GEOMETRIC SIMPLICITY THEORY

BYUNGHAN KIM

The class of *simple* structures, properly containing that of *stable* structures is introduced by Shelah [51]. Specific algebraic simple unstable structures are studied by Hrushovski and others [8][9][24][27][29]. After the author establishes in mid 90s the symmetry and transitivity of forking [30][31], and with Pillay [38], type-amalgamation over models, an enormous amount of works has been made on *simplicity theory*, the study of simple structures, by generalizing the results on stability theory(=the study of stable structures), discovering more complicated nature of simple structures, and producing the interplay between stability theory and simple structures.

The initial developments of simplicity theory, mostly focused on general foundational issues, are surveyed in [39] and also an expository book [56] by F. O. Wagner is published in 2000.

During the last several years, the research trend of simplicity theory has moved toward so called *geometric* simplicity theory, primarily concerned about combinatorial geometric or algebraic sides of simplicity theory, parallel to *geometric stability theory* [45]. It is well-known geometric stability theory is a major technical bridge connecting pure model theory and its applications to algebraic geometry and number theory. Hrushovski's full resolution of the function field version Mordell-Lang conjecture in number theory [21] is a spectacular example of such.

In this note, we try to summarize (a part of) the recent works on geometric simplicity theory. It should be noted that we have absolutely no intention to make an encyclopedic presentation, which by now is an almost impossible task as too much are developed to completely absorb (even present) all. The results in this paper are selected (within the author's narrow knowledge) mainly to expedite reaching the explanations of current works on geometric simplicity theory. The content is a skimmed skeleton of the author's book which will be published soon [36]. This is also a summary of the author's tutoring lectures at Asian Logic Conference 10, Kobe University, Japan, September 1-6, 2008. We thank the conference organizers, in particular Toshiyasu Arai and

The author was supported by KRF grant 2009-0075931.

Hirotaka Kikyo, for their hospitality, and encouragements for writing this note.

We take standard terminology. T is a complete theory in the language $\mathcal{L}$. As usual we fix a saturated model $\mathcal{M}$ of T having cardinality $\bar{\kappa}$. We indeed work in $\mathcal{M}^{\mathrm{eq}}$. Unless mentioned otherwise all tuples and sets $A, B, C, ...$, are from $\mathcal{M}^{\mathrm{eq}}$ of sizes $< \bar{\kappa}$. Models M, N are also elementary submodels of small sizes. We will deal with infinite tuples but mostly $a, b, x, y, ..$ are finite tuples. For a type p, $\mathrm{dom}(p)$ is the parameter set for p. Abusive notation may be used such as AB, or Ab for $b = (b_1, ..., b_n)$, which indeed means $A \cup B$ or $A \cup \{b_1, ..., b_n\}$ respectively. When we say *bounded* sets, or *bounded* indexes, we mean they are small, i.e. $< \bar{\kappa}$. We write $a \equiv_A b$ to denote $tp(a/A) = tp(b/A)$. Recall that

$$\mathrm{acl}(A) := \{b \in \mathcal{M}^{\mathrm{eq}} | \ b \text{ is algebraic over } A, \text{ i.e. there are only finitely many solutions to } \mathrm{tp}(b/A)\}, \text{ and}$$

$$\mathrm{dcl}(A) := \{b \in \mathcal{M}^{\mathrm{eq}} | \ b \text{ is definable over } A, \text{ i.e. } b \text{ is the unique solution to } \mathrm{tp}(b/A)\}.$$

When we write a formula as $\psi(x)$, it may contain parameters, but if written $\varphi(x; y)$ it usually is an $\mathcal{L}$-formula, and $\varphi(x, c)$ with some parameters c is said to be an *instance* of φ. Recall that the *name* of $\psi(x)$ is the canonical code $e \in \mathcal{M}^{\mathrm{eq}}$ such that for any automorphism f of $\mathcal{M}^{eq}$, f fixes the solution set of $\psi(x)$ iff it fixes e. For a rank R, by $R(a/A, ...)$ we mean $R(\mathrm{tp}(a/A), ...)$.

1. Forking and simplicity

In this section we present general basic facts on forking and simple theories. Most of the results here are mentioned in [38], and their proofs can be found in [56].

Definition 1.1. (1) For an integer $k \geq 1$, and a formula $\varphi(x, c)$, we say $\varphi(x, c)$ *k-divides over* a set A if there is a sequence $\langle c_i | i \in \omega \rangle$ such that
  (a) $\mathrm{tp}(c/A) = \mathrm{tp}(c_i/A)$ for all i, and
  (b) $\{\varphi(x, c_i) | i \in \omega\}$ is k-inconsistent, i.e. any finite subset of size k is inconsistent.

(2) A type $p(x)$, k-*divides over* A if there is a formula $\varphi(x, a)$ such that $p \vdash \varphi(x, a)$, and $\varphi(x, a)$ k-divides over A. The type p *divides over* A if it k-divides over A for some k. (Conventionally, any inconsistent formula divides over any set.)

(3) p *forks over* A if there are formulas $\varphi_1(x, a_1), ..., \varphi_n(x, a_n)$ such that
  (a) $p \vdash \bigvee_{1 \leq i \leq n} \varphi_i(x, a_i)$, and

(b) $\varphi_i(x, a_i)$ divides over A for each i.

(4) For $A \subseteq B$, if a complete type p over B does not fork (divide, resp.) over A, then we say p is a *nonforking (nondividing, resp.) extension* of $p\lceil A$, and conversely the latter an *unforked (undivided, resp.) restriction* of the former.

(5) Given $p \in S(A)$, we say $I = \langle a_i | \ i < \omega \rangle$ is a *Morley sequence* of p if I is an A-indiscernible sequence, and for $i \in \omega$, $\text{tp}(a_{i+1}/Aa_0...a_i)$ is a nonforking extension of p.

From the definition, the following easily follows.

Fact 1.2. (Finite character) *Let $A \subseteq B$. Then* $\text{tp}(a/B)$ *does not fork/divide (resp.) over A iff for any finite $b \in B$,* $\text{tp}(a/Ab)$ *does not fork/divide (resp.) over A.*

The prototypical example of dividing appears in the theory of equivalence relation E having infinitely many infinite equivalence classes. There, xEa 2-divides over the empty set. There is an example of k-dividing for $k > 2$. Consider $(\mathbb{Q}^2, E)$ where, for any $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{Q}^2$, $E(a, b)$ iff ($a_1 = a_2$ or $b_1 = b_2$). Here xEb 3-divides (but does not 2-divide) over $\emptyset$.

Intuitively dividing is the right notion for dependence: For type-definable non-disjoint sets $X, Y \subseteq \mathcal{M}^n$, we may temporarily say that Y *k-divides* X if there are arbitrarily many automorphisms f_i fixing X setwise such that $\{Y_i = f_i(Y)\}_i$ is k-inconsistent. Then $X_i = X \cap Y_i \neq \emptyset$ as well. It intuitively means that the set Y chops up X into unboundedly many pieces X_i by conjugates Y_i. It follows that $\text{tp}(c/Ab)$ divides over A iff the set Y of realizations of $\text{tp}(c/Ab)$ *divides* X, that of $\text{tp}(c/A)$. This, in a certain sense, means that c is satisfying more relations (or dependent) with Ab than it is with A.

Why, then is forking introduced over dividing? With forking, we have the following extension axiom which is one of crucial factors establishing further arguments. From now on we write $A \mathop{\smile\!\!\!\!\!\!\mid}_B C$ if for any finite $a \in A$, $tp(a/BC)$ does not divide over B.

Fact 1.3. [52] (Extension) *Let $A \subseteq B \subseteq C$. If $p \in S(B)$ does not fork over A, then there is an extension $q \in S(C)$ of p, which does not fork over A; equivalently, if $c \mathop{\smile\!\!\!\!\!\!\mid}_A B$, then there is $c' \equiv_B c$ such that $c' \mathop{\smile\!\!\!\!\!\!\mid}_A C$.*

The following is the core fact of forking.

Theorem 1.4. [31][34] (Fundamental theorem of forking) *Let T be arbitrary. The following are equivalent.*

(1) *Forking*[1] *satisfies symmetry: For any* A, B, C, $A \downfree_B C$ *iff* $C \downfree_B A$.
(2) *Forking satisfies transitivity: For any* $A \subseteq B \subseteq C$ *and* D, $D \downfree_A C$ *iff* $D \downfree_A B$ *and* $D \downfree_B C$.
(3) *Forking satisfies local character: For any set* A *and finite* d, *there is* $A_d(\subseteq A)$ *of size* $\leq |\mathcal{L}|$ *such that* $d \downfree_{A_d} A$.

Definition 1.5. (1) T is said to be *simple* if one of the above equivalent properties of forking holds.
(2) T is *unstable* if there is $\varphi(x, y)$ and $a_i, b_i \in \mathcal{M}$ $(i < \omega)$ such that $\mathcal{M} \models \varphi(a_i, b_j)$ iff $i < j$.
(3) T is *stable* if it is not unstable.

Fact 1.6. (1) *If* T *is stable, then it is simple.*
(2) [31] *Assume* T *is simple. Then any complete type has a Morley sequence. Moreover, the two notions of forking and dividing coincide. In fact the following are equivalent.*
  (a) $\varphi(x, a)$ *divides over* A.
  (b) *For any Morley sequence* I *of* $\mathrm{tp}(a)$, $\{\varphi(x, a')|\ a' \in I\}$ *is inconsistent.*
  (c) *For some Morley sequence* I *of* $\mathrm{tp}(a)$, $\{\varphi(x, a')|\ a' \in I\}$ *is inconsistent.*
  (d) $\varphi(x, a)$ *forks over* A.
(3) [51] T *is simple iff dividing satisfies local character for 1-types, i.e. for any* $p \in S_1(A)$, *there is* $A_d \subseteq A$ *with* $|A_d| \leq |\mathcal{L}|$ *such that* p *does not divides over* A_d.

Now in simple T, we say a family of sets $\mathcal{C} = \{C_i|\ i \in I\}$ is A-*independent* (equivalently saying, *independent over* A) if

$$C_i \underset{A}{\downfree} \bigcup \{C_j|\ j(\neq i) \in I\}.$$

holds.

It is a good exercise, using symmetry and transitivity, to check that $\mathcal{C} = \{C_i|\ i \in \omega\}$ is A-*independent* iff $C_{i+1} \downfree_A C_0 \ldots C_i$ for all $i \in \omega$. In particular, I is a Morley sequence of p iff I, a sequence in p, is indiscernible and independent over $\mathrm{dom}(p)$.

Definition & Fact 1.7. (1) We say T is *supersimple* if for any set A and finite d, there is finite $A_d(\subseteq A)$ such that $d \downfree_{A_d} A$.
(2) We say T is *superstable* if it is supersimple and stable.
(3) Recall that countable T is said to be ω-*stable* if there are only countably many complete types over a countable model.

[1]That forking satisfies the property X is the same amount of saying nonforking satisfies X.

(4) Fact: T is supersimple iff for any $p \in S_1(A)$, p does not divide over some finite $A_d(\subseteq A)$.

We present some examples of simple structures.

Example 1.8. Classification chart of simple structures:

$$\aleph_1\text{-categorical} \subset \omega\text{-stable} \subset \text{superstable} \begin{matrix} \subset \text{supersimple} \\ \\ \subset \text{stable} \end{matrix} \subset \text{simple}$$

Examples in (n) below is not in the category of (m) for $m < n$. Applying quantifier elimination to 1.6.3, 1.7.4 is the key idea verifying the membership of the example in the given category. For algebraic examples, analysis of definable sets (in the case of pseudo-finite fields [9]) or giving an independence notion to check type-amalgamation over a model (in the case of ACFA, smoothly approximated structures [8][24]) is the due course to verify (super)simplicity.

