ for all $k \in [n+2]$. On the other hand, assume that there exist $u_j^k$ ($k = 1, \ldots, n+2$, $j = 1, \ldots, n+1$) such that for each $k \in [n+2]$, $(u_1^k, \ldots, u_{n+1}^k)$ is compatible from $P_n$; $u_j^k = u_j^{k+1}$ for all $1 \leq k \leq j \leq n+1; \text{ and } \pi(x_j^k) = (d_1, \ldots, \hat{d}_j, \ldots, d_{n+1})$ where $d_1 \ldots d_{n+1} = a_1 \ldots \hat{a}_k \ldots a_{n+2}$. Moreover assume $Q'(u_1^k, \ldots, u_{n+1}^k)$ holds for each $k \in [n+1]$. We want to verify that it holds as well when $k = n+2$ (the other cases being similar).

Note first that due to $(\leq n)$-uniqueness for $q$, it follows $v_1 \ldots v_n = B u_1^k \ldots u_n^k$ for all $k \in [n+2]$. Since $Q'$ isolates $\text{tp}(v_{n+1}/v_1 \ldots v_n B)$, this implies

$$v_1 \ldots v_{n+1} = B u_1^k \ldots u_{n+1}^k$$

for all $k \in [n+1]$. Furthermore, again by $(\leq n)$-uniqueness, we have

$$\langle v_1^k \ldots v_{n+1}^k \mid k = 1, \ldots, n \rangle = B \langle u_1^k \ldots u_{n+1}^k \mid k = 1, \ldots, n \rangle.$$

Then due to the $Q'$-relation on $(v_1^{n+1}, \ldots, v_{n+1}^{n+1}) = (v_1, \ldots, v_{n+1})$, and on $(u_1^{n+1}, \ldots, u_{n+1}^{n+1}) = (u_1, \ldots, u_{n+1})$, it clearly follows that

$$\langle v_1^k \ldots v_{n+1}^k \mid k = 1, \ldots, n+1 \rangle = B \langle u_1^k \ldots u_{n+1}^k \mid k = 1, \ldots, n+1 \rangle.$$

Then since $Q'_u(v_1^{n+2}, \ldots, v_{n+1}^{n+2})$ holds, so does $Q'_u(u_1^{n+2}, \ldots, u_{n+1}^{n+2}).$
Now in order to show Theorem 0.1, throughout we fix \( n \geq 2 \) and we assume \( p \) has \((\leq n)\)-uniqueness, or equivalently \((n+1)\)-CA. So for any \( c' \in \text{acl}(c) \) with \( c \models p \), \( \text{tp}(c') \) has \((n+1)\)-CA as well. Due to Fact 1.6, if \( p \) additionally has \((n+1)\)-uniqueness over \( B \) then both \( H_n(p), \Gamma_n(p) \) are trivial. Hence in the rest we assume \( p \) does not have \((n+1)\)-uniqueness over \( B \).

Remark 2.11. In what follows, we will cite some facts which were proved in the previous paper [6] under the assumption that \( p \) does not have \((n+1)\)-uniqueness but the entire theory \( T \) has \((n)\)-uniqueness. However, in all steps of the proofs which used \((n)\)-uniqueness of \( T \), all that was actually needed was the “local” assumption of \((\leq n)\)-uniqueness for \( p^{(k)} \) for every \( k \). So by Fact 1.7 above, there is no problem in applying these results to our case.

We recall a definition and a fact from [6].

Definition 2.12. An \((n-1)\)-symmetric system for \( \langle c_1, \ldots, c_{n+1} \rangle \) is a collection of (generally infinite) tuples \( \{ \bar{c}_s : s \in [n+1]^{\langle n \rangle} \} \) such that:

1. \( \bar{c}_s \) is a tuple enumerating \( \text{acl} \{ a_i : i \in s \} \);
2. for any permutation \( \sigma \) of \([n+1] \), there is an automorphism \([\sigma] \) such that
   \[ [\sigma](\bar{c}_s) = \bar{c}_{\sigma(s)} \]
   for every \( s \in [n+1]^{\langle n \rangle} \); and
3. for any \( s \in [n+1]^{\langle n \rangle} \) and any two permutations \( \sigma \) and \( \tau \) of \([n+1] \),
   \[ [\sigma] \circ [\tau](\bar{c}_s) = [\sigma \circ \tau](\bar{c}_s). \]

Fact 2.13. There is an \((n-1)\)-symmetric system for \( \langle c_1, \ldots, c_{n+1} \rangle \).

Let \( \sigma_j \) (\( j \in [n] \)) be the permutation of \([n+1] \) sending \( (1, \ldots, n; n+1) \) to \( (1, \ldots, j, \ldots n+1; j) \). We also fix corresponding automorphisms \([\sigma_j] \) witnessing an \((n-1)\)-symmetric system for \( \langle c_1, \ldots, c_{n+1} \rangle \), as in Fact 2.13.

Definition 2.14. (1) We say a Morley sequence \( \langle a_1, \ldots, a_{n+1} \rangle \) with \( a_i \in \bar{c}_i \), is compatible with \([\sigma_j] \)'s if
   \[ [\sigma_j](a_1, \ldots, a_n; a_{n+1}) = (a_1, \ldots, \hat{a}_j, \ldots, a_{n+1}; a_j) \]
   for each \( j \in [n] \).

