Geometric Simplicity Theory

Byunghan Kim

Dept. Math. Yonsei University

ALC 10
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Outline

1. n-amalgamation
2. $\mathcal{M}^{\text{geq}}$
3. n-simplicity
4. Fields in simple theories

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Let $\mathcal{C}_T$ be a category of the algebraically closed substructures of $\mathcal{M}$. Recall that any poset is a category.
For $n \in \omega$, write $\mathcal{P}(n)^- := \mathcal{P}(n) - \{n\}$.

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Definition

- A functor $a : W(\subseteq \mathcal{P}(n)) \to \mathcal{C}_T$ is said to be *independence preserving (i.p.)* if
  1. for any $w_0, w_1 \subseteq w \in W$, $a_{w_0} \mathop{\smile\!\!\!\!\!\!\!\;\,|}_{a_{w_0 \cap w_1}} a_{w_1}$ holds within a_w;
  2. for $w \in W$, $a_w = \mathrm{acl}(\bigcup\{a_{\{i\}} \mid i \in w\})$.
- We say T has *n-amalgamation* if any i.p. functor $a : \mathcal{P}(n)^- \to \mathcal{C}_T$ can be extended to i.p. $\hat{a} : \mathcal{P}(n) \to \mathcal{C}_T$.

3-amalgamation is type-amalgamation (the Independence Theorem). Hence any simple T has 3-amalgamation.

3-amalgamation

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4-amalgamation

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4-amalgamation over acl base set

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Example

Consider $[A]^2 = \{\{a, b\} | a \neq b \in A\}$ where A infinite. Let $B = [A]^2 \times \{0, 1\}$ where $\{0, 1\} = \mathbb{Z}_2$.
Also let $E \subseteq A \times [A]^2$ be a membership relation, and let P be a subset of B^3 such that $((w_1, \delta_1)(w_2, \delta_2)(w_3, \delta_3)) \in P$ iff there are distinct $a_1, a_2, a_3 \in A$ such that for $\{i, j, k\} = \{1, 2, 3\}$, $w_i = \{a_j, a_k\}$, and $\delta_1 + \delta_2 + \delta_3 = 0$.
Let $M = (A, [A]^2, B;\ E, P;\ \mathrm{Pr}_1 : B \to [A]^2)$. Then M is stable.

The stable example does not have 4-amalgamation over $\emptyset = \mathrm{acl}(\emptyset)$.

Why

Note first that $\mathrm{dcl}(\emptyset) = \mathrm{acl}(\emptyset)$, and for $a \in A$, $\mathrm{dcl}(a) = \mathrm{acl}(a)$. Now choose distinct $a_1, a_2, a_3, a_4 \in A$. For $\{i, j, k\} \subseteq \{1, 2, 3, 4\}$, fix an enumeration $\overline{a_i a_j} = (b_{ij}, \ldots)$ of $\mathrm{acl}(a_i a_j)$ where $b_{ij} = (\{a_i, a_j\}, \delta) \in B = [A]^2 \times \{0, 1\}$. Let $r_{ij}(x_{ij}) = \mathrm{tp}(\overline{a_i a_j})$, and let x^1_{ij} be the variable for b_{ij}. Note that $b_{ij} = (\{a_i, a_j\}, \delta)$ and $b'_{ij} = (\{a_i, a_j\}, \delta + 1)$ have the same type over $a_i a_j$. Hence there is $(\overline{a_i a_j})' = (b'_{ij}, \ldots)$ also realizing $r_{ij}(x_{ij})$. Therefore we have complete types $r_{ijk}(x_{ijk})$, $r'_{ijk}(x'_{ijk})$ both extending $r_{ij}(x_{ij}) \cup r_{ik}(x_{ik}) \cup r_{jk}(x_{jk})$ realized by some enumerations of $\mathrm{acl}(a_i a_j a_k)$ such that $P(x^1_{ij} x^1_{ik} x^1_{jk}) \in r_{ijk}$ whereas $\neg P(x^1_{ij} x^1_{ik} x^1_{jk}) \in r'_{ijk}$. Then it is easy to see that $r_{123} \cup r_{124} \cup r_{134} \cup r'_{234}$ is inconsistent.