(1) $\aleph_1$-categorical structures: An infinite set. A vector space $(V, +, 0, \{r : V \to V\}| \ r \in F\})$ (each r is a scalar multiplication) over a field F. Algebraically closed fields.
(2) ω-stable structures: Differentially closed fields. A theory of equivalence relations E_i $(i \in \omega)$ such that E_i refines each E_{i+1}-class into infinitely many infinite E_i-classes.
(3) Superstable structures: $(\mathbb{N}, +, 0)$. A theory of equivalence relations E_i $(i \in \omega)$ such that E_{i+1} refines each E_i-class into two infinite E_{i+1}-classes.
(4) Stable structures: Separably (but not algebraically) closed fields. A theory of equivalence relations E_i $(i \in \omega)$ such that E_{i+1} refines each E_i-class into infinitely many infinite E_i-classes.
(5) Supersimple structures: The theory of the random graph. The theory of the bipartite random graph. Pseudo-finite fields(= infinite models of the theory of all finite fields). ACFA(=the model companion of the theory of fields equipped with an automorphism). (Unstable) smoothly approximated structures such as a vector space over a finite field (as in (1)) equipped with a non-degenerated bilinear form. Any (non-separably closed) pseudo-algebraically closed fields (see 8.1.1) is perfect and bounded.
(6) Simple structures: (Imperfect bounded) Pseudo-algebraically closed fields.

In section 8, we will get back to those field examples more in detail.

We now introduce various notions of ranks for simple theories.

Definition 1.9. Let $\varphi(x; y)$ be an $\mathcal{L}$-formula, and let k be an integer ≥ 2. For any type $p(x)$, $D(p(x), \varphi(x, y), k) = D_{\varphi,k}(p)$ (either a natural number or ∞) is recursively defined as follows:

(1) $D(p, \varphi, k) \geq 0$ for any consistent type p.
(2) $D(p, \varphi, k) \geq n + 1$ if for some a, $\varphi(x, a)$ k-divides over $\mathrm{dom}(p)$, and $D(p \cup \{\varphi(x; a)\}, \varphi, k) \geq n$.
(3) $D(p, \varphi, k) = n$ if $D(p, \varphi, k) \geq n$ and $D(p, \varphi, k) \not\geq n + 1$.
(4) $D(p, \varphi, k) = \infty$ if $D(p, \varphi, k) \geq n$ for all $n \in \omega$.

Definition 1.10. For a type $p(x)$, $D(p(x), \varphi(x, y)) = D_\varphi(p)$ (either an ordinal or ∞) is defined as:

$D(p, \varphi) \geq \alpha + 1$ if there is a such that $\varphi(x, a)$ divides over $\mathrm{dom}(p)$ and $D(p \cup \{\varphi(x, a)\}, \varphi) \geq \alpha$.

For limit δ, $D(p, \varphi) \geq \delta$ if $D(p, \varphi) \geq \alpha$ for all $\alpha < \delta$. The other part of the definition is similar to above.

Definition 1.11. For a formula $\psi(x)$ (with parameters), we define $D(\psi(x)) \geq \alpha + 1$ if there is a formula $\varphi(x)$ dividing over $\mathrm{dom}(\psi(x))$, and $D(\psi(x) \wedge \varphi(x)) \geq \alpha$; The other part of the definition is similar.

For a type, we let

$$D(p(x)) := \min\{D(\psi_1(x) \wedge ... \wedge \psi_n(x)) |\ \psi_i(x) \in p(x)\}.$$

Definition 1.12. For $p \in S(A)$, the crucial clause $SU(p) \geq \alpha+1$ holds if p has a forking complete extension q such that $SU(q) \geq \alpha$.

Fact 1.13. *The following are equivalent.*

(1) $\varphi(x, y)$ *has the* k-tree property ($k \geq 2$)*, i.e. there exist tuples* c_α *with* $\alpha \in \omega^{<\omega}$ *such that*
 (a) *for any* $\alpha \in \omega^{<\omega}$*,* $\{\varphi(x, c_{\alpha \frown n}) | n \in \omega\}$ *is* k*-inconsistent,*
 (b) *for all* $\beta \in \omega^\omega$*,* $\{\varphi(x, c_{\beta\lceil n}) | n \in \omega\}$ *is consistent.*
(2) *For* φ, k*,* $D(x = x, \varphi, k) = \infty$.

Each of $D_{\varphi,k}/D$ or SU-rank characterizes simplicity/supersimplicity, respectively.

Fact 1.14. (1) *The following are equivalent.*
 (a) T *is simple.*
 (b) $D(x = x, \varphi, k) < \infty$ *for any* φ, k.
 (c) $D(x = x, \varphi, k) < \omega$ *for any* φ, k.
 (d) T *does not have the tree property i.e. there does not exist* φ *satisfying some* k*-tree property.*
(2) $D(x = x, \varphi, k) \leq D(x = x, \varphi) \leq D(x = x)$.
(3) $SU(p) \leq D(p)$.

Fact 1.15. *The following are equivalent.*

(1) *T is supersimple.*
(2) *$D(x = x) < \infty$.*
(3) *$SU(p) < \infty$ for any complete p.*

The rank $D_{\varphi,k}$ (SU, resp.) represents forking too for simple (supersimple, resp.) theories.

Fact 1.16. (1) *T simple. $c \mathop{\smile\!\!\!\!\!\!\mid}_A B$ iff $D(c/A, \varphi, k) = D(c/AB, \varphi, k)$ for any φ, k.*
(2) *T supersimple. $c \mathop{\smile\!\!\!\!\!\!\mid}_A B$ iff $SU(c/AB) = SU(c/A)$.*

D-rank reflects forking for superstable T, but not for supersimple T in general. Namely, if T is supersimple, then that $SU(c/AB) = SU(c/A)$ implies $c \mathop{\smile\!\!\!\!\!\!\mid}_A B$. If T is superstable, then the converse holds as well.

By 1.14.2, if T is supersimple, then $D_\varphi(x = x) < \infty$. But there is a simple theory where $D_\psi(x = x) = \infty$ for some ψ [6]. Interesting class defined by D_ψ-rank is the class of *low* theories.

Definition & Fact 1.17. (1) T is said to be *low* if $D(x = x, \varphi) < \omega$ for any $\varphi(x, y)$. There even exists a non-low supersimple theory [5].
(2) Fact: T is low iff T is simple, and for any $\varphi(x, y)$, there is k such that whenever $\varphi(x, a)$ divides over $\emptyset$, it k-divides. Hence if T is low then for any $\varphi(x, y)$ and a set A, there is a set of formulas $\Sigma(y)$ over A such that $\models \Sigma(a)$ iff $\varphi(x, a)$ divides over A.

We finish this section by stating the extension of Lachlan's classical result on superstable theories to the context of supersimple theories. Recall that $I(T, \kappa)$ is the number of nonisomorphic models of T having size κ.

Theorem 1.18. (1) *(Lachlan [41]) If T is countable superstable, then either $I(T, \omega) = 1$ or $\geq \omega$.*
(2) [33] *If T is countable supersimple, then either $I(T, \omega) = 1$ or $\geq \omega$.*

2. Type amalgamation and canonical bases

Before Theorem 1.4 is established, historically Shelah already has proved in 1970s [52] that various forking axioms mentioned in section 1 hold for stable theories, and which has initiated stability theory, the study of stable structures.

Fact 2.1. *Let T be stable. Then two notions of forking and dividing are equivalent (see 1.6.2). Nonforking also satisfies the following.*

1. (Finite character) *1.2.*
2. (Symmetry) *1.4.1.*
3. (Transitivity) *1.4.2.*
4. (Extension) *1.3.*
5. (Local character) *1.4.3.*
6. (Uniqueness over a model) *Assume $p(x) \in S(M)$. For $M \subseteq A$, there is a unique nonforking extension $q(x) \in S(A)$ of p.*

Moreover the six axioms characterize stability and forking. Namely if, in a theory T, an automorphism-invariant relation between complete types and sets satisfies all the axioms (1) to (6), then the theory is stable and the relation has to be nonforking.

It is then natural to ask, for simple theories, what axiom can substitute for (Uniqueness over a model). The following are responded in [38].

Theorem 2.2.
1. (Type amalgamation[2] over a model) T *is simple. Given a model and sets $M \subseteq A_0, A_1$, if $c_0 \equiv_M c_1$, $A_0 \mathop{\smile\!\!\!\!\!\!|}_M A_1$, and $c_i \mathop{\smile\!\!\!\!\!\!|}_M A_i$ $(i = 0, 1)$, then there is $c \equiv_{A_i} c_i$ such that $c \mathop{\smile\!\!\!\!\!\!|}_M A_0A_1$.*
2. *The five basic axioms in 2.1 together with the above axiom characterize simplicity and forking.*

Now we explain the notions of various strong types introduced by Shelah and Lascar. Recall that an equivalence relation $E(x, y)$ is said to be *finite* if it has finitely many classes, *bounded* if it has $< \bar{\kappa}$ many classes. E is said to be *invariant* over A if for any A-automorphism f, $E(a, b)$ holds iff $E(f(a), f(b))$ holds.

Definition 2.3.
1. We say finite tuples a, b have the same *strong type* over A (write $a \equiv^s_A b$, or $\mathrm{stp}(a/A) = \mathrm{stp}(b/A)$) if $E(a, b)$ holds, for any finite definable equivalence relation $E(x, y)$ over A. When a, b are infinite, $a \equiv^s_A b$ holds if $\mathrm{stp}(a'/A) = \mathrm{stp}(b'/A)$ for any finite subtuples $a' \subseteq a$, $b' \subseteq b$ of corresponding indexes.
2. We say a, b possibly infinite, have the same KP (Kim-Pillay) type over A (write $a \equiv^{KP}_A b$), if $E(a, b)$ holds, for any bounded type-definable equivalence relation $E(x, y)$ over A.
3. We say a, b possibly infinite, have the same Lascar (strong) type over A (write $a \equiv^L_A b$, or $\mathrm{Ltp}(a/A) = \mathrm{Ltp}(b/A)$ if $E(a, b)$ holds, for any bounded invariant equivalence relation $E(x, y)$ over A.

[2]It is originally called *the Independence Theorem.*

The notion of Lascar type is introduced by Lascar [42]. Note that $a \equiv_A b \Rightarrow a \equiv^s_A b \Rightarrow a \equiv^{KP}_A b \Rightarrow a \equiv^L_A b$ holds. In general, $a \equiv^s_A b$ iff $a \equiv_{\mathrm{acl}(A)} b$ (where acl is taken in $\mathcal{M}^{eq}$). If T is simple, Lascar type is KP type. Indeed more is true.

Proposition 2.4. [32] *Suppose that T is simple. Then T is G-compact, i.e. for tuples a, b of arbitrary length, $a \equiv^L_A b$ holds iff $a' \equiv^{KP}_A b'$ holds for any finite subtuples $a' \subseteq a$, $b' \subseteq b$ of corresponding indexes.*

Recall now that in stable theories, the uniqueness axiom holds over algebraically closed sets too.

Fact & Definition 2.5. Let T be stable.

(1) *(Uniqueness for strong types)* For $B \subseteq C$, if $a \equiv^s_B a'$ and $a \mathop{\smile\!\!\!\!\!\!\mid}_B C$, $a' \mathop{\smile\!\!\!\!\!\!\mid}_B C$, then $a \equiv_C a'$.

(2) A complete type $\mathrm{tp}(c/A)$ is said to be *stationary* if it satisfies the uniqueness axiom, equivalently $\mathrm{tp}(c/A)$ and $\mathrm{tp}(c/\mathrm{acl}(A))$ have the same set of realizations.

Now in simple theories, the following substitute the uniqueness axiom.

Theorem 2.6. (Type amalgamation for Lascar types) *For $B \subseteq C_0, C_1$, if $a_0 \equiv^L_B a_1$, $C_0 \mathop{\smile\!\!\!\!\!\!\mid}_B C_1$, and $a_i \mathop{\smile\!\!\!\!\!\!\mid}_B C_i$ $(i = 0, 1)$, then there is $a \equiv_{C_i} a_i$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_B C_0C_1$.*

Then again it is natural to question whether type amalgamation holds for strong types too. This is still an open question. But Buechler affirmatively answer for low theories.

Theorem 2.7. [3] *If T is low, then Lascar types are strong types. Equivalently, type amalgamation holds for strong types.*

For supersimple T, Buechler, Pillay, and Wagner also give a positive answer. Indeed they prove more, namely elimination of hyperimaginaries for supersimple theories. This important issue comes next.

Definition 2.8. (1) T has *elimination of imaginaries* if for any definable equivalence class a/E, there is finite $b \in \mathcal{M}$ such that they are interdefinable (i.e. for any automorphism fixes the class a/E setwise iff it fixes the tuple b).