We say a Morley sequence \( \langle a_1, \ldots, a_i \rangle \) with \( a_i \in \bar{c}_i \) (\( 1 \leq i \leq n \)), is compatible with \([\sigma_j] \)'s if there is \( a_{i+1} \ldots a_{n+1} \) so that
Given such a compatible Morley sequence \( \langle a_1, \ldots, a_{n+1} \rangle \) is compatible with \([\sigma_j]'s\). Obviously if such compatible \( \langle a_1, \ldots, a_i \rangle \) is given, then \( a_{i+1} \ldots a_{n+1} \) is uniquely determined.

(2) Given such a compatible Morley sequence \( \langle a_1, \ldots, a_{n+1} \rangle \) in (a), suppose that \((P_1, P_2, \ldots, P_n, Q)\) is a generic \(n\)-ary groupoid in \(\text{tp}(a_1)\). Then we say \(u \in P_n(a_1, \ldots, a_n)\) is compatible with \([\sigma_j]'s\) if \([\sigma_1](u), \ldots, [\sigma_n](u), u\) is a compatible \((n+1)\)-tuple from \(P_n\).

In (2) above even if \(u \in P_n(a_1, \ldots, a_n)\) is compatible with \([\sigma_j]'s\), some \(u' \in P_n(a_1, \ldots, a_n)\) need not be compatible with \([\sigma_j]'s\).

The following is essentially proved in [6].

**Fact 2.15.** Assume \( \langle a_1, \ldots, a_n \rangle \) is a Morley sequence \( (a_i \in \overline{a}) \) compatible with \([\sigma_j]'s\). Let

\[
\langle a_1 \ldots a_n \rangle \setminus \partial(a_1 \ldots a_n),
\]

\((a_1\) and \(u\) are all finite tuples) be given. Then there is a Morley sequence \( \langle a'_1, \ldots, a'_n \rangle \) compatible with \([\sigma_j]'s\) where \(a_i \in \text{dcl}(a'_i) \subseteq \overline{a} \), and some \(\emptyset\)-type-definable generic \(n\)-ary groupoid \((P_1, P_2, \ldots, P_n, Q)\) in \(\text{tp}(a'_1)\) such that \(u \in \text{dcl}(u')\) for some \(u' \in P_n(a'_1, \ldots, a'_n)\) compatible with \([\sigma_j]'s\).

We let

\[
F_p := \{ u \in \langle c_1 \ldots c_n \rangle : (I_u, \ldots, (P_u)_n; Q_u; \pi_u) \text{ is a generic } n\text{-ary groupoid in } \text{tp}((c_u)_1) \text{ where } (c_u)_i \in \overline{c_i}, \text{ such that } u \in \text{dcl}(c_u)_n((c_u)_1, \ldots, (c_u)_n) \text{ is compatible with } [\sigma_j]'s \},
\]

which is non-empty since \(p\) does not have \((n + 1)\)-uniqueness.

Given \(u \in F_p\), since \((c_u)_i \in \text{dcl}(u)\) we may write the finite tuple \(u = ((c_u)_1, \ldots, (c_u)_n, u)\) (assuming that \(u\) contains \((c_u)_1 \ldots (c_u)_n\) when there is no chance of confusion. Hence the fact that \(u\) is compatible with \([\sigma_j]'s\) implies the same for \((c_u)_1, \ldots, (c_u)_n\). Now we let \(p_u := \text{tp}((c_u)_1)\), and \(q_u := \text{tp}(u/\partial((c_u)_1 \ldots (c_u)_n))\). As in Fact 2.9, let \(G_u := \text{Aut}(q_u)\); let \(\Pi_u := \text{the finite solution set of } q_u\); and we write \([G_u]_u := G_{q_u}\), the finite abelian biding group isomorphic to each \(G_{u'}\) with \(u' \equiv u\).

**Remark 2.16.** Notice that similarly to the case \(n = 2\) in [4] or [5], by Fact 2.15, \((F_p, \leq)\) with \(u \leq u'\) iff \(u \in \text{dcl}(u')\) and \(\pi_u(u) \in \text{dcl}(\pi_{u'}(u'))\), is a directed set. Now due to our choice of \(F_p\), always \((c_u)_1(c_u)_{j+1} = (c_u)_j(c_u)_j\) for \(j \in [n]\). Hence, \(\pi_{u}(u) \in \text{dcl}(\pi_{u'}(u'))\) iff \((c_u)_1 \in \text{dcl}((c_u')_1)\).

We claim now that for \(u \leq u' \in F_p\), and for any non-empty tuple \(s\) of increasing elements from \([n]\), it follows that \((\pi_u)_s(c_u) \in \text{dcl}((\pi_{u'})_s(c_u'))\) (see the notation explained in Definition 2.1): We only show this when \(s = [n - 1]\), as the other cases will be similar. Suppose not, say there are \(x \neq v := (\pi_u)[n-1](u)\) such that \(x \equiv u' v\) where \(w' := (\pi_u)[n-1](u')\).
Now let $x_jv_jw_j := [\sigma_j](xvw')$ for $j \in [n - 1]$. Then by the choice of $F_p$, $\pi_u^n(u') = (w'_1, \ldots, w'_{n-1}, w)$, and $\pi_u^n(u) = (v_1, \ldots, v_{n-1}, v)$. Moreover due to $n$-uniqueness, there is an elementary map $\sigma$ over $\pi_u^n(u')$ sending $v_1 \ldots v_{n-1}v$ to $x_1 \ldots x_{n-1}x$. Since $v_1 \ldots v_{n-1}v \in \text{dcl}(u')$, clearly $u' \neq u'' := \sigma(u)$. On the other hand, from the definition of symmetric witnesses, it follows

$$\text{tp}(u''/\pi_u^n(u')) = \text{tp}(u'/\pi_u^n(u')) \vdash \text{tp}(u'/\partial(\pi_u^n(u')))$$

In particular,

$$u' \equiv \partial(\pi_u^n(u')) u'' \text{, and } u' \equiv v_1 \ldots v_{n-1}v, x_1 \ldots x_{n-1}x u'',$$

since $v_1 \ldots v_{n-1}v, x_1 \ldots x_{n-1}x \in \partial(\pi_u^n(u'))$. Hence

$$u'v_1 \ldots v_{n-1}v \equiv u''x_1 \ldots x_{n-1}x \equiv u'x_1 \ldots x_{n-1}x,$$

which contradicts $v_1 \in \text{dcl}(u')$. We have verified the claim.

