Geometric Simplicity Theory

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Outline

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Again fix a solution set D of a strong type of rank 1.
Recall that if D is ω-categorical non-trivial superstable, then Zilber showed it is 1-based and a vector space over a finite field having the same geometry can definably be recovered.
Later Hrushovski extended the result.

Fact

If D is 1-based non-trivial stable, then there is a type-definable group $(G, +)$ such that $(V = G/G_0, +)$ forms a vector space over a division ring of the definable endomorphisms of V.
Moreover there $\perp$ is linear independence.

To get the group, the group configuration theorem is used.

Definition

By *a group configuration* we mean a 6-tuple $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ with a tuple e such that, for $\{i, j, k\} = \{1, 2, 3\}$,

- $f_i \in \mathrm{acl}(f_j, f_k; e)$,
- $x_i \in \mathrm{acl}(f_j, x_k; e)$,
- all other triples from C are independent over e.

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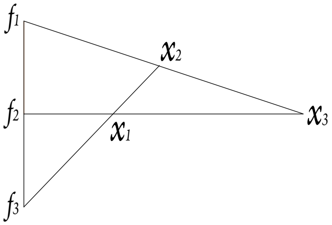

group configuration

If there is $C' = (f_1', f_2', f_3', x_1', x_2', x_3')$ such that $\mathrm{acl}(f_i e) = \mathrm{acl}(f_i' e), \mathrm{acl}(x_i e) = \mathrm{acl}(x_i' e)$, then C' is also a group configuration over e. In this case, we say C and C' are *equivalent* over e.

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Definition

We say $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ is a *group configuration from a type-definable homogeneous space* (G, X) if (G, X) and the group action of G on X are all type-definable, and $f_i \in G$, $x_i \in X$ generic elements with $f_1^{-1} f_2 = f_3$, $x_2 = f_3 x_1$, $x_3 = f_2 x_1$.

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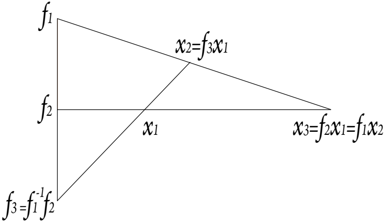

group configuration from type-definable homogeneous space

The group configuration theorem

(Hrushovski) Assume T stable. A group configuration $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ is given. Then it is equivalent to some group configuration from a type-definable homogeneous space (G, X) over $e(\bigcup C)$.

ω-categoricity or stationarity is strongly used.

What can we say in the context of simple theories.

We may additionally assume T has Q.E.
Let $\mathcal{C}_T$ be a category of the algebraically closed substructures of $\mathcal{M}$. Recall that any poset is a category.
For $n \in \omega$, write $\mathcal{P}(n)^- := \mathcal{P}(n) - \{n\}$.

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Definition

- A functor $a : W(\subseteq \mathcal{P}(n)) \to \mathcal{C}_T$ is said to be *independence preserving (i.p.)* if
  1. for any $w_0, w_1 \subseteq w \in W$, $a_{w_0} \downarrow_{a_{w_0 \cap w_1}} a_{w_1}$ holds within a_w;
  2. for $w \in W$, $a_w = \operatorname{acl}(\bigcup\{a_{\{i\}} | \ i \in w\})$.
- We say T has *n-amalgamation* if any i.p. functor $a : \mathcal{P}(n)^- \to \mathcal{C}_T$ can be extended to i.p. $\hat{a} : \mathcal{P}(n) \to \mathcal{C}_T$.

3-amalgamation is type-amalgamation (the Independence Theorem). Hence any simple T has 3-amalgamation.

3-amalgamation

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4-amalgamation

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Let $\{A_0, A_1, A_2\}$ be independent over $A (\subseteq A_i = \mathrm{acl}(A_i))$.
For $\{i, j, k\} = \{0, 1, 2\}$, let $c_i \equiv_{A_j} c_k$ and $c_i \mathop{\smile\!\!\!\!\!\!|}_A A_j A_k$.
Then there is $c \equiv_{A_j A_k} c_i$ such that $c \mathop{\smile\!\!\!\!\!\!|}_A A_0 A_1 A_2$.