(2) T has *elimination of hyperimaginaries* if for any type-definable equivalence class a/E (a possibly infinite), there are definable equivalence classes $\{a_i/E_i\}_i$ such that they are interdefinable. In other words, for any type-definable equivalence relation $E(x, y)$ over $\emptyset$, and any complete type $p(x)$ over $\emptyset$, there are $\emptyset$-definable

equivalence relations $E_i(x_i, y_i)$ (finite $x_i(\subseteq x)$, $y_i(\subseteq x)$ have corresponding subindexes), such that

$$p(x) \wedge p(y) \models E(x,y) \leftrightarrow \bigwedge_i E_i(x_i, y_i).$$

By working in $\mathcal{M}^{eq}$, there is no harm to assume that any theory has elimination of imaginaries.

Theorem 2.9. *(Buechler, Pillay, Wagner* [4]*) If T is supersimple, then T has elimination of hyperimaginaries.*

Therefore if T is supersimple, then $a \equiv^L_A b$ implies $a \equiv^s_A b$ (so $a \equiv_{\mathrm{acl}(A)} b$) even for infinite tuples, and we can identify $\mathrm{Ltp}(a/A) = \mathrm{stp}(a/A) = \mathrm{tp}(a/\mathrm{acl}(A))$.

We do not know still whether every simple T has elimination of hyperimaginaries. Hence in general, to go further we need to introduce the notion of *hyperimaginary* types and *the bounded closures of hyperimaginaries*. But for the rest of this note we assume, for convenience, T **is simple having elimination of hyperimaginaries**, unless stated otherwise. Hence we freely use, in simple T, type amalgamation over an algebraically closed set. Most results stated afterward indeed hold for arbitrary simple T as hyperimaginary versions.

The next issue is about canonical base, which is immensely important notion in developing further deeper works in simplicity theory.

Definition 2.10. (1) We say $\mathrm{tp}(a/A)$ is an *amalgamation base* if type amalgamation holds for $\mathrm{tp}(a/A)$, equivalently both $\mathrm{tp}(a/A)$ and $\mathrm{tp}(a/\mathrm{acl}(A))$ have the same set of realizations.

(2) Let $p = \mathrm{tp}(a/A)$ be an amalgamation base. We say $A_0(\subseteq A)$ is the *canonical base* of p, written $\mathrm{Cb}(p) = \mathrm{Cb}(a/A) = \mathrm{Cb}(a/\mathrm{acl}(A))$, if
   (a) p does not fork over A_0,
   (b) $p\lceil A_0$ is an amalgamation base, and
   (c) whenever $A'(\subseteq A)$ satisfies (a),(b), then $A_0 \subseteq \mathrm{dcl}(A)$.

In stable T, p is stationary iff it is amalgamation base.

Theorem 2.11. [18] *Any amalgamation base has the canonical base.*

Due to the canonical base notion, forking satisfies more axiom as such. Let $B_i = \mathrm{acl}(B_i) \subseteq B$ $(i = 0, 1)$. If $a \mathop{\smile\!\!\!\!\!\!\,|}\,_{B_0} B$ and $a \mathop{\smile\!\!\!\!\!\!\,|}\,_{B_1} B$, then $a \mathop{\smile\!\!\!\!\!\!\,|}\,_{B_0 \cap B_1} B$: Let $C = \mathrm{Cb}(a/B)$. Then $C \subseteq B_i$, Therefore by transitivity, it follows $a \mathop{\smile\!\!\!\!\!\!\,|}\,_C B_0 \cap B_1$ and $a \mathop{\smile\!\!\!\!\!\!\,|}\,_{B_0 \cap B_1} B$.

There now is a definability of type issue to do with the canonical base. Different from the stable case, amalgamation base p need not satisfy the uniqueness axiom. But it does if we restrict our attention to p-stable formulas. Recall that given a complete type $p(x)$ and $\varphi(x,y) \in \mathcal{L}$, φ-type $p\lceil\varphi$ is the set of formulas in p, each of which is equivalent in T to some Boolean combination of instances of φ.

Definition 2.12. Let $p(x) \in S(A)$ be an amalgamation base. We call $\varphi(x,y) \in \mathcal{L}$, *p-stable* if the φ-type $p\lceil\varphi$ has a unique nonforking φ-type extension over any $B \supseteq A$.

Fact 2.13.
1. *$\varphi(x,y)$ is stable iff $\varphi(x,y)$ is p-stable for any amalgamation base $p(x)$.*
2. *If $\varphi(x,y)$ is p-stable, then there is $\psi_\varphi(y)$ over* $\mathrm{Cb}(p)$*, called the* φ-definition *of p, such that $\models \psi_\varphi(b)$ holds iff $\varphi(x,b)$ is in some nonforking extension of p.*

Theorem 2.14. [34] *Assume T is supersimple, and $p(x)$ is an amalgamation base. Then*

$$\mathrm{Cb}(p) = \{\textit{names of } \psi_\varphi(y) \mid \varphi(x,y) \textit{ is } p\textit{-stable}\}.$$

Lastly we state the relationship between Morley sequences and canonical bases.

Fact 2.15. *Given amalgamation base p,* $\mathrm{Cb}(p) \subseteq I$ *where I is a Morley sequence of p.*

3. Modularity

In this section we introduce another important subclass, called *modular*. Studying such class is one of main themes of geometric simplicity theory.

Consider the example in 1.8 of vector space $(V, +, 0, \{r : V \to V\} \mid r \in F\})$ over a field F. Here the nonforking independence $\mathop{\smile\!\!\!\!\!\!\,|}$ is exactly linear independence, i.e. $A \mathop{\smile\!\!\!\!\!\!\,|}_B C$ iff for any finite $a \in A$, $\dim(a/B) = \dim(a/BC)$, and SU-rank measures the basis dimension. Moreover $\mathrm{acl}(A)$ is exactly the linear span of A. For two subspaces A, B, $A \mathop{\smile\!\!\!\!\!\!\,|}_{A\cap B} B$ holds. When they are finite dimensional, it means $\dim(A \cup B) = \dim(A) + \dim(B) - \dim(A \cap B)$.

We say T is *modular* if nonforking shares the dimensional property as mentioned.

Definition 3.1.
1. T is called *modular* if for any A, B, we have
$$A \mathop{\smile\!\!\!\!\!\!\,|}_{\mathrm{acl}(A)\cap\mathrm{acl}(B)} B.$$
2. T is said to be *1-based* if for any indiscernible sequence $I = \langle a_i \mid i < \omega\rangle$, we have $I - \{a_0\}$ is Morley over a_0.

(3) Similarly, for a type p over C, we say p is *modular* (*1-based* resp.) if after naming C, the above holds for any subsets or tuples from the solution set of p.

By so far developed techniques we can easily show the equivalence of modularity and 1-basedness.

Fact 3.2. *The following are equivalent.*[3]

(1) *T is 1-based.*
(2) *For an amalgamation base $p = \mathrm{tp}(c_0/A)$, $\mathrm{Cb}(c_0/A) \subseteq \mathrm{acl}(c_0)$.*
(3) *T is modular.*

Proof. (1)⇒(2) We can assume $A = \mathrm{acl}(A)$. Let $C := \mathrm{Cb}(c_0/A)(\subseteq A)$. Now there is a Morley sequence $J = \langle c_i | i \leq \omega \rangle$ of p. Then $c_\omega \mathop{\smile\!\!\!\!\!\mid}_A I$ where $I = J - \{c_\omega\}$. Since $c_\omega \mathop{\smile\!\!\!\!\!\mid}_C AI$ and $C \subseteq I$, we have $c_\omega \mathop{\smile\!\!\!\!\!\mid}_I A$. Hence $C = \mathrm{Cb}(c_\omega/A) = \mathrm{Cb}(c_\omega/I)$. By 1-basedness, $c_\omega \mathop{\smile\!\!\!\!\!\mid}_{c_0} I$. Hence $C \subseteq \mathrm{acl}(c_0)$.

(2)⇒(3) Given A, B, we have $A \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{Cb}(A/B)} B$ and $\mathrm{Cb}(A/B) \subseteq \mathrm{acl}(A)$. Hence by transitivity, $A \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}(A) \cap \mathrm{acl}(B)} B$.

(3)⇒(1) Indiscernible $I = \langle c_i | \, i < \omega \rangle$ is given. (c) implies $c_0 \mathop{\smile\!\!\!\!\!\mid}_A c_1 c_2 ...$ where $A = \mathrm{acl}(c_0) \cap \mathrm{acl}(c_1 c_2 ...)$. Now, $tp(c_1 c_2 .../\,\mathrm{acl}(c_0)) = tp(c_2 c_3 .../\,\mathrm{acl}(c_0))$. Hence $A = \mathrm{acl}(c_0) \cap \mathrm{acl}(c_2 c_3 ...)$. Again, $tp(c_0/\,\mathrm{acl}(c_2 c_3 ...)) = tp(c_1/\,\mathrm{acl}(c_2 c_3 ...))$, and thus $A = \mathrm{acl}(c_1) \cap \mathrm{acl}(c_2 c_3 ..)$. Then, due to (c) again, $c_1 \mathop{\smile\!\!\!\!\!\mid}_A c_2 c_3 ....$ From transitivity, we have $c_1 \mathop{\smile\!\!\!\!\!\mid}_{c_0 A} c_2 c_3 ....$ Then $c_1 \mathop{\smile\!\!\!\!\!\mid}_{c_0} c_2 c_3 ...$ since $A \subseteq \mathrm{acl}(c_0)$. Thus (a) follows. □

4. Geometries

Now we substantially get into geometric simplicity theory.

We begin this section by defining a *geometry*, a combinatorial generalization of the basis notion.

Definition 4.1. Let S be a set. If an operation $cl : \mathcal{P}(S) \to \mathcal{P}(S)$ satisfies the following properties, then we say that (S, cl) is a *geometry*[4]

(1) For $A \subseteq S$, $A \subseteq cl(A) = cl(cl(A))$.
(2) For $A \subseteq B \subseteq S$, $cl(A) \subseteq cl(B)$.
(3) For $A \subseteq S$ and $a, b \in S$, if $a \in cl(Ab) \smallsetminus cl(A)$, then $b \in cl(Aa)$.
(4) If $a \in cl(A)$, then $a \in cl(A_0)$, for some finite $A_0 \subseteq A$.

[3]The same proof works for a type.

[4]It indeed is called a pregeometry and a *geometry* is meant to be a pregeometry where each closure is a singleton. But here we try not to distinguish.

Let (S, cl) be a geometry. It is said to be *homogeneous* if for any closure X of a finite set and $a, b \in S \smallsetminus X$, there is an automorphism of (S, cl) which fixes X pointwise and moves a to b. We say $(S, cl) = (S_0, cl_0)$ is equivalent to (S_1, cl_1) if there is a bijection $f : S'_0/\sim_0 \rightarrow S'_1/\sim_1$ preserving closure relations, where for $a, b \in S'_i := S_i - cl_i(\emptyset)$, $a \sim_i b$ iff $cl_i(a) = cl_i(b)$.

We say $A(\subseteq S)$ is *independent* if $a \notin cl((A \smallsetminus \{a\})$ for all $a \in A$. Given B, a subset $B_0 \subseteq B$ is said to be a *basis* for B if $B \subseteq cl(B_0)$ and B_0 is independent. It follows that any two bases for B have the same cardinality, denoted by $dim(B)$. Any $A \subseteq S$ gives a *localized* geometry on S defined by $cl_A(B) = cl(A \cup B)$, and a notion of dimension *over* A $(dim(-/A))$. For $A, B, C \subseteq S$, if $dim(A'/C) = dim(A'/B \cup C)$ for any finite $A' \subseteq A$, then we say that A is *independent from* B *over* C.

Definition 4.2. Let (S, cl) be a geometry.

(1) (S, cl) is *trivial* if $cl(A) = \bigcup\{cl(\{a\}) : a \in A\}$ for all $A \subseteq S$.
(2) (S, cl) is *modular* if X is independent from Y over $X \cap Y$ for all closed X, Y, or equivalently, if $dim(X) + dim(Y) = dim(X \cup Y) + dim(X \cap Y)$ for finite dimensional closed X and Y.
(3) (S, cl) is *locally modular* if it is modular over some point in S.
(4) (S, cl) is *locally finite* if the closure of a finite set is finite.

Each of the properties is invariant under localization. Thus modularity implies local modularity.