Now for $u \leq u' \in F_p$, any $\sigma' \in G_{u'}$ fixes $\partial(\pi_u(u))$ pointwise (here that $\pi_u(u) \in \text{dcl}(\pi_u(u'))$ is used). Hence $\sigma'(u) \in \Pi_u$, and we write $(\sigma' \restriction u)$ to denote the unique $\sigma \in G_u$ such that $\sigma(u) = \sigma'(u)$; and $\chi^u_v : G_v \to G_u$ to denote the group homomorphism sending $\sigma'$ to $(\sigma' \restriction u)$. Clearly $\chi^u_v = \text{id}$. Then

$$G := \{G_u \mid u \in F_p, \{\chi^u_v \mid u \leq u' \in F_p\}\}$$

forms a directed system of finite abelian groups. The profinite abelian group $\Gamma(=\text{the inverse limit of } G)$ is isomorphic to $\Gamma_n(p)$: We define a homomorphism $\varphi : \Gamma_n(p) \to \Gamma$ by sending $\sigma \in \Gamma_n(p)$ to the element in $\Gamma$ represented by the function $u \in F_p \mapsto (\sigma \restriction u) \in G_u$. By Fact 2.15, this embedding is one-to-one. On the other hand, suppose that $g \in \Gamma$ is represented by a sequence $\langle \sigma_u : u \in F_p \rangle$ with $\sigma_u \in G_u$. Then it follows from compactness that there is some $\sigma \in \Gamma_n(p)$ such that $\varphi(\sigma) = g$, that is, $\varphi$ is surjective.

Now we state a key lemma (relying on Lemma 2.10) used in showing that the maps $\chi^u_v$ commute with augmentation maps that will be defined in section 3 (Lemma 3.4). In the following lemma, we keep using the notation $u = ((c_u)_1, \ldots, (c_u)_n, u) \in F_p$ with the fixed $c_1, \ldots, c_{n+1} \models p$. We write $(c_u)_{n+1}$ to denote the unique element such that $((c_u)_1, \ldots, (c_u)_{n+1})$ is compatible with $[\sigma_j]$’s.

**Lemma 2.17.** There is a family of $\mathcal{L}(B)$-formulas $\{Q'_u : u \in F_p\}$ such that: for every $u \in F_p$, $(I_u, \ldots, (P_u)_n; Q'_u; \pi^u_1)$ still forms a generic groupoid in $p_u$; and for any $u_1, \ldots, u_m \in F_p$, and any $v^1_1 \ldots v^c_m \equiv
We let \((c, v^2_k, \ldots, v^n_k, u_k)\) be a compatible \(n\)-tuple from \((P_{u_k})_n\) for each \(k \in [m]\), we also have

\[ v^1_1 \ldots v^1_m \equiv u_1 \ldots u_m, \]

where \(\models Q'_u(v^1_k, \ldots, v^n_k, u_k)\).

**Proof.** We let \(\Psi(F_p, [\sigma_j](F_p))\) be a type with variables for all \(u \in F_p\) and \([\sigma_j](u)\), saying that \(u \equiv [\sigma_j](u)\). Note also, for \(u \in F_p\) and for the type \(\Phi_u := \Phi_{p_u}(x_1, \ldots, x_{n+1})\) in Lemma 2.10, we can assume the corresponding Morley sequence is \(\langle c_u \rangle_1, \ldots, (c_u)_n, 1\), and \(x_{n+1}\) is the variable for \(u \in F_p\) (so we write \(x^u\) for \(x_{n+1}\)), and \(x_j\) is the variable for \([\sigma_j](u)\) (so we write it as \(x^{[\sigma_j](u)}\) or \(x^u\)).

**Claim.** \(\bigcup \{\Phi_u(x^{[\sigma_1](u)}, \ldots, x^{[\sigma_n](u)}, x^u) \mid u \in F_p\} \cup \bigcup_{j=1}^n \Psi(F_p, [\sigma_j](F_p))\) is consistent.

**Proof of Claim.** Due to compactness, it suffices to see that for \(u_1, \ldots, u_m \in F_p\), \(F \land \Psi\) is consistent, where

\[ \Phi := \bigwedge_{k \in [m]} \Phi_{u_k}(x^u_1, \ldots, x^n_k, x^{u_k}) \quad \text{and} \quad \Psi := \bigwedge_{j \in [n]} x^{u_1} \ldots x^{u_m} \equiv x^{u_1}_1 \ldots x^{u_m}_j. \]

By Fact 2.15, there are \(v \in F_p\) with \(u_1, \ldots, u_m \leq v\) \((k \in [m])\), and a generic groupoid \((L_v, \ldots, (P_v)_n; Q_v, \pi_v)\) in \(p_v\) such that \(v = \langle (c_v)_1, \ldots, (c_v)_n, v\rangle\) where \((c_v)_j \in \mathcal{C}_v\). Hence in particular \((\bar{c}, v_2, \ldots, v_n, v)\) is a compatible \(n\)-tuple from \((P_v)_n\) where \(v_j := [\sigma_j](v)\), and there is \(v_1\) such that \(Q_v(v_1, \ldots, v_n, v)\) holds. Then there are \(u_1^j \ldots u_m^j v_j \equiv u_1 \ldots u_m v\) for \(j \in [n]\). Now an assignment of \(u_1 \ldots u_m \mapsto x^{u_1} \ldots x^{u_m}\) and \(u_1^j \ldots u_m^j \mapsto x^{u_1}_j \ldots x^{u_m}_j\) clearly realizes \(\Psi\).

