Geometric Simplicity Theory

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ALC 10

Dept. Math. Yonsei University
http://math.yonsei.ac.kr/bkim
September 1-6, 2008
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Again fix a solution set $D$ of a strong type of rank 1. Recall that if $D$ is $\omega$-categorical non-trivial superstable, then Zilber showed it is 1-based and a vector space over a finite field having the same geometry can definably be recovered. Later Hrushovski extended the result.

**Fact**

*If $D$ is 1-based non-trivial stable, then there is a type-definable group $(G, +)$ such that $(V = G/G_0, +)$ forms a vector space over a division ring of the definable endomorphisms of $V$. Moreover there is linear independence.*
To get the group, the group configuration theorem is used.

**Definition**

By a **group configuration** we mean a 6-tuple $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ with a tuple $e$ such that, for $\{i, j, k\} = \{1, 2, 3\}$,

- $f_i \in \text{acl}(f_j, f_k; e)$,
- $x_i \in \text{acl}(f_j, x_k; e)$,
- all other triples from $C$ are independent over $e$. 
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- all other triples from $C$ are independent over $e$. 

![group configuration](attachment:image.png)
If there is $C' = (f'_1, f'_2, f'_3, x'_1, x'_2, x'_3)$ such that
$\text{acl}(f_i e) = \text{acl}(f'_i e), \text{acl}(x_i e) = \text{acl}(x'_i e)$, then $C'$ is also a group configuration over $e$. In this case, we say $C$ and $C'$ are equivalent over $e$. 
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**Definition**

We say $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ is a group configuration from a type-definable homogeneous space $(G, X)$ if $(G, X)$ and the group action of $G$ on $X$ are all type-definable, and $f_i \in G, x_i \in X$ generic elements with $f_1^{-1}f_2 = f_3, x_2 = f_3x_1, x_3 = f_2x_1$. 
If there is $C' = (f'_1, f'_2, f'_3, x'_1, x'_2, x'_3)$ such that $\text{acl}(f_i e) = \text{acl}(f'_i e), \text{acl}(x_i e) = \text{acl}(x'_i e)$, then $C'$ is also a group configuration over $e$. In this case, we say $C$ and $C'$ are equivalent over $e$.

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The group configuration theorem

(Hrushovski) Assume $T$ stable. A group configuration $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ is given. Then it is equivalent to some group configuration from a type-definable homogeneous space $(G, X)$ over $e(\downarrow C)$.

$\omega$-categoricity or stationarity is strongly used.

What can we say in the context of simple theories.
We may additionally assume $T$ has Q.E.
Let $C_T$ be a category of the algebraically closed substructures of $M$. Recall that any poset is a category.
For $n \in \omega$, write $\mathcal{P}(n)^- := \mathcal{P}(n) - \{n\}$. 
We may additionally assume $T$ has Q.E. Let $C_T$ be a category of the algebraically closed substructures of $M$. Recall that any poset is a category. For $n \in \omega$, write $\mathcal{P}(n)^- := \mathcal{P}(n) - \{n\}$.

**Definition**

- A functor $a : W(\subseteq \mathcal{P}(n)) \to C_T$ is said to be independence preserving (i.p.) if
  1. for any $w_0, w_1 \subseteq w \in W$, $a_{w_0} \downarrow_{a_{w_0} \cap w_1} a_{w_1}$ holds within $a_w$;
  2. for $w \in W$, $a_w = acl(\bigcup \{a_i \mid i \in w\})$.

- We say $T$ has $n$-amalgamation if any i.p. functor $a : \mathcal{P}(n)^- \to C_T$ can be extended to i.p. $\hat{a} : \mathcal{P}(n) \to C_T$.

3-amalgamation is type-amalgamation (the Independence Theorem). Hence any simple $T$ has 3-amalgamation.
3-amalgamation
4-amalgamation
Let \( \{A_0, A_1, A_2\} \) be independent over \( A(\subseteq A_i = \text{acl}(A_i)) \).
For \( \{i, j, k\} = \{0, 1, 2\} \), let \( c_i \equiv_{A_j} c_k \) and \( c_i \downarrow_A A_j A_k \).
Then there is \( c \equiv_{A_j A_k} c_i \) such that \( c \downarrow_A A_0 A_1 A_2 \).
Let \( \{A_0, A_1, A_2\} \) be independent over \( A(\subseteq A_i = acl(A_i)) \).
For \( \{i, j, k\} = \{0, 1, 2\} \), let \( c_i \equiv_{A_j} c_k \) and \( c_i \downarrow_A A_j A_k \).
Then there is \( c \equiv_{A_j A_k} c_i \) such that \( c \downarrow_A A_0 A_1 A_2 \).

The above candidate for the definition of 4-amalgamation does not work!, although it looks like a very natural one for higher dimensional IT.
Let \( \{ A_0, A_1, A_2 \} \) be independent over \( A(\subseteq A_i = acl(A_i)) \).
For \( \{ i, j, k \} = \{ 0, 1, 2 \} \), let \( c_i \equiv_{A_j} c_k \) and \( c_i \upharpoonright A_j A_k \).
Then there is \( c \equiv_{A_j A_k} c_i \) such that \( c \upharpoonright A A_0 A_1 A_2 \).