Resolution

In his recent preprint [http://arxiv.org/abs/math/0603413v1], Hrushovski showed that if $\mathcal{M} \models T$ is stable, then there is $CM^* \models T^*$ in $\mathcal{L}^*(\supseteq \mathcal{L})$ such that $\mathcal{M}$ is *stably embedded* into $\mathcal{M}^*$, and $\mathcal{M}^*$ has n-amalgamation over *any acl* bases.
We may write $\mathcal{M}^*$ as $\mathcal{M}^{\mathrm{geq}}$.
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Open Problem

Can we construct such $\mathcal{M}^*$ for simple T? If yes, then possibly we can remove the assumption of 4-amalgamation in the group configuration theorem.

In the following notions of n-simplicity and $K(n)$-simplicity, for convenience, we use imprecise definitions good enough however representing the essence of notions.
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Fact

Let $I = \langle a_i | \ i \in \omega \rangle$ be Morley, and let $b \mathop{\smile\!\!\!\!\!\mid} a_0$. Then there is $b' \equiv_{a_0} b$ such that I is Morley over b' and $b' \mathop{\smile\!\!\!\!\!\mid} I$.

Above fact is a particular case of 3-amalgamation (IT), and used crucially in showing IT for simple T.

Definition

- T is $K(n)$-simple if for $k \leq n$ and any Morley $I = \langle a_i | \ i \in \omega \rangle$, whenever $I_k = \langle a_i | i < k \rangle$ is Morley over b with $b \mathop{\smile\!\!\!\!\!\mid} I_k$, there is $b' \equiv_{I_k} b$ such that I is Morley over b' and $b' \mathop{\smile\!\!\!\!\!\mid} I$.

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- We say T has n-CA if T has k-amalgamation for all $k \leq n$.

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Both n-simplicity, $K(n)$-simplicity are particular cases of $(n+2)$-CA.

Question

Are those 3 notions equivalent?

simple = 1-simple = K(1)-simple = 3-amalgamation = 3-CA

Yes

(Kolesnikov) 2-simple = K(2)-simple = 4-amalgamation = 4-CA

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Yes and No

(K, Kolesnikov, Tsuboi) Yes:
n-simple $= (n+2)$-CA

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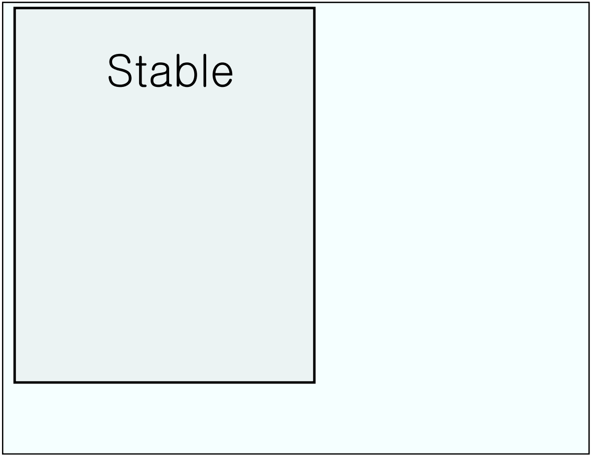
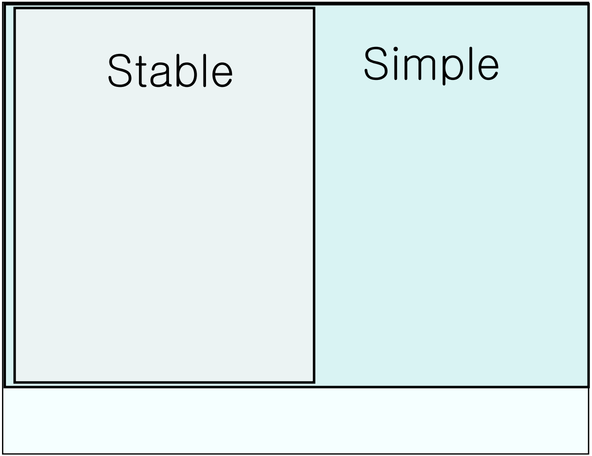
(K, Kolesnikov, Tsuboi) Yes:
n-simple = $(n+2)$-CA