Moreover the equivalent conditions in Lemma 2.10 imply that

\[ \Phi_u(v_1, \ldots, v_n, v) \]

holds. Now due to the description of the type \(\Phi_u\) in Lemma 2.10, clearly the assignment realizes \(\Phi\) as well. We have proved the Claim.

The rest of the proof now follows easily from the claim, together with \((\leq n)\)-uniqueness of \(p\) and the choice of \(F_p\). Due to the Claim and Lemma 2.10, for \(u \in F_p\), we can choose \(Q'_u\) as in 2.10. Now assume \(u_1, \ldots, u_m \in F_p\) and \(v^1 \ldots v^m\) are given satisfying the hypothesis of Lemma 2.17. As in the proof of Claim, there are \(u_1^j \ldots u_m^j\), with \(u_1 \ldots u_m\) realizing \(\Phi \land \Psi\). Then due to \((\leq n)\)-uniqueness, in fact it follows that

\[ v^2_1 \ldots v^2_m \ldots v^n_1 \ldots v^n_m u_1 \ldots u_m \equiv u^2_1 \ldots u^2_m \ldots u^n_1 \ldots u^n_m u_1 \ldots u_m. \]
Due to Lemma 2.17, in the remainder of this paper we can and do assume that our generic groupoids \((I_u, \ldots, (P_u)_n; Q_u, \pi_u^i)\) for \(u \in F_p\) satisfy the property described in the lemma. More generally due to \((\leq n)\)-uniqueness, the following holds: Suppose that \(u_1, \ldots, u_m \in F_p\), and \(v^1_i \ldots v^1_m \equiv u_1^1 \ldots u_m^1 \equiv u_1^{n+1} \ldots u_m^{n+1}\), as well. Therefore \(v^1_1 \ldots v^1_m \equiv u_1^1 \ldots u_m^1 \equiv u_1^\ell \ldots u_m^\ell\).

\[\text{for} \quad i < n\]

Remark 2.18. Due to Lemma 2.17, in the remainder of this paper we can and do assume that our generic groupoids \((I_u, \ldots, (P_u)_n; Q_u, \pi_u^i)\) for \(u \in F_p\) satisfy the property described in the lemma. More generally due to \((\leq n)\)-uniqueness, the following holds: Suppose that \(u_1, \ldots, u_m \in F_p\), and \(v^1_i \ldots v^1_m \equiv u_1^1 \ldots u_m^1 (\ell \in [n + 1] \setminus \{j\})\) are given such that \((v^1_k, \ldots, v^1_k, \ldots, v^1_k)\) is a compatible \(n\)-tuple from \((P_u)_n\) for each \(k \in [m]\). Then we also have

\[v^1_1 \ldots v^1_m \equiv u_1^1 \ldots u_m^1\]

where \(= Q_u (v^1_1, \ldots, v^1_k)\).

3. Proof of Theorem 0.1

The initial procedure of the proof of Theorem 0.1 will be similar to that of the case of \(n = 2\) in [5], but we need to take more care in selecting the choice maps (the functions \(\alpha^i_u\) and \(\epsilon_u\) below). Given a non-zero tuple \(s\) of increasing numbers from \([\nu]\), let \(A_s\) be some enumeration of \(\{(\pi_u)_s(c_u)\} = ((c_u)_1, \ldots, (c_u)_n, u) \in F_p\}.\) Note that with our fixed automorphisms \([\sigma]\), it follows that \([\sigma](A_s) = A_{\sigma(s)}\) if \(n \notin s\).

For simplices \(h \in S_i(p) (i < n)\), we will choose elementary maps \(\tau_h : A_{(i, i+1)} \rightarrow h(s)\) where \(s := \text{supp}(h)\), in such a manner that \(h^j \circ \tau_{\partial^j h} = \tau_h \circ [\sigma_{j+1}]\) on \(A_i\), where \(h^j (j \leq i)\) is the transition map \(h^j\) with \(s_j := \text{supp}(\partial^j h)\), \(\quad (\dagger)\). We can do this inductively using \((\leq n)\)-uniqueness, as follows. If \(h \in S_0(p)\), then take \(\tau_h\) to be any elementary embedding sending \(A_{(i)}\) onto \(\text{supp}(h)\). Assume now that for \(i < n\), we have chosen \(\tau_h\)'s for \(h \in S_{i-1}(p)\) satisfying (\(\ddagger)\) too. Indeed by \((\leq n)\)-uniqueness, \(\bigcup_{j \leq i} h^j \circ \tau_{\partial^j h} \circ [\sigma_{j+1}]^{-1}\) is an elementary map on \(\bigcup_{j \leq i} A_{\sigma_{j+1}(i)}\), so we can extend to some elementary map \(\tau_h\) on \(A_{i+1}\), which obviously works.