Fact 4.3. *(S, cl) is modular iff for any subset A and $b \in S$, whenever $c \in cl(Ab)$, then $c \in cl(ab)$ for some $a \in cl(A)$.*

Now we putting the set theoretical notion of geometry to the model theoretic context. Assume T is strongly minimal (i.e. any definable subset (with parameters) of $\mathcal{M}^1$ is either finite or co-finite) so that stable. Then it is easy to see that $(\mathcal{M}^1, \mathrm{acl})$ forms a geometry, (where for a set $A \subseteq \mathcal{M}^1$, the closure is $\mathrm{acl}(A)(\cap\mathcal{M}^1)$). There, the forking independence in section 1 coincides with the dimensional independence. The following are the typical examples of such strongly minimal geometries.

Example 4.4. (1) Infinite set (having no relation other than equality): a trivial geometry.

(2) (a) Vector space $(V, +, 0, r)_{r \in F}$ as mentioned in section 3: a modular geometry. The corresponding projective space has the equivalent geometry.

(b) Affine space $(V, \lambda_r, G)_{r \in F}$ where $\lambda_r(u, v) = ru + (1 - r)v$, $G(u, v, w) = u - v + w$: a locally modular geometry. But this geometry is not modular as two parallel lines (each having dimension[5] 2) in a plain (having dimension 3) do not satisfy the equation in 4.2.(2).

(3) Complex field $\mathbb{C} = (\mathbb{C}, +, -, \times, 0, 1)$: a non-modular geometry.

Zilber's Principle (early 80s) roughly says that above are more or less all the examples of strongly minimal structures. Namely, any strongly minimal structure is either trivial, locally modular, or interpreting an infinite field (which must be algebraically closed field by Macintyre's result [43]).

Hrushovski (early 90s) constructed the counterexamples of Zilber's Principle using his ingenious construction method, which are not locally modular where no infinite group is interpretable(=definable in $\mathcal{M}^{eq}$)[20]. But later on, he and Zilber suggested the so-called 'Zariski condition', and *proved* Zilber's Principle under the constraint[28]. It is well-known that Hrushovski (mid 90s) solved function field version Mordell-Lang conjecture in number theory, by the striking applications of the mentioned results[21].

Zilber's Principle also makes sense in the o-minimal theory context, and Starchenko and Peterzil solved it fully and positively[44].

Now if $\mathcal{M}^1$ is strongly minimal, then there is a unique complete 1-type p over $\emptyset$ having SU-rank 1. p indeed is a strong type. If D is the solution set of the type[6], then (D, cl) where $cl(-) = \mathrm{acl}(-) \cap D$ has the equivalent geometry to $(\mathcal{M}^1, \mathrm{acl})$. Hence it suffices to pay our attention to (D, cl) to comprehend the geometry of the strongly minimal structure. Indeed for *any* SU-rank 1 strong type p (over $\emptyset$ after suppressing $\mathrm{dom}(p)$) with the solution set D, the same closure relation gives the geometry to D. Moreover inside D, $\dim(-/A) = SU(-/A)$ and $\smile\!\!\!\!\!\!\;|$ = dimensional independence.

For the rest of this section, we fix such $D = (D, cl)$ in a simple theory. Understanding the geometry is the suitable generalization of the study of strong minimal structures. In the spirit of Zilber's principle for the strong minimality context, we come up with the following for the simple theory context.

Question 4.5. (1) Relationship between D being 1-based(=modular in the sense of 3.1.1) and its geometry being (locally) modular[7]

(2) If (D, cl) is locally modular or more generally D is 1-based, then can we recover a vector space from D?

[5]Here the dimension is that of the geometry.

[6]Here $|D| = \bar{\kappa}$.

[7]For the moments, not to confuse with 3.1.1, when we refer to modularity in 4.2, we either say D has the modular geometry, or (D, cl) is modular.

(3) If D is not modular, then when we get a field and what kind of a field we get?

Indeed describing how much we have accomplished to solve the questions is the whole theme of this paper. Let us first see the answers of 4.5.1 for the stable case.

Definition 4.6. We call D (equivalently p) is *k-linear* ($k \geq 1$) if for any $a, b \in D^1$, and $A \subseteq D$, if $SU(ab/A) = 1$, then $SU(e) \leq k$ where $e = \mathrm{Cb}(ab/A)$. It is called *linear* if it is 1-linear.

Fact 4.7. [45, 2.2.6] *T stable. The following are equivalent.*

(1) *D has the locally modular geometry (i.e. for $A, B \subseteq D$ and $d \in D^1$, $A \downarrow_{\mathrm{acl}(Ad)\cap \mathrm{acl}(Bd)\cap D} B$).*
(2) *D is linear.*
(3) *(D, cl_A) is modular for some A.*
(4) *D is 1-based (i.e. for $A, B \subseteq D$, $A \downarrow_{\mathrm{acl}(A)\cap \mathrm{acl}(B)} B$).*

Above Fact 4.7 no longer holds in general if D is simple. Consider $V_P := (V; P)$, where V is the Affine space (4.4.2) and P is the unary generic predicate on V as in [10]. Note that the independence relation $\downarrow$ computed on V and (V, P) are equal. Hence $D =$(the solution set of)$P(x)$ is also 1-based. But it not locally modular: Choose $a, b, c, d \in P$ such that both $a \neq b$, $c \neq d$ forms two parallel lines. Choose also $e \in P$ not in the plain generated by $\{a, b, c, d\}$. By the genericity of P, we can assume that $\{e\} = \mathrm{acl}(abe) \cap \mathrm{acl}(cde) \cap P$. Then

$$3 = \dim(abcd/e) \neq \dim(ab/e) + \dim(cd/e) - \dim(e/e) = 4,$$

and as P is the solution set of $\mathrm{tp}(e)$ in V_P, it is not locally modular.

To resolve above nuisance, we have to get missing points. For this we introduce the following notion for simple theories.

Definition 4.8. $G(D)$ denotes the collection of all SU-rank 1 elements in $D^{eq} := \mathrm{dcl}(D)$.

$D \subseteq G(D)$ and $(G(D), cl)$ also forms a geometry where again $cl(-) = \mathrm{acl}(-) \cap G(D)$.

Fact 4.9. [54] [13] *D simple. The following are equivalent.*

(1) *D is 1-based.*
(2) *D is linear.*
(3) *$G(D)$ has a modular geometry.*
(4) *$(G(D), cl_A)$ is modular, for any (some) set A.*

Above fact gives more clear picture to the geometry of 1-based type. Identifying it as a locally modular geometry is rather a partial (such as only for the stable case) and mislead approach. Only after the study of general simple 1-based type, we have better and correct understanding of the type. This also answers the first question of 4.5.

4.1. **ω-categorical case.** Now we move to the second question of 4.5. For the rest of section 4, we restrict out attention mostly to the ω-categorical case, and review answers or counterexamples related to the question (2) of 4.5. Again we see the stable context first.

Theorem 4.10. *(Zilber[58]) D stable and ω-categorical.*

(1) *Then D is 1-based.*
(2) *If (D, cl) is non-trivial, then a vector space over a finite field having the same geometry can definably be recovered.*

Remark 4.11. Above theorem is no longer true for simple theories. In [22], Hrushovski constructs non 1-based ω-categorical simple D by using a variation of his construction method (to produce stable counterexamples of Zilber's Principle) mentioned after 4.4.

Fact 4.12. [15] *D stable.*

(1) *If D has the non-trivial modular geometry, then it is equivalent to that of some projective space over a division ring.*
(2) *If ω-categorical D has the non-modular but locally modular geometry, then it is equivalent to that of some Affine space over a finite field.*

The fact is (not) generalized for the simple context as follows.

Fact 4.13. [13] *D simple.*

(1) *If D has the non-trivial modular geometry, then again it is a projective geometry over some division ring. More generally, if D is nontrivial 1-based, then $G(D)$ has a projective geometry over some division ring.*
(2) *(de Piro) The second fact of 4.12 in ω-categorical case is not generalized to the simple context. Namely there is a simple ω-categorical D whose geometry is non-modular and locally modular, but not equivalent to that of any Affine space over a finite field.*

Even if simple D is ω-categorical, $G(D)$ need not be definable. But it is essentially so when D is 1-based. This is the topic of next subsection.

4.2. **1-based case.**

Fact 4.14. [13] *D 1-based. If* $u \in G(D)$*, then* $u \in \mathrm{acl}(u_0u_1u_2)$ *for some* $u_0, u_1, u_2 \in D$*. More precisely if* (x, y) *is a fixed independent pair from D. Then* $u \in acl(x'y'z)$ *where* $x', y', z \in D$ *and* $\mathrm{tp}(xy) = \mathrm{tp}(x'y')$*.*

Corollary 4.15. *If D is ω-categorical and non-trivial 1-based, then D has a strongly minimal reduct, and a vector space over a finite field can definably be recovered.*

Proof of Fact 4.14. Suppose that $u \in \mathrm{dcl}(a_1...a_{n+1})$ $(a_i \in D)$ with the induction hypothesis for n. We will verify 4.14 for $n+1$. We can assume that $\{a_1, ..., a_{n+1}\}$ is independent. Now let $g = \mathrm{Cb}(ua_1/a_2...a_{n+1})$. Thus $SU(g) = 1$ and $g \in G(D)$. Since $a_1 \notin \mathrm{acl}(g) \subseteq \mathrm{acl}(a_2...a_{n+1})$, u is in the line generated by $\{g, a_1\}$ (*). Then, since $g \in \mathrm{acl}(a_2...a_{n+1})$, by the induction hypothesis, $g \in \mathrm{acl}(bcd)$ for some $c, d \in D$ such that $tp(cd) = tp(xy)$. If g is already in $\mathrm{acl}(cd)$, then from (*), $u \in \mathrm{acl}(ga_1) \subseteq \mathrm{acl}(cda_1)$. Thus, in this case, 4.14 holds.

Therefore we only need to consider the case when $g \notin \mathrm{acl}(cd)$. We can also clearly assume that $g \notin \mathrm{acl}(b)$ (otherwise we are done). Then, since $G(D)$ is modular, we can find $v \in \mathrm{acl}(cd) \cap \mathrm{acl}(bg) \cap G(D)$ such that $g \in \mathrm{acl}(bv)$ (†), and $\dim(vg) = 2$. Also, at least one of $\{v, c\}$ or $\{v, d\}$ (say $\{v, d\}$) is independent (**). Now, as $d, a_1 \in D$, $\mathrm{Ltp}(d) = \mathrm{Ltp}(a_1)$. Hence, we can find v' such that $\mathrm{Ltp}(vd) = \mathrm{Ltp}(v'a_1)$ ($\star$). In particular, $\mathrm{Ltp}(v) = \mathrm{Ltp}(v')$. Then by (*),(**),($\star$), we can amalgamate types $\mathrm{tp}(v'/a_1)$ and $\mathrm{tp}(v/g)$, so that we obtain $v'' \models \mathrm{tp}(v'/a_1) \cup \mathrm{tp}(v/g)$. Hence, there are b', c' such that $\mathrm{tp}(vgb) = \mathrm{tp}(v''gb')$, $\mathrm{tp}(vdc) = \mathrm{tp}(v''a_1c')$ (by ($\star$)), and so by (†), $\mathrm{tp}(c'a_1) = \mathrm{tp}(xy)$, $v'' \in \mathrm{acl}(c'a_1)$ and $g \in \mathrm{acl}(b'v'')$ (‡). Then by (*),(‡),

$$u \in \mathrm{acl}(ga_1) \subseteq \mathrm{acl}(ga_1v'') \subseteq \mathrm{acl}(a_1b'v'') \subseteq \mathrm{acl}(c'a_1b').$$

Since $\mathrm{tp}(c'a_1) = \mathrm{tp}(xy)$, the $(n+1)$th induction hypothesis for Fact 4.14 is deduced. □

We can impose triviality to the whole theory.

Definition 4.16. T is *trivial* if for a, b, c, A, whenever $\{a, b, c\}$ is pairwise independent over A, then $\{a, b, c\}$ is independent over A.

Fact 4.17. *Suppose that T is 1-based. Then T is trivial if and only if all SU-rank 1 types are trivial (i.e. having the trivial geometries).*

Corollary 4.18. *Let T be non-trivial, 1-based and ω-categorical. Then an infinite dimensional vector space over a finite field, in particular the infinite additive group, is definable.*

4.3. **ω-categorical supersimple case.** In this subsection we describe related important issues of the finite rank property and CM-triviality.