The above candidate for the definition of 4-amalgamation does not work!, although it looks like a very natural one for higher dimensional IT.

**Why**

Let \( M \) be the random graph in \( L = \{ R \} \). Choose distinct \( a_i, b_i, d_i \in M \) and imaginary elements \( c_i = \{ b_i, d_i \} \) \((i = 0, 1, 2) \).
We can additionally assume that
\[
R(b_2, a_0) \land R(d_2, a_1) \land \neg R(b_2, a_1) \land \neg R(d_2, a_0)
\]
and
\[
\text{tp}(b_2 d_2; a_0 a_1) = \text{tp}(b_1 d_1; a_0 a_2) = \text{tp}(b_0 d_0; a_1 a_2).
\]
Now it follows that \( \text{Ltp}(c_1/a_0) = \text{Ltp}(c_2/a_0), \text{Ltp}(c_0/a_1) = \text{Ltp}(c_2/a_1) \) and
\( \text{Ltp}(c_0/a_2) = \text{Ltp}(c_1/a_2) \). However
\( \text{Ltp}(c_0/a_1 a_2), \text{Ltp}(c_1/a_0 a_2), \text{Ltp}(c_2/a_0 a_1) \) have no common realization.
Correct definition for 4-amalgamation is then the one defined using functors.

Equivalently: Let \( \{A_0, A_1, A_2\} \) be independent over \( A(\subseteq A_i = \text{acl}(A_i)) \).

For \( \{i,j,k\} = \{0, 1, 2\} \), let \( \text{acl}(c_i A_j) \equiv_{A_j} \text{acl}(c_k A_j) \) and \( c_i \downarrow_A A_j A_k \), where \( \text{acl}(c_i A_j) \) is a subsequence of a fixed sequence \( \text{acl}(c_i A_j A_k) \).

Then there are \( c \) and an enumeration \( \text{acl}(cA_0 A_1 A_2) \) such that \( \text{acl}(cA_j A_k) \equiv_{\text{acl}(A_j A_k)} \text{acl}(c_i A_j A_k) \) and \( c \downarrow_A A_0 A_1 A_2 \).
Theorem

(Ben-Yaacov, Tomasic, Wagner) \( T \) simple. A group configuration \( C = (f_1, f_2, f_3, x_1, x_2, x_3) \) is given. Then it is equivalent to some group configuration from an invariant homogeneous space \((G, X)\) over \(e(\downarrow C)\).
Theorem

(Ben-Yaacov, Tomasic, Wagner) $T$ simple. A group configuration $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ is given. Then it is equivalent to some group configuration from an invariant homogeneous space $(G, X)$ over $e(\downarrow C)$.

Theorem

(de Piro, K, J. Millar: JML '07) Assume $T$ simple having 4-amalgamation. A group configuration $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ is given. Then there is a type-definable group $G$ having generics equivalent to $f_i$. 
Corollary

$T$-1-based nontrivial having 4-amalgamation. Then there is a type-definable group $(G, +)$ such that $(V = G/G_0, +)$ forms a vector space over a division ring of the type-definable endomorphisms of $V$, and $\downarrow$ is linear independence there.
Corollary

Let $T$ be 1-based nontrivial having 4-amalgamation. Then there is a type-definable group $(G, +)$ such that $(V = G/G_0, +)$ forms a vector space over a division ring of the type-definable endomorphisms of $V$, and $\downarrow$ is linear independence there.

The group configuration theorem

(K) Assume $T$ simple having 4-amalgamation. A group configuration $C = (f_1, f_2, f_3, x_1, x_2, x_3)$ is given. Then it is equivalent to some group configuration from a type-definable homogeneous space $(G, X)$ over $e(\downarrow C)$. 
$G = G(x)$ a group type-defined over $\emptyset$ in simple $T$. There is a smallest $\emptyset$-type-definable group $G^0$ having bounded index in $G$. $G^0$ must be normal.

**Example**

Let $G = (G(= \mathbb{F}_2^\omega), +; \langle, \rangle; \mathbb{F}_2)$, where 
\[\langle (a_i | i < \omega), (b_i | i < \omega) \rangle := \sum_{i<\omega} a_i b_i \in \mathbb{F}_2.\]
For $A \subseteq V$, $G^0_A = \{g \in G | \langle g, a \rangle = 0 \text{ for all } a \in A\}$. 
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Let $G = (G(= F_2^\omega), +; \langle, \rangle; F_2)$, where
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\langle (a_i \mid i < \omega), (b_i \mid i < \omega) \rangle := \sum_{i<\omega} a_i b_i \in F_2.
\]
For $A \subseteq V$, $G^0_A = \{g \in G \mid \langle g, a \rangle = 0 \text{ for all } a \in A\}$.