Now using the chosen elementary maps, we select functions on \(S_{n-1}(p)\), by which we will define a group homomorphism from \(H_n(p)\) to \(\Gamma_n(p)\). Let \(h \in S_i(p)\) be given \((i < n)\), with the elementary embedding \(\tau_h\) sending \(A_{i+1}\) into \(\text{supp}(h)\). Then for each \(u \in F_p\), we let \(\alpha^i_u : S_i(p) \rightarrow (P_u)_{i+1}\) such that \(\alpha^i_u(h) = \tau_h((\pi_u)_{i+1}(u))\). In particular, if \(i = n - 1\), then \(\alpha^{n-1}_u(h) = \tau_h(u)\). These selections are compatible with the transition maps of \(h\). Namely for \(s \subseteq t \subseteq w \subseteq \text{supp}(h)\),

\[\text{THE HUREWICZ CORRESPONDENCE 17}\]
Given finite tuple $\mathcal{H}$ holds. Thus the claim follows.

Lastly, if $h \in \mathcal{S}_n(p)$ with $t = \text{supp}(h) = \{k_0 < \ldots < k_n\}$, we write a finite tuple

$$h^u_j := h^{\langle j \rangle} \circ s^{\langle j \rangle} (\alpha u^{-1} (\partial^j h)), \text{ where } j \leq n.$$ Notice that due to (*) above, $(h^u_{k_0}, \ldots, h^u_{k_n})$ is a compatible tuple of $(n + 1)$-elements of $(P_u)_n$.

As pointed out in Remark 2.16, $\Gamma_n(p)$ is the inverse limit of the directed system $\{[G]_u \mid u \in \mathcal{F}_p\}$ of finite abelian groups. We begin to define a group isomorphism between $H_n(p)$ and $\Gamma_n(p)$.

**Definition 3.1.** Given $u \in \mathcal{F}_p$, we define $\epsilon_u : \mathcal{S}_n(p) \to [G]_u$ as follows: Let $f \in \mathcal{S}_n(p)$ with $t = \text{supp}(f) = \{k_0 < \ldots < k_n\}$. Then (due to the connectedness of the witness) there is a unique $u_n$ such that

$$Q_u(f_{k_0}^u, \ldots, f_{k_{n-1}}^u, u_n)$$ holds. We let $\epsilon_u(f) = [(u_n, f_{k_0}^u)] \in [G]_u$. This is well-defined due to Fact 2.9. Finally, we extend $\epsilon_u$ linearly to $C_u(p)$.

**Remark 3.2.** Notice that $\epsilon_u(f)$ only depends on $\text{tp}(f_{k_0}^u, \ldots, f_{k_n}^u)$. Moreover, $f_{k_n}^u \in \text{dcl}(f_{k_0}^u, \ldots, f_{k_{n-1}}^u)$. But $(f_{k_0}^u, \ldots, f_{k_{n-1}}^u)$ does not determine $\epsilon_u(f)$ (though it determines $u_n$), since it neither does $\text{tp}(f_{k_n}^u/f_{k_0}^u, \ldots, f_{k_{n-1}}^u)$. Indeed, $\epsilon_u(f)$ depends on $\text{tp}(u_n, f_{k_n}^u)$. Note that $u_n$ and $f_{k_n}^u$ are inter-definable over $[G]_u \subseteq \text{acl}(\emptyset) = \emptyset$.

Moreover if we let $u_j$ be the unique element such that

$$Q_u(f_{k_0}^u, \ldots, f_{k_{j-1}}^u, u_j, f_{k_{j+1}}^u, \ldots, f_{k_n}^u)$$ holds. Then we claim that $\epsilon_u(f) = (-1)^{n-j}[(u_j, f_{k_j}^u)]$: Let $\mu = \epsilon_u(f)$. Hence by Fact 2.9(2),

$$Q_u(f_{k_0}^u, \ldots, f_{k_{j-1}}^u, (-1)^{n-j} \mu.f_{k_j}^u = u_j, f_{k_{j+1}}^u, \ldots, f_{k_{n-1}}^u, \mu.u_n = f_{k_n}^u)$$ holds. Thus the claim follows.

**Lemma 3.3.** If $d \in B_n(p)$ then for every $u \in \mathcal{F}_p$, we have $\epsilon_u(d) = 0$.

**Proof.** This can be proved by Fact 2.15 and 2.9(2). It suffices to show when $d = \partial g$ for $g \in \mathcal{S}_{n+1}(p)$ with $t = \text{supp}(g) = \{m_0 < \ldots < m_{n+1}\}$, so $d = \sum_{j \leq n+1} (-1)^j g_j$ where $g_j := \partial^j g$. We write a tuple $g_{jk} := (g_j)_m^u (j \neq k \leq n+1)$. Due to Remark 3.2, there is no harm in assuming that all the transition maps of $g$ are inclusion maps. Hence $g_{jk} = g_{kj} \subseteq g(t)$. Note that for $j \leq n$ there is a unique tuple $u_j$ such that

$$Q_u(g_{0j}, \ldots, g_{kj}, \ldots, g_{nj}, u_j)$$
Suppose that

\[ \epsilon : A \to G \]

is a group homomorphism. Then due to associativity of the operation \( \cdot \) in Fact 2.15, we have that \( Q_u(u_0, \ldots, u_n) \); and \( \epsilon_u(g_j) = [(u_j, g_{n+1,j})] \) \( (j \leq n) \). Also write \( \mu_j := \epsilon_u(g_j) \) \( (j \leq n + 1) \).