Fact 4.19. *Let T be supersimple and ω-categorical. Then T is 1-based if and only if T has finite SU-rank (i.e. every complete type has finite SU-rank) and all SU-rank 1 types are 1-based.*

As pointed out, Theorem 4.10 no longer holds for simple theories. But still the following question is sensible to ask and its answer is still unknown.

Fact 4.20. *(Cherlin, Harrington, Lachlan [11]) Assume T is ω-categorical superstable. Then it has finite SU-rank. Hence due to Zilber's theorem and above Fact 4.19, T is 1-based.*

Open Problem 1. If T is ω-categorical supersimple, then does it have finite SU-rank?

Moreover all the stable and simple counterexamples mentioned in 4.11 have the following property.

Definition 4.21. T is said to be *CM-trivial* if for acl-sets A, B, C, whenever $\mathrm{acl}(A\cup C)\cap \mathrm{acl}(A\cup B) = A$ then $\mathrm{Cb}(C/A) \subseteq \mathrm{acl}(\mathrm{Cb}(C/A\cup B))$.

Any 1-based theory is CM-trivial. As said, Hrushovski's example of non 1-based ω-categorical supersimple theory is CM-trivial. Hence naturally we question below.

Open Problem 2. If T is ω-categorical supersimple, then is it CM-trivial?

The solution of Problem 1 is positive, in the case of groups or if the answer of Problem 2 is yes. Recall that a group G is said to be P-by-Q if G has a normal subgroup N having the property P such that G/N has the property Q. As usual, G' denotes the commutator subgroup of G.

Theorem 4.22. *(Evans, Wagner [17])*

1. *If a group G (possibly with extra relations) over $\emptyset$ is ω-categorical supersimple, then it is (finite-by-Abelian)-by-finite (i.e. $(G^0)'$ is finite, where G^0 is the smallest $\emptyset$-definable subgroup of G having finite index, which must be normal (see an explanation after 7.1)), and it has finite SU-rank.*
2. *If T is ω-categorical supersimple and CM-trivial, then it has finite SU-rank.*

Now in the following section we deal with the question 4.5.2 in the more general (not necessarily ω-categorical) context.

5. Group configuration

We review some terminology for the type-definable groups in simple theories. Note that the notion of generic types plays central role. Roughly saying a type p in a group G (i.e. $p(x) \models G(x)$ which type-defines G) is said to be *generic* if p and G have the same ($D_{\varphi,k}$ or SU-)rank.

Definition 5.1. Let $(G, ., 1_G)$ be a group in T (so there may exist additional relations on G) type-defined say by $G(x)$, $.(x, y, z)$ over $\emptyset$ (after suppressing the domains). By $S_G(A)$, we mean the collection of complete types over A extending $G(x)$.

A type $p(x) \in S_G(A)$ is said to be *left-generic* of G over A if for any $g \models p$ and $a \in G$ with $g \mathop{\smile\!\!\!\!\!\!\mid}_A a$, we have $a.g \mathop{\smile\!\!\!\!\!\!\mid}_A a$. We say g is *left-generic* over A for G, if $\mathrm{tp}(g/A)$ is so. Right-generic types are defined similarly. A (partial) type $q(x)$ over A implying $G(x)$ is said to be *(left/right-)generic* if some generic g over A realizes $q(x)$. More generally when $(G, X, \centerdot)$ is a type-definable homogeneous space over $\emptyset$, then $x \in X$ is *generic* over A, if $a \centerdot x \mathop{\smile\!\!\!\!\!\!\mid}_A a$ holds for $a \in G$ with $a \mathop{\smile\!\!\!\!\!\!\mid}_A x$.

For a partial type $q(x) \models G(x)$ over A, and $g \in G$, we write $g.q$ to denote the left translation (of the solution set) of q by g, and it is easy to see that $g.q$ is a partial type over $A \cup \{g^{-1}\}$.

Fact 5.2. *We take the same notation in 5.1.*

1. *There is generic $p(x) \in S_G(A)$.*
2. *$g \in G$ is left-generic over A iff it is right-generic over A.*
3. *The following are equivalent.*
   - (a) *The type p is generic.*
   - (b) *For any $g \in G$, the type $g.p$ does not fork over $\emptyset$.*
   - (c) *For any $g \in G$, the type $g.p$ does not fork over A.*
   - (d) *For any $g \in G$, $g.p$ is generic.*

We now again fix a solution set D of a strong type over $\emptyset$ of SU-rank 1. In the previous section we have seen that if D is ω-categorical non-trivial superstable, then Zilber showed it is 1-based and a vector space over a finite field having the same geometry can definably be recovered. Later Hrushovski extends the result as follows.

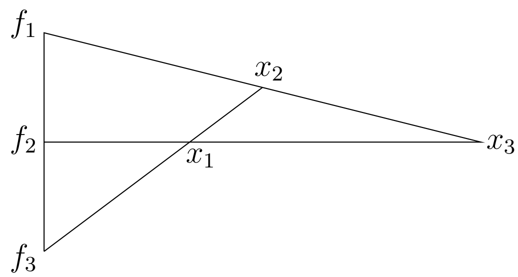
Fact 5.3. *If D is 1-based non-trivial stable, then there is a type-definable group $(G, +)$ such that $(V = G/G_0, +)$, where $G_0 = \mathrm{acl}(\mathrm{dom}(G)) \cap G$, forms a vector space over the division ring of the definable endomorphisms of V. Moreover there $\mathop{\smile\!\!\!\!\!\!\mid}$ is linear independence.*

In other words, Hrushovski gives positive answers to 4.5.2 for stable theories even in the non ω-categorical context. When he constructs the group, the group configuration theorem plays a pivotal role.

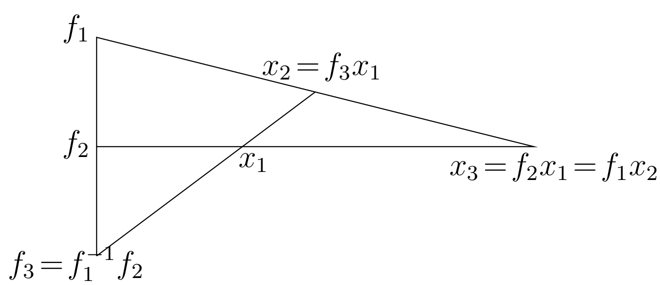
Definition 5.4. By *a group configuration* over e we mean a 6-tuple $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ with a tuple e such that, for $\{i, j, k\} = \{1, 2, 3\}$,

(1) $f_i \in \mathrm{acl}(f_j, f_k; e)$,
(2) $x_i \in \mathrm{acl}(f_j, x_k; e)$, and
(3) all other triples from C are independent over e.

If there is $C' = (f'_1, f'_2, f'_3, x'_1, x'_2, x'_3)$ such that $\mathrm{acl}(f_i e) = \mathrm{acl}(f'_i e), \mathrm{acl}(x_i e) = \mathrm{acl}(x'_i e)$, then C' is also a group configuration over e. In this case, we say C and C' are *equivalent* over e.



Definition 5.5. We say $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ is a *group configuration from a type-definable homogeneous space* (G, X) if (G, X) and the group action of G on X are all type-definable, and $f_i \in G$, $x_i \in X$ generic elements with $f_1^{-1} f_2 = f_3$, $x_2 = f_3 x_1$, $x_3 = f_2 x_1$.



Theorem 5.6. *(Hrushovski* [19]*) (The group configuration theorem for stable theories) Assume T stable. A group configuration $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ is given. Then a type-definable homogeneous space (G, X) over $e(\downarrow C)$ is constructed so that C is equivalent over e to some group configuration from (G, X).*

In the proof of above theorem, the stationarity of strong types is crucially used. As stationarity is no longer available, how to generalize the group configuration theorem to the simple theory context remains difficult problem to the researchers. Surprisingly there is a deep connection between this question and the so called notion of generalized amalgamation. We will discuss the issue in the following section.

6. Generalized amalgamation

We introduce the concept of generalized amalgamation. This will be used to achieve the group configuration theorem for simple theories, but the notion itself is very intriguing in its own right. *In this section, tuples and the number of variables are possibly infinite.*

Definition 6.1. Let $I = \mathcal{P}(n)^-(= \mathcal{P}(n) \setminus \{n\})$, ordered by inclusion. Let $(\{A_i\}_{i\in I}, \{\pi^i_j\}_{i\leq j\in I})$ be a directed family. Namely, each $\pi^i_j : A_i \to A_j$ is an elementary map; $\pi^i_i = \mathrm{id}_{A_i}$; and $\pi^j_k \circ \pi^i_j = \pi^i_k$ for $i \leq j \leq k \in I$. We say T has *n-amalgamation* if whenever

(1) $\{\pi^{\{i\}}_u(A_{\{i\}}) : i \in u\}$ is $\pi^{\emptyset}_u(A_\emptyset)$-independent,
(2) $A_u = \mathrm{acl}(\bigcup_{i\in u} \pi^{\{i\}}_u(A_{\{i\}}))$,

for any $u \in I$, then we can extend the direct family to the one indexed by $\mathcal{P}(n)$ (by finding A_n and π^j_n) so that (1) and (2) hold for n, too. We say T has *n-amalgamation* over A, if $A_\emptyset = \mathrm{acl}(A) = A$.

One can restate the definition in terms of functorial terminology. Recall that by a category $\mathcal{C} = (\mathrm{Ob}(\mathcal{C}), \mathrm{Mor}(\mathcal{C}))$, we mean a class $\mathrm{Ob}(\mathcal{C})$ of members called *objects* of the category; equipped with a class $\mathrm{Mor}(\mathcal{C}) = \{\mathrm{Mor}(a, b)|\ a, b \in \mathrm{Ob}(\mathcal{C})\}$ where $\mathrm{Mor}(a, b) = \mathrm{Mor}_{\mathcal{C}}(a, b)$ is the class of *morphisms* between objects a, b (we write $f : a \to b$ to denote $f \in \mathrm{Mor}(a, b)$); and composition maps

$$\circ : \mathrm{Mor}(a, b) \times \mathrm{Mor}(b, c) \to \mathrm{Mor}(a, c)$$

for each $a, b, c \in \mathrm{Ob}(\mathcal{C})$ such that

(1) (Associativity) if $f : a \to b$, $g : b \to c$ and $h : c \to d$ then $h \circ (g \circ f) = (h \circ g) \circ f$ holds, and
(2) (Identity) for each object c, there exists a morphism $1_c : c \to c$ called the identity morphism for c, such that for $f : a \to b$, we have $1_b \circ f = f = f \circ 1_a$.

Note that any ordered set $(P, \leq)$ is a category where objects are members of P, and $\mathrm{Mor}(a, b) = \{(a, b)\}$ if $a \leq b$; $= \emptyset$ otherwise.

Now we recall a functor F between two categories $\mathcal{C}, \mathcal{D}$. The functor F sends an object $c \in \mathrm{Ob}(\mathcal{C})$ to $F(c) \in \mathrm{Ob}(\mathcal{D})$; and a morphism $f \in \mathrm{Mor}_{\mathcal{C}}(a, b)$ to $F(f) \in \mathrm{Mor}_{\mathcal{D}}(F(a), F(b))$ in such a way that

(1) (Associativity) $F(g \circ f) = F(g) \circ F(f)$ for $f : a \to b$, $g : b \to c$;
(2) (Identity) $F(1_c) = 1_{F(c)}$.

Definition 6.2. Let $\mathcal{C}_T$ be the category of algebraically closed substructures with elementary embeddings.

(1) A functor $a : W(\subseteq \mathcal{P}(n)) \to \mathcal{C}_T$ is given, where W is closed under subsets. For $u \subseteq w \in W$, we write $a(w) = a_w$, and a'_u (within a_w) means $a((u,w))(a_u)$. Now a is said to be *independence preserving (i.p.)* if for $w \in W$,
   (a) $\{a'_{\{i\}} | i \in w\}$ is $a'_\emptyset$-independent (within a_w);
   (b) $a_w = \operatorname{acl}(\bigcup\{a'_{\{i\}} |\ i \in w\})$.

(2) We say T has *n-complete amalgamation* (n-CA) if any i.p. functor $a : W \to \mathcal{C}_T$ can be extended to i.p. $\hat{a} : \mathcal{P}(n) \to \mathcal{C}_T$.

Equivalently we can define in terms of types.