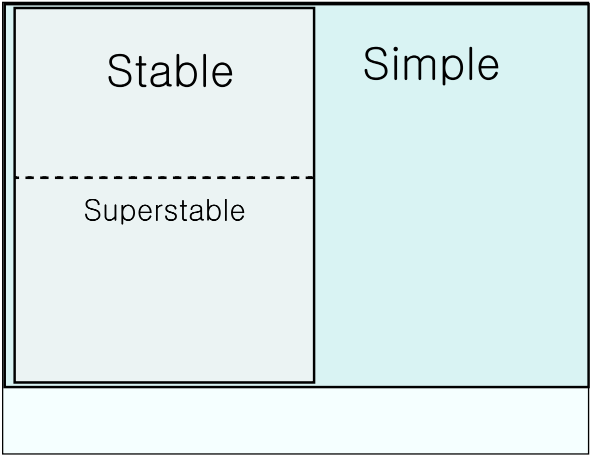
No: For each $n \geqslant 3$, there is an example of $K(n)$-simple but not having $(n+2)$-CA.
$\mathcal{L} = \{R\}$, R is a n-ary relation.

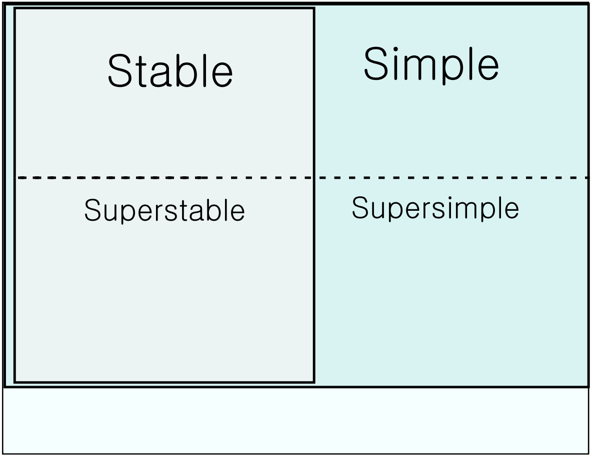
$$\mathcal{K} := \{A \mid A \quad \begin{array}{l} \text{is a finite } R\text{-structure;} \\ R \text{ is symmetric and irreflexive;} \\ \text{for any } A_0 \subseteq A \text{ with } |A_0| = n+1, \text{ the no. of} \\ n\text{-element subsets of } A_0 \text{ holding } R \text{ is even } \} \end{array}$$

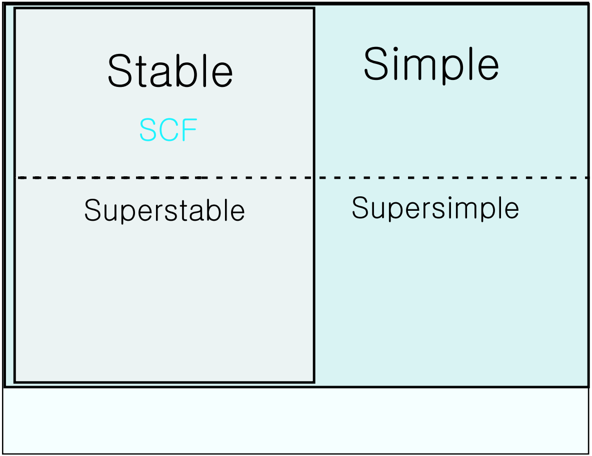
The Fraïssé limit of $\mathcal{K}$ is the simple ω-categorical example.