Now due to the connectedness of the witness, for some \( \tau \in [G]_u \)

\[ \vert Q_u(\mu_0, u_0, \mu_1, u_1, \ldots, \mu_{n-1}, u_{n-1}, \tau, u_n) \]

holds, i.e., \( \vert Q_u(g_{n+1,0}, g_{n+1,1}, \ldots, g_{n+1,n-1}, \tau, u_n) \) \( (*) \). Moreover due to Fact 2.9(2),

\[ (-1)^n \tau + \sum_{0 \leq j < n} (-1)^j \mu_j = 0. \]

Hence \( \tau = (-1)^{n-1} \sum_{j<n} (-1)^j \mu_j \). Now (*) implies

\[ \mu_{n+1} = \epsilon_u(g_{n+1}) = [((\tau, u_n, g_{n+1,n})] = \mu_n - \tau = \mu_n + (-1)^n \sum_{j<n} (-1)^j \mu_j. \]

Then it easily follows that \( \epsilon_u(d) = \sum_{j \leq n+1} (-1)^j \mu_j = 0. \)

Due to the previous lemma, each \( \epsilon_u \) induces a well-defined map \( \tilde{\epsilon}_u : H_n(p) \to [G]_u \). And then by the following lemma, the maps \( \tilde{\epsilon}_u \) induce a group homomorphism \( \epsilon : H_n(p) \to \Gamma_n(p) \).

**Lemma 3.4.** Suppose that \( v \leq u \in F_p \), and \( f \in S_n(p) \). Then

\[ \chi_n^u(\epsilon_u(f)) = \epsilon_v(f). \]

**Proof.** Let \( \text{supp}(f) = \{k_0 < \cdots < k_n\} \). There are \( u'_n \equiv u \) and \( v'_n \equiv v \) such that

\[ Q_u(f_{k_0}^u, \ldots, f_{k_{n-1}}^u, u'_n), \text{ and } Q_v(f_{k_0}^v, \ldots, f_{k_{n-1}}^v, v'_n) \]

hold. Now \( \epsilon_u(f) = [(u'_n, f_{k_n}^u)] \in [G]_u \), and \( \epsilon_v(f) = [(v'_n, f_{k_n}^v)] \in [G]_v \). Note that both \( (f_{k_0}^u, \ldots, f_{k_{n-1}}^u, \hat{\gamma}) \) and \( (f_{k_0}^v, \ldots, f_{k_{n-1}}^v, \hat{\gamma}) \) are compatible \( n \)-tuples from \( (P_u)_n \) and \( (P_v)_n \), respectively. Hence due to \( (\leq n) \)-uniqueness, we have that

\[ u_1 \ldots u_n \equiv f_{k_0}^u \ldots f_{k_{n-1}}^u, \text{ and } v_1 \ldots v_n \equiv f_{k_0}^v \ldots f_{k_{n-1}}^v, \]

where \( u_j v_j = [\sigma_j](uv) \) \( (j = 1, \ldots, n) \). Thus obviously \( v_j u_j \equiv vu \). Therefore due to Lemma 2.17 and Remark 2.18, for the unique elements \( u' \) and \( v' \) such that

\[ Q_u(u_1, \ldots, u_n, u'), \text{ and } Q_v(v_1, \ldots, v_n, v'), \]

we have that \( u'v' \equiv uv \), as well. Moreover

\[ u_1 \ldots u_n u' \equiv f_{k_0}^u \ldots f_{k_{n-1}}^u, \text{ and } v_1 \ldots v_n v' \equiv f_{k_0}^v \ldots f_{k_{n-1}}^v. \]

On the other hand, as pointed out in (***) before Definition 3.1, \( uv \equiv f_{k_j}^u f_{k_j}^v \) for every \( j = 0, \ldots, n \). Then since \( v \in \text{dcl}(u) \) and \( v'_n \in \text{dcl}(f_{k_0}^v \ldots f_{k_{n-1}}^v) \),
The homomorphism $\epsilon$ due to Theorem 1.9, it suffices to show the case when $g$ such that $f = f_{k_0} \cdots f_{k_{n-1}} u_n'$, we have that

$$u_1 \ldots u_n u'; v_1 \ldots v_n v' \equiv f_{k_0}^u \cdots f_{k_{n-1}}^u u_n'; f_{k_0}^v \cdots f_{k_{n-1}}^v v'_n,$$

holds. In particular $u_n' v_n' \equiv u' v' \equiv w$. Now as defined in Remark 2.16, if we consider $\epsilon_u(f) \in G_{u_n}$, then $\chi^u_n(\epsilon_u(f))(v_n')$ is the unique $x$ such that $(x, \epsilon_u(f)(u_n')) = (x, f_{k_n}^u) \equiv (v, u)$. Hence $x = f_{k_n}^v$. Therefore $\chi^u(\epsilon_u(f)) = \epsilon_v(f)$.

\[\square\]

**Lemma 3.5.** The homomorphism $\epsilon$ is injective, i.e., if $d \in \mathbb{Z}_n(p)$ and $\epsilon_u(d) = 0$ for each $u \in \mathcal{F}_p$, then $d \in \mathcal{B}_n(p)$.

**Proof.** Due to Theorem 1.9, it suffices to show the case when $d = \sum_{i \leq n+1} (-1)^i f_i$ is an $n$-shell, where $f_i$ is an $n$-simplex in $p$ with support $(n+2) \setminus \{i\}$. We shall construct an $(n+1)$-simplex $g$ with support $n+2$ such that $\partial g = d$, i.e. $\partial^* g = f_i$. Note that $d$ induces a functor $g^d$ with domain $\bigcup_{i \leq n+1} p((n+2) \setminus \{i\})$ such that $g^d \mid p((n+2) \setminus \{i\}) = f_i$.