Definition 6.3. We say T has *n-complete amalgamation* (n-CA) over a set $B = \operatorname{acl}(B)$ if the following holds: Let W be a collection of subsets of $\mathcal{P}(n)$, closed under subsets. For each $w \in W$, a complete type $r_w(x_w)$ over B is given where x_w is possibly an infinite set of variables. Suppose that

(1) for $w \subseteq w'$, we have $x_w \subseteq x_{w'}$ and $r_w \subseteq r_{w'}$.

Moreover for any $a_w \models r_w$,

(2) $\{a_{\{i\}} | i \in w\}$ is B-independent,
(3) a_w is as a set $\operatorname{acl}(\bigcup_{i\in w} a_{\{i\}}B)$ (and the map $a_w \to x_w$ is a bijection).

Then there is a complete type $r_n(x_n)$ over B such that (1),(2),(3) hold for all $w \in W \cup \{n\}$. We say T has n-CA if it has n-complete amalgamation over any algebraically closed set.

It is not hard to see that T has n-CA over B iff T has m-amalgamation over B for all $m \leq n$. Moreover the usual type-amalgamation (2.6) is indeed equivalent to 3-amalgamation. Hence any simple theory has 3-amalgamation.

One may wonder why such a bit complicated amalgamation is taken as the generalized version of type-amalgamation. Recall that the usual type-amalgamation (or the independence theorem) is as follows: For $A_0 \mathop{\smile\!\!\!\!\!\!|}_A A_1$ with $A \subseteq A_0, A_1$, if $c_0 \equiv^L_A c_1$ and $c_i \mathop{\smile\!\!\!\!\!\!|}_A A_i$ ($i = 0, 1$), then there is $c \equiv^L_{A_i} c_i$ such that $c \mathop{\smile\!\!\!\!\!\!|}_A A_0A_1$. Note that we can think of A_0, A_1 (after naming A) as two vertices of a base edge of a triangle and c a top vertex, and $\operatorname{Ltp}(c_0/A_0), \operatorname{Ltp}(c_1/A_1)$ are the 2 (edge) types to be amalgamated. One would expect generalized amalgamation to be a natural generalization of this amalgamation(=3-amalgamation), using a tetrahedron and higher dimensional simplices instead of a triangle (*). For example the candidate for 4-amalgamation would be: Let

$\{A_0, A_1, A_2\}$ be independent over $A(\subseteq A_i = \mathrm{acl}(A_i))$. For $\{i,j,k\} = \{0,1,2\}$, let $c_i \equiv_{A_j} c_k$ and $c_i \mathop{\smile\!\!\!\!\!\mid}_A A_jA_k$. Then there is $c \equiv^L_{A_jA_k} c_i$ such that $c \mathop{\smile\!\!\!\!\!\mid}_A A_0A_1A_2$ (**).

However the above candidate for the definition of 4-amalgamation does not work, although it looks natural.

Example 6.4. Let M be the random graph in $\mathcal{L} = \{R\}$. Choose distinct $a_i, b_i, d_i \in M$ and imaginary elements $c_i = \{b_i, d_i\}$ $(i = 0,1,2)$. We can additionally assume that $R(b_2, a_0) \wedge R(d_2, a_1) \wedge \neg R(b_2, a_1) \wedge \neg R(d_2, a_0)$ and $\mathrm{tp}(b_2d_2; a_0a_1) = \mathrm{tp}(b_1d_1; a_0a_2) = \mathrm{tp}(b_0d_0; a_1a_2)$. Now it follows that $\mathrm{Ltp}(c_1/a_0) = \mathrm{Ltp}(c_2/a_0), \mathrm{Ltp}(c_0/a_1) = \mathrm{Ltp}(c_2/a_1)$ and $\mathrm{Ltp}(c_0/a_2) = \mathrm{Ltp}(c_1/a_2)$. However $\mathrm{Ltp}(c_0/a_1a_2)$, $\mathrm{Ltp}(c_1/a_0a_2)$, $\mathrm{Ltp}(c_2/a_0a_1)$ have no common realization.

In above example, $\{c_0, c_1, c_2\}$ can be considered as a base triangle, and $\mathrm{Lstp}(d_0/c_0c_1)$, $\mathrm{Lstp}(d_1/c_1c_2)$, $\mathrm{Lstp}(d_2/c_2c_0)$ form other 3 triangles attached to the base triangle. The example shows that even if the edges of the 3 triangles are compatible over the base vertices, there is no common vertex joining the 3 triangles. On the other hand, due to the nature of the random graph if we only work in the home-sort, then any desired 3 types attached on a base triangle with compatible edges will be realized. As we want the notion of generalized amalgamation to be preserved in interpreted theories, we can not take above (**) as 4-amalgamation. Moreover when we use inductive arguments for example, we often have to consider not only the algebraic closures of vertices of amalgamated types but also those of higher dimensional surfaces as well, since after naming parameters the surface dimension is increasing.

Conclusively, the accurate definition for 4-amalgamation is then the one defined in 6.1-3. But note that indeed the essential philosophy of n-amalgamation mentioned in (*) is the same. We simply need extra care not only to compatible surfaces but to the (enumerations of) algebraic closures of all the lower dimensional surfaces.

The correct 4-amalgamation can be written in this manner (***): Let $\{A_0, A_1, A_2\}$ be independent over $A(\subseteq A_i = \mathrm{acl}(A_i))$. For $\{i,j,k\} = \{0,1,2\}$, let $\overrightarrow{\mathrm{acl}(c_iA_j)} \equiv_{A_j} \overrightarrow{\mathrm{acl}(c_kA_j)}$ and $c_i \mathop{\smile\!\!\!\!\!\mid}_A A_jA_k$, where $\overrightarrow{\mathrm{acl}(c_iA_j)}$ is a subsequence of a fixed sequence $\overrightarrow{\mathrm{acl}(c_iA_jA_k)}$.
Then there are c and an enumeration $\overrightarrow{\mathrm{acl}(cA_0A_1A_2)}$ such that

$$\overrightarrow{\mathrm{acl}(cA_jA_k)} \equiv_{\overrightarrow{\mathrm{acl}(A_jA_k)}} \overrightarrow{\mathrm{acl}(c_iA_jA_k)}$$

and $c \mathop{\smile\!\!\!\!\!\mid}_A A_0A_1A_2$.

6.1. **Group configuration for simple theories.** We are now ready to speak about the group configuration theorem for simple theories. Without generalized amalgamation, I. Ben-Yaacov, I. Tomasic, and F. O. Wagner obtained the group configuration theorem as follows.

Theorem 6.5. *(Ben-Yaacov, Tomasic, Wagner* [2][53]*) T simple. A group configuration* $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ *is given. Then it is equivalent to some group configuration from an* invariant *homogeneous space* (G, X) *over* $e(\smile C)$.

The proof of the theorem is very ingenious and deep. The group and the homogeneous space they produce are *not* type-definable but just automorphism invariant, hence do not completely fit into the first order context. However in the ω-categorical context, the group turns out to be definable, and using this they resolved the conjecture positively that if ω-categorical D is k-linear (4.6), then it is linear [53]. This is known to be true for stable D without the assumption of ω-categoricity, and it is still unknown whether the same holds for an arbitrary simple D.

Later the author together with T. dePiro and J. Millar obtained the type-definable group configuration theorem under the generalized amalgamation property.

Theorem 6.6. *(de Piro, Kim, Millar* [14]*) Assume T simple having 4-amalgamation. A group configuration* $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ *is given. Then there is a type-definable group* G *over* $e(\smile C)$ *with generics* g_i *such that* $\mathrm{acl}(f_i e) = \mathrm{acl}(g_i e)$.

Theorem 6.7. [33] *(The group configuration theorem for simple theories) Assume T simple having 4-amalgamation. A group configuration* $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ *is given. Then it is equivalent to some group configuration from a type-definable homogeneous space* (G, X) *over* $e(\smile C)$.

In section 7, as promised we will speak about the application of theorems 6.6,7 to answer the question (2) of 4.5 similarly to the stable case 5.3. Before getting into it, we speak further issues on generalized amalgamation in the next subsections.

6.2. **Generalized imaginaries.** In this subsection we restrict our attention mainly to stable theories. Unfortunately, even in stable theories, n-amalgamation need not hold over an algebraically closed set, though it always holds over a model (thus 6.6,7 can cover all stable theories if we assume 4-amalgamation over models). There is an example below.

Example 6.8. Consider $[A]^2 = \{\{a,b\} | a \neq b \in A\}$ where A infinite. Let $B = [A]^2 \times \{0,1\}$ where $\{0,1\} = \mathbb{Z}/2\mathbb{Z}$. Also let $E \subseteq A \times [A]^2$ be a membership relation, and let P be a subset of B^3 such that $((w_1,\delta_1)(w_2,\delta_2)(w_3,\delta_3)) \in P$ iff there are distinct $a_1, a_2, a_3 \in A$ such that for $\{i,j,k\} = \{1,2,3\}$, $w_i = \{a_j, a_k\}$, and $\delta_1 + \delta_2 + \delta_3 = 0$.

Let $M = (A, [A]^2, B;\ E, P;\ \mathrm{Pr}_1 : B \to [A]^2)$. Then since M is a reduct of $(A, \mathbb{Z}/2\mathbb{Z})^{\mathrm{eq}}$, M is stable. We work in M^{eq} and argue that the example does not have 4-amalgamation. Note first that $\mathrm{dcl}(\emptyset) = \mathrm{acl}(\emptyset)$, and for $a \in A$, $\mathrm{dcl}(a) = \mathrm{acl}(a)$. Now choose distinct $a_1, a_2, a_3, a_4 \in A$. For $\{i,j,k\} \subseteq \{1,2,3,4\}$, fix an enumeration $\overline{a_i a_j} = (b_{ij}, ...)$ of $\mathrm{acl}(a_i a_j)$ where $b_{ij} = (\{a_i, a_j\}, \delta) \in B = [A]^2 \times \{0,1\}$. Let $r_{ij}(x_{ij}) = \mathrm{tp}(\overline{a_i a_j})$, and let x^1_{ij} be the variable for b_{ij}. Note that $b_{ij} = (\{a_i, a_j\}, \delta)$ and $b'_{ij} = (\{a_i, a_j\}, \delta + 1)$ have the same type over $a_i a_j$. Hence there is $(\overline{a_i a_j})' = (b'_{ij}, ...)$ also realizing $r_{ij}(x_{ij})$. Therefore we have complete types $r_{ijk}(x_{ijk})$, $r'_{ijk}(x'_{ijk})$ both extending $r_{ij}(x_{ij}) \cup r_{ik}(x_{ik}) \cup r_{jk}(x_{jk})$ realized by some enumerations of $\mathrm{acl}(a_i a_j a_k)$ such that $P(x^1_{ij} x^1_{ik} x^1_{jk}) \in r_{ijk}$ whereas $\neg P(x^1_{ij} x^1_{ik} x^1_{jk}) \in r'_{ijk}$. Then it is easy to see that $r_{123} \cup r_{124} \cup r_{134} \cup r'_{234}$ is inconsistent.

In his recent preprint [25], Hrushovski remedies the deficiency of generalized amalgamation for stable theories by introducing the so called generalized imaginaries. More precisely, if $\mathcal{M} \models T$ is stable, then he is able to construct $(\mathcal{M} \subseteq)\mathcal{M}^* \models T^*$ in some $\mathcal{L}^*(\supseteq \mathcal{L})$ such that $\mathcal{M}$ is *stably embedded* into $\mathcal{M}^*$ (i.e. every subset of $\mathcal{M}$ definable in $\mathcal{M}^*$ is already definable in $\mathcal{M}$ within $\mathcal{L}(\mathcal{M})$), and $\mathcal{M}^*$ has n-amalgamation for all n over *any* algebraically closed sets. As the new elements in $\mathcal{M}^*$ can be called *generalized imaginaries*, we may write $\mathcal{M}^*$ as $\mathcal{M}^{\mathrm{geq}}$. In short, like $\mathcal{M}^{\mathrm{eq}}$, without loss of generality, we can freely work in $\mathcal{M}^{\mathrm{geq}}$ and assume to have n-amalgamation when T is stable.

Open Problem 3. Can we construct such $\mathcal{M}^*$ for simple T? If yes, then possibly we can remove the assumption of 4-amalgamation in the group configuration theorem.