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Why

Let M be the random graph in $\mathcal{L} = \{R\}$. Choose distinct $a_i, b_i, d_i \in M$ and imaginary elements $c_i = \{b_i, d_i\}$ $(i = 0, 1, 2)$. We can additionally assume that $R(b_2, a_0) \wedge R(d_2, a_1) \wedge \neg R(b_2, a_1) \wedge \neg R(d_2, a_0)$ and $\mathrm{tp}(b_2 d_2; a_0 a_1) = \mathrm{tp}(b_1 d_1; a_0 a_2) = \mathrm{tp}(b_0 d_0; a_1 a_2)$. Now it follows that $\mathrm{Ltp}(c_1/a_0) = \mathrm{Ltp}(c_2/a_0), \mathrm{Ltp}(c_0/a_1) = \mathrm{Ltp}(c_2/a_1)$ and $\mathrm{Ltp}(c_0/a_2) = \mathrm{Ltp}(c_1/a_2)$. However $\mathrm{Ltp}(c_0/a_1 a_2), \mathrm{Ltp}(c_1/a_0 a_2), \mathrm{Ltp}(c_2/a_0 a_1)$ have no common realization.

Correct definition for 4-amalgamation is then the one defined using functors.

Equivalently: Let $\{A_0, A_1, A_2\}$ be independent over $A(\subseteq A_i = \mathrm{acl}(A_i))$.
For $\{i,j,k\} = \{0,1,2\}$, let $\overrightarrow{\mathrm{acl}(c_iA_j)} \equiv_{A_j} \overrightarrow{\mathrm{acl}(c_kA_j)}$ and $c_i \mathop{\smile\!\!\!\!\!\!\!\;|}_A A_jA_k$, where $\overrightarrow{\mathrm{acl}(c_iA_j)}$ is a subsequence of a fixed sequence $\overrightarrow{\mathrm{acl}(c_iA_jA_k)}$.
Then there are c and an enumeration $\overrightarrow{\mathrm{acl}(cA_0A_1A_2)}$ such that $\overrightarrow{\mathrm{acl}(cA_jA_k)} \equiv_{\overrightarrow{\mathrm{acl}(A_jA_k)}} \overrightarrow{\mathrm{acl}(c_iA_jA_k)}$ and $c \mathop{\smile\!\!\!\!\!\!\!\;|}_A A_0A_1A_2$.

Theorem

(Ben-Yaacov, Tomasic, Wagner) T simple. A group configuration $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ *is given. Then it is equivalent to some group configuration from an* invariant *homogeneous space* (G, X) *over* $e(\mathop{\smile}\limits^{|} C)$.

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Theorem

(de Piro, K, J. Millar: JML '07) Assume T simple having 4-amalgamation. A group configuration $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ is given. Then there is a type-definable group G having generics equivalent to f_i.

Corollary

T 1-based nontrivial having 4-amalgamation. Then there is a type-definable group $(G, +)$ such that $(V = G/G_0, +)$ forms a vector space over a division ring of the type-definable endomorphisms of V, and $\smile$ is linear independence there.

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$G = G(x)$ a group type-defined over $\emptyset$ in simple T.
There is a smallest $\emptyset$-type-definable group G^0 having bounded index in G. G^0 must be normal.

Example

Let $G = (G(= \mathbb{F}_2^{\omega}), +;\ \langle , \rangle;\ \mathbb{F}_2)$, where
$\langle (a_i|\ i < \omega), (b_i|\ i < \omega) \rangle := \sum_{i<\omega} a_i b_i \in \mathbb{F}_2$.
For $A \subseteq V$, $G^0_A = \{g \in G|\ \langle g, a \rangle = 0 \text{ for all } a \in A\}$.

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$\langle (a_i \mid i < \omega), (b_i \mid i < \omega) \rangle := \sum_{i<\omega} a_i b_i \in \mathbb{F}_2$.
For $A \subseteq V$, $G_A^0 = \{g \in G \mid \langle g, a \rangle = 0 \text{ for all } a \in A\}$.

For the rest, G is 1-based.

Now reset G by G^0. Hence G has no proper $\emptyset$-type-definable subgroup of bounded index.

Theorem

(Wagner)

- $G' = [G, G]$, *the commutator subgroup of* G *is bounded. Indeed,* $G' \leq Z(G)$.
- *Hence* G *is bounded-by-abelian, and* $G_0 = G \cap \mathrm{acl}(\emptyset)(\geq G')$ *is normal.*

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Proof: Note that $Z(G)$ need not be type-definable where as $\tilde{Z} = \tilde{Z}(G) := \{g \in G | \ [G : C_G(g)] \text{ is bounded } \}$ is.