What is left is to construct transition maps $g^i := g_{n+2}^{(n+2)-\{i\}}$ in order to extend $g^d$ to a desired $g$. Now due to $(n+1)$-CA of $p$, we can amalgamate $f_0, \ldots, f_n$. Namely we can find compatible transition maps $g_0, \ldots, g_n$ such that $g_i(f_i((n+2) \setminus \{i\})) = a_0 \ldots a_{n+1}$ for some independent realizations $a_0, \ldots, a_{n+1}$ of $p$. From now on, we write $f := f_{n+1}$, and for a subset $s$ of $n+1$, $f^s := (f_{n+1})^s$. It remains to show that there is a transition map $g^{n+1}$ sending $f(n+1)$ to $\text{acl}(a_0 \ldots a_n)$ compatible with the other transition maps. Notice that there is no harm in assuming that for each $s = (n+1) \setminus \{j\}$, as sets,

$$f^s(s) = a_0 \ldots \hat{a}_j \ldots a_n.$$

Moreover due to $n$-uniqueness, we can further assume/select that, for each $n$-subset $s \subset n+1$,

$$g_s := g^{n+1} \mid f^s(s)$$

is the identity map (except when $s = n$). Hence it suffices to show the following claim, since then we can take $g^{n+1}$ an $A$-elementary map sending the left to the right, and we are done. Obviously $a_{<n} := a_0 \ldots a_{n-1}$.

**Claim:** $\bar{a}_{<n} \equiv_A h(\bar{a}_{<n})$, where

$$h := g^n \circ (f_n)^{n+2} \setminus \{n\} \circ (f^n)^{-1},$$

and $A = \bigcup\{a_0 \ldots \hat{a}_j \ldots a_n \mid j < n\}$.

**Proof of Claim.** Suppose not, then by compactness it is not hard to find some $u \in \mathcal{F}_p$ such that $f_n^u(= (f_{n+1})^u_n)$ (see the notation before Definition 3.1) and $h(f_n^u)$ do not have the same type over $f := (f_0^u, \ldots, f_{n+1}^u)$
The map $c$. Let $f_j := g^j((f_j)_n^u)$ (j ≠ k ≤ n + 1, but (j, k) ≠ {n, n + 1, (n + 1, 1)}). Notice that $g_{jk} = g_{kj}$. We put $g_{n,n+1} = g_{n,n+1} := g_n^u((f_n)_n^u)$ Thus in fact, $g_{j,n+1} = f_j^u$ for $j < n$; and $g_{n,n+1} = h(f_n^u)$. For the rest we use a similar idea as in the proof of Lemma 3.3. In particular, for $j ≤ n$, we can find a unique tuple $u_j$ such that

$$Q_u(g_{u_j}, ..., g_{u_j}, ..., g_{u_j}, u_j).$$

Then due to associativity, $Q_u(u_0, ..., u_n)$ holds; and $μ_j := ε_u(f_j) = [(u_j, g_{n+1,j})] (j ≤ n)$. We let $μ_{n+1} := [(u_{n}', g_{n+1,n})]$, where $u_{n}'$ is the unique element such that $Q_u(f; u_{n})$ holds. Then due to Lemma 3.3, we have that

$$\sum_{j ≤ n+1} (-1)^j μ_j = 0.$$

Now since

$$ε_u(d) = \sum_{j ≤ n} (-1)^j μ_j + (-1)^{n+1}[(u_{n}', f_{n}^u)] = 0,$$

it follows that $σ_{n+1} = [(u_{n}', f_{n}^u)]$ and $f_{n}^u = g_{n+1,n} = h(f_n^u)$. This contradicts (*) above.

**Lemma 3.6.** The map $ε$ is surjective.

*Proof.* Let $σ ∈ Γ_n(p) = Aut(c_1...c_n/∂(c_1...c_n))$. Then $τ$ is represented by mapping $u ∈ F_p \mapsto [(u, ε_u)] ∈ G_n$ such that whenever $v ≤ u$ then $χ_n^u([(u, ε_u)]) = [(v, ε_v)]$. In fact, $ε_u = σ(u)$ for all $u ∈ F_p$ (*).

Now it suffices to find an $n$-simplex $f$ in $p$ such that $ε_u(f) = [(u, ε_u)]$ (*)&. We build the $f$ as follows. Consider again the chosen dependent $c_1, ..., c_{n+1} \vdash p$. We take $supp(f) = \{0, ..., n\} = n + 1$, and for every $j ≤ n$, $∂^j f$ is the untwisted functor on $c_{j+1}$ where $s_{j+1} := [n + 1] \setminus \{j + 1\}$ (see the notation in Definition 2.12(1)), i.e., $∂^j f(s_{j+1}) = c_{j+1}$ and all the transition maps of $∂^j f$ are identity maps. Now for $j < n$, we also take the identity map as a transition map $f_{n+1}^{(n+1),(j)}$. Hence it remains to find a transition map $f_{n+1}^n$, some $∂(c_1...c_{n+1})$-elementary map sending $c_1...c_{n+1}$ onto itself, such that (*) holds.