6.3. **n-simplicity.** Notice that it is very cumbersome working with terminology for the full definition of n-amalgamation. On the other hand, the usage of the natural simpler notations as pointed out in (**) before 6.4 (not bothered by the enumerations of all the lower dimensional surfaces), although not 100% correct ones, might be more transparent to conceptualize.

Hence in the following definitions of n-simplicity and $K(n)$-simplicity, for convenience, we use imprecise notations good enough however representing the essence of concepts. (The reader will be able to fully recover the correct definitions as the same manners in (***) below Example 6.4.) Also for simplicity, we describe the notions only over $\emptyset = \mathrm{acl}(\emptyset)$.

Fact 6.9. *Let $I = \langle a_i |\ i \in \omega \rangle$ be Morley, and let $b \smile\!\!\!\!| \, a_0$. Then there is $b' \equiv_{a_0} b$ such that I is Morley over b' and $b' \smile\!\!\!\!| \, I$.*

Above fact is a particular case of type amalgamation(=the independence theorem=3-amalgamation), and used crucially in showing it for simple T. Namely, showing particular case of 3-amalgamation is the way of fully obtaining 3-amalgamation. One may ask whether the same holds for n-amalgamation. In this spirit, Kolesnikov [40] introduces the notion of *n-simplicity* by generalizing above Fact 6.9.

Definition 6.10. (1) T is $K(n)$-*simple* if for $k \leq n$ and any Morley sequence $I = \langle a_i |\ i \in \omega \rangle$, whenever $I_k = \langle a_i | i < k \rangle$ is Morley over b with $b \smile\!\!\!\!| \, I_k$, there is $b' \equiv_{I_k} b$ such that I is Morley over b' and $b' \smile\!\!\!\!| \, I$.
(2) T is *n-simple* if for $k \leq n$ and any Morley $I = \langle a_i |\ i \leq k \rangle$, whenever $I_k = \langle a_i | i < k \rangle$ is Morley over b with $b \smile\!\!\!\!| \, I_k$, there is $b' \equiv_{I_k} b$ such that I is Morley over b' and $b' \smile\!\!\!\!| \, I$.
(3) Recall that T has n-CA if T has k-amalgamation for all $k \leq n$.

As in simplicity(=1-simplicity), both n-simplicity, $K(n)$-simplicity are particular cases of $(n+2)$-CA. It then is natural to imagine that $K(n)$-simplicity, n-simplicity, and $(n+2)$-CA might be all equivalent. Type amalgamation already says that is the case when $n = 1$. Indeed Kolesnikov show in his thesis and [40] that is the case for $n = 2$ as well and leave the case $n > 2$ as questions. He also supplies examples of simple structures having n-CA but not having $(n+1)$-CA for each $n \geq 3$.

However in the author's joint paper with Kolesnikov and Tsuboi [37], it is revealed that the harmony breaks up when $n \geq 3$. Namely, they first show that the equivalence of n-simplicity and $(n+2)$-CA for all n. But they also build, for each $n \geq 3$, an example of $K(n)$-simple but not having $(n+2)$-CA. (Indeed this is the reason why 6.10.1 is called $K(n)$-*simple* rather than *n-simple*, albeit it is more canonical generalization of 6.9): Let $\mathcal{L} = \{R\}$ where R is an n-ary relation. Let

$$\begin{aligned}\mathcal{K} := \{A| \quad & A \text{ is a finite } R\text{-structure; } R \text{ is symmetric and irreflexive;} \\ & \text{for any } A_0 \subseteq A \text{ with } |A_0| = n+1, \text{ the no. of} \\ & n\text{-element subsets of } A_0 \text{ satisfying } R \text{ is even}\}\end{aligned}$$

Then $\mathcal{K}$ has the Fraïssé limit, and that is the simple ω-categorical counterexample.

7. 1-based groups

As promised in this section we see the analogous result of 5.3 for the simple case as the answer for 4.5.2. This is obtained by an application of 6.6 or 6.7.

Theorem 7.1. *T 1-based nontrivial having 4-amalgamation. Then there is a type-definable group $(G, +)$ say over $\emptyset = \mathrm{dom}(G)$ such that $(V = G/G_0, +)$, where $G_0 = G \cap \mathrm{acl}(\emptyset)$, forms a vector space over a division ring of the type-definable endomorphisms of V, and $\downarrow$ is linear independence there.*

For the rest of this section, we sketch to explain how to obtain the vector space from the non-trivial modular structure. Note that by 4.17, there exists a non-trivial SU-rank 1 strong type p. For convenience, name $\mathrm{dom}(p)$ and assume $p \in S(\emptyset)$. Since p is non-trivial, there exists $\{a, b, c\}$ realizing p such that b, c is independent and $a \in \mathrm{acl}(b, c) \setminus (\mathrm{acl}(b) \cup \mathrm{acl}(c))$. Choose yx realizing $\mathrm{tp}(ab/c)$ with $yx \downarrow_c ab$. Then $\dim(ay/bx) = 1$ as $y \in \mathrm{acl}(abx)$ and $a \downarrow bx$. Let $z = \mathrm{Cb}(\mathrm{Ltp}(ay/bx))$, then by 1-basedness, $z \in \mathrm{acl}(ay) \cap \mathrm{acl}(bx)$. Moreover, by a straightforward rank calculation, we have $SU(z) = 1$. This gives a group configuration $C = (a, b, c, x, y, z)$. Now by Theorem 6.6, we obtain a type-definable group G over $e(\downarrow C)$ such that the generic types have SU-rank 1. The group G is 1-based since the underlying theory is 1-based.

To proceed, we now need to study type-definable (1-based) groups in simple theories. Hence fix the group type-definable group $G = G(x)$ over $\emptyset$ (after suppressing e). Note that for a set A, by intersecting all the type-definable subgroups over A having bounded index, we get the smallest A-type-definable group G^0_A, called the connected component of G over A, having bounded index in G (i.e. $[G : G^0_A] = \lambda < \bar{\kappa}$ no matter how large saturated model $\mathcal{M}$ we take). It is not hard to see that G^0_A is a normal subgroup. It is well-known that in the case of stable theories, $G^0_A = G^0_\emptyset$ for any A. But for the simple case, it need not be true. For example, if G is a vector space over a finite field equipped with a bilinear map such as an inner product, then G^0_A is a subspace orthogonal to all the vectors in A.

Now reset G by $G^0 = G^0_\emptyset$. Hence G *has no proper $\emptyset$-type-definable subgroup of bounded index* $(\star)$. *For the rest of this section, G is 1-based.* We recall some more notation.

- For $g, h \in G$, we write $h^g := g^{-1}.h.g$.

- The *commutator* of $g, h \in G$ is $[g,h] = g^{-1}.h^{-1}.g.h$. For $A, B \subseteq G$, $[A,B] := \{[g,h] |\ g \in A,\ h \in B\}$. Recall that the commutator subgroup G' is the subgroup generated by $[G,G]$.
- The *center* of G is $Z(G) = \{g \in G|\ g.h = h.g \text{ for all } h \in G\}$.
- The *centralizer* of $g \in G$ is $C_G(g) = \{h \in G | g.h = h.g\}$.
- We say two type-definable groups H_0, H_1 are *commensurate* if the index $[H_i : H_0 \cap H_1]$ is bounded for each $i = 0, 1$. That two groups are commensurate is an equivalence relation.

Results in this subsection are due to F. O. Wagner [55].

Fact 7.2. *(G 1-based) Any type-definable subgroup of G^n is commensurate with one over* $\mathrm{acl}(\emptyset)$.

Theorem 7.3. (1) *G' is bounded. Indeed, $G' \leq Z(G)$.*
(2) *Hence G is bounded-by-Abelian, and the group $G_0 = G \cap \mathrm{acl}(\emptyset) (\geq G')$ is normal.*

Proof. We sketch the proof. (2) is automatically deduced from (1). So we will show (1). Note that $Z(G)$ need not be type-definable where as its approximation $\tilde{Z} = \tilde{Z}(G) := \{g \in G|\ [G : C_G(g)] \text{ is bounded}\}$ is. It follows $\tilde{Z}$ is a normal subgroup of G.

Claim 1. $\tilde{Z}' \subseteq \mathrm{acl}(\emptyset)$, and $\tilde{Z}' \leq Z(G)$: For $g \in \tilde{Z}$, $|[g,G]| = |[G,g]| = [G : C_G(g)]$ is bounded, hence $[g,G] \subseteq \mathrm{acl}(g)$. Then for $h, g \in \tilde{Z}$ with $h \mathop{\smile\!\!\!\!\!\!\mid}\, g$, we have $[h,g] \in \mathrm{acl}(g) \cap \mathrm{acl}(h) = \mathrm{acl}(\emptyset)$. For arbitrary $h, g \in \tilde{Z}$, there are $h_i (\mathop{\smile\!\!\!\!\!\!\mid}\, g)$ such that $h = h_1.h_2$, and the repeated applications yield $[h,g] \in \mathrm{acl}(\emptyset)$ as well. Hence $\tilde{Z}' \subseteq \mathrm{acl}(\emptyset)$. Moreover since $\tilde{Z}$ is normal in G, so is $\tilde{Z}'$. Hence for $h \in \tilde{Z}'$, we have $[G,h] \subseteq \tilde{Z}'$. Therefore $[G : C_G(h)] = |[G,h]|$ is bounded and due to the assumption ($\star$) above, $C_G(h) = G$. Thus $\tilde{Z}' \subseteq Z(G)$ and Claim 1 is proved.

It suffice to show the following claim.

Claim 2. $\tilde{Z}$ has bounded index in G. Hence $\tilde{Z} = G$: Note that for $h, g, g' \in G$, we have

$$
\begin{array}{lll}
 & h^g = h^{g'} & \text{iff} \quad hg'g^{-1} = g'g^{-1}h \\
\text{iff} & h \in C_G(g'g^{-1}) & \text{iff} \quad (h, h^g) = (h, h^{g'}) \in H_g \cap H_{g'};
\end{array}
$$

where $H_g = \{(h, h^g)|\ h \in G\}$ is a type-definable subgroup of $G \times G$. It follows H_g and $H_{g'}$ are commensurate iff $\{h \in G|\ h^g = h^{g'}\} = C_G(g'.g^{-1})$ has bounded index in G iff $g'.g^{-1} \in \tilde{Z}(G)$ iff g', g are in the same right coset of $Z(G)$.

Now due to 7.2, there are only boundedly many commensurate equivalence classes of subgroups of G^2. Hence $Z(G)$ must have bounded index. Therefore Claim 2 follows. □

We have shown that G/G_0 is Abelian. *We now use the fact that* $SU(g) = 1$ *for generic* $g \in G \quad (\star\star)$.

Definition 7.4. By an *endogeny* of G/G_0, we mean an endomorphism f of G/G_0 such that the graph of f is *type-definably induced* over $\operatorname{acl}(\emptyset)$, i.e. there is a type-definable subset S_f over $\operatorname{bdd}(\emptyset)$ of $G \times G$ such that

$$\{(a + G_0, b + G_0)|\ (a, b) \in S_f\}$$

forms the graph of f.

Let $\operatorname{End}(G) := \{f|\ f \text{ is an endogeny of } G/G_0\}$. Using $(\star\star)$, it is not hard to check $\operatorname{End}(G)$ with addition and composition forms a division ring.

Theorem 7.5. *For* $a_0, ..., a_n \in G \smallsetminus G_0$*, the following are equivalent.*

(1) $\{a_0, ..., a_n\}$ *is (forking) dependent.*
(2) $a_i \in \operatorname{acl}(\{a_0, ..., a_n\} \smallsetminus \{a_i\})$ *for some* $i \leq n$.
(3) *There are* $f_0, ..., f_n \in \operatorname{End}(G)$ *not all zero such that* $f_0(a_0 + G_0) + ... + f_n(a_n + G_0) = 0$ *in* G/G_0.

Hence $(V, +)$ *with* $V := G/G_0$ *forms a vector space over the division ring* $\operatorname{End}(G)$*, in which* $\downarrow$ = *linear independence.*

To prove this we need the important fact below.

Fact 7.6.
- *Any strong type in* G^n *is generic for some coset of a type-definable subgroup of* G^n.

The analogous result for the stable case is:

- *(Hrushovski, Pillay* [26]*) (T stable.) Any type-definable subset of* G^n *is a finite Boolean combination of cosets of type-definable subgroups of* G^n.

Proof sketch of 7.5. (3)⇒(2)⇒(1) Clear.