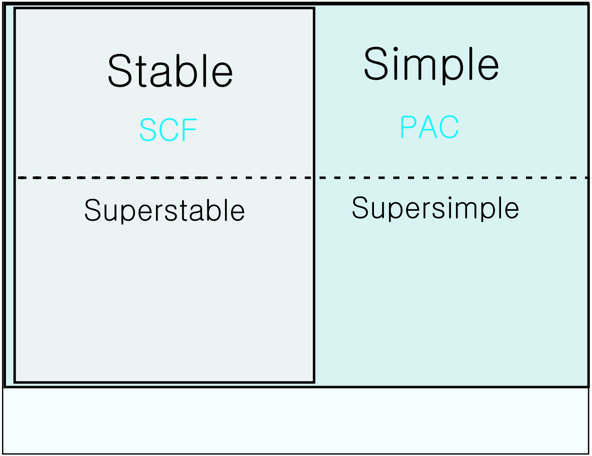

Stable

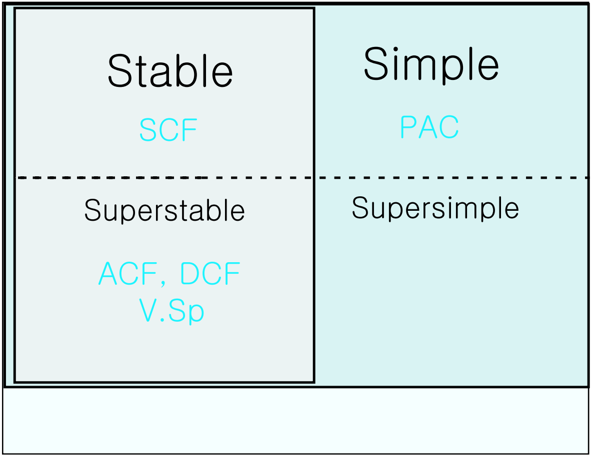

Stable
Simple

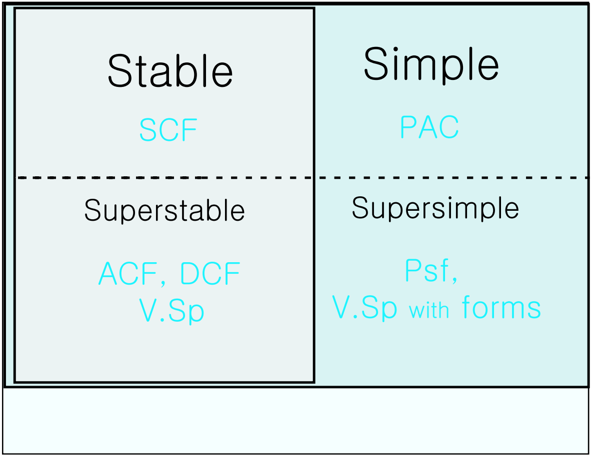

Stable
Simple
Superstable


Stable
Simple
Superstable
Supersimple


Stable
SCF
Superstable
Simple
Supersimple


Stable
SCF
Simple
PAC
Superstable
Supersimple


Stable
SCF
Simple
PAC
Superstable
ACF, DCF
V.Sp
Supersimple


Stable
SCF
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Psf,
V.Sp with forms

Definition

F a field.

- F is said to be PAC if any absolutely irreducible algebraic set V defined over F has F-rational point.
- F is *pseudo-finite* if it is an infinite field of the theory of all finite fields.
- An extension field E of F is *separable* if it is algebraic over F, and for each $e \in E$, $\mathrm{irr}_F(e)$ has no multiple roots.
- F *perfect* if every algebraic extension is separable; equivalently either $\mathrm{Char}(F) = 0$ or $x \mapsto x^p$ where $p = \mathrm{Char}(F) > 0$ is onto.
- F is *bounded* for each $n > 1$, there are only finitely many separably algebraic extensions of degree n.
- F is *separably closed* if it has no proper separable extension.

Fact

F PAC.

- *(Chatzidakis) F is simple iff F is bounded.*
- *(Hrshovski, Pillay, Poizat) F is supersimple iff F is perfect and bounded.*
- *(Duret, Wood) F is stable iff F is separably closed.*

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Fact

(Ax)

A field F is pseuo-finite iff

1. *it is perfect*
2. *it is PAC, and*
3. *for each n it has exactly one algebraic extension of degree n*

Hence pseuo-finite fields are unstable supersimple (of SU-rank 1).

Fact

- *(Macintyre; Cherlin, Shelah) Superstable division ring is an algebraically closed field.*
- *(Pillay, Scanlon, Wagner) Supersimple division ring is a field.*

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Open Problem

- Is any stable field separably closed?
- Is any supersimple field PAC?