Claim 1) $[\tilde{Z}, \tilde{Z}]$ is bounded, and $\leq Z(G)$: For $g \in \tilde{Z}$, $|[g, G]| = |[G, g]| = |[G : C_G(g)]|$ is bounded, hence $[g, G] \subseteq \mathrm{acl}(g)$. Then for $h \downarrow g \in \tilde{Z}$, $[h, g] \in \mathrm{acl}(g) \cap \mathrm{acl}(h) = \mathrm{acl}(\emptyset)$. For arbitrary $h, g \in \tilde{Z}$, there is $h_i \downarrow g$ such that $h = h_1.h_2$. The repeated applications yield Claim 1. Moreover for $h \in \tilde{Z}'$, $[G, h] \subseteq \tilde{Z}'$. $\therefore$ $C_G(h) = G$, and $\tilde{Z}' \subseteq Z(G)$.

Claim 2) $\tilde{Z}$ has bounded index in G. Hence $\tilde{Z} = G$: Note that for $h, g, g' \in G$,

$$\begin{aligned} & h^g = h^{g'} && \text{iff} \quad hg'g^{-1} = g'g^{-1}h \\ \text{iff} \quad & h \in C_G(g'g^{-1}) && \text{iff} \quad (h, h^g) = (h, h^{g'}) \in H_g \cap H_{g'}; \end{aligned}$$

where $H_g = \{(h, h^g) | \ h \in G\}$ be a type-definable subgroup of $G \times G$. It follows H_g and $H_{g'}$ are commensurate iff $[G : C_G(g'.g^{-1})]$ bounded iff $g'.g^{-1} \in \tilde{Z}(G)$.

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Fact

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$\therefore$ Claim 2 follows. $\square$

We have shown that G/G_0 is abelian.
Additionally assume $SU(G) = 1$.

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Definition

By an *endogeny* of G/G_0, we mean an endomorphism f of G/G_0 such that the graph of f is *type-definably induced* over $\mathrm{acl}(\emptyset)$, i.e. there is a type-definable subset S_f over $\mathrm{bdd}(\emptyset)$ of $G \times G$ such that

$$\{(a + G_0, b + G_0) | \ (a, b) \in S_f\}$$

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Let $\mathrm{End}(G) = \{f | \ f \text{ is an endogeny of } G/G_0\}$. With addition and composition, it is easy to check $\mathrm{End}(G)$ forms a division ring.

Structure Theorem

(Wagner) For $a_0, \ldots, a_n \in G \setminus G_0$, the following are equivalent.

1. $\{a_0, \ldots, a_n\}$ *is dependent.*
2. $a_i \in \mathrm{acl}(\{a_0, \ldots, a_n\} \setminus \{a_i\})$ *for some* $i \leq n$.
3. *There are* $f_0, \ldots, f_n \in \mathrm{End}(G)$ *not all zero such that* $f_0(a_0 + G_0) + \ldots + f_n(a_n + G_0) = 0$ *in* G/G_0.

Hence $(V, +)$ *with* $V := G/G_0$ *forms a vector space over the division ring* $\mathrm{End}(G)$, *where* $\perp$ =*linear independence.*

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Fact

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- *(Hrushovski, Pillay) (T stable.) Any type-definable subset of* G^n *is a finite Boolean combination of cosets of type-definable subgroups of* G^n.

Proof Sketch of Structure Theorem: Suffices to show (1)⇒(3). By Fact, there is $H(\leq G^{n+1})$ a type-definable subgroup over $\mathrm{acl}(\emptyset)$ such that $\mathrm{stp}(a_0, ..., a_n)$ is generic for some $H + \bar{h}$.

For $i < n$, let

$$F_i := \{(x_i, x_n) \in G \times G |\ (0, ..., 0, x_i, 0, ..., 0, x_n) \in H\}.$$

Then one can show

$$f_i := \{(a + G_0, b + G_0) |\ (a, b) \in F_i\}$$

is the graph of a nonzero endogeny of G/G_0, and

$$f_0(a_0 + G_0) + ... + f_{n-1}(a_{n-1} + G_0) = -a_n + G_0.$$