(1)⇒(3). By induction we can suppose that any proper subtuple of $\bar{a} = (a_0, ..., a_n)$ is independent. Due to Fact 7.6, there is $H(\leq G^{n+1})$ a type-definable subgroup over $\operatorname{acl}(\emptyset)$ such that $\operatorname{tp}(a_0, ..., a_n/\operatorname{acl}(\emptyset))$ is generic for some $H + \bar{h}$. For $i < n$, let

$$F_i := \{(x_i, x_n) \in G \times G|\ (0, ..., 0, x_i, 0, ..., 0, x_n) \in H\}.$$

Then one can show

$$f_i := \{(a + G_0, b + G_0)|\ (a, b) \in F_i\}$$

is the graph of a nonzero endogeny of G/G_0, and

$$f_0(a_0 + G_0) + ... + f_{n-1}(a_{n-1} + G_0) = -a_n + G_0. \qquad \square$$

8. Fields having simple theories

As said in section 4 regarding to the question 4.5.3 for stable theories, although there are counterexamples to Zilber's Principle in its face value, Hrushovski and Zilber show that under the so called 'Zariski constraint' if a strongly minimal structure is not modular, then an algebraically closed field is interpretable. Strongly minimal sets in many important algebraic structures such as DCF, ACFA (1.8) are Zariski, so there Zilber's Principle hold. Lots of spectacular applications have been made to algebraic geometry and number theory [21][23][48][49][50] using this trichotomy principle. Arguably the solution to Zilber's Principle in the Zariski context and its applications to core mathematics is the most culminant events in the history of model theory.

Almost nothing has been done in respect to the generalization of Hrushovski and Zilber trichotomy to the simple theory context, i.e. the study for the question (3) of 4.5 in unstable simple theories is not cultivated yet. It is even not known yet what kinds of fields should be supersimple. We are only a little soothed by the achievement of the group configuration theorem for simple theories which, in the stable case, plays a pivotal role. In this last section, hence instead we summarize so far known results on fields having simple theories. In particular when a field is PAC, the suitable classification is made in the context of subclasses of simple theories.

Definition 8.1. Let F be a field.

(1) F is said to be *pseudo-algebraically closed* (PAC) if any absolutely irreducible algebraic set V defined over F has F-rational point.
(2) F is *pseudo-finite* if it is an infinite field of the theory of all finite fields.
(3) An extension field E of F is *separable* if it is algebraic over F, and for each $e \in E$, the irreducible polynomial of e over F has no multiple roots.
(4) F *perfect* if every algebraic extension is separable; equivalently either the characteristic of F is 0, or $x \mapsto x^p$ is onto where p is the characteristic of F.
(5) F is *bounded* for each $n > 1$, in its algebraic closure, there are only finitely many separably algebraic extensions of degree n.
(6) F is *separably closed* if it has no proper separable extension.

Fact 8.2. *Assume F is PAC.*

(1) *(Chatzidakis [7]) F is simple iff F is bounded.*

(2) *(Hrshovski* [24]*; Pillay, Poizat* [46]*) F is supersimple iff F is perfect and bounded.*
(3) *(Duret* [16]*, Wood* [57]*) F is stable iff F is separably closed.*

Historically the first simple unstable fields studied seriously are pseudo-finite fields by J. Ax [1] in late 60s.

Fact 8.3. *A field F is pseudo-finite if and only if it is perfect, PAC, and for each n it has exactly one algebraic extension of degree n.*

Hence pseuo-finite fields are unstable supersimple (of SU-rank 1).

Fact 8.4. (1) *(Macintyre* [43]*; Cherlin, Shelah* [12]*) Superstable division ring is an algebraically closed field.*
(2) *(Pillay, Scanlon, Wagner* [47]*) Supersimple division ring is a field.*

Open Problem 4. (1) Is any stable field separably closed?
(2) Is any supersimple field PAC?

References

[1] J. Ax, 'The elementary theory of finite fields', *Annals of Math.* 88 (1968) 239-271.

[2] I. Ben-Yaacov, I. Tomasic and F. Wagner, 'Constructing an almost hyperdefinable group', *Journal of Math. Logic* 4 (2004) 181-212.

[3] S. Buechler, 'Lascar strong types in some simple theories', *J. of Symbolic Logic* 64 (1999) 817-824.

[4] S. Buechler, A. Pillay and F. O. Wagner 'Supersimple theories', *J of AMS* 14 (2000) 109-124.

[5] E. Casanovas and B. Kim, 'A supersimple nonlow theory', *Notre Dame Journal of Formal Logic* 39 (1998) 507-518.

[6] E. Casanovas and F. O. Wagner, 'Local supersimplicity and related concepts', *Journal of Symbolic Logic* 67 (2002) 744-758.

[7] Z. Chatzidakis, 'Simplicity and Independence for Pseudo-algebraically closed fields', in *Models and Computability*, London Math. Soc. Lecture Notes Series 259, Cambridge University Press, Cambridge (1999) 41-61.

[8] Z. Chatzidakis and E. Hrushovski, 'Model theory of difference fields', *Transactions of AMS* 351 (1999) 2997-3071.

[9] Z. Chatzidakis, A. Macintyre and L. van den Dries, 'Definable sets in finite fields', *J. reine angew. Math.* 427 (1992) 107-135.

[10] Z. Chatzidakis and A. Pillay, 'Generic structures and simple theories', *Ann. Pure and Applied Logic* 95 (1998) 71-92.

[11] G. Cherlin, L. Harrington, and A. H. Lachlan, '$\aleph_0$-categorical, $\aleph_0$-stable structures', *Ann. Pure and Applied Logic* 28 (1985) 103-135.

[12] G. Cherlin and S. Shelah, 'Superstable fields and groups', *Ann. of Math. Logic* 18 (1980) 227-270.

[13] T. de Piro and B. Kim, 'The geometry of 1-based minimal types', *Transactions of American Math. Soc.* 355 (2003) 4241-4263.

[14] T. de Piro, B. Kim and J. Millar 'Constructiong the hyperdefinable group from the group configuration', *Journal of Math. Logic* 6 (2006) 121-139.

[15] J. Doyen and X. Hubaut, 'Finite regular locally projective spaces', *Math. Z.* 119 (1971) 83-88 .

[16] J. L. Duret, 'Les corps faiblement algébriquement clos non séparablement clos ont la propriété d'indépendance', in *Modle theory of algebra and arithmetic*, Springer Lecture Note 834 (1980) 135-157.

[17] D. Evans and F. O. Wagner, 'Supersimple ω-caterorical groups and theories', *J. of Symbolic Logic* 65 (2000) 767-776.

[18] B. Hart, B. Kim, and A. Pillay, 'Coordinatization and canonical bases in simple theories', *J. of Symbolic Logic* 65 (2000) 293-309.

[19] E. Hrushovski, 'Unimodular minimal theories', *Journal of London Math. Soc.* 46 (1992) 385-396.

[20] E. Hrushovski, 'A new strongly minimal set', *Ann. Pure and Applied Logic* 62 (1993) 147-166.

[21] E. Hrushovski, 'The Mordell-Lang conjecture for function fields', *J. of AMS* 9 (1996) 667-690.

[22] E. Hrushovski, 'Simplicity and the Lascar group', preprint (1998).

[23] E. Hrushovski, 'Difference fields and Manin-Mumford conjecture', *Ann. of Pure and Applied Logic* 112 (2001) 43 -115.

[24] E. Hrushovski, 'Pseudo-finite fields and related structures', *Quaderni di Matematica* 11 (2002) 151-212.

[25] E. Hrushovski, 'Groupoids, imaginaries and internal covers', preprint (2006).

[26] E. Hrushovski and A. Pillay, 'Weakly normal groups', in *Logic Colloquium 85*, North Holland, Amsterdam (1987) 233-244.

[27] E. Hrushovski and A. Pillay, 'Groups definable in local fields and pseudo-finite fields', *Israel J. Math.* 85 (1994) 203-262.

[28] E. Hrushovski and B. Zilber, 'Zariski geometries', *Journal of AMS* 9 (1996) 1-56.

[29] W. M. Kantor, M. W. Liebeck, and H. D. Macpherson, '$\aleph_0$-categorical structures smoothly approximated by finite structures', *Proc. London Math. Soc.* (3) 59 (1989) 439-463.

[30] B. Kim, *Simple first order theories*, Ph. D. Thesis, Univ. of Notre Dame, 1996.

[31] B. Kim, 'Forking in simple unstable theories', *J. of London Math. Soc.* 57 (1998) 257-267.

[32] B. Kim, 'A note on Lascar strong types in simple theories', *J. of Symbolic Logic* 63 (1998) 926-936.

[33] B. Kim, 'On the number of countable models of a countable supersimple theory', *J. of London Math. Soc.* (2) 60 (1999) 641-645.

[34] B. Kim, 'Simplicity, and stability in there', *Journal of Symbolic Logic* 66 (2001) 822-836.

[35] B. Kim 'Recovering the hyperdefinable group action in the group configuration theorem', will appear in *J. of Symbolic Logic*.

[36] B. Kim, *Simplicity theory*, will be published by Oxford Univ. Press.

[37] B. Kim, A. Kolesnikov and A. Tsuboi 'Generalized amalgamation and n-simplicity,' *Annals of Pure and Applied Logic* 155 (2008) 97-114.

[38] B. Kim and A. Pillay, 'Simple theories', *Ann. Pure and Applied Logic* 88 (1997) 149-164.

[39] B. Kim and A. Pillay, 'From stability to simplicity', *Bulletin of Symbolic Logic* 4 (1998) 17-36.

[40] A. Kolesnikov, 'n-simple theories', *Ann. Pure and Applied Logic* 131 (2005) 227-261.

[41] A. H. Lachlan, 'On the number of countable models of a countable superstable theory', in *Logic, Methodology, Philosophy of Science IV*, North Holland, Amsterdam (1973) 45-56.

[42] D. Lascar, 'On the category of models of a complete theory', *J. of Symbolic Logic* 47 (1982) 249-266.

[43] A. Macintyre, 'On ω_1-categorical theories of fields' *Fundamenta Mathematicae* 71 (1971) 1-25.

[44] Y. Peterzil and S. Starchenko, 'A trichotomy theorem for o-minimal structures', *Proceedings of London Math. Soc.* (3) 77 (1998) 481-523.

[45] A. Pillay, *Geometric stability theory*, Oxford University Press, Oxford (1996).

[46] A. Pillay and B. Poizat, 'Corps et chirurgie' *J of Symbolic Logic* 60 (1995) 528-533.

[47] A. Pillay, T. Scanlon and F. O. Wagner, 'Supersimple fields and division rings', *Mathematical Research Letters* 5 (1998) 473-483.

[48] T. Scanlon, 'The *abc*-theorem for commutative algebraic groups in characteristic p', *International Math. Research Notices* 18 (1997) 881-898.

[49] T. Scanlon, 'p-adic distance from torsion points of semi-abelian varieties', *J. reine angew. Math.* 499 (1998) 225-236.

[50] T. Scanlon, 'Local André-Oort conjecture for the universal abelian variety', *Inventiones mathematicae* 163 (2006) 191-211.

[51] S. Shelah, 'Simple unstable theories', *Ann. Math. Logic* 19 (1980) 177-203.

[52] S. Shelah, *Classification theory*, 2nd edn, North-Holland, Amsterdam (1990).

[53] I. Tomasic and F. O. Wagner, 'Applications of the group configuration theorem in simple theories', *Journal of Math. Logic* 3 (2003) 239-255.

[54] E. Vassiliev, 'Generic pairs of *SU*-rank 1 structures', *Ann. Pure and Applied Logic* 120 (2003) 103-149.

[55] F. O. Wagner, 'Groups in simple theories', in *Logic Colloquium 2001*, Lecture Notes in Logic 20, A K Peters, Wellesley, USA (2005) 440-467.

[56] F. O. Wagner, *Simple theories*, Kluwer Academic Publishers, Dordrecht (2000).

[57] C. Wood, 'Notes on the stability of separably closed fields', *J. of Symbolic Logic* 44 (1979) 412-416.

[58] B. I. Zilber, *Uncountably categorical theories*, AMS Translations of Mathematical Monographs, Vol. 117.

Department of Mathematics, Yonsei University, 134 Shinchon-dong, Seodaemun-gu, Seoul 120-749, South Korea

E-mail address: bkim@yonsei.ac.kr