**For the rest, $G$ is 1-based.**

Now reset $G$ by $G^0$. Hence $G$ has no proper $\emptyset$-type-definable subgroup of bounded index.
Theorem

(Wagner)

- $G' = [G, G]$, the commutator subgroup of $G$ is bounded. Indeed, $G' \leq Z(G)$.
- Hence $G$ is bounded-by-abelian, and $G_0 = G \cap \text{acl}(\emptyset)(\geq G')$ is normal.
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- Hence $G$ is bounded-by-abelian, and $G_0 = G \cap \text{acl}(\emptyset)(\geq G')$ is normal.

Proof: Note that $Z(G)$ need not be type-definable where as $	ilde{Z} = \tilde{Z}(G) := \{g \in G| [G : C_G(g)]$ is bounded } is.

Claim 1) $[\tilde{Z}, \tilde{Z}]$ is bounded, and $\leq Z(G)$: For $g \in \tilde{Z}$, 
$|[g, G]| = |[G, g]| = |[G : C_G(g)]|$ is bounded, hence 
$[g, G] \subseteq \text{acl}(g)$. Then for $h \downarrow g \in \tilde{Z}$, 
$[h, g] \in \text{acl}(g) \cap \text{acl}(h) = \text{acl}(\emptyset)$. For arbitrary $h, g \in \tilde{Z}$, there is 
$h_i \downarrow g$ such that $h = h_1.h_2$. The repeated applications yield Claim 1. Moreover for $h \in \tilde{Z}'$, $[G, h] \subseteq \tilde{Z}'$. \therefore $C_G(h) = G$, and 
$\tilde{Z}' \subseteq Z(G)$. 
Claim 2) \(\tilde{Z}\) has bounded index in \(G\). Hence \(\tilde{Z} = G\): Note that for \(h, g, g' \in G\),

\[h^g = h^{g'} \quad \text{iff} \quad hg'g^{-1} = g'g^{-1}h \]

\[\text{iff} \quad h \in C_G(g'g^{-1}) \quad \text{iff} \quad (h, h^g) = (h, h^{g'}) \in H_g \cap H_{g'};\]

where \(H_g = \{(h, h^g) | h \in G\}\) be a type-definable subgroup of \(G \times G\). It follows \(H_g\) and \(H_{g'}\) are commensurate iff

\([G : C_G(g'.g^{-1})]\) bounded iff \(g'.g^{-1} \in \tilde{Z}(G)\).
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**Fact**

*(1-based) Any type-definable subgroup of \(G\) is commensurate with 
one over \(\text{acl}(\emptyset)\).*
Claim 2) \( \tilde{Z} \) has bounded index in \( G \). Hence \( \tilde{Z} = G \): Note that for \( h, g, g' \in G \),

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\begin{align*}
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**Fact**

*(1-based) Any type-definable subgroup of \( G \) is commensurate with one over \( acl(\emptyset) \).*

\( \therefore \) Claim 2 follows. \( \square \)
We have shown that $G/G_0$ is abelian. Additionally assume $SU(G) = 1$. 
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**Definition**

By an *endogeny* of $G/G_0$, we mean an endomorphism $f$ of $G/G_0$ such that the graph of $f$ is *type-definably induced* over $acl(\emptyset)$, i.e. there is a type-definable subset $S_f$ over $bdd(\emptyset)$ of $G \times G$ such that

$\{(a + G_0, b + G_0) | (a, b) \in S_f\}$

forms the graph of $f$. 
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forms the graph of $f$.

Let $\text{End}(G) = \{f | f$ is an endogeny of $G/G_0\}$. With addition and composition, it is easy to check $\text{End}(G)$ forms a division ring.
Structure Theorem

(Wagner) For $a_0, \ldots, a_n \in G \setminus G_0$, the following are equivalent.

1. $\{a_0, \ldots, a_n\}$ is dependent.
2. $a_i \in acl(\{a_0, \ldots, a_n\} \setminus \{a_i\})$ for some $i \leq n$.
3. There are $f_0, \ldots, f_n \in \text{End}(G)$ not all zero such that $f_0(a_0 + G_0) + \cdots + f_n(a_n + G_0) = 0$ in $G/G_0$.

Hence $(V, +)$ with $V := G/G_0$ forms a vector space over the division ring $\text{End}(G)$, where $\downarrow = \text{linear independence}$. 
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(Wagner) Any strong type in $G^n$ is generic for some coset of a type-definable subgroup of $G^n$. 
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(Hrushovski, Pillay) (T stable.) Any type-definable subset of $G^n$ is a finite Boolean combination of cosets of type-definable subgroups of $G^n$. 
Proof Sketch of Structure Theorem: Suffices to show (1) $\Rightarrow$ (3). By Fact, there is $H(\leq G^{n+1})$ a type-definable subgroup over acl($\emptyset$) such that stp($a_0, ..., a_n$) is generic for some $H + \bar{h}$.

For $i < n$, let

$$F_i := \{(x_i, x_n) \in G \times G | (0, ..., 0, x_i, 0, ..., 0, x_n) \in H\}.$$  

Then one can show

$$f_i := \{(a + G_0, b + G_0) | (a, b) \in F_i\}$$  

is the graph of a nonzero endogeny of $G/G_0$, and

$$f_0(a_0 + G_0) + ... + f_{n-1}(a_{n-1} + G_0) = -a_n + G_0.$$