Note now that with out loss of generality, we can assume that the fixed elementary maps (in the beginning of section 3) $τ_{∂^j f} = [σ_{j+1}]$ for $j ≤ n$ (where $[σ_{n+1}]$ is the identity map). Hence $f_{j}^u = [σ_{j+1}(u)]$. In particular $f_{n+1}^n = u$. Now by Lemma 2.17 and Remark 2.18, there is a $∂(c_1...c_{n+1})$-elementary map $μ$ such that $Q_u(f_1^u,..., f_{n+1}^u; μ(u))$ hold for all $u ∈ F_p$. Moreover due to (*) above, there is another $∂(c_1...c_{n})$-elementary map $σ'$ such that $(μ(u), σ' ∘ μ(u))$ holds for all $u ∈ F_p$. Hence if we take $f_{n+1}^n$ as $σ' ∘ μ$, it clearly works. □
4. A FAMILY OF EXAMPLES

In this brief section, we discuss a family of examples which show that in a stable theory, for any \( n \geq 2 \), it is possible to have a type \( p \) with \( H_i(p) = 0 \) for every \( i < n \) but \( H_n(p) \neq 0 \), and in fact given any profinite abelian group \( G \), there are such examples where \( H_n(p) \cong G \).

When \( G \) is a finite nontrivial abelian group, the examples are the complete theories \( T_{G,n} \) of a connected \( n \)-ary polygroupoid with “inverse maps” from section 4 of [6]. Briefly, the idea was to construct totally categorical theories with a sort \( I_\Pi \) for a totally indiscernible set, a single fiber in each \( P_i \) (for \( 2 \leq i < n \)) over each ordered \( i \)-element subset of \( i \), and fibers in \( P_n \) on which there are regular actions of \( G \). Furthermore, for quantifier elimination, we added “inverse maps” \( \iota_\sigma : P_n \to P_n \) to the language for each \( \sigma \in S_n \) such that \( \iota_\sigma \) induces bijections between the fibers \( P_n(a_1, \ldots, a_n) \) and \( P_n(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) \).

In [6], we gave a complete axiomatization of \( T_{G,n} \) and showed that it eliminated quantifiers, is totally categorical, has weak elimination of imaginaries (after adding constants for the group \( G \)), and characterized algebraic closure and nonforking.

Proposition 4.1. If \( p \in S(\text{acl}(\emptyset)) \) is the unique 1-type in the sort \( I \) in \( T_{G,n} \), then \( H_i(p) = 0 \) for every \( i \in \{2, \ldots, n-1\} \) and \( H_n(p) \cong G \).

Proof. The fact that \( H_i(p) = 0 \) for \( i < n \) follows from the fact that \( T_{G,n} \) has \((\leq n)\)-uniqueness (see Proposition 4.13 of [6]).

The calculation of \( H_n(p) \) is simplest to see by means of \( \Gamma_n(p) \), which we now know is congruent to \( H_n(p) \). Choose any \( n \) distinct elements \( c_1, \ldots, c_n \) from \( I \) (which are automatically independent, by results from [6]). The set \( \partial(c_1, \ldots, c_n) \) is interdefinable with the fibers in \( P_2, \ldots, P_{n-1} \) above tuples of length less than \( n \) from \( \{c_1, \ldots, c_n\} \), and so clearly no element of \( P_n((c_1, \ldots, c_n)) \) can lie in \( \partial(c_1, \ldots, c_n) \).

By the characterization of algebraic closure in [6], there can be no other elements in \( \text{acl}(c_1, \ldots, c_n) \) other than those in the definable closure of the fibers of \( P_i \) over tuples from \( \{c_1, \ldots, c_n\} \), so any element of \( \overline{c_1, \ldots, c_n} \) must also be definable from such fibers. Now for any permutation \( \sigma \in S_n \), each \( f \in P_n((c_{\sigma(1)}, \ldots, P_{\sigma(n)})) \) lies in \( \overline{c_1, \ldots, c_n} \): this is clear due to the \( Q \) relation, since for any \( c_{n+1} \in I \setminus \{c_1, \ldots, c_n\} \), we can pick elements \( f_i \in P_n((c_{\sigma(1)}, \ldots, c_{\sigma(i)}, \ldots, c_{\sigma(n)}, c_{n+1})) \) such that

\[
\models Q(f_1, \ldots, f_n, f),
\]

and so \( f \in \text{dcl}(f_1, \ldots, f_n) \).
Due to the interdefinability of the fibers $P((c_{\sigma(1)}, \ldots, c_{\sigma(n)}))$ for various $\sigma$ (via the inverse maps $t_{\sigma}$) and the remarks above, in fact

$$\Gamma_n(p) \cong \text{Aut}(f/\partial(c_1, \ldots, c_n))$$

for some (any) $f \in P_n((c_1, \ldots, c_n))$. But by the characterization of automorphism groups given by Proposition 4.4 and Corollary 4.8 of [6], $\text{Aut}(f/\partial(c_1, \ldots, c_n)) \cong G$, and we are done. \hfill \Box

Finally, note that for any profinite abelian group $G$ which is a limit of an inverse system $\{G_i : i \in J\}$ of finite abelian groups $G_i$, we can use the same procedure as described in Remark 2.29 of [5] to embed all of the theories $T_{G_i}$ (for $i \in J$) into a single multisorted theory $T_G$ with separate sorts for each $T_{G_i}$ and the natural functions $f_{i,j} : T_{G_i} \to T_{G_j}$ induced by the maps $\varphi_{i,j} : G_i \to G_j$ in the inverse system $\{G_i : i \in J\}$. Then if $p$ is the 1-type over $\text{acl}(\emptyset)$ of a generic point in any of the $I$ sorts of any of the $T_{G_i}$, it can be checked that $H_n(p) \cong G$.

\textbf